FORMULAS FOR LAGRANGIAN AND ORTHOGONAL DEGENERACY LOCI;

The $\widetilde{Q}$-Polynomials Approach

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Contents

Introduction

In this paper we give formulas for the fundamental classes of Schubert subschemes in Lagrangian and orthogonal Grassmannians of maximal rank subbundles as well as some globalizations of them. Our motivation to deal with this subject came essentially from 3 examples where such degeneracy loci appear in algebraic geometry: 1^o The Brill-Noether loci for Prym varieties, as defined by Welters [W], 2^o The loci of curves with sufficiently many theta characteristics, as considered by Harris [Har], 3^o Some "higher" Brill-Noether loci in the moduli spaces of higher rank vector bundles over curves, considered by Bertram and Feinberg [B-F] and, independently, by Mukai [Mu].

[1] Research carried out during the author's stay at the Max-Planck-Institut für Mathematik as a fellow of the Alexander von Humboldt Stiftung.

The common denominator of these 3 situations is a simple and beautiful construction of Mumford [M]. With a vector bundle over a curve equipped with a nondegenerate quadratic form with values in the sheaf of 1-differentials, Mumford associates an even dimensional vector space endowed with a nondegenerate quadratic form and 2 maximal isotropic subspaces such that the space of global sections of the initial bundle is the intersection of the two isotropic subspaces. A globalization of this construction allows one to present in a similar way the varieties in 1^o, 2^o above as loci where two isotropic rank n subbundles of a certain rank $2n$ bundle equipped with a quadratic nondegenerate form, intersect in dimension exceeding a given number. On the other hand, the locus in 3^o admits locally this kind of presentation using an appropriate symplectic form.

These varieties are particular cases of Schubert subschemes in Lagrangian and orthogonal Grassmannian bundles and their globalizations. The formulas for such loci are the main theme of this paper. More specifically, given a vector bundle V on a variety X, endowed with a nondegenerate symplectic or orthogonal form, we pick E and $F_1 \subset F_2 \subset ... \subset F_n = F$ - isotropic subbundles of V ($rank\ E = n, rank\ F_i = i$) and for a given sequence $1 \leqslant a_1 < ... < a_k \leqslant n$ we look at the locus:

$$D(a.) := \left\{ \ x \in X |\ dim\left(E \cap F_{a_p}\right)_x \geqslant p,\ p = 1, ..., k \right\}.$$

We distinguish three cases:
1. Lagrangian: $rank\ V = 2n$, the form is symplectic;
2. odd orthogonal: $rank\ V = 2n+1$, the form is orthogonal;
3. even orthogonal: $rank\ V = 2n$, the form is orthogonal.

(In the latter case the definition of $D(a.)$ must be slightly modified - see Section 9.)

Let us remark that the loci $D(a.)$ (for the Lagrangian case) admit an important specialization to the loci introduced by Ekedahl and Oort in the moduli space of abelian varieties with fixed dimension and polarization, in characteristic p (see, e.g. [O], the references therein and [E-vG]). This comes from certain filtrations on the de Rham cohomology defined with the help of the Frobenius- and "Verschiebung"-maps. The formulas of the present paper are well suited to computations of the fundamental classes of such loci in the Chow groups of the moduli spaces - for details see a forthcoming paper by T. Ekedahl and G. van der Geer [E-vG].

The goal of this paper is to give an algorithm for computing the fundamental classes of $D(a.)$ as polynomials in the Chern classes of E and F_i. Formulas given here can be thought of as Lagrangian and orthogonal analogs of the formulas due independently to Kempf-Laksov [K-L] and Lascoux [L] (notice, however, that the formulas given in [K-L] are proved under a weaker assumption of "expected" dimension). The strategy here is similar to that in [K-L] and uses a certain desingularization of Lagrangian and orthogonal Schubert subschemes. The main technical difference between [K-L] and our approach is that the class of our desingularization in the Lagrangian and orthogonal cases seems - to the best of our knowgledge and attempts - not to be given by the top Chern class of some vector bundle. This makes a significant difference and additional difficulty. We overcome this obstruction by using the classes of the *diagonals* of Isotropic Grassmannian bundles. To establish formulas for the classes of these diagonals, we use the results of [P1,2] where the classes of Schubert subvarieties in Lagrangian and orthogonal Grassmannians were described with the help of a family of symmetric polynomials introduced by I.Schur [S] in 1911 and then forgotten for a long time. The importance of these Q- and P-polynomials to algebraic geometry was discovered by the first named author in [P1] and then developed in [P2]. In fact in [P2, Sect.6], a variant of these polynomials was used to give a full description of Schubert Calculus on Grassmannians of maximal dimensional isotropic subspaces associated with a nondegenerate symplectic and orthogonal form. These familes of symmetric polynomials are called $\widetilde{Q}$- and $\widetilde{P}$-polynomials in the present paper. The results of [P2, Sect.6], recalled in Theorem 2.1 below, are a natural source of the ubiquity of $\widetilde{Q}$- and $\widetilde{P}$-polynomials in various formulas of this paper. As a general rule, these are $\widetilde{Q}$-polynomials that appear in the Lagrangian case and $\widetilde{P}$-polynomials that appear in the orthogonal cases.

In general, our approach gives an efficient algorithm for finding formulas for Lagrangian and orthogonal Schubert subschemes. In several cases, however, we are able to give compact expressions. At first, these are the cases of one (i.e. $k = 1$) and two Schubert conditions (the case of one Schubert condition is usually referred to as a *special* Schubert subscheme). The corresponding formulas are given in Theorem 6.1 and 7.4.

The derivation of those formulas uses a formula for the Gysin pushforward of $\widetilde{Q}$- and $\widetilde{P}$-polynomials (Theorems 5.10, 5.14, 5.16) in Isotropic Grassmannian bundles. For instance, in the Lagrangian case, $\pi : LG_nV \to X$ with the tautological subbundle R, the element $\widetilde{Q}_I R^\vee$ has a nonzero image under π_* only if each number p, $1 \leqslant p \leqslant n$, appears as a part of I with an odd multiplicity m_p. If the latter condition takes place then

$$\pi_*\widetilde{Q}_I R^\vee = \prod_{p=1}^{n} \left((-1)^p c_{2p} V\right)^{(m_p-1)/2}.$$

Occasionally, we also give formulas for Gysin pushforward of S-polynomials (Theorems 5.13, 5.15, 5.17) in Isotropic Grassmannian bundles. For example, in the Lagrangian case, the element $s_I R^\vee$ has a nonzero image under π_* only if the partition I is of the form $2J + \rho_n$ for some partition J (here, $\rho_n = (n, n-1, ..., 1)$). If $I = 2J + \rho_n$ then

$$\pi_* s_I R^\vee = s_J^{[2]} V \quad ,$$

where the right hand side is defined as follows: if $s_J = P(e.)$ is a unique presentation of s_J as a polynomial in the elementary symmetric functions e_i, $E-$ a vector bundle, then $s_J^{[2]}(E) := P$ with e_i replaced by $(-1)^i c_{2i}E$, $i = 1, 2, \ldots$.

Another case (corresponding to the Schubert condition $a. = (n-k+1, ..., n)$) that leads to compact formulas is the variety of maximal rank isotropic subbundles that intersect a fixed maximal rank isotropic subbundle, in the Grassmannian of such subbundles, in dimension exceeding a given number (Proposition 3.2 and its analogs). Thanks to the

Cohen-Macaulayness of Schubert subschemes in isotropic Grassmannians proved in [DC-L], one gets globalizations of those formulas (as well as the other ones) to more general loci. For instance, the latter case $a. = (n-k+1, ..., n)$ globalizes to the Mumford type locus discussed above where two maximal rank isotropic subbundles E and F intersect in dimension exceeding k, say.

Our formulas (see Theorems 9.1, 9.5 and 9.6) are quadratic expressions in $\widetilde{Q}$- and $\widetilde{P}$-polynomials of the subbundles. More explicitly in the corresponding cases we have

1. Lagrangian: $\sum \widetilde{Q}_I E^\vee \cdot \widetilde{Q}_{(k,k-1,\dots,1)\setminus I} F^\vee$;
2. odd orthogonal: $\sum \widetilde{P}_I E^\vee \cdot \widetilde{P}_{(k,k-1,\dots,1)\setminus I} F^\vee$;
3. even orthogonal: $\sum \widetilde{P}_I E^\vee \cdot \widetilde{P}_{(k-1,k-2,\dots,1)\setminus I} F^\vee$;

where in 1. and 2. the sum is over all subsequences I in $(k, k-1, \dots, 1)$, and in 3. the sum is over all subsequences I in $(k-1, k-2, \dots, 1)$.

Formula 3. has been recently used by C. De Concini and the first named author in [DC-P] to compute the fundamental classes of the Brill-Noether loci V^r for the Prym varieties (see [W]), thus solving a problem of Welters, left open since 1985. The formula of [DC-P] asserts that if either V^r is empty or of pure codimension $r(r+1)/2$ in the Prym variety then its fundamental class in the numerical equivalence ring, or its cohomology class is equal to

$$2^{r(r-1)/2} \prod_{i=1}^{r} ((i-1)!/(2i-1)!) \; [\Xi]^{r(r+1)/2},$$

where Ξ is the theta divisor on the Prym variety.

Finally, in the Appendix we collect a number of useful results about Quaternionic Grassmannians. We use them to reprove some results proved earlier using different methods and to show how some problems concerning Grassmannians of nonmaximal Lagrangian subspaces can be reduced to those of maximal Lagrangian subspaces; this sort of application we plan to develop elsewhere.

Some of the results of this paper were announced in [P-R1].

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Background

Several results of this paper: e.g. Proposition 3.2, its odd orthogonal analog and Proposition 3.6 as well as their globalizations in Theorems 9.1, 9.5 and 9.6 were obtained already in Spring 1993 when we tried to deduce formulas for the loci $D(a.)$ by combining the ideas of the paper of Kempf and Laksov [K-L] with the Q-polynomials technique developed in [P1,2]. These results were announced together with outlines of their proofs in [P-R1].

In summer '93, we received an e-mail message from Professor W. Fulton informing us about his (independent) work on the same subject and announcing another expressions for the loci considered in Proposition 3.2 and 3.6 of the present paper. In February '94 we obtained from Professor W. Fulton his preprints [F1,2] containing details of his e-mail announcement. Both the form of the formulas obtained as well as the approach used in [F1,2] are totally different from the content of our work and just a simple comparison of the results of [F1,2] with ours leads to very nontrivial new identities which are interesting in themselves. It would be desirable to develop, in a systematic way, the comparison of formulas given in [F1,2] from one side with those in the present paper and [P-R1] - from the other one.

Conventions

Partitions are weakly decreasing sequences of positive integers (as in [Mcd1] and are denoted by capital Roman letters (as in [L-S1]). We identify partitions with their Ferrers' diagrams. The relation "$\subset$" for partitions is induced from that for diagrams.

For a given partition $I = (i_1, i_2, \dots)$ we denote by $|I|$ (the *weight* of I) the partitioned number (i.e. the sum of all parts of I) and by $l(I)$ (the *length* of I) the number of nonzero parts of I. Moreover, $I^\sim$ denotes the dual partition of I, i.e. $I^\sim = (j_1, j_2, \dots)$ where $j_p = card\{h | i_h \geqslant p\}$, and $(i)^k$ - the partition $(i, \dots, i)$ (k-times).

Given sequences $I = (i_1, i_2, \dots)$ and $J = (j_1, j_2, \dots)$ we denote by $I \pm J$ the sequence $(i_1 \pm j_1, i_2 \pm j_2, \dots)$.

By *strict* partitions we mean those whose (positive) parts are all different.

In this paper, we denote by $s_i(E)$ the complete symmetric polynomial of degree i with variables specialized to the Chern roots of a vector bundle E.

The reader should be careful with our notion of $\widetilde{Q}$-polynomials here. Namely, since we are mainly interested in the polynomials in the Chern classes of vector bundles, we introduce $\widetilde{Q}$-polynomials given by the Pfaffian of an antisymmetric matrix whose entries are quadratic expressions in the elementary symmetric polynomials rather than in the "one row" Schur's Q-polynomials. Therefore these polynomials are different from the original Schur's Q-polynomials. Note that nonzero $\widetilde{Q}$-polynomials $\widetilde{Q}_I(x_1, \dots, x_n)$ are indexed by "usual" partitions I but the parts of these partitions cannot exceed the number of variables; on the contrary, nonzero Schur's Q-polynomials $Q_I(x_1, \dots, x_n)$ are indexed by strict partitions I only but the parts of these partitions can be bigger than the number of variables.

Also, the specialization of $\widetilde{Q}_I(x_1, \ldots, x_n)$ with (x_i) equal to the sequence of the Chern roots of a rank n vector bundle E, denoted here - accordingly - by $\widetilde{Q}_I E$, is a different cohomology class than the one associated with E in [P1] and [P2, Sect.3 and 5], and denoted by $Q_I E$ therein. (Notice, however, that the $\widetilde{Q}$-polynomials appeared already in an implicit way in [P2, Sect.6].) The reader should make a proper distinction between Schur's Q-polynomials and $\widetilde{Q}$-polynomials that are mainly used in the present paper.

By $G_n V$ we denote the usual Grassmannian (of n-dimensional subspaces in V), by $LG_n V$ - the Lagrangian Grassmannian and by $OG_n V$ - the orthogonal one. Moreover, $\mathbb{P}(V) = G_1 V$. We follow mostly [F] for the terminology in algebraic geometry. In many situations when the notation starts to be too cumbersome, we omit some pullback-indices of the induced vector bundles.

1. Schubert subschemes and their desingularizations

We start with the Lagrangian case. Let K be an arbitrary ground field.

Assume that V is a rank $2n$ vector bundle over a smooth scheme X over K equipped with a nondegenerate symplectic form. Moreover, assume that a flag $V_. : V_1 \subset V_2 \subset \ldots \subset V_n$ of Lagrangian (i.e. isotropic) subbundles w.r.t. this form is fixed, with $rank\ V_i = i$. Let $\pi : LG_n(V) \to X$ denote the Grassmannian bundle parametrizing the Lagrangian rank n subbundles of V. $G = LG_n(V)$ is endowed with the tautological Lagrangian bundle $R \subset V_G$. Given a sequence $a_. = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$ we consider in G a closed subset:

$$\Omega(a_.) = \Omega(a_.; V_.) = \{g \in G |\ dim\big(R \cap V_{a_i}\big)_g \geqslant i,\ i = 1, \ldots, k\}.$$

The locus $\Omega(a_.)$, called a *Schubert subscheme* is endowed with a reduced scheme structure induced from the reduced one of the corresponding Schubert subscheme in the Grassmannian $G_n V$ - this is discussed in detail , e.g., in [L-Se].

The following desingularization of $\Omega = \Omega(a_.)$ should be thought of as a Lagrangian analogue of the construction used in [K-L]. Let $\mathcal{F} = \mathcal{F}(a_.) = \mathcal{F}(V_{a_1} \subset \ldots \subset V_{a_k})$ be the scheme parametrizing flags $A_1 \subset A_2 \subset \ldots \subset A_k \subset A_{k+1}$ such that $rank\ A_i = i$ and $A_i \subset V_{a_i}$ for $i = 1, \ldots, k$; $rank\ A_{k+1} = n$ and A_{k+1} is Lagrangian. $\mathcal{F}$ is endowed with the tautological flag $D_1 \subset D_2 \ldots \subset D_k \subset D_{k+1}$, where $rank\ D_i = i$, $i = 1, \ldots, k$ and $rank\ D_{k+1} = n$. We will write D instead of D_{k+1}.

We have a fibre square:

$$\begin{array}{ccc} G \times_X \mathcal{F} & \xrightarrow{p_2} & \mathcal{F} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ G & \longrightarrow & X \end{array}$$

Let $\alpha : \mathcal{F} \to G$ be the map defined by: $(A_1 \subset A_2 \subset \ldots \subset A_{k+1}) \mapsto A_{k+1}$, in other words α is a "classifying map" such that $\alpha^* R = D$. It is easily verified that α maps $\mathcal{F}$ onto

Ω and α is an isomorphism over the open subset of Ω parametrizing rank n Lagrangian subbundles A of V such that $rank(A \cap V_{a_i}) = i$, $i = 1, \dots, k$. Moreover, α induces a section s of p_2. Set $Z := s(\mathcal{F}) \subset G \times_X \mathcal{F}$. Alternatively, we can describe Z as $(1 \times \alpha)^{-1}(\Delta)$ where Δ is the diagonal in $G \times_X G$. The map p_1 restricted to Z is a desingularization of Ω. Therefore $[\Omega] = (p_1)_*([Z])$. On the other hand, $[Z] = (1 \times \alpha)^*([\Delta])$ (because, e.g., of [K-L, Lemma 9]). Note that $\mathcal{F}$ is obtained as a composition of the following Flag- and Grassmannian bundles. Let $Fl = Fl(a.) = Fl(V_{a_1} \subset \dots \subset V_{a_k})$ be the "usual" Flag bundle parametrizing flags $A_1 \subset \dots \subset A_k$ where $rank\ A_i = i$ and $A_i \subset V_{a_i}$, $i = 1, \dots, k$. Let $C_1 \subset \dots \subset C_k$ be the tautological flag on Fl. We will write C instead of C_k. Then $\mathcal{F}$ is the Lagrangian Grassmannian bundle $LG_{n-k}(C^\perp/C)$ over Fl, where $C^\perp$ is the subbundle of V_{Fl} consisting of all v that are orthogonal to C w.r.t. the given symplectic form. Note that $C \subset C^\perp$ because C is Lagrangian, $rank(C^\perp/C) = 2(n-k)$ and the vector bundle $C^\perp/C$ is endowed with a nondegenerate symplectic form induced from the one on V. Of course the tautological Lagrangian rank $n-k$ subbundle on $LG_{n-k}(C^\perp/C)$ is identified with $D/C_{\mathcal{F}}$. In other words, $\mathcal{F}$ is a composition of a Flag bundle (with the fiber being $Fl(K^{a_1} \subset \dots \subset K^{a_k})$ and a Lagrangian Grassmanian bundle (with the fiber being $LG_{n-k}(K^{2(n-k)})$. In particular,

$$dim\ \Omega = dim\ \mathcal{F} = dim\ Z = \sum_{i=1}^{k}(a_i - i) + \binom{n-k+1}{2} + dim\ X.$$

The following particular cases will be treated in a detailed way in the sequel of this paper: $a. = (n-k+1,\ n-k+2, \dots, n)$ (then $\Omega(a.)$ parametrizes Lagrangian rank n subbundles L of V such that $rank(L \cap V_n) \geqslant k$); $a. = (n+1-i)$, i.e. $k = 1$; and $a. = (n+1-i, n+1-j)$, i.e. $k = 2$.

Now consider the orthogonal case. Let K be a ground field of characteristic different from 2. Assume, that V is a rank $2n+1$ vector bundle over a smooth scheme X over a field K equipped with a nondegenerate orthogonal form. All definitions, notions and notation with the following exceptions are used mutatis mutandis: the Grassmannian bundle parametrizing the rank n isotropic subbundles of V is denoted OG_nV, instead of "symplectic" use "orthogonal" and instead of "Lagrangian" use "isotropic". Of course, $\mathcal{F}$ is now a composition of the same Flag bundle Fl and the odd orthogonal Grassmannian bundle $OG_{n-k}(C^\perp/C)$, where C is the rank k tautological subbundle on Fl.

Assume now that V is rank $2n$ vector bundle over a smooth connected scheme X over K equipped with a nondegenerate orthogonal form. The scheme parametrizing isotropic rank n subbundles of V breaks up into two connected components denoted OG'_nV and OG''_nV. Let V_n be a fixed rank n isotropic subbundle of V. Then OG'_nV (resp. OG''_nV) parametrizes rank n isotropic subbundles $E \subset V$ such that $dim(E \cap V_n)_x \equiv n (mod\ 2)$ (resp. $dim(E \cap V_n)_x \equiv n+1 (mod\ 2)$) for every $x \in X$. Write $G' := OG'_nV$ and $G'' := OG''_nV$. Two isotropic rank n subbundles are in the same component iff they intersect fiberwise in dimension congruent to n modulo 2.

Fix now a flag $V_1 \subset V_2 \subset \ldots \subset V_n$ of isotropic subbundles of V with $rank\ V_i = i$. Given a sequence $a_{.} = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$ such that $k \equiv n\ (mod\ 2)$, we consider in G' a Schubert subvariety:

$$\Omega(a_.) = \Omega(a_.; V_.) = \left\{ g \in G' |\ dim(R \cap V_{a_i})_g \geqslant i,\ i = 1, \ldots, k \right\}$$

($R \subset V_G$ is here the tautological bundle). Similarly, given a sequence $a_{.} = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$ such that $k \equiv n+1\ (mod\ 2)$, we consider in G'' a Schubert subvariety

$$\Omega(a_.) = \Omega(a_.; V_.) = \left\{ g \in G'' |\ dim(R \cap V_{a_i})_g \geqslant i,\ i = 1, \ldots, k \right\}.$$

(Over a point, say, the interiors of the $\Omega(a.)$'s form a cellular decomposition of G' and respectively G''.) Here, the definition of the scheme structure is more delicate than in the previous two cases (roughly speaking, instead of minors one should use the Pfaffians of the "coordinate" antisymmetric matrix of G' and G''). We refer the reader for details to [L-Se] and references therein.

The Schubert subvarieties $\Omega(a_.)$ in G' (resp. $\Omega(a_.)$ in G'') are desingularized using the same construction as above but instead of the scheme $\mathcal{F}$ one must now use the following scheme $\mathcal{F}'$ (resp. $\mathcal{F}''$). Let $\mathcal{F}' = \mathcal{F}'(a_.) = \mathcal{F}'(V_{a_1} \subset \ldots \subset V_{a_k})$ be a scheme parametrizing flags $A_1 \subset A_2 \subset \ldots \subset A_k \subset A_{k+1}$ such that $rank\ A_i = i$ and $A_i \subset V_{a_i}$ for $i = 1, \ldots, k$; $rank\ A_{k+1} = n$, A_{k+1} is isotropic and $rank(A_{k+1} \cap V_n)_x \equiv n\ (mod\ 2)$ for any $x \in X$. The definition of $\mathcal{F}'' = \mathcal{F}''(a_.)$ is the same with exception of the last condition now replaced by: $rank(A_{k+1} \cap V_n)_x \equiv n+1\ (mod\ 2)$ for any $x \in X$. Let $p' : \mathcal{F} \to X$ (resp. $p'' : \mathcal{F} \to X$) denote the projection maps. Of course, $\mathcal{F}'$ (resp. $\mathcal{F}''$) now is a composition of the same Flag bundle Fl and the even orthogonal Grassmannian bundle $OG'_{n-k}(C^{\perp}/C)$ (resp. $OG''_{n-k}(C^{\perp}/C)$), where C is the rank k tautological subbundle on Fl.

The formula for dimension now is different:

$$dim\ \mathcal{F}' = dim\ \mathcal{F}'' = \sum_{i=1}^{k}(a_i - i) + \binom{n-k}{2} + dim\ X.$$

We finish this Section with the following lemma which will be of constant use in this paper.

Lemma 1.1. *Consider cases 1., 2., 3. of a vector bundle endowed with a nondegenerate form Φ that are specified in the Introduction. Let $C \subset V$ be an isotropic subbundle and $C^{\perp}$ be the subbundle of V consisting of all $v \in V$ such that $\Phi(v, c) = 0$ for any $c \in C$.*

1. Then one has an exact sequence

$$0 \longrightarrow C^{\perp} \longrightarrow V \xrightarrow{\ \phi\ } C^{\vee} \longrightarrow 0$$

where the map ϕ is defined by $v \mapsto \Phi(v, -)$. In particular, in the Grothendieck group, $[V] = [C^{\perp}] + [C^{\vee}]$, $[C^{\perp}/C] = [V] - [C] - [C^{\vee}]$ and the Chern classes of $C^{\perp}/C$ are the same as the ones of the element $[V] - [C \oplus C^{\vee}]$ in the Grothendieck group.

2. Assume now that C is a maximal isotropic subbundle of V. Then in cases 1. and 3. we have $C = C^{\perp}$ and $c_{.}(V) = c_{.}(C \oplus C^{\vee})$; in case 2. one has $rank(C^{\perp}/C) = 1$ and $2c_{.}(V) = 2c_{.}(C \oplus C^{\vee})$.

The latter equality of assertion 2 in case 2. follows from the fact that the form Φ induces an isomorphism $(C^{\perp}/C)^{\otimes 2} \simeq \mathcal{O}_X$. This assertion will be used in the proof of Theorem 5.14 and 5.15 and is well suited for this purpose because of the appearance of the factor "2^n" on the right hand side of the formulas of the theorems.

2. Isotropic Schubert Calculus and the class of the diagonal

Let us first recall the following result on Lagrangian and orthogonal Schubert Calculus from [P1,2]. We work here in the Chow rings; all results, however, are equally valid in the cohomology rings.

We need two families of polynomials in the Chern classes of a vector bundle E over a smooth variety X. Their construction is inspired by I. Schur's paper [S]. The both families will be indexed by partitions (i.e. by sequences $I = (i_1 \geqslant \ldots \geqslant i_k \geqslant 0)$ of integers). Set, in the Chow ring $A^*(X)$ of X, for $i \geqslant j \geqslant 0$:

$$\widetilde{Q}_{i,j}E := c_iE \cdot c_jE + 2\sum_{p=1}^{j}(-1)^p c_{i+p}E \cdot c_{j-p}E,$$

so, in particular $\widetilde{Q}_iE := \widetilde{Q}_{i,0}E = c_iE$ for $i \geqslant 0$. In general, for a partition $I = (i_1, \ldots, i_k)$, $k-$even (by putting $i_k = 0$ if necessary), we set in $A^*(G)$:

$$\widetilde{Q}_IE := Pf\left(\widetilde{Q}_{i_p,i_q}E\right)_{1 \leqslant p < q \leqslant k},$$

where Pf means the Pfaffian of the given antisymmetric matrix.

The member of the second family, associated with a partition I, is defined by

$$\widetilde{P}_IE := 2^{-l(I)}\widetilde{Q}_IE.$$

Observe that in particular $\widetilde{P}_iE = c_iE/2$ (so here we must assume that c_iE is divisible by 2), and

$$\widetilde{P}_{i,j}E = \widetilde{P}_iE \cdot \widetilde{P}_jE + 2\sum_{p=1}^{j-1}(-1)^p \widetilde{P}_{i+p}E \cdot \widetilde{P}_{j-p}E + (-1)^j \widetilde{P}_{i+j}E.$$

It should be emphasise that $\widetilde{Q}$- and $\widetilde{P}$-polynomials are especially important and useful for isotropic (sub)bundles.

The following result from [P1, (8.7)] and [P2, Sect.6], gives a basic geometric interpretation of $\widetilde{Q}$- and $\widetilde{P}$-polynomials.

Theorem 2.1. [P2, Theorems 6.17, 6.17']

(i) *Let V be a $2n-$dimensional vector space over a field K endowed with a nondegenerate symplectic form. Then, one has in $A^*(LG_nV)$,*

$$[\Omega(a.)] = \widetilde{Q}_I R^\vee \ ,$$

where R is the tautological subbundle on LG_nV and $i_p = n+1-a_p$, $p=1,\dots,k$.

(ii) *Let V be a $(2n+1)-$dimensional vector space over a field K of char. $\neq 2$ endowed with a nondegenerate orthogonal form. Then, one has in $A^*(OG_nV)$,*

$$[\Omega(a.)] = \widetilde{P}_I R^\vee \ ,$$

where R is the tautological subbundle on OG_nV and $i_p = n+1-a_p$, $p=1,\dots,k$.

(iii) *Let V be a $2n-$dimensional vector space over a field K of char. $\neq 2$ endowed with a nondegenerate orthogonal form. Then one has in $A^*(OG'_nV)$ (resp. $A^*(OG''_nV)$),*

$$[\Omega(a.)] = \widetilde{P}_I R^\vee \ ,$$

where R is the tautological subbundle on OG'_nV (resp. OG''_nV) and $i_p = n-a_p$, $p = 1,...,k$. (Notice that the indexing family of I's runs here over all strict partitions contained in ρ_{n-1}.)

Observe that by Lemma 1.1, $R^\vee$ is the tautological quotient bundle on LG_nV, OG'_nV and OG''_nV. Moreover, the Chern classes of the tautological quotient bundle on OG_nV and $R^\vee$ are equal.

Notice that a new proof of this result has been given recently by Billey and Haiman in [B-H]. We stress that [P2, Theorems 6.17, 6.17'] contain stronger variants of those assertions. For instance, consider in case (iii) the assignment

$$P_I \mapsto [\Omega(a.)] \text{ for } I \subset \rho_{n-1}, \ -zero, \text{ otherwise,}$$

where P_I is the Schur's P-function (see [S] with $P_I := 2^{-l(I)}Q_I$) and $a.$ is obtained from I by reversing the rule in (iii) and adding an n at the end (if necessary) to achieve the correct parity. It was shown in loc. cit. that this assignment is a *ring* homomorphism which allows one to identify the Chow ring of $A^*(OG'_nV)$ (resp. $A^*(OG''_nV)$) with the quotient ring of the ring of Schur's P-functions modulo the ideal $\oplus\mathbb{Z}P_I$, the sum over all strict partitions I not contained in ρ_{n-1}.

Assume now that V is a vector bundle over a smooth variety X and $V.$ is a flag of isotropic bundles on X. Then, using Noetherian induction, one shows that $\{\widetilde{Q}_I R^\vee\}_{I\subset\rho_n}$, $\{\widetilde{P}_I R^\vee\}_{I\subset\rho_n}$ and $\{\widetilde{P}_I R^\vee\}_{I\subset\rho_{n-1}}$ are $A^*(X)$-bases respectively of $A^*(LG_nV)$, $A^*(OG_nV)$ and $A^*(OG'_nV)$ (resp. $A^*(OG''_nV)$). Moreover, there is an expression for $\Omega(a.;V.)$ as a polynomial in the Chern classes of $R^\vee$ and V_i. (This follows, e.g., from the existence of desingularizations given in Section 1 and formulas for Gysin push forwards - for "usual" Flag bundles they are obtained by iterating a well known Projective bundle case; for isotropic Grassmannian bundles, they are given for the first time in Section 5 of the present paper). Then the maximal degree term in $c.(R^\vee)$ of this expression, in respective cases (i), (ii), (iii), coincides with that in Theorem 2.1. We will call it *the dominant term* (w.r.t. R).

Let G_1, G_2 be two copies of the Lagrangian Grassmannian bundle LG_nV over a smooth variety X, equipped with the tautological subbundles R_1 and R_2. Write $GG := G_1 \times_X G_2$. Consider the following diagonal

$$\Delta = \Big\{(g_1, g_2) \in GG | \; \big((R_1)_{GG}\big)_{(g_1,g_2)} = \big((R_2)_{GG}\big)_{(g_1,g_2)}\Big\}.$$

Our goal is to write down a formula for the class of this diagonal. We first record:

Lemma 2.2. *Let G be a smooth complete variety such that the "$\times$-map"* (cf. [F, end of Sect.1]) *gives an isomorphism $A^*(G \times G) \simeq A^*(G) \otimes A^*(G)$. Assume that there exists a family $\{b_\alpha\}$, $b_\alpha \in A^{n_\alpha}(G)$, such that $A^*(G) = \oplus \mathbb{Z} b_\alpha$, and for every α there is a unique α' such that $n_\alpha + n_{\alpha'} = dim\ G$ and $\int_X b_\alpha \cdot b_{\alpha'} \neq 0$. Let $\int_X b_\alpha \cdot b_{\alpha'} = 1$. Then the class $[\Delta]$ in $A^*(G \times G)$ is given by $\sum\limits_\alpha b_\alpha \times b_{\alpha'}$.*

Proof. It follows from the assumptions that in $A^*(G \times G)$, $[\Delta] = \sum m_{\alpha\beta} b_\alpha \times b_\beta$, for some integers $m_{\alpha\beta}$ and $n_\alpha + n_\beta = dim\ G$ for all pairs (α, β) indexing the sum. We have by a standard property of intersection theory for $g, h \in A^*(G)$

$$\int_{X \times X} [\Delta] \cdot (g \times h) = \int_X g \cdot h.$$

Hence the coefficients $m_{\alpha\beta}$ satisfy:

$$m_{\alpha\beta} = \int_{X \times X} [\Delta] \cdot (b_{\alpha'} \times b_{\beta'}) = \int_X b_{\alpha'} \cdot b_{\beta'}.$$

The latter expression, according to our assumption is not zero only if $\alpha' = (\beta')'$ i.e. $\beta = \alpha'$, when it equals 1. This proves the lemma. □

For a given positive integer k, put $\rho_k = (k, k-1, \dots, 2, 1)$. For a strict partition $I \subset \rho_k$ (i.e. $i_1 \leqslant k$, $i_2 \leqslant k-1, \dots$) we denote by $\rho_k \setminus I$ the strict partition whose parts complement the parts of I in the set $\{k, k-1, \dots, 2, 1\}$.

The Lagrangian Grassmannian (over a point, say) satisfies the assumptions of the lemma with $\{\widetilde{Q}_I R^\vee\}_{strict\ I \subset \rho_n}$ playing the role of $\{b_\alpha\}$ and for $\alpha = I$ we have $\alpha' = \rho_n \setminus I$. This is a direct consequence the existence of a well known cellular decomposition of such a Grassmannian into Schubert cells and the results of [P2] recalled in Theorem 2.1(i) together with a description of Poincaré duality in $A^*(LG_nV)$ from loc.cit. Thus in this situation we get by the lemma:

Lemma 2.3. *The class of the diagonal Δ of the Lagrangian Grassmannian equals*

$$[\Delta] = \sum \widetilde{Q}_I(R_1^\vee) \times \widetilde{Q}_{\rho_n \setminus I}(R_2^\vee),$$

the sum over all strict $I \subset \rho_n$.

We will now show that the same formula holds true for an arbitrary smooth base space X of a vector bundle V.

Lemma 2.4. *Let $\pi : G \to X$ be a proper morphism of smooth varieties such that π^* makes $A^*(G)$ a free $A^*(X)$-module, $A^*(G) = \oplus A^*(X) \cdot b_\alpha$, where $b_\alpha \in A^{n_\alpha}(G)$ and for any α there is a unique α' such that $n_\alpha + n_{\alpha'} = \dim G - \dim X$ and $\pi_*(b_\alpha \cdot b_{\alpha'}) \neq 0$; let $\pi_*(b_\alpha \cdot b_{\alpha'}) = 1$. Moreover, denoting by $p_i : G \times_X G \to G$ ($i = 1,2$) the projections, assume that, for a smooth $G \times_X G$, the homomorphism $A^*(G) \otimes_{A^*(X)} A^*(G) \to A^*(G \times_X G)$, defined by $g \otimes h \mapsto p_1^* g \cdot p_2^* h$, is an isomorphism. Then the class of the diagonal Δ in $G \times_X G$ equals*

$$[\Delta] = \sum_{\alpha\beta} m_{\alpha\beta} b_\alpha \otimes b_\beta,$$

where, for any α, β, $m_{\alpha\beta} = P_{\alpha\beta}(\{\pi_*(b_\gamma \cdot b_\delta\})$ for some polynomial $P_{\alpha\beta} \in \mathbb{Z}[\{x_{\gamma\delta}\}]$.

Proof. Denote by $\delta : G \to G \times_X G$, $\delta' : G \to G \times_K G$ (the Cartesian product) the diagonal embeddings and by γ the morphism $\pi \times_X \pi : G \times_X G \to X$. For $g, h \in A^*(G)$ we have

$$\pi_*(g \cdot h) = \pi_*\Big((\delta')^*(g \times h)\Big) = \pi_*\Big(\delta^*(g \otimes h)\Big) = \gamma_*\delta_*\Big(\delta^*(g \otimes h)\Big) = \gamma_*\Big([\Delta] \cdot (g \otimes h)\Big),$$

using $\pi = \gamma \circ \delta$ and standard properties of intersection theory from [F]. Hence, by the assumptions, we get

$$\begin{aligned}\pi_*(b_{\alpha'} \cdot b_{\beta'}) = \gamma_*([\Delta] \cdot (b_{\alpha'} \otimes b_{\beta'}) &= (\pi_* \otimes \pi_*)\Big(\big(\sum m_{\gamma\delta} b_\gamma \otimes b_\delta\big) \cdot (b_{\alpha'} \otimes b_{\beta'})\Big) \\ &= m_{\alpha\beta} + \sum m_{\gamma\delta} \pi_*(b_\gamma \cdot b_{\alpha'}) \cdot \pi_*(b_\delta \cdot b_{\beta'})\end{aligned}$$

where the degree of $m_{\gamma\delta} \in A^*(X)$ is less then the degree of $m_{\alpha\beta}$. The assertion now follows by induction on the degree of $m_{\alpha\beta}$. $\square$

Let now $G = (LG_n V \to X)$ be a Lagrangian Grassmannian bundle, and use the same letter to denote the total space of $LG_n V$.

Proposition 2.5. *The class of the diagonal of the Lagrangian bundle in $A^*(G \times_X G)$ equals*

$$[\Delta] = \sum \widetilde{Q}_I(R_1^\vee)_{GG} \cdot \widetilde{Q}_{\rho_n \setminus I}(R_2^\vee)_{GG},$$

the sum over all strict $I \subset \rho_n$, $GG = G \times_X G$ and R_i, $i = 1, 2$, are the tautological (sub)bundles on the corresponding factors.

Proof. Consider the family $\{b_\alpha\}_\alpha = \{\widetilde{Q}_I R^\vee\}_I$ where the indexing set of the α's runs over the set of all strict partitions $I \subset \rho_n$ and $n_\alpha = |I|$. This family satisfies the assumptions of Lemma 2.4. Observe that the required properties w.r.t. π_* follow from the case $X = point$ by invoking the universal character of formulas for $\pi_*(\widetilde{Q}_I R^\vee \cdot \widetilde{Q}_J R^\vee)$. Indeed, it will follow from results of Section 5 and 4 (obtained independently) that for any I, J, $\pi_*(\widetilde{Q}_I R^\vee \cdot \widetilde{Q}_J R^\vee)$ is given by some universal polynomial expression in the Chern classes of $V^\vee$. Also, arguing by Noetherian induction we get $A^*(G) \otimes_{A^*(X)} A^*(G) \simeq A^*(G \times_X G)$. It follows from the proof of Lemma 2.4 that the polynomials $P_{\alpha\beta}$ are the same for any $\pi : LG_n V \to X$. Hence we have by Lemma 2.4

$$[\Delta] = \sum m_{IJ}\, p_1^* \widetilde{Q}_I(R_1^\vee) \cdot p_2^* \widetilde{Q}_J(R_2^\vee),$$

where m_{IJ} are universal polynomial expressions in the Chern classes of $V^\vee$.

As we have already noticed, the summands occuring on the right hand side for $|I|+|J|=n(n-1)/2$ are the same as in the case $X = point$. On the other hand, for $|I|+|J|<n(n-1)/2$ we wish to show that the corresponding summands occur with vanishing universal coefficients. Instead of proceeding directly we use the following specialization argument. Consider the Grassmannian $X = G_n(\mathbb{C}^N)$, $N \gg 0$. Then using the tautological vector bundle S on X we put $V := S\oplus S^\vee$ and equip it with a nondegenerate symplectic form , e.g. , the one corresponding to the antisymmetric map $S^\vee\oplus S \to S\oplus S^\vee$ given by the matrix

$$\begin{pmatrix} 0 & id_S \\ -id_{S^\vee} & 0 \end{pmatrix}.$$

Take two copies $G_1 \to X_1$ and $G_2 \to X_2$ of $G \to X$ (endowed with the tautological Lagrangian rank n subbundles R_1 and R_2). Our goal is to show that the fibre product diagonal $\Delta \subset G_1 \times_X G_2$ parametrizing the points (x, g_1, g_2) with $g_1 = g_2$, has the desired class. This will finish the proof because with $N \to \infty$ the bundle $V = S \oplus S^\vee$ has the generic Chern classes of a bundle endowed with a nondegenerate symplectic form, and hence an appearance of a nonzero universal coefficient m_{IJ} for $|I|+|J| < n(n-1)/2$ would show up in this situation. Let Δ' in $G_1 \times_K G_2$ (i.e. in the Cartesian product) be another diagonal parametrizing the points (x_1, g_1, x_2, g_2) with $x_1 = x_2$, $g_1 = g_2$. Note that for the natural map $i : G_1 \times_X G_2 \to G_1 \times_K G_2$, one has: $i_*[\Delta] = [\Delta']$. If S_1 is the tautological subbundle on X_1 and Q_2 is the tautological quotient bundle on X_2, then the subscheme $G_1 \times_X G_2$ in $G_1 \times_K G_2$ is identified with the scheme $Zeros(S_1 \to \mathbb{C}^N \to Q_2)$.

We now investigate

$$[\Delta] = \sum d_{I_1,I_2} \widetilde{Q}_{I_1}(R_1^\vee) \cdot \widetilde{Q}_{I_2}(R_2^\vee)$$

in $A^*(G_1 \times_X G_2)$, where $d_{I_1,I_2} \in A^*(X)$ (we omit writing "p_i^*" as well as the pullback-indices, for brevity). We have

$$\begin{aligned}[\Delta'] = i_*[\Delta] &= \sum_{I_1,I_2} d_{I_1,I_2} \widetilde{Q}_{I_1}(R_1^\vee) \cdot \widetilde{Q}_{I_2}(R_2^\vee) \cdot \Big(class\ of\ G_1 \times_X G_2 \text{ in } G_1 \times_K G_2\Big) \\ &= \sum_{I_1,I_2,J} d_{I_1,I_2} \widetilde{Q}_{I_1}(R_1^\vee) \cdot \widetilde{Q}_{I_2}(R_2^\vee) \cdot s_J(S_1^\vee) \cdot s_{J^*}(Q_2).\end{aligned}$$

Here $s_J(-)$ denotes the Schur polynomial of the indicated vector bundles (see e.g. [F], [P1,2]); J runs over partitions contained in $(N-n, \dots, N-n)$ (n times); given such a partition J, by J^* we denote the dual of the partition $(N-n-j_n, \dots, N-n-j_1)$.

Observe that G satisfies the assumptions of Lemma 2.2 because the bundle S is trivial over every Schubert cell of X. Indeed, the given Schubert cell

$$\{L \in X |\ dim(L \cap \mathbb{C}^{k_p} = p,\ p = 1, \dots, n\}$$

is contained in the complement of the hypersurface defined by the $(n \times n)$-minor spanned by the columns $k_1, \ldots, k_n$, where X is now identified with the quotient space of the space of $(n \times N)$ nonsingular complex matrices modulo $SL(n, \mathbb{C})$ acting by left multiplication, and over such a complement S is trivial. Hence the total space of G has a cellular decomposition formed of the products of the cells of X and those of the fiber of $G \to X$.

Therefore, using Lemma 2.2 with respect to (the total space of) G, we have

$$[\Delta'] = \sum s_J(S_1^\vee) \cdot \widetilde{Q}_I(R_1^\vee) \cdot s_{J^\bullet}(Q_2) \cdot \widetilde{Q}_{\rho_n \setminus I}(R_2^\vee),$$

the sum over $J \subset (N-n)^n$ and $I \subset \rho_n$.

Comparison of the two developments gives $d_{I_1, I_2} \neq 0$ iff $I_2 = \rho_n \setminus I_1$ and $d_{I, \rho_n \setminus I} = 1$ for every I. This finishes the proof of the proposition. □

Corollary 2.6. *With the notation of Section 1 and $G\mathcal{F} := G \times_X \mathcal{F}$, the class of Z in $A^*(G\mathcal{F})$ (i.e. the image of the class of the diagonal of $G \times_X G$ via $(1 \times \alpha)^*$) equals*

$$\sum_{strict\ I \subset \rho_n} \widetilde{Q}_I(D_{G\mathcal{F}}^\vee) \cdot \widetilde{Q}_{\rho_n \setminus I}(R_{G\mathcal{F}}^\vee).$$

Thus the problem of computing the class of Ω is essentially that of calculation $p_*(\widetilde{Q}_I D^\vee)$ where $p : \mathcal{F} \to X$ is the projection map; then we use a base change.

Consider now the case of the orthogonal Grassmannian of rank n subbundles of V, where *rank* $V = 2n+1$. The results of Lemma 2.3, Proposition 2.5 and Corollary 2.6 translate mutatis mutandis to this case with $\widetilde{Q}$-polynomials replaced by $\widetilde{P}$-polynomials (in virtue of Theorem 2.1(ii)). Thus the problem of computing the class of Ω is essentially that of calculation $p_*(\widetilde{P}_I D^\vee)$ where $p : \mathcal{F} \to X$ is the projection.

Finally, consider two connected components of the orthogonal Grassmannian of rank n subbundles of V, where *rank* $V = 2n$, as defined in Section 1. The results of Lemma 2.3, Proposition 2.5 and Corollary 2.6 translate mutatis mutandis to this case with ρ_n replaced by ρ_{n-1} and $\widetilde{Q}$-polynomials replaced by $\widetilde{P}$-polynomials (in virtue of Theorem 2.1(iii)). Thus the problem of computing the class of Ω is essentially that of calculation $p'_*(\widetilde{P}_I D^\vee)$ and $p''_*(\widetilde{P}_I D^\vee)$ where $p' : \mathcal{F}' \to X$ and $p'' : \mathcal{F}'' \to X$ are the projection maps.

3. Subbundles intersecting an n-subbundle in dim $\geqslant k$

We will now show an explicit computation in case $a_. = (n-k+1, \ldots, n)$. This computation relies on a simple linear algebra argument. The results of this Section will be reproved in Section 8 using the algebra of divided differences operators.

We start with the Lagrangian case and follow the notation from Section 1. The results here are stated in the Chow rings but they are equally valid in the cohomology rings.

Proposition 3.1. *Assume* $a_{\cdot} = (n-k+1, \dots, n)$. *Let* $I \subset \rho_n$ *be a strict partition. If* $(n, n-1, \dots, k+1) \not\subset I$, *then* $p_* \widetilde{Q}_I D^\vee = 0$. *In the opposite case, write* $I = (n, n-1, \dots, k+1, j_1, \dots, j_l)$, *where* $j_l > 0$ *and* $l \leqslant k$. *Then* $p_* \widetilde{Q}_I D^\vee = \widetilde{Q}_{j_1, \dots, j_l} V_n^\vee$.

Proof. It suffices to prove the formula for a vector bundle $V \to B$ endowed with a symplectic form, X equal to $LG_n V$ and V_n equal to the tautological subbundle on $LG_n V$. (Recall that $\Omega(n-k+1, \dots, n; V_\cdot)$ depends only on V_n ; more precisely, it parametrizes Lagrangian rank n subbundles L of V such that $rank(L \cap V_n) \geqslant k$.) The variety $\mathcal{F}$ in this case parametrizes triples (L, M, N) of vector bundles over B such that L and N are Lagrangian rank n subbundles of V and M is a rank k subbundle of $L \cap N$. Let $W_\cdot$: $W_1 \subset W_2 \subset \dots \subset W_n$ be a flag of Lagrangian subbundles of V with $rank\ W_i = i$. For a partition $J = (j_1 > \dots > j_l > 0) \subset \rho_k$,

$$\alpha_J = \Omega(n+1-j_1, \dots, n+1-j_l; W_\cdot) = \{L \in X |\ rank(L \cap W_{n+1-j_h}) \geqslant h,\ h = 1, \dots, l\}$$

defines a Schubert cycle whose class has the dominant term (w.r.t. V_n) equal to $\widetilde{Q}_J V_n^\vee \in A^*(X)$. It is well known that α_J is an irreducible subvariety of X provided B is irreducible.

Similarly, for a partition $I = (i_1 > \dots > i_l > 0) \subset \rho_n$, $q : \mathcal{F} \to LG_n V$ the projection on the third factor,

$$A_I = q^* \Omega(n+1-i_1, \dots, n+1-i_l; W_\cdot) =$$
$$= \{(L, M, N) \in \mathcal{F} |\ rank(N \cap W_{n+1-i_h}) \geqslant h,\ h = 1, \dots, l\}$$

defines a cycle whose class has the dominant term (w.r.t. D) equal to $\widetilde{Q}_I D^\vee \in A^*(\mathcal{F})$. Also, A_I is an irreducible subvariety of $\mathcal{F}$ provided B is irreducible.

We will show (the pushforward is taken on the cycles level) that:

1) If $I \not\supset (n, n-1, \dots, k+1)$ then $p_* A_I = 0$. Passing to the rational equivalence classes, this implies $p_* \widetilde{Q}_I D^\vee = 0$.

2) If $I \supset (n, n-1, \dots, k+1)$ i.e. $I = (n, n-1, \dots, k+1, j_1, \dots, j_l)$, where $j_l > 0$ and $l \leqslant k$, then $p_* A_I = \alpha_J$ where $J = (j_1, ..., j_l)$. Then, passing to the rational equivalence classes (and using the projection formula), we get the following equality involving the dominant terms: $p_* \widetilde{Q}_I D^\vee = \widetilde{Q}_J V_n^\vee$.

Observe that **1)** holds if $l(I) \leqslant n-k$ because we then have $codim_{\mathcal{F}} A_I = |I| < n + (n-1) + \dots + (k+1)$, which is the dimension of the fiber of p. We will need the following:

<u>Claim</u> Let $I \subset \rho_n$ be a strict partition. Let $l = card\{h \mid i_{n-k+h} \neq 0\}$. Assume that $l > 0$. Then one has

$$(*) \qquad p(A_I) \subset \alpha_{i_{n-k+1}, i_{n-k+2}, \dots, i_{n-k+l}}.$$

Indeed, for $(L, -, N) \in A_I$, since $rank(L \cap N) \geqslant k$, the inequality $rank(N \cap W_r) \geqslant h$ implies $rank(L \cap W_r) \geqslant h - (n-k)$ for every h, r; this gives $(*)$.

1) To prove this assertion we first use (*) (by the above remark we can assume that $l(I) > n-k$) and thus get

$$codim_{\mathcal{F}} A_I - codim_X p(A_I) \leqslant (i_1 + \ldots + i_n) - (i_{n-k+1} + \ldots + i_n) = i_1 + \ldots + i_{n-k}.$$

Then, since $I \not\supset (n, n-1, \ldots, k+1)$, we have

$$i_1 + \ldots + i_{n-k} < n + \ldots + (k+1),$$

where the last number is the dimension of the fiber of p. Hence comparison of the latter inequality with the former yields $p_* A_I = 0$.

2) To prove this, it suffices to show $p(A_I) \subset \alpha_J$, $dim\ A_I = dim\ \alpha_J$; and if $p_* A_I = d \cdot \alpha_J$ for some $d \in \mathbb{Z}$ then $d = 1$. We have:

$p(A_I) \subset \alpha_J$: this is a direct consequence of (*).

$dim\ A_I = dim\ \alpha_J$: this results from comparison of the following three formulas $dim\ \mathcal{F} = dim\ X + k(n-k) + (n-k)(n-k+1)/2$, $codim_X \alpha_J = |J|$, and $codim_{\mathcal{F}} A_I = n + \ldots + (k+1) + |J|$.

Therefore $p_* A_I = d \cdot \alpha_J$ for some integer d. To show $d = 1$ it suffices to find an open subset $U \subset \alpha_J$ such that $p|_{p^{-1}U} : p^{-1}U \to U$ is an isomorphism. We define the open subset U in question as $\alpha_J \setminus \Omega(n-k; W_.)$. More explicitly, U is defined by the conditions:

$$rank(L \cap W_{n+1-j_1}) \geqslant 1, \ \ldots, \ rank(L \cap W_{n+1-j_l}) \geqslant l \ \text{ and } \ L \cap W_{n-k} = (0).$$

Observe that these conditions really define an open nonempty subset of α_J because $\Omega(n+1-j_1, \ldots, n+1-j_l; W_.) \not\subset \Omega(n+1-(k+1); W_.)$ for $J \subset \rho_k$. (Recall that for $I = (i_1 > \ldots > i_l > 0)$, $J = (j_1 > \ldots > j_{l'} > 0)$ one has $\Omega(n+1-i_1, \ldots, n+1-i_l; W_.) \subset \Omega(n+1-j_1, \ldots, n+1-j_{l'}; W_.)$ iff $I \supset J$.)

Since our problem of showing that $d = 1$ is of local nature, we can assume that B is a point and deal with vector spaces instead of vector bundles. Let us choose a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ such that, denoting the symplectic form by Φ, we have $\Phi(e_i, e_j) = 0 = \Phi(f_i, f_j)$ and $\Phi(e_i, f_j) = -\Phi(f_j, e_i) = \delta_{i,j}$. Assume that W_i is generated by the first i vectors $\{e_j\}$. Let W^i be the subspace generated by the last i vectors $\{e_j\}$. Moreover, let $\widetilde{W_i}$ be the subspace generated by the first i vectors $\{f_j\}$ and $\widetilde{W^i}$ be the subspace generated by the last i vectors $\{f_j\}$.

Observe that for a strict partition $\rho_n \supset I \supset (n, n-1, \ldots, k+1)$ a necessary condition for "$(-,-,N) \in A_I$" is "$N \supset W_{n-k}$". (This corresponds to the first $(n-k)$ Schubert conditions defining A_I.) On the other hand, if $L \in U$ then $L \cap W_{n-k} = (0)$ and consequently L must contain $\widetilde{W_{n-k}}$ (from the rest, i.e. $W^k \oplus \widetilde{W^k}$, we can get at most $k-$dimensional isotropic subspace). Hence also $|L \cap (W^k \oplus \widetilde{W^k})| = k$ ($| - |$ denotes the dimension). We conclude that a necessary choice for an n-dimensional Lagrangian subspace N such that $(L, M, N) \in A_I$ for some M, is

$$N := W_{n-k} \oplus L \cap \left(W^k \oplus \widetilde{W}^k\right).$$

It follows from the above discussion that N is really a Lagrangian subspace of dimension n and it satisfies the first $(n-k)$ Schubert conditions defining A_I. N also satisfies the last l ($\leqslant k$) Schubert conditions defining A_I : since $|L \cap W_{n+1-j_h}| \geqslant h$ and $L \cap W_{n-k} = (0)$, we have $|N \cap W_{n+1-j_h}| = |W_{n-k}| + h \geqslant n - k + h$ for $h = 1, \ldots, l$.

Moreover, since $|L \cap N| = k$, the subspace M above is determined uniquely: $M = L \cap N$.

Summing up, we have shown that $d = 1$; the ends the proof of **2)**.

Thus the proposition has been proved. □

Proposition 3.2. *One has in $A^*(G)$,*

$$[\Omega(n-k+1, \ldots, n-1, n)] = \sum_{strict\ I \subset \rho_k} \widetilde{Q}_I(V_n^\vee)_G \cdot \widetilde{Q}_{\rho_k \setminus I}(R^\vee)\ .$$

Proof. This formula is obtained directly by pushing forward via $(p_2)_*$ the class of Z in $G \times_X \mathcal{F}$ given by

$$\sum_{strict\ I \subset \rho_n} \widetilde{Q}_I(D^\vee_{G\mathcal{F}}) \cdot \widetilde{Q}_{\rho_n \setminus I}(R^\vee_{G\mathcal{F}})$$

(see Corollary 2.5), with the help of Proposition 3.1. □

Example 3.3. For successive k (and any n) the formula reads (with $D = D_{G\mathcal{F}}$, $R = R_{G\mathcal{F}}$ for brevity):

k=1 $\widetilde{Q}_1 D^\vee + \widetilde{Q}_1 R^\vee$;

k=2 $\widetilde{Q}_{21} D^\vee + \widetilde{Q}_2 D^\vee \cdot \widetilde{Q}_1 R^\vee + \widetilde{Q}_1 D^\vee \cdot \widetilde{Q}_2 R^\vee + \widetilde{Q}_{21} R^\vee$;

k=3 $\widetilde{Q}_{321} D^\vee + \widetilde{Q}_{32} D^\vee \cdot \widetilde{Q}_1 R^\vee + \widetilde{Q}_{31} D^\vee \cdot \widetilde{Q}_2 R^\vee + \widetilde{Q}_{21} D^\vee \cdot \widetilde{Q}_3 R^\vee + \widetilde{Q}_3 D^\vee \cdot \widetilde{Q}_{21} R^\vee + \widetilde{Q}_2 D^\vee \cdot \widetilde{Q}_{31} R^\vee + \widetilde{Q}_1 D^\vee \cdot \widetilde{Q}_{32} R^\vee + \widetilde{Q}_{321} R^\vee$.

In the odd orthogonal case, the analogs of Propositions 3.1 and 3.2 are obtained by replacing $\widetilde{Q}$-polynomials through $\widetilde{P}$-polynomials. The proofs are essentially the same. In particular, α_J and A_I are defined in the same way. Obviously, in the proof of the analog of Proposition 3.1 one should now choose a basis $e_1, \ldots, e_n, f_1, \ldots, f_n, g$ such that the matrix of the orthogonal form w.r.t. this basis is

$$\left(\begin{array}{c|c|c} 0 & I_n & 0 \\ \hline I_n & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

where I_n is the $(n \times n)$-identity matrix. Then W_i, W^i, $\tilde{W}_i$ and $\tilde{W}^i$ are defined in the same way as in the Lagrangian case and the same proof goes through with $\widetilde{P}$-polynomials replacing $\widetilde{Q}$-polynomials.

Let us pass now to the even orthogonal case. So let $V \to X$ (X is connected) be a rank $2n$ vector bundle endowed with a nondegenerate quadratic form. Fix an isotropic rank n subbundle V_n of V. Recall that for $k \equiv n \ (mod\ 2)$ by $p' : \mathcal{F}' \to X$ we denote the Flag bundles parametrizing the flags $A_1 \subset A_2$ of subbundles of V such that $rank\ A_1 = k$, $rank\ A_2 = n$, $A_1 \subset V_n$ and A_2 is isotropic with $dim(A_2 \cap V_n)_x \equiv n \ (mod\ 2)$ for every $x \in X$. Similarly, for $k \equiv n+1 \ (mod\ 2)$ by $p'' : \mathcal{F}'' \to X$ we denote the Flag bundle parametrizing the flags $A_1 \subset A_2$ of subbundles of V such that $rank\ A_1 = k$, $rank\ A_2 = n$, $A_1 \subset V_n$ and A_2 is isotropic with $dim(A_2 \cap V_n)_x \equiv n+1 \ (mod\ 2)$ for every $x \in X$.

Remark 3.4. The component parametrizing rank n isotropic subbundles A of V with $dim(A \cap V_n)_x \equiv n \ (mod\ 2)$, $x \in X$, is isomorphic to that parametrizing rank n isotropic subbundles A of V with $rank(A \cap V_n)_x \equiv n+1 \ (mod\ 2)$, $x \in X$. For this well-known result we refer, e.g., to [G-Z, Lemma 18] where the space of all isotropic rank n subbundles of V is presented as a double unramified cover of $OG_{n-1}W$ with two sheets equal to OG'_nV and OG''_nV, where W is a rank $(2n-1)$ subbundle of V such that the form restricted from V to W is nondegenerate. Any such an isomorphism induces an isomorphism of $\mathcal{F}'$ and $\mathcal{F}''$ as schemes over X and the pullback via this isomorphism of the rank n tautological bundle on $\mathcal{F}'$ is equal to the rank n tautological bundle on $\mathcal{F}''$. Hence it is clear that the pushforward formulas for $p'_*\widetilde{P}_ID^\vee$ and $p''_*\widetilde{P}_ID^\vee$ are the same.

In the even orthogonal case the analog of Proposition 3.1 reads

Proposition 3.5. *Let $I \subset \rho_{n-1}$ be a strict partition. If $(n-1, n-2, \dots, k) \not\subset I$ then $p'_*\widetilde{P}_ID^\vee = 0$. In the opposite case, write $I = (n-1, n-2, \dots, k, j_1, \dots, j_l)$, where $j_l > 0$ and $l \leqslant k-1$. Then*

$$p'_*\widetilde{P}_ID^\vee = \widetilde{P}_{j_1,\dots,j_l}V_n^\vee.$$

The same formula is valid for p''_*.

Proof. We consider the case of p'_*. It suffices to prove the formula for a rank $2n$ vector bundle $V \to B$ (we assume that B is irreducible) endowed with a nondegenerate orthogonal form, X equal to OG'_nV or OG''_nV and V_n equal to the tautological subbundle on X. Then the variety $\mathcal{F}'$ parametrizes triples (L, M, N) such that $dim(L \cap N)_b \equiv n \ (mod\ 2)$ for every $b \in B$ (i.e. L and N either belong together to OG'_nV or together to OG''_nV) and M is a rank k subbundle of $L \cap N$.

We will now prove the proposition for $X = OG'_nV$. (Obvious modifications lead to a proof in the case $X = OG''_nV$.) Since the strategy of proof is the same as in the Lagrangian case, we will skip those parts of the reasoning which have appeared already in the proof of Proposition 3.1. Let $W_\cdot : W_1 \subset W_2 \subset \dots \subset W_n$ be an isotropic flag in V.

For $J = (j_1 > \dots > j_l > 0) \subset \rho_{k-1}$ we define

$$\alpha_J = \Omega(n-j_1, \dots, n-j_l; W_\cdot) \quad if \quad l \equiv n \ (mod\ 2) \ and$$
$$\alpha_J = \Omega(n-j_1, \dots, n-j_l, n; W_\cdot) \quad if \quad l \equiv n+1 \ (mod\ 2).$$

Similarly for $I = (i_1 > \dots > i_l > 0) \subset \rho_{n-1}$, $q : \mathcal{F}' \to OG'_nV$ the projection on the third factor, we define

$$A_I = q^*\Omega(n-i_1, \dots, n-i_l; W_\cdot) \quad if \quad l \equiv n \ (mod\ 2) \ and$$
$$A_I = q^*\Omega(n-i_1, \dots n-i_l, n; W_\cdot) \quad if \quad l \equiv n+1 \ (mod\ 2).$$

It is known that α_J and A_I are irreducible subvarieties provided B is. The dominant terms of the classes of α_J and A_I are equal to $\widetilde{P}_J V_n^\vee$ and $\widetilde{P}_I D^\vee$ respectively.

The proposition now follows from:

1) If $I \not\supset (n-1, n-2, \dots, k)$ then $p_* A_I = 0$.
2) If $I \supset (n-1, n-2, \dots, k)$ i.e $I = (n-1, n-2, \dots, k+1, k, j_1, \dots, j_l)$, where $j_l > 0$ and $l \leqslant k-1$, then $p_* A_I = \alpha_J$ where $J = (j_1, \dots, j_l)$.

Assertion 1) (being obvious if $l(I) < n-k$) is a consequence of:

Claim: For every strict partition $I \subset \rho_{n-1}$, let $l = card\{h \mid i_{n-k+h} \neq 0\}$. Assume that $l > 0$. Then one has

$$(*) \qquad p'(A_I) \subset \alpha_{i_{n-k+1}, \dots, i_{n-k+l}}.$$

Inclusion (*) also implies $p'(A_I) \subset \alpha_J$ in 2). The equality $dim\ p'(A_I) = dim\ \alpha_J$ now follows from: $dim\ \mathcal{F}' = dim\ X + k(n-k) + (n-k)(n-k-1)/2$, $codim_X \alpha_J = |J|$ and $codim_{\mathcal{F}'} A_I = (n-1) + \dots + k + |J|$.

Therefore $p'_* A_I = d \cdot \alpha_J$ for some integer d. To prove that $d = 1$ it is sufficient to show an open subset $U \subset \alpha_J$ such that $p'|_{(p')^{-1}U}$; $(p')^{-1}U \to U$ is an isomorphism. The open subset U in question is defined as

$$\alpha_J \setminus \Omega(n-k; W.)\ if\ \ n\ \ is\ odd\ \ and\ \ \alpha_J \setminus \Omega(n-k, n; W.)\ if\ \ n\ \ is\ even.$$

The problem being local, we can assume that B is a point. Let $e_1, \dots, e_n, f_1, \dots, f_n$ be a basis of V such that denoting the orthogonal form by Φ we have $\Phi(e_i, e_j) = \Phi(f_i, f_j) = 0$, $\Phi(e_i, f_j) = \Phi(f_j, e_i) = \delta_{i,j}$ and W_i is spanned by $e_1, \dots, e_i$. Define $W^i, \widetilde{W_i}$ and $\widetilde{W^i}$ as in the Lagrangian case.

Now, given $L \in U$, the unique N such that $(L, M, N) \in A_I$ for some M, is defined as in the proof of Proposition 3.1. This N satisfies the last $l(\leqslant k-1)$ Schubert conditions defining A_I: since $|L \cap W_{n-j_h}| \geqslant h$ and $L \cap W_{n-k} = (0)$, we have $|N \cap W_{n-j_h}| = |W_{n-k}| + h \geqslant n-k+h$ for $h = 1, \dots, l$. Finally, the M above is determined uniquely: $M = L \cap N$, and $p'|_{(p')^{-1}U}$ is an isomorphism. The proposition follows. □

Proposition 3.6. *If $k \equiv n\ (mod\ 2)$ (resp. $k \equiv n+1\ (mod\ 2)$) then one has in $A^*(OG'_n V)$ (resp. in $A^*(OG''_n V)$):*

$$[\Omega(n-k+1), \dots, n-1, n)] = \sum_{strict\ I \subset \rho_{k-1}} \widetilde{P}_I(V_n^\vee)_G \cdot \widetilde{P}_{\rho_{k-1} \setminus I}(R^\vee).$$

Proof. This formula is obtained directly by pushing forward via p'_* (resp. p''_*) the class of Z in $G \times_X \mathcal{F}'$ (resp. $G \times_X \mathcal{F}''$) given by

$$\sum_{strict\ I \subset \rho_{n-1}} \widetilde{P}_I(D^\vee_{G\mathcal{F}}) \cdot \widetilde{P}_{\rho_{n-1} \setminus I}(R^\vee_{G\mathcal{F}})$$

using Proposition 3.5. □

4. $\widetilde{Q}$-polynomials and their properties

In this Section we define a new family of symmetric polynomials modelled on the Schur's Q-polynomials. In Schur's Pfaffian-definition, we replace Q_i by e_i – the i-th elementary symmetric polynomial. It turns out that after this modification one gets an interesting family of symmetric polynomials whose properties are studied in this Section and then applied in the next ones.

Let $X = (x_1, x_2, \dots)$ be a sequence of independent variables. Denote by X_n the subsequence $(x_1, \dots, x_n)$. We set $\widetilde{Q}_i(X_n) := e_i(X_n)$. Given two nonnegative integers i, j we define

$$\widetilde{Q}_{i,j}(X_n) = \widetilde{Q}_i(X_n)\widetilde{Q}_j(X_n) + 2\sum_{p=1}^{j}(-1)^p \widetilde{Q}_{i+p}(X_n)\widetilde{Q}_{j-p}(X_n)$$

Finally, for any (i.e. not necessary strict) partition $I = (i_1 \geqslant i_2 \geqslant \dots \geqslant i_k \geqslant 0)$, with even k (by putting $i_k = 0$ if necessary), we set

$$\widetilde{Q}_I(X_n) = Pf\Big(\widetilde{Q}_{i_p,i_q}(X_n)\Big)_{1\leqslant p<q\leqslant k}.$$

Equivalently, $\widetilde{Q}_I(X_n)$ is defined recurrently on $l(I)$, by putting for odd $l(I)$,

$$(*) \qquad \widetilde{Q}_I(X_n) = \sum_{j=1}^{l(I)}(-1)^{j-1}\widetilde{Q}_{i_j}(X_n)\widetilde{Q}_{I\setminus i_j}(X_n),$$

and for even $l(I)$,

$$(**) \qquad \widetilde{Q}_I(X_n) = \sum_{j=2}^{l(I)}(-1)^{j}\widetilde{Q}_{i_1,i_j}(X_n)\widetilde{Q}_{I\setminus\{i_1,i_j\}}(X_n).$$

The latter case, with $l = l(I)$, can be rewritten as

$$(***) \qquad \widetilde{Q}_I(X_n) = \sum_{j=1}^{l-1}(-1)^{j-1}\widetilde{Q}_{i_j,i_l}(X_n)\widetilde{Q}_{I\setminus\{i_j,i_l\}}(X_n).$$

Note that assuming formally $i_l = 0$, the relation (***) specializes to (*). We will refer to the above equations as to Laplace-type developements or simply recurrent formulas.

We warn the reader that, with this definition, it is <u>not</u> true either that $\widetilde{Q}_{i,j}(X_n) = -\widetilde{Q}_{j,i}(X_n)$ or that the only nonzero polynomials $\widetilde{Q}_I(X_n)$ are those associated with strict partitions I.

We start with a useful linearity-type formula for $\widetilde{Q}$-polynomials indexed by <u>strict</u> partitions.

Proposition 4.1. *For any strict partition I one has*

$$\widetilde{Q}_I(X_n) = \sum_{j=0}^{l(I)} x_n^j \Big(\sum_{|I|-|J|=j} \widetilde{Q}_J(X_{n-1}) \Big),$$

where the sum is over all partitions $J \subset I$ such that $I \setminus J$ has at most one box in every row.

Proof. We use induction on $l(I)$.

1° $l(I) = 1$. Since we have: $e_i(X_n) = e_i(X_{n-1}) + x_n e_{i-1}(X_{n-1})$, the assertion follows.

2° $l(I) = 2$. We have for $i > j > 0$ and with $e_i = e_i(X_n)$, $\bar{e}_i = e_i(X_{n-1}), \bar{e}_{-1} = 0$,

$$\widetilde{Q}_{i,j}(X_n) = e_i e_j + 2 \sum_{p=1}^{j} (-1)^p e_{i+p} e_{j-p} =$$

$$= (\bar{e}_i + x_n \bar{e}_{i-1})(\bar{e}_j + x_n \bar{e}_{j-1}) + 2 \sum_{p=1}^{j} (-1)^p (\bar{e}_{i+p} + x_n \bar{e}_{i+p-1})(\bar{e}_{j-p} + x_n \bar{e}_{j-p-1})$$

$$= \Big(\bar{e}_i \bar{e}_j + 2 \sum_{p=1}^{j} (-1)^p \bar{e}_{i+p} \bar{e}_{j-p}\Big) + x_n \Big[\Big(\bar{e}_{i-1} \bar{e}_j + 2 \sum_{p=1}^{j} (-1)^p \bar{e}_{i-1+p} \bar{e}_{j-p}\Big) +$$

$$+ \Big(\bar{e}_i \bar{e}_{j-1} + 2 \sum_{p=1}^{j-1} (-1)^p \bar{e}_{i+p} \bar{e}_{j-1-p}\Big)\Big] + x_n^2 \Big(\bar{e}_{i-1} \bar{e}_{j-1} + 2 \sum_{p=1}^{j-1} (-1)^p \bar{e}_{i-1+p} \bar{e}_{j-1-p}\Big)$$

$$= \widetilde{Q}_{i,j}(X_{n-1}) + x_n \Big[\widetilde{Q}_{i-1,j}(X_{n-1}) + \widetilde{Q}_{i,j-1}(X_{n-1})\Big] + x_n^2 \widetilde{Q}_{i-1,j-1}(X_{n-1}).$$

3° By the remarks before the proposition, to prove the assertion in general it suffices to show it by using the recurrent relation (***). (Note that the R.H.S. of the formula of the proposition specializes after the formal replacement $i_l := 0$ ($l = l(I)$) to the expression asserted for $(i_1 > i_2 > \ldots > i_{l-1})$.

So, let us assume that l is even and set $\widetilde{Q}_I := \widetilde{Q}_I(X_n)$, $\bar{Q}_I = \widetilde{Q}_I(X_{n-1})$. Moreover, let $\mathcal{P}(I, j)$ be the set of all partitions $J \subset I$ such that $I \backslash J$ has at most one box in every row and $|I| - |J| = j$. We have by induction on l,

$$\widetilde{Q}_{I \backslash \{i_j, i_l\}} = \sum_{r=0}^{l-2} x_n^r \left(\sum_{J \in \mathcal{P}(I \backslash \{i_j, i_l\}, r)} \bar{Q}_J \right)..$$

Therefore, using 2° we have

$$\widetilde{Q}_I = \sum_{j=1}^{l-1} (-1)^{j-1} \Big[\bar{Q}_{j_j, i_l} + x_n (\bar{Q}_{i_j - 1, i_l} + \bar{Q}_{i_j, i_l - 1}) + x_n^2 \bar{Q}_{i_j - 1, i_l - 1}\Big]$$
$$\times \left[\sum_{r=0}^{l-2} x_n^r \Big(\sum_{J \in \mathcal{P}(I \backslash \{i_j, i_l\}, r)} \bar{Q}_J \Big) \right]$$

On the other hand, apply the relation (***) to the R.H.S. of the formula in the proposition. One gets:

$$\sum_{j=0}^{l} x_n^j \Big(\sum_{J \in \mathcal{P}(I,j)} \bar{Q}_J \Big) = \sum_{j=0}^{l} x_n^j \left[\sum_{J \in \mathcal{P}(I,j)} \Big(\sum_{q=1}^{l-1} (-1)^{q-1} \bar{Q}_{j_q, j_l} \cdot \bar{Q}_{J \setminus \{j_q, j_l\}} \Big) \right].$$

It is straightforward to verify that these both sums contain $2^l(l-1)$ terms of the form

$$(-1)^s x^j \bar{Q}_{a,b} \bar{Q}_{c_1, \ldots, c_{l-2}},$$

and such a term appears in both sums if and only if

$$(c_1, \ldots, c_s, a, c_{s+1}, \ldots, c_{l-2}, b) \in \mathcal{P}(I, j).$$

Thus the assertion follows and the proof of the proposition is complete. □

Proposition 4.2. : $\widetilde{Q}_{i,i}(X_n) = e_i(x_1^2, \ldots, x_n^2)$.

Proof. By definition we have $\big(e_i = e_i(X_n)\big)$:

$$\widetilde{Q}_{i,i}(X_n) = e_i e_i - 2e_{i-1}e_{i+1} + 2e_{i+2}e_{i-2} - \ \ldots = \sum_{p=0}^{i} (-1)^{p+i} e_p e_{2i-p}.$$

On the other hand, with an indeterminate t, we have

$$(1 + x_1 t) \ldots (1 + x_n t)(1 - x_1 t) \ldots (1 - x_n t) = (1 - x_1^2 t^2) \ldots (1 - x_n^2 t^2),$$

or equivalently,

$$\Big(\sum e_p t^p \Big) \Big(\sum (-1)^q e_q t \Big) = \sum (-1)^i e_i(x_1^2, \ldots, x_n^2) t^2.$$

This implies

$$(-1)^i e_i(x_1^2, \ldots, x_n^2) = \sum_{p=0}^{i} (-1)^p e_p \cdot e_{2i-p}.$$

Comparison of those two expressions gives the assertion. □

Proposition 4.3. *For partitions $I' = (i_1, i_2, \ldots, j, j, \ldots, i_{k-1}, i_k)$ and $I = (i_1, \ldots, i_k)$, the following equality holds*

$$\widetilde{Q}_{I'}(X_n) = \widetilde{Q}_{j,j}(X_n) \widetilde{Q}_I(X_n).$$

Proof. Write $\widetilde{Q}_I$ for $\widetilde{Q}_I(X_n)$. We use induction on k. For $k = 0$, the assertion is obvious. For $k = 1$, we have $\widetilde{Q}_{i,j,j} = \widetilde{Q}_i \widetilde{Q}_{j,j}$ and $\widetilde{Q}_{j,j,i} = \widetilde{Q}_{j,j} \widetilde{Q}_i$ by the Laplace type developements, so the assertion follows.

In general, it suffices to show the assertion inductively, using the relation (***), if the marked "j" does not appear on the last place; and independently, to prove it (inductively) for $I' = (i_1, \dots, i_k, j, j)$. In both instances k is assumed to be even.

In the former case, using (***) we get

$$\begin{aligned}\widetilde{Q}_{I'} =&\widetilde{Q}_{i_2,\dots,j,j,\dots,i_{k-1}} - \dots \pm \widetilde{Q}_{j,i_k}\widetilde{Q}_{i_1,i_2,\dots,j,\dots,i_k} \\ &\mp \widetilde{Q}_{j,i_k}\widetilde{Q}_{i_1,i_2,\dots,j,\dots,i_k} \pm \dots - \widetilde{Q}_{i_{k-1},i_k}\widetilde{Q}_{i_1,\dots,j,j,\dots,i_{k-2}}\end{aligned}$$

and the assertion follows from the induction assumption by using the relation (***) w.r.t. $\widetilde{Q}_{i_1,\dots,i_k}$ once again.

In the latter case we use the relation (**). We have

$$\begin{aligned}\widetilde{Q}_{i_1,\dots,i_k,j,j} =&\widetilde{Q}_{i_1,i_2}\widetilde{Q}_{i_3,\dots,i_k,j,j} - \dots + \widetilde{Q}_{i_1,i_k}\widetilde{Q}_{i_2,\dots,i_{k-1},j,j} \\ &- \widetilde{Q}_{i_1,j}\widetilde{Q}_{i_2,\dots,i_k,j} + \widetilde{Q}_{i_1,j}\widetilde{Q}_{i_2,\dots,i_k,j}\end{aligned}$$

and the assertion follows from the induction assumption by using the relation (**) w.r.t. $\widetilde{Q}_{i_1,\dots,i_k}$ once again. □

Lemma 4.4. *Let $I = (i_1, i_2, \dots, i_k)$ be a partition. If $i_1 > n$ then $\widetilde{Q}_I(X_n) = 0$.*

Proof. We use induction on $l(I)$. For $l(I) = 1, 2$ the assertion is obvious because $e_p(x_1, \dots, x_n) = 0$ for $p > n$. For bigger $l(I)$ one uses induction on the length and the recurrent formulas, which immediately imply the assertion. □

Let $SPol(X_n)$ denote the ring of symmetric polynomials in X_n. Similarly, we denote by $QPol(X_n)$ the ring generated by Schur's Q-polynomials in X_n. Let $\mathcal{J}$ denote the ideal in $SPol(X_n)$ generated by $e_i(x_1^2, \dots, x_n^2)$, $1 \leqslant i \leqslant n$. We now invoke a corollary of [P2, Theorem 6.17] combined with [B-G-G, Theorem 5.5] and [D2, 4.6(a)]: there is a ring isomorphism $SPol(X_n)/\mathcal{J} \to QPol(X_n)/\oplus \mathbb{Z}Q_I(X_n)$, where I runs over all strict partitions $I \not\subset \rho_n$, given by $e_i(X_n) \mapsto Q_i(X_n)$ (see the remark after Theorem 6.17 in [P2, pp.181-182]).

Proposition 4.5. *The set $\{\widetilde{Q}_I(X_n)\}$ indexed by all partitions such that $i_1 \leqslant n$ forms an additive basis of $SPol(X_n)$.*

Proof. By the remark above we have that $\widetilde{Q}_I(X_n)$ with I strict (and $l(I) \leqslant n$) form an additive basis of the quotient ring $SPol(X_n)/\mathcal{J}$. Thus every polynomial in $SPol(X_n)$ has the form $\sum \alpha_I \widetilde{Q}_I(X_n) + f \cdot g$ where $f \in \mathcal{J}$, $g \in SPol(X_n)$ and its degree is less than the one of the initial polynomial. Arguing by induction on degree, we can assume that g is a $\mathbb{Z}-$combination of the $\widetilde{Q}_I(X_n)$'s (observe that for the degree one symmetric polynomials the assertion obviously holds). As a consequence of Propositions 4.2 and 4.3 we get that every symmetric polynomial is a $\mathbb{Z}-$combination of the $\widetilde{Q}_I(X_n)$'s. Since the cardinality (for each degree) of the looked at family is the same as the one of the $\mathbb{Z}-$basis $\{e_I(X_n)\}$ of $SPol(X_n)$ (see [Mcd1]), the final assertion follows. □

Corollary-Definition 4.6. *For every $k \leqslant n$, every strict partition I and every (non necessary strict) partition $J \subset I$, there exist uniquely defined polynomials*

$$\widetilde{Q}_{I/J}(x_{k+1},\dots,x_n) \in SPol(x_{k+1},\dots,x_n)$$

such that the following equality holds

$$\widetilde{Q}_I(X_n) = \sum_{J \subset I} \widetilde{Q}_J(X_k)\widetilde{Q}_{I/J}(x_{k+1},\dots,x_n).$$

Proof. Since $SPol(X_n) \subset SPol(X_k) \otimes SPol(x_{k+1},\dots,x_n)$, the assertion follows from the previous proposition. □

Example 4.7. In 1° and 2° we set $\widetilde{Q}_I := \widetilde{Q}_I(X_n)$ for brevity. The following equalities hold:

1° $\widetilde{Q}_{5544441} = \widetilde{Q}_{55}\widetilde{Q}_{44441} = \widetilde{Q}_{55}\widetilde{Q}_{44}\widetilde{Q}_{441} = \widetilde{Q}_{55}\widetilde{Q}_{44}\widetilde{Q}_{44}\widetilde{Q}_1 = \widetilde{Q}_{55}\widetilde{Q}_{4444}\widetilde{Q}_1$;

2° $\widetilde{Q}_{5554443331} = \widetilde{Q}_{55}\widetilde{Q}_{44}\widetilde{Q}_{33}\widetilde{Q}_{5431} = \widetilde{Q}_{554433}\widetilde{Q}_{5431}$;

3° Here, we set $\bar{Q}_I := \widetilde{Q}_I(x_1,x_2)$, $\bar{Q}'_I := \widetilde{Q}_I(x_3)$. Then

$$\begin{aligned}
&\widetilde{Q}_{321}(x_1,x_2,x_3) = \\
&\quad = x_3\bar{Q}_{221} + x_3^2(\bar{Q}_{211} + \bar{Q}_{22}) + x_3^3\bar{Q}_{21} = x_3\bar{Q}_{22}\bar{Q}_1 + x_3^2(\bar{Q}_{11}\bar{Q}_2 + \bar{Q}_{22}) + x_3^3\bar{Q}_{21} \\
&\quad = x_3 e_2(x_1^2,x_2^2)(x_1+x_2) + x_3^2\big[e_1(x_1^2,x_2^2)x_1x_2 + e_2(x_1^2,x_2^2)\big] + x_3^3\big(x_2\bar{Q}'_{11} + x_2^2\bar{Q}'_1\big) \\
&\quad = x_3(x_1^2x_2^2)(x_1+x_2) + x_3^2\big[(x_1^2+x_2^2)x_1x_2 + x_1^2x_2^2\big] + x_1^3\big(x_2x_3^2 + x_2^2x_3\big).
\end{aligned}$$

By iterating the linearity formula for $\widetilde{Q}_I(X_n)$ (Proposition 4.1), we get the following algorithm for decomposition of $\widetilde{Q}_I = \widetilde{Q}_I(X_n)$ into a sum of monomials:

1. If I is not strict, we factorize

$$\widetilde{Q}_I = \widetilde{Q}_{j_1,j_1} \cdot \widetilde{Q}_{j_2,j_2} \cdot \dots \cdot \widetilde{Q}_{j_l,j_l} \cdot \widetilde{Q}_L,$$

where L is strict (we use Proposition 4.3).

2. We apply the linearity formula to $\widetilde{Q}_L(X_n)$ and x_n. Also, we decompose

$$\begin{aligned}
\widetilde{Q}_{j_p,j_p}(X_n) &= e_{j_p}\big(x_1^2,\dots,x_n^2\big) \\
&= e_{j_p}\big(x_1^2,\dots,x_{n-1}^2\big) + e_{j_p-1}\big(x_1^2,\dots,x_{n-1}^2\big)x_n^2 \\
&= \widetilde{Q}_{j_p,j_p}(X_{n-1}) + \widetilde{Q}_{j_p-1,j_p-1}(X_{n-1})x_n^2.
\end{aligned}$$

We then repeat 1 and 2 with the so obtained $\widetilde{Q}_I(X_{n-1})$'s, thus extracting x_{n-1}; then, we proceed similarly with the so obtained $\widetilde{Q}_I(X_{n-2})$'s etc.

Note that if we stop this procedure after extracting the variables $x_n, x_{n-1}, \ldots, x_{k+1}$ we get a developement:

$$\widetilde{Q}_I(X_n) = \sum_J \widetilde{Q}_J(X_k) F_J(x_{k+1}, \ldots, x_n),$$

where the sum is over $J \subset I$ (this follows from the linearity formula; J are not necessary strict). Moreover, $F_J(x_{k+1}, \ldots, x_n) = \widetilde{Q}_{I/J}(x_{k+1}, \ldots, x_n)$.

Of course, a similar set of formulas can be written for $\widetilde{P}$-polynomials $\widetilde{P}_I(X_n) := 2^{-l(I)} \widetilde{Q}_I(X_n)$. We leave it to the (interested in) reader.

Given a rank n vector bundle E with the Chern roots $(e_1, \ldots, e_n)$ we set $\widetilde{Q}_I E := \widetilde{Q}_I(X_n)$ with x_i specialized to e_i. Similarly, we define $\widetilde{Q}_{I/J}E$, $\widetilde{P}_I E$ and $\widetilde{P}_{I/J}E$. Note that this notation is consistent with that used in Section 2 and 3.

We finish this Section with the following example.

Example 4.8. Let $n = 5$, $\widetilde{Q}_I = \widetilde{Q}_I(X_5)$ and $s_J = s_J(X_5)$. We have:

$$\widetilde{Q}_{54} = s_{22221} \quad \widetilde{Q}_{53} = s_{22211} \quad \widetilde{Q}_{52} = s_{22111} \quad \widetilde{Q}_{51} = s_{21111}$$

$$\widetilde{Q}_{43} = s_{2221} - s_{22111} \quad \widetilde{Q}_{42} = s_{2211} - s_{21111} \quad \widetilde{Q}_{41} = s_{2111} - s_{11111}$$

$$\widetilde{Q}_{32} = s_{221} - s_{2111} + s_{11111} \quad \widetilde{Q}_{31} = s_{211} - s_{1111}$$

$$\widetilde{Q}_{21} = s_{21} - s_{111}$$

$$\widetilde{Q}_{543} = s_{33321} - s_{33222} \quad \widetilde{Q}_{542} = s_{33221} - s_{32222} \quad \widetilde{Q}_{541} = s_{32221} - s_{22222}$$

$$\widetilde{Q}_{532} = s_{33211} - s_{32221} + s_{22222} \quad \widetilde{Q}_{531} = s_{32211} - s_{22221}$$

$$\widetilde{Q}_{521} = s_{32111} - s_{22211}$$

$$\widetilde{Q}_{432} = s_{3321} - s_{3222} - s_{33111} \quad \widetilde{Q}_{431} = s_{3221} - s_{32111} - s_{2222}$$

$$\widetilde{Q}_{421} = s_{3211} - s_{31111} - s_{2221}$$

$$\widetilde{Q}_{321} = s_{321} - s_{3111} - s_{222}$$

$$\widetilde{Q}_{5432} = s_{44321} - s_{44222} - s_{43331} \quad \widetilde{Q}_{5431} = s_{43321} - s_{43222} - s_{33331}$$

$$\widetilde{Q}_{5421} = s_{43221} - s_{42222} - s_{33321}$$

$$\widetilde{Q}_{5321} = s_{43211} - s_{42221} - s_{33311}$$

$$\widetilde{Q}_{4321} = s_{4321} - s_{43111} - s_{4222} - s_{3331} + s_{32221} - 2s_{22222}$$

$$\widetilde{Q}_{54321} = s_{54321} - s_{54222} - s_{53331} - s_{44421} + s_{43332} - 2s_{33333}.$$

5. Divided differences and isotropic Gysin push forwards

Let $V \to X$ be a vector bundle of rank $2n$ endowed with a nondegenerate symplectic form. Let $\pi : LG_n(V) \to X$ and $\tau : LFl(V) \to X$ denote respectively the Grassmannian bundle of all Lagrangian subbundles of V and the Flag bundle of total flags of Lagrangian subbundles of V. We have $\tau = \pi \circ \omega$ where $\omega : LFl(V) \to LG_n(V)$ is the projection map. The main goal of this section is to derive several formulas for the Gysin push forward $\pi_* : A^*(LG_n(V)) \to A^*(X)$ if X is a smooth algebraic variety, or, $\pi_* : H^*(LG_n(V), \mathbb{Z}) \to H^*(X, \mathbb{Z})$ if X is a topological manifold.

We start with by recalling the Weyl group W_n of type C_n. This group is isomorphic to $S_n \ltimes \mathbb{Z}_2^n$. We write a typical element of W_n as $w = (\sigma, \tau)$ where $\sigma \in S_n$ and $\tau \in \mathbb{Z}_2^n$; so that if $w' = (\sigma', \tau')$ is another element, their product in W_n is $w \cdot w' = (\sigma \circ \sigma', \delta)$ where "$\circ$" denotes the composition of permutations and $\delta_i = \tau_{\sigma'(i)} \cdot \tau'_i$. To represent elements of W_n we will use the standard "barred-permutation" notation, writing them as permutations equipped with bars on those places (numbered with "i") where $\tau_i = -1$. Instead of using a standard system of generators of W_n given by simple reflections $s_i = (1, 2, \ldots, i+1, i, \ldots, n)$, $1 \leqslant i \leqslant n-1$, and $s_n = (1, 2, \ldots, n-1, \bar{n})$, we will use the following system of generators $S = \{s_o = (\bar{1}, 2, \ldots, n), \ \ s_1, \ldots, s_{n-1}\}$ corresponding to the basis: $(-2\varepsilon_1), \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n$. It is easy to check that (W_n, S) is a Coxeter system of type C_n. This "nonstandard" system of generators has several advantages over the standard one: it leads to easier reasonings by induction on n and the divided differences associated with it produce "stable" symplectic Schubert type polynomials (for the details concerning the latter topic - consult a recent work of S.Billey and M.Haiman [B-H]). Let us record first the formula for the length of an element $w = (\sigma, \tau) \in W_n$ w.r.t. S. This formula can be proved by induction on $l(w)$ and we leave this to the (interested in) reader.

Lemma 5.1. $l(w) = \sum\limits_{i=1}^{n} a_i + \sum\limits_{\tau_j = -1} (2b_j + 1)$, *where* $a_i := card\{p | \ p > i \ \& \ \sigma_p < \sigma_i\}$ *and* $b_j := card\{p | \ p < j \ \& \ \sigma_p < \sigma_j\}$.

In the sequel, whenever we will speak about the "length" of an element $w \in W_n$, we will have in mind the length w.r.t. S.

Let $X_n = (x_1, \ldots, x_n)$ be a sequence of indeterminates.

We now define symplectic divided differences $\partial_i : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$, $i = 0, 1, \ldots, n-1$, setting

$$\partial_0(f) = (f - s_0 f)/(-2x_1)$$
$$\partial_i(f) = (f - s_i f)/(x_i - x_{i+1}) \quad , \qquad i = 1, \ldots, n-1 \quad ,$$

where s_0 acts on $\mathbb{Z}[X_n]$ by sending x_1 to $-x_1$ and s_i - by exchanging x_i with x_{i+1} and leaving the remaining variables invariant. For every $w \in W_n$, $l(w) = l$, let $s_{i_1} \cdot \ldots \cdot s_{i_l}$ be a reduced decomposition w.r.t. S. Following the theory in [B-G-G] and [D1,2] we define $\partial_w := \partial_{s_{i_1}} \cdot \ldots \cdot \partial_{s_{i_l}}$. By loc.cit. we get a well-defined operator of degree $-l(I)$ acting on $\mathbb{Z}[X_n]$ (here, "well-defined" means: independent on the reduced decomposition chosen).

We want first to study the operator ∂_{w_o} where $w_o = (\bar{1}, \bar{2}, \ldots, \bar{n})$ is the maximal length element of W_n. To this end we need some preliminary considerations. We record the following (apparently new) identity in the ring $QPol(X_n)$ of Schur's Q-polynomials in X_n. In 5.2 - 5.4 below we will write: $e_i = e_i(X_n)$, $s_I = s_I(X_n)$, $Q_I = Q_I(X_n)$ and $\widetilde{Q}_I = \widetilde{Q}_I(X_n)$ for brevity.

Proposition 5.2. *In $QPol(X_n)$,*

$$Q_{\rho_k} = Det(a_{i,j})_{1\leqslant i,j\leqslant k},$$

where $a_{i,j} = Q_{k+1+j-2i}$ if $k+1+j-2i \neq 0$ (with $Q_i = 0$ for $i < 0$) and $a_{i,j} = 2$ if $k+1+j-2i = 0$.

Proof. We have from the theory of symmetric polynomials (see [P2] and the references therein),

$$Q_{\rho_k} = 2^k s_{\rho_k} = Det\Big(2e_{k+1+j-2i}\Big)_{1\leqslant i,j\leqslant k}.$$

By using elementary operations on successive rows (with the help of the Pieri formula - see [Mcd1] and [L-S1]), the latter determinant can be rewritten as

$$Det\Bigg(2 \sum_{\substack{hooks\ I,\\ |I|=k+1+j-2i}} s_I\Bigg)_{1\leqslant i,j\leqslant k}.$$

The degree 0 entries in this determinant are equal to 2 and the negative degree entries vanish. Since $Q_i = 2 \sum_{hooks\ I, |I|=i} s_I$, the assertion follows. □

Invoking the remark just before Proposition 4.5, we thus get:

Corollary 5.3. *In $SPol(X_n)$, $\widetilde{Q}_{\rho_k}$ is congruent to $Det(b_{i,j})_{1\leqslant i,j\leqslant k}$ modulo $\mathcal{J}$, where $b_{i,j} = e_{k+1+j-2i}$ if $k+1+j-2i \neq 0$ (with $e_i = 0$ for $i < 0$) and $b_{i,j} = 2$ if $k+1+j-2i = 0$.*

We now state:

Lemma 5.4. *In $SPol(X_n)$, $\widetilde{Q}_{\rho_n} \equiv e_n e_{n-1} \dots e_1 \equiv s_{\rho_n}$ (mod $\mathcal{J}$).*

Proof. By the corollary it is sufficient to prove that $Det(b_{i,j})_{1\leqslant i,j\leqslant n} \equiv e_n e_{n-1} \dots e_1 \equiv s_{\rho_n}$ (mod $\mathcal{J}$). Recall that $s_{\rho_n} = Det(c_{i,j})_{1\leqslant i,j\leqslant n}$ where $c_{i,j} = e_{n+1+j-2i}$ if $n+1+j-2i \neq 0$ and $c_{i,j} = 1$ if $n+1+j-2i = 0$, i.e. the matrices $(b_{i,j})$ and $(c_{i,j})$ are the same modulo the degree 0 entries.

Let us write the determinants $Det(b_{i,j})$ and $Det(c_{i,j})$ as the sums of the standard $n!$ terms (some of them are zero). It is easy to see that apart of the "diagonal" term $e_n e_{n-1} \dots e_1$, every other term appearing in both sums is divisible by $e_n e_{n-1} \dots e_{p+1} e_p^2$ for some $p \geqslant 1$. We claim that, $e_n e_{n-1} \dots e_{p+1} e_p^2 \in \mathcal{J}$. Indeed, $e_n^2 \in \mathcal{J}$ and suppose, by induction, that we have shown $e_n e_{n-1} \dots e_{q+1} e_q^2 \in \mathcal{J}$ for $q > p$. Then

$$e_n e_{n-1} \dots e_{p+1} e_p^2 = e_n e_{n-1} \dots e_{p+1} \left[\widetilde{Q}_{p,p} + 2\sum_{i=1}^{p} (-1)^{i-1} e_{p+i} e_{p-i}\right]$$

belongs to $\mathcal{J}$ by the induction assumption, because $\widetilde{Q}_{p,p} \in \mathcal{J}$ (see Proposition 4.2). This shows that

$$Det(b_{i,j}) \equiv e_n e_{n-1} \dots e_1 \equiv Det(c_{i,j}) \ (mod\ \mathcal{J}).$$

Thus the lemma is proved. $\square$

Of course the last three results and their proofs are equally valid for countable many variables.

The following known result (see, e.g., [D1]) is accompanied by a proof for the reader's convenience.

Proposition 5.5. *One has for $f \in \mathbb{Z}[X_n]$,*

$$\partial_{w_o}(f) = (-1)^{n(n+1)/2}\Big(2^n x_1 \cdot \ldots \cdot x_n \prod_{i<j}(x_i^2 - x_j^2)\Big)^{-1} \sum_{w \in W_n} (-1)^{l(w)} w(f).$$

Proof. By the definition of ∂_{w_o} we infer that $\partial_{w_o}(f) = \sum\limits_{w \in W_n} \alpha_w w$ where the coefficients α_w are rational functions in $x_1, \dots, x_n$. Since w_o is the maximal length element in W_n, $\partial_i \circ \partial_{w_o} = 0$ for all $i = 0, 1, \dots, n-1$. Consequently $s_i \partial_{w_o} = \partial_{w_o}$ for $i = 0, 1, \dots, n-1$ and hence $v\partial_{w_o} = \partial_{w_o}$ for all $v \in W_n$. In particular, for every $v \in W_n$, $\partial_{w_o} = \sum\limits_{w \in W_n} v(\alpha_w) vw$. Thus $\alpha_{vw} = v(\alpha_w)$ for all $v, w \in W_n$, and we see that, e.g., α_{w_o} determines uniquely all the α_w's.

<u>Claim</u> $\alpha_{w_o} = (-1)^{l(w_o)}\Big(2^n x_1 \cdot \ldots \cdot x_n \prod\limits_{i<j}(x_i^2 - x_j^2)\Big)^{-1}$.

Proof of the claim: Denote now the maximal length element in W_n by $w_o^{(n)}$. We argue by induction on n. For $n = 1$, we have $\alpha_{w_o^{(1)}} = -\frac{1}{2x_1}$. We now record the following equality:

$$w_o^{(k+1)} = s_k \cdot s_{k-1} \cdot \ldots \cdot s_1 \cdot s_0 \cdot s_1 \cdot \ldots \cdot s_{k-1} \cdot s_k \cdot w_o^{(k)},$$

that implies

$$\partial_{w_o^{(k+1)}} = \partial_k \circ \partial_{k-1} \circ \ldots \circ \partial_1 \circ \partial_0 \circ \partial_1 \circ \ldots \circ \partial_{k-1} \circ \partial_k \circ \partial_{w_o^{(k)}}.$$

It follows easily from the latter equality that

$$\alpha_{w_o^{(k+1)}} = (-1)^{k+1}\Big(2x_{k+1} \prod_{i \leqslant k}(x_i - x_{k+1}) \prod_{i \leqslant k}(x_i + x_{k+1})\Big)^{-1} \alpha_{w_o^{(k)}}.$$

This allows us to perform the induction step $n \to n+1$, thus proving the claim. Finally, for arbitrary $w \in W_n$,

$$\alpha_w = ww_o(\alpha_{w_o}) = (-1)^{n(n+1)/2 + l(w)}\Big(2^n x_1 \cdot \ldots \cdot x_n \prod_{i<j}(x_i^2 - x_j^2)\Big)^{-1}. \quad \square$$

Corollary 5.6. *(i)* $\partial_{w_o}(x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}) = 0$ *if* α_p *is even for some* $p = 1, \dots, n$.
(ii) If all α_p *are odd then* $\partial_{w_o}(x^\alpha) = s_{\rho_n}(X_n)^{-1}\partial(x^\alpha)$*, where here and in the sequel* ∂ *denotes the Jacobi symmetrizer* $\left(\sum_{\sigma\in S_n}(-1)^{l(\sigma)}\sigma(-)\right)/\prod_{i<j}(x_i - x_j)$.

Proof. (i) Let us fix $\sigma \in S_n$ and look at all elements of W_n of the form (σ, τ) where $\tau \in \mathbb{Z}_2^n$. Then, writing x^α for $x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}$, we have

$$\sum_\tau (-1)^{l(\sigma,\tau)}(\sigma,\tau)x^\alpha = (-1)^{l(\sigma)}x^\alpha \sum_\tau (-1)^{card\{p|\ \tau_p=-1\}}\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n},$$

because (see Lemma 5.1) $l(\sigma,\tau) = \sum a_i + \sum_{\tau_j=-1}(2b_j+1) \equiv l(\sigma) + card\{p|\ \tau_p = -1\}$ $(mod\ 2)$. Suppose that some numbers among $\alpha_1, \dots, \alpha_n$ are even. We will show that this implies

$$\sum_\tau (-1)^{card\{p|\ \tau_p=-1\}}\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n} = 0.$$

We can assume that $\alpha_1, \dots, \alpha_k$ are odd and $\alpha_{k+1}, \dots, \alpha_n$ are even for some $k < n$ (by permuting the τ_p's if necessary). We have

$$\begin{aligned}
&\sum_\tau (-1)^{card\{p|\ \tau_p=-1\}}\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n} = \\
&\qquad = \sum_\tau (-1)^{card\{p|\ \tau_p=-1\}}(-1)^{card\{p|\ \tau_p=-1,\ p\leqslant k\}} \\
&\qquad = \sum_\tau (-1)^{card\{p|\ \tau_p=-1,\ p>k\}} \\
&\qquad = 2^k\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i} = 2^k(1-1)^{n-k} = 0.
\end{aligned}$$

(ii) Let us now compute $\partial_{w_o}(x_1^{\alpha_1}\dots x_n^{\alpha_n})$ where all α_p are odd. Then

$$\sum_\tau (-1)^{card\{j|\ \tau_j=-1\}}\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n} = 2^n, \text{ and}$$

$$\begin{aligned}
\partial_{w_o}(x^\alpha) &= \Big(2^n x_1\dots x_n\prod_{i<j}(x_i^2 - x_j^2)\Big)^{-1} 2^n \sum_{\sigma\in S_n}(-1)^{l(\sigma)}\sigma(x^\alpha) \\
&= s_{\rho_n}(x_n)^{-1}\partial(x^\alpha). \quad \square
\end{aligned}$$

We now record the following properties of the operator $\nabla = \partial_{(\bar{n},\dots,\bar{2},\bar{1})}$.

Lemma 5.7. *(i) If* $f \in SPol(x_1^2, \dots, x_n^2)$ *then* $\nabla(f\cdot g) = f\cdot\nabla(g)$.
(ii) $\nabla(\widetilde{Q}_{\rho_n}(X_n)) = (-1)^{n(n+1)/2}$.

Proof. (i) This assertion is clear because every polynomial in $SPol(x_1^2, \dots, x_n^2)$ is W_n-invariant. Observe that it implies that if $f \equiv g$ (mod $\mathcal{I}$) then $\nabla(f) \equiv \nabla(g)$ (mod $\mathcal{I}$).

(ii) In this part we will use the following properties of the Jacobi symmetrizer ∂ (see [L-S2], [Mcd2]):

1. If $f \in SPol(X_n)$, $g \in \mathbb{Z}[X_n]$ then $\partial(f \cdot g) = f \cdot \partial(g)$.
2. For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\partial x^\alpha = s_{\alpha - \rho_{n-1}}(X_n)$. In particular, if $\alpha_i = \alpha_j$ for some $i \neq j$ then $\partial x^\alpha = 0$.
3. $\partial = \partial_{(n,n-1,\dots,1)}$.

Let $e_i = e_i(X_n)$. Since $\widetilde{Q}_{\rho_n}(X_n) = e_n e_{n-1} \dots e_1$ (mod $\mathcal{I}$), we have

$$\nabla(\widetilde{Q}_{\rho_n}(X_n)) = \nabla(e_n e_{n-1} \dots e_1) = (\nabla \circ \partial)(x^{\rho_{n-1}} e_n e_{n-1} \dots e_1).$$

by properties 1 and 2 above. Since

$$(\overline{n}, \overline{n-1}, \dots, \overline{1}) \circ (n, n-1, \dots, 1) = w_0,$$

the latter expression equals $\partial_{w_0}(x^{\rho_{n-1}} e_n e_{n-1} \dots e_1)$ by property 3. The degree of the polynomial $x^{\rho_{n-1}} e_n e_{n-1} \dots e_1$ is n^2. Assuming that $\alpha_1 + \dots + \alpha_n = n^2$, we have $\partial_{w_0}(x^\alpha) \neq 0$ only if $x^\alpha = x_{w(1)}^{2n-1} x_{w(2)}^{2n-3} \dots x_{w(n)}$ for some $w \in S_n$. Indeed, it follows from Corollary 5.6(i) that $\partial_{w_0}(x^\alpha) \neq 0$ only if all the α_i's are odd. Moreover, they must be all different; otherwise $\partial x^\alpha = 0$ (and consequently $\partial_{w_o} x^\alpha = 0$) by property 2. We conclude that $\{\alpha_1, \dots, \alpha_n\} = \{2n-1, 2n-3, \dots, 1\}$. But there is only one such a monomial x^α in $x^{\rho_{n-1}} e_n e_{n-1} \dots e_1$, namely the one with $(\alpha_1, \dots, \alpha_n) = (2n-1, 2n-3, \dots, 1)$. Therefore

$$\partial_{w_o}(x^{\rho_{n-1}} e_n e_{n-1} \dots e_1) = \partial_{w_0}(x_1^{2n-1} x_2^{2n-3} \dots x_n) = (-1)^{n(n+1)/2}$$

by Corollary 5.6(ii) and property 2. □

We now pass to a geometric interpretation of the operator ∇.

Proposition 5.8. *Specializing the variables $x_1, \dots, x_n$ to the Chern roots $q_1, \dots, q_n$ of the tautological quotient vector bundle on LG_nV (which is isomorphic to $R^\vee$), one has the equality*

$$\pi_*(f(q_1, \dots, q_n)) = \left(\partial_{(\overline{n}, \overline{n-1}, \dots, \overline{2}, \overline{1})} f\right)(q_1, \dots, q_n),$$

where $f(-)$ is a polynomial in n variables.

Proof. We have, e.g. by comparing the results of [A-C] and [D1], the equalities:

$$\tau_*(f(q_1, \dots, q_n)) = \left(\partial_{(\overline{1}, \overline{2}, \dots, \overline{n})} f\right)(q_1, \dots, q_n) \quad \text{and}$$

$$\omega_*(f(q_1, \dots, q_n)) = \left(\partial_{(n,n-1,\dots,1)} f\right)(q_1, \dots, q_n)$$

Since

$$(\overline{1}, \overline{2}, \dots, \overline{n}) = (\overline{n}, \overline{n-1}, \dots, \overline{1}) \circ (n, n-1, \dots, 1),$$

we get

$$\partial_{(\overline{1}, \overline{2}, \dots, \overline{n})} = \partial_{(\overline{n}, \overline{n-1}, \dots, \overline{1})} \circ \partial_{(n,n-1,\dots,1)}.$$

Of course, $\tau_* = \pi_* \circ \omega_*$. Since ω_* is surjective, comparison of the latter equation with the former implies the desired assertion about π_*. □

We now show how to compute the images via π_* of $\widetilde{Q}$-polynomials in the Chern classes of $R^\vee$. Let us write $X_n^\vee = (-x_1, \dots, -x_n)$ for brevity.

Proposition 5.9. *One has $\nabla(\widetilde{Q}_I(X_n^\vee)) \neq 0$ iff the set of parts of I is equal to $\{1,2,\dots,n\}$ and each number p $(1 \leqslant p \leqslant n)$ appears in I with an odd multiplicity m_p. Then, the following equality holds in $\mathbb{Z}[X_n]$,*

$$\nabla\big(\widetilde{Q}_I(X_n^\vee)\big) = \prod_{p=1}^{n} e_p(x_1^2,\dots,x_n^2)^{(m_p-1)/2}.$$

Proof. By Proposition 4.3 we can express $\widetilde{Q}_I(X_n^\vee)$ as

$$\widetilde{Q}_I(X_n^\vee) = \widetilde{Q}_{j_1,j_1}(X_n^\vee)\ \dots\ \widetilde{Q}_{j_l,j_l}(X_n^\vee)\widetilde{Q}_L(X_n^\vee),$$

where L is a strict partition. (We divide the elements of the multiset I into pairs of equal elements and the set L whose elements are all different.) Some of the j_p's can be mutually equal.

By Proposition 4.2, $\widetilde{Q}_{j,j}(X_n^\vee) = e_j(x_1^2,\dots,x_n^2)$ is a scalar w.r.t. ∇.

By Lemma 4.4, $\widetilde{Q}_L(X_n^\vee) \neq 0$ only if $L \subset \rho_n$. On the other hand, for a strict partition $L \subset \rho_n$, $\nabla(\widetilde{Q}_L(X_n^\vee)) \neq 0$ only if $L = \rho_n$, when it is equal to 1 (see Lemma 5.7(ii)).

Putting this information together, the assertion follows. □

Consequently, specializing (x_i) to the Chern roots (r_i) of the tautological subbundle on $LG_n(V)$ we have

Theorem 5.10. *The element $\widetilde{Q}_I R^\vee$ has a nonzero image under $\pi_* : A^*(LG_nV) \to A^*(X)$ (resp. $\pi_* : H^*(LG_nV,\mathbb{Z}) \to H^*(X,\mathbb{Z})$) only if each number p, $1 \leqslant p \leqslant n$, appears as a part of I with an odd multiplicity m_p. If the latter condition takes place then*

$$\pi_*\widetilde{Q}_I R^\vee = \prod_{p=1}^{n}\big((-1)^p c_{2p}V\big)^{(m_p-1)/2}.$$

Proof. This follows from Proposition 5.9 and the equality $c_{2p}V = (-1)^p e_p(r_1^2,\dots,r_n^2)$. □

Our next goal will be to show how to compute the images via π_* of S-polynomials in the Chern classes of the tautological Lagrangian bundle. To this end we record the following identity of symmetric polynomials. We have found this simple and remarkable identity during our work on isotropic Gysin pushforwards and have not seen it in the literature.

Proposition 5.11. *For every partition $I = (i_1,\dots,i_n)$ and any positive integer p, one has in $SPol(X_n)$,*

$$s_I\big(x_1^p,\dots,x_n^p\big)\cdot s_{(p-1)\rho_{n-1}}(X_n) = s_{pI+(p-1)\rho_{n-1}}(X_n).$$

where, given a partition $I = (i_1,i_2,\dots)$, we write $pI = (pi_1,pi_2,\dots)$.

Proof. We use the Jacobi presentation of a Schur polynomial as a ratio of two alternants (see [Mcd1], [L-S1]). We have:

$$s_I(x_1^p, \dots, x_n^p) = \frac{Det(x_k^{(i_l+n-l)p})_{1\leqslant k,l\leqslant n}}{Det(x_k^{p(n-l)})_{1\leqslant k,l\leqslant n}}$$

$$= \frac{Det(x_k^{pi_l+(n-l)(p-1)+(n-l)})_{1\leqslant k,l\leqslant n}}{Det(x_k^{n-l})_{1\leqslant k,l\leqslant n} \cdot \left(\frac{Det(x_k^{(p-1)(n-l)+(n-l)})_{1\leqslant k,l\leqslant n}}{Det(x_k^{n-l})_{1\leqslant k,l\leqslant n}} \right)}$$

$$= \frac{s_{pI+(p-1)\rho_{n-1}}(X_n)}{s_{(p-1)\rho_{n-1}}(X_n)}. \quad \square$$

Corollary 5.12. *For $p = 2$ we get*

$$s_I(x_1^2, \dots, x_n^2) \cdot s_{\rho_{n-1}}(X_n) = s_{2I+\rho_{n-1}}(X_n).$$

(For another derivation of this identity with the help of Quaternionic Grassmannians see the Appendix.)

Our goal is to give a geometric translation of the latter formula, or rather its consequence

$$(*) \qquad s_I(x_1^2, \dots, x_n^2) \cdot s_{\rho_n}(X_n) = s_{\rho_n+2I}(X_n).$$

Theorem 5.13. *The element $s_I R^\vee$ has a nonzero image under π_* only if the partition I is of the form $2J + \rho_n$ for some partition J. If $I = 2J + \rho_n$ then*

$$\pi_* s_I R^\vee = s_J^{[2]} V \quad ,$$

where the right hand side is defined as follows: if $s_J = P(e.)$ is a unique presentation of s_J as a polynomial in the elementary symmetric functions e_i, E – a vector bundle, then $s_J^{[2]}(E) := P$ with e_i replaced by $(-1)^i c_{2i} E \quad (i = 1, 2, \dots)$.

Proof. Since $s_I R^\vee = \omega_*(q^{I+\rho_{n-1}})$ where $q = (q_1, \dots, q_n)$ are the Chern roots of $R^\vee$ (see [P1,2], for instance), we infer from Corollary 5.6(i) that $s_I R^\vee$ has a nonzero image under π_* only if all parts of $I + \rho_{n-1}$ are odd. This implies that $l(I) = n$ and I is strict thus of the form $I' + \rho_n$ for some partition I'. Finally all parts of $I' + \rho_n + \rho_{n-1}$ are odd iff $I' = 2J$ for some partition J, as required.

Assume now that $I = 2J + \rho_n$ and specialize the identity (*) by replacing the variables (x_i) by the Chern roots (q_i). The claimed formula now follows since: $s_I(q_1^2, \dots, q_n^2)$ is a scalar w.r.t. π_*, $\pi_* s_{\rho_n}(q_1, \dots, q_n) = 1$ by Lemma 5.7(ii) combined with Lemma 5.4; finally $(-1)^i c_{2i} V = e_i(q_1^2, \dots, q_n^2)$ because of Lemma 1.1(2). $\square$

Observe that the theorem contains an explicit calculation of the ratio in Corollary 5.6(ii).

We now pass to the odd orthogonal case. The Weyl group W_n of type B_n. is isomorphic to $S_n \ltimes \mathbb{Z}_2^n$ and its elements are "barred-permutations". We use the following system of generators of W_n: $S = \{s_o = (\bar{1}, 2, \ldots, n),\quad s_1, \ldots, s_{n-1}\}$ corresponding to the basis $(-\varepsilon_1), \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, ..., \varepsilon_{n-1} - \varepsilon_n$. Consequently, the divided differences ∂_i, $i = 1, \ldots, n-1$, are the same but $\partial_0(f) = (f - s_0 f)/(-x_1)$.

Theorem 5.14. *The element $\widetilde{Q}_I R^\vee$ has a nonzero image under $\pi_* : A^*(OG_nV) \to A^*(X)$ (resp. $\pi_* : H^*(OG_nV, \mathbb{Z}) \to H^*(X, \mathbb{Z})$) only if each number p, $1 \leqslant p \leqslant n$, appears as a part of I with an odd multiplicity m_p. If the latter condition takes place then*

$$\pi_* \widetilde{Q}_I R^\vee = 2^n \prod_{p=1}^{n} \left((-1)^p c_{2p} V\right)^{(m_p - 1)/2}.$$

This holds because the calculation in Proposition 5.9 now goes as follows: with the notation from the proof of Proposition 5.9, the polynomial

$$\widetilde{Q}_I(X_n^\vee) = 2^n \widetilde{Q}_{j_1, j_1}(X_n^\vee) \ \ldots\ \widetilde{Q}_{j_l, j_l}(X_n^\vee) \widetilde{P}_{\rho_n}(X_n^\vee)$$

is mapped via $\partial_{(\overline{n}, \overline{n-1}, \ldots, \overline{1})}$ to

$$2^n \prod_{h=1}^{l} e_{j_h}(x_1^2, ..., x_n^2),$$

since $\partial_{(\overline{n}, \overline{n-1}, \ldots, \overline{1})}(\widetilde{P}_{\rho_n}(X_n^\vee)) = 1$. (The proof of the last statement is the same as that of Lemma 5.7(ii).)

The analog of Proposition 5.5 reads

$$\partial_{w_0}(f) = (-1)^{n(n+1)/2} \Big(x_1 \cdot \ldots \cdot x_n \prod_{i<j}(x_i^2 - x_j^2)\Big)^{-1} \sum_{w \in W_n} (-1)^{l(w)} w(f).$$

The analog of Theorem 5.13 now reads:

Theorem 5.15. *The element $s_I R^\vee$ has a nonzero image under π_* only if the partition I is of the form $2J + \rho_n$ for some partition J. If $I = 2J + \rho_n$ then*

$$\pi_* s_I R^\vee = 2^n s_J^{[2]} V,$$

where $s_J^{[2]}(-)$ is defined as in Theorem 5.13.

This holds because $s_{\rho_n}(X_n^\vee)$ is congruent to $2^n \widetilde{P}_{\rho_n}(X_n^\vee)$ modulo $\mathcal{J}$ (Lemma 5.4) and $\pi_* \widetilde{P}_{\rho_n} R^\vee = 1$. Also, we use Lemma 1.1(2).

The even orthogonal case can be deduced from the odd one as follows. Let V be a vector bundle of rank $2n$ over X (X is connected) endowed with a nondegenerate orthogonal form. Let W denote a rank $(2n-1)$ subbundle of V such that the form restricted from V to W is nondegenerate. Then we have $OG'_n V \simeq OG_{n-1} W$ and similarly $OG''_n V \simeq OG_{n-1} W$ (see [G-Z, Lemma 18]). Via these identifications $\widetilde{Q}_I(R_V^\vee)$ corresponds to $\widetilde{Q}_I(R_W^\vee)$ where R_V denotes the tautological subbundle on $OG'_n V$ and $OG''_n V$, and R_W denotes the tautological subbundle on $OG_{n-1} W$ (thus I runs over partitions $\subset \rho_{n-1}$). The analogs of Theorems 5.10 and 5.13 now read with $R = R_V$ and $\pi : OG'_n V \to X$ or $\pi : OG''_n \to X$.

Theorem 5.16. *The element* $\widetilde{Q}_I R^\vee$ $(I \subset \rho_{n-1})$ *has a nonzero image under* π_* *only if each number* p, $1 \leqslant p \leqslant n-1$, *appears as a part of* I *with an odd multiplicity* m_p. *If the latter condition takes place then*

$$\pi_* \widetilde{Q}_I R^\vee = 2^{n-1} \prod_{p=1}^{n-1} \Big((-1)^p c_{2p} V \Big)^{(m_p - 1)/2}.$$

Theorem 5.17. *The element* $s_I R^\vee$ $(l(I) \leqslant n-1)$ *has a nonzero image under* π_* *only if the partition* I *is of the form* $2J + \rho_{n-1}$ *for some partition* J $(l(J) \leqslant n-1)$. *If* $I = 2J + \rho_{n-1}$, *then*

$$\pi_* s_I R^\vee = 2^{n-1} s_J^{[2]} V,$$

where $s_J^{[2]}(-)$ *is defined as in Theorem 5.13.*

Remark 5.18. 1. Our desingularizations of Schubert subschemes are compositions of Flag- and Isotropic Grassmannian bundles (see Section 1). Therefore Corollary 2.6, the algebra of $\widetilde{Q}$-polynomials together with formulas for Gysin push forwards (Theorem 5.10 for Lagrangian Grassmannians and a well known formula for Projective bundles) give an explicit algorithm for calculation the fundamental classes of Schubert subschemes in the Lagrangian Grassmannian bundles. One has analogous algorithms in the orthogonal cases. Examples of such calculations are given in Section 6 and 7.

2. In case X is singular, by interpreting polynomials in Chern classes as operators acting on Chow groups (see [F]) or singular homology groups, the same formulas hold (after an obvious adaptation of them to the operator setup).

6. Special Schubert subschemes

We consider the Lagrangian case $G = LG_n V$ and follow the notation introduced in Section 1. The result here is stated in the Chow rings but it is equally valid in the cohomology rings.

Proposition 6.1. *The class of* $\Omega(a)$ *in* $A^*(G)$, *where* $a = n + 1 - i$, *is given by the formula*

$$\big[\Omega(a)\big] = \sum_{p=0}^{i} c_p(R^\vee) \cdot s_{i-p}(V_a^\vee).$$

Proof. The desingularization $\mathcal{F}$ of $\Omega(a) \subset G$ is given by the composition:

$$\mathcal{F} = LG_{n-1}(C^\perp / C) \xrightarrow{\pi_1} \mathbb{P}(V_a) \xrightarrow{\pi_2} G\ ,$$

where π_1 and π_2 denote the corresponding projection maps. By Corollary 2.5 we have

$$(*) \qquad [Z] = \sum_{strict\ I \subset \rho_n} \widetilde{Q}_I D^\vee \cdot \widetilde{Q}_{\rho_n \setminus I} R^\vee.$$

Let S be the tautological rank $n-1$ bundle on $\mathcal{F}$; $S = D/C_{\mathcal{F}}$. Let $c = c_1(C_{\mathcal{F}}^{\vee})$. Then, by Proposition 4.1,

$$(^{**}) \qquad \widetilde{Q}_I D^{\vee} = \sum_{k=0}^{n} c^k \cdot \sum_J \widetilde{Q}_J S^{\vee} \ ,$$

the sum over all partitions $J \subset I$ of weight $|J| = |I| - k$ and $I \setminus J$ has at most one box in each row. By Theorem 5.10 the only I's in (*) for which $(\pi_1)_* \widetilde{Q}_I D^{\vee} \neq 0$, are those one containing ρ_{n-1}, i.e.

$$I = \big(n, n-1, \ldots, p+1, p-1, \ldots, 1\big)$$

for some $p = 0, 1, \ldots, n$. (Note that $[D^{\vee}] = [S^{\vee}] + [C_{\mathcal{F}}^{\vee}]$ and $J \subset I$.) Then the only term in (**) which contributes nontrivially is the one with $J = \rho_{n-1}$ and $k = n - p$.

Since, by a well-known push forward formula for Projective bundles, we have

$$(\pi_2)_*\big(c^{n-p}\big) = s_{n-p-(n-i)}\big(V_a^{\vee}\big) = s_{i-p}\big(V_a^{\vee}\big) \ ,$$

we infer that only $p = 0, 1, \ldots, i$ give a nontrivial contribution from (**) (with $k = n-p$). Finally, we get

$$\big[\Omega(a)\big] = (\pi_2\pi_1)_*[Z] = \sum_{p=0}^{i} \widetilde{Q}_p(R^{\vee}) \cdot s_{i-p}(V_a^{\vee}) = \sum_{p=0}^{i} c_p(R^{\vee}) \cdot s_{i-p}(V_a^{\vee})$$

as asserted. □

A similar formula can be deduced in the orthogonal cases. We leave this to the (interested in) reader.

7. Two Schubert conditions

We consider the Lagrangian case. The results here are stated in the Chow rings but they are equally valid in the cohomology rings. Our desingularization in case $a. = (n+1-i, n+1-j)$ is the composition (we use the notation of Section 1, *rank* $C = 2$):

$$\mathcal{F} = LG_{n-2}(C^{\perp}/C) \xrightarrow{\pi_1} Fl(V_a \subset V_b) \xrightarrow{\pi_2} G,$$

where $(a, b) = (n+1-i, n+1-j)$ and the element to be push forwarded via $(\pi_2\pi_1)_*$ is $\sum \widetilde{Q}_I D^{\vee} \cdot \widetilde{Q}_{\rho_n \setminus I} R^{\vee}$, the sum over all strict $I \subset \rho_n$. Let S be the tautological rank $(n-2)$ bundle on $LG_{n-2}(C^{\perp}/C)$; $S = D/C_{\mathcal{F}}$. It follows from Theorem 5.10, using $[D^{\vee}] = [S^{\vee}] + [C_{\mathcal{F}}^{\vee}]$, that the unique I's for which $(\pi_1)_* \widetilde{Q}_I(D^{\vee}) \neq 0$ are of the form $I = \rho_n$, $I = (n, n-1, \ldots, \hat{p}, \ldots, 1) =: I_p$, $I = (n, n-1, \ldots, \hat{p}, \ldots, \hat{q}, \ldots, 1) =: I_{p,q}$ (here, p and q run over $\{1, \ldots, n\}$ and the symbol " ^ " indicates the corresponding omission).

We need the following technical lemma.

Lemma 7.1. *If rank* $C = 2$ *then*

(i) $\widetilde{Q}_{I_p/\rho_{n-2}}(C^\vee) = s_{n-1,n-p}(C^\vee)$;

(ii) *For* $q < p$, $\widetilde{Q}_{I_{p,q}/\rho_{n-2}}(C^\vee) = s_{n-q-1,n-p}(C^\vee)$;

(iii) *For* $0 \leqslant v \leqslant n-1$, $\widetilde{Q}_{\rho_n/(\rho_{n-2}+(2)^v)^\sim}(C^\vee) = s_{n-v,n-v-1}(C^\vee)$.

Proof. The proof is an easy application of the linearity formula from Proposition 4.1 and is given here in case (i) (the proofs of (ii) and (iii) being similar).

Denote the Chern roots of $C^\vee$ by x_1, x_2. We apply first Proposition 4.1 w.r.t. x_1 and then – w.r.t. x_2. Consider the skew Ferrers' diagram of I_p/ρ_{n-2} and fill up the boxes, whose subtraction correspond to the summands in Proposition 4.1 w.r.t. x_1, with "1". Then fill up the boxes, whose subtraction correspond to summands in Proposition 4.1 w.r.t. x_2, with "2". Of course it is impossible to have two "1" or two "2" in one row. Also, the following configuration cannot appear:

$$\begin{array}{cc} \mathrm{x} & 1 \\ 2 & \end{array}$$

where the box "x" belong to $D_{\rho_{n-2}}$ (Having two equal rows ending with $\begin{smallmatrix} x \\ 2 \end{smallmatrix}$ we use Proposition 4.3, thus we must subtract both boxes instead of the lower one only). For example, for $n = 6$, $p = 3$ we get two Ferrers' diagrams, one contained in another (depicted with "." and "x"):

$$\begin{array}{cccccc} \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \cdot & \cdot \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \cdot & \cdot & \\ \mathrm{x} & \mathrm{x} & \cdot & \cdot & & \\ \mathrm{x} & \cdot & & & & \\ \cdot & & & & & \end{array}$$

and we have 3 possibilities:

$$\begin{array}{cccccc} & & & & 2 & 1 \\ & & & 2 & 1 & \\ & & 2 & 1 & & \\ & 2 & & & & \\ 2 & & & & & \end{array} \qquad \begin{array}{cccccc} & & & & 2 & 1 \\ & & & 2 & 1 & \\ & & 2 & 1 & & \\ & 2 & & & & \\ 1 & & & & & \end{array} \qquad \begin{array}{cccccc} & & & & 2 & 1 \\ & & & 2 & 1 & \\ & & 2 & 1 & & \\ & 1 & & & & \\ 1 & & & & & \end{array}$$

giving $Q_{I_3/\rho_4}(x_1, x_2) = (x_1x_2)^3(x_1^2 + x_1x_2 + x_2^2) = s_{5,3}(x_1, x_2)$. In general, arguing in the same way, we get

$$\begin{aligned} Q_{I_p/\rho_{n-2}}(x_1, x_2) &= (x_1x_2)^{n-p}(x_1^{p-1} + x_1^{p-2}x_2 + \ldots + x_2^{p-1}) = \\ &= e_2(x_1, x_2)^{n-p}s_{p-1}(x_1, x_2) = s_{n-1,n-p}(x_1, x_2). \quad \square \end{aligned}$$

Lemma 7.2. *With the above notation we have:*

(i) For $q < p$, $(\pi_1)_*(\widetilde{Q}_{I_{p,q}}D^\vee) = s_{n-q-1,n-p}(C^\vee)$;

(ii) $(\pi_1)_*(\widetilde{Q}_{I_p}D^\vee) = s_{n-1,n-p}(C^\vee)$;

(iii) $$(\pi_1)_*(\widetilde{Q}_{\rho_n}D^\vee) = \sum_{k=0}^{n-2}(-1)^k c_{2k}V \cdot \Big[s_{n-k,n-k-1}(C^\vee) - s_{n-k+1,n-k-2}(C^\vee) + \dots$$
$$\dots + (-1)^{n-k}s_{2(n-k-1),1}(C^\vee)\Big].$$

Proof. Assertions (i) and (ii) follow immediately from Lemma 7.1(i),(ii) and Theorem 5.10. As for (iii), we have (below, $(\pi_1)_*($ *other terms* $) = 0$):

$$(\pi_1)_*(\widetilde{Q}_{\rho_n}D^\vee) =$$

$$= (\pi_1)_*\Big[\sum_{v=0}^{n-2}\widetilde{Q}_{(\rho_{n-2}+(2)^v)^\sim}(S^\vee)\cdot\widetilde{Q}_{\rho_n/(\rho_{n-2}+(2)^v)^\sim}(C^\vee_{\mathcal{F}}) + \ (\textit{other terms})\Big]$$

$$= \sum_{v=0}^{n-2}(-1)^v c_{2v}(C^\perp/C)\cdot\widetilde{Q}_{\rho_n/(\rho_{n-2}+(2)^v)^\sim}(C^\vee)$$

$$= \sum_{v=0}^{n-2}(-1)^v\Big[\sum_{k+l=v}c_{2k}V\cdot s_{2l}(C\oplus C^\vee)\Big]\cdot s_{n-v,n-v-1}(C^\vee)$$

$$= \sum_{k=0}^{n-2}(-1)^k c_{2k}V\cdot\Big[\sum_{l=0}^{n-2-k}(-1)^l s_{2l}(C\oplus C^\vee)\cdot s_{n-k-l,n-k-l-1}(C^\vee)\Big]$$

$$= \sum_{k=0}^{n-2}(-1)^k c_{2k}V\cdot\Big[s_{n-k,n-k-1}(C^\vee) - s_{n-k+1,n-k-2}(C^\vee) + \dots$$

$$\dots + (-1)^{n-k}s_{2(n-k-1),1}(C^\vee)\Big],$$

where the above equalities follow from: Theorem 5.10, Lemma 1.1 and Pieri's formula ([Mcd1], [L-S1]); recall that *rank* $C = 2$. $\square$

Lemma 7.3. *Let $a < b$ and $k \geqslant l$ be arbitrary positive integers. Let C be the rank 2 tautological (sub)bundle of $\tau : Fl(a,b) \to X$. Then*

$$\tau_* s_{k,l}(C^\vee) = s_{l-(a-1)}(V_a^\vee)\cdot s_{k-(b-2)}(V_b^\vee) - s_{k-(a-2)}(V_a^\vee)\cdot s_{l-(b-1)}(V_b^\vee).$$

Proof. Let $C_1 \subset C_2 = C$ be the tautological subbundles on $Fl(a,b)$, $C_1 \subset V_a$, $C_2 \subset V_b$; *rank* $C_h = h$, $h = 1,2$. Let $x_1 = c_1(C_1^\vee)$ and $x_2 = c_1((C_2/C_1)^\vee)$. Consider the presentation of $\tau : Fl(a,b) \to X$ in the form of the composition:

$$\mathbb{P}((V_b/C_1)^\vee) \xrightarrow{\pi_1} \mathbb{P}(V_a^\vee) \xrightarrow{\pi_2} X.$$

We have

$$\tau_* s_{k,l}(C^\vee) = \tau_*\Big[(x_1x_2)^l(x_1^{k-l} + x_1^{k-l-1}x_2 + \ldots + x_1x_2^{k-l-1} + x_2^{k-l})\Big].$$

The assertion now follows by applying to all summands the well known formulas:

$$(\pi_1)_*(x_2^p) = s_{p-(b-2)}(V_b^\vee/C_1^\vee) = s_{p-(b-2)}(V_b^\vee) - s_{p-(b-2)-1}(V_b^\vee)\cdot x_1,$$
$$(\pi_2)_*(x_1^p) = s_{p-(a-1)}(V_a^\vee)$$

and simplifying. □

Theorem 7.4. *For $i > j > 0$ one has in $A^*(G)$ with $a = n+1-i$, $b = n+1-j$,*

$$[\Omega(a,b)] = \sum_{\substack{p>q\geqslant 0\\ p\leqslant i,q\leqslant j}} \widetilde{Q}_{p,q}R^\vee\cdot\big(s_{i-p}(V_a^\vee)\cdot s_{j-q}(V_b^\vee) - s_{i-q}(V_a^\vee)\cdot s_{j-p}(V_b^\vee)\big)+$$
$$+\sum_{k=0}^{i-1}\sum_{p\geqslant 1}(-1)^{k+p-1}c_{2k}V\cdot\big(s_{-k+i-p}(V_a^\vee)\cdot s_{-k+j+p}(V_b^\vee) - s_{-k+i+p}(V_a^\vee)\cdot s_{-k+j-p}(V_b^\vee)\big),$$

where we assume $s_h(-) = 0$ for $h < 0$.

Proof. It follows from Lemma 7.2 that

$$[\Omega(a,b)] = \sum_{0\leqslant q<p}(\pi_2)_*\big(s_{n-q-1,n-p}(C^\vee)\big)\cdot\widetilde{Q}_{p,q}R^\vee +$$
$$+\sum_{k=0}^{n-2}(-1)^k c_{2k}V\cdot(\pi_2)_*\Big[s_{n-k,n-k-1}(C^\vee) - s_{n-k+1,n-k-2}(C^\vee) + \ldots$$
$$+(-1)^{n-k}s_{2(n-k-1),1}(C^\vee)\Big].$$

Applying Lemma 7.3 to $\pi_2 : Fl(a,b)\to X$, the assertion follows. □

Example 7.5. 1. For $i = 2, j = 1$ and any n the formula reads:

$$\widetilde{Q}_{21}R^\vee + \widetilde{Q}_2R^\vee\cdot s_1V_n^\vee + \widetilde{Q}_1R^\vee\cdot(s_1V_{n-1}^\vee\cdot s_1V_n^\vee - s_2V_{n-1}^\vee)+$$
$$+(s_1V_{n-1}^\vee\cdot s_2V_n^\vee - s_3V_{n-1}^\vee - s_3V_n^\vee - c_2V\cdot s_1V_n^\vee) =$$
$$= \widetilde{Q}_{21}R^\vee + \widetilde{Q}_2R^\vee\cdot\widetilde{Q}_1V_n^\vee + \widetilde{Q}_1R^\vee\cdot\widetilde{Q}_2V_n^\vee + \widetilde{Q}_{21}V_n^\vee.$$

2. For $i = 3, j = 1$ and any n one obtains, with $\widetilde{Q}_{p,q} = \widetilde{Q}_{p,q}R^\vee$, $s_k = s_k(V_{n-2}^\vee)$ and $s'_k = s_k(V_n^\vee)$, the expression:

$$\widetilde{Q}_{31} + \widetilde{Q}_3 \cdot s_1' + \widetilde{Q}_{21} \cdot s_1 + \widetilde{Q}_2 \cdot s_1 \cdot s_1' + \widetilde{Q}_1 \cdot (s_2 \cdot s_1' - s_3) + \\ + s_2 \cdot s_2' - s_4 - s_1 \cdot s_3' + s_4' - c_2 V \cdot (s_1 \cdot s_1' - s_2') + c_4 V.$$

3. For $i = 3, j = 2$ and any n one obtains, with $\widetilde{Q}_{p,q} = \widetilde{Q}_{p,q} R^\vee$ and $s_{k,l} = s_{k,l}(V_{n-1}^\vee)$, the expression:

$$\widetilde{Q}_{32} + \widetilde{Q}_{31} \cdot s_1 + \widetilde{Q}_3 \cdot s_2 + \widetilde{Q}_{21} \cdot s_{11} + \widetilde{Q}_2 \cdot s_{21} + \widetilde{Q}_1 \cdot s_{22} + s_{32} - s_{41} + s_5 - c_2 V \cdot (s_{21} - s_3) + c_4 V \cdot s_1.$$

More generally we have:

Corollary 7.6. *With the above notation and $j = i - 1$, $s_{k,l} = s_{k,l}(V_{n+2-i}^\vee)$, the class $[\Omega(a,b)]$ equals*

$$\sum_{i \geqslant p > q \geqslant 0} \widetilde{Q}_{p,q} R^\vee \cdot s_{i-1-q,i-p} + \sum_{k=0}^{i-1} (-1)^k c_{2k} V \cdot \sum_{h=0}^{i-1-k} (-1)^h s_{-k+i+h,-k-1+i-h}.$$

Similar formulas can be deduced in the orthogonal cases. We leave this to the (interested in) reader.

8. Section 3 revisited via the operator approach

The goal of this Section is to provide another proofs of the main results of Section 3 by using divided differences operators. We start with the Lagrangian case. The methods here are used in the context of the Chow rings but they equally work in the cohomology rings. Let $X_n = (x_1, \dots, x_n)$ be a sequence of indeterminates. Recall (see Section 5) that the symplectic Weyl group W_n is isomorphic to $S_n \ltimes \mathbb{Z}_2^n$ and the elements of W_n are identified with "barred permutations": if $w = (\sigma, \tau)$, $\sigma \in S_n$, $\tau \in \mathbb{Z}_2^n$ then we write w as the sequence $(w_1, \dots, w_n)$ endowed with bars on places where $\tau_i = -1$. In particular, $w_0 = (\overline{1}, \overline{2}, \dots, \overline{n})$ is the longest element of W_n. Consider in W_n the poset $W^{(n)}$ of minimal length left coset representatives of W_n modulo its subgroup generated by reflections corresponding to the simple roots $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n$ (in the standard notation):

$$W^{(n)} = \Big\{ (\overline{z}_1 > \overline{z}_2 > \dots > \overline{z}_l; y_1 < \dots < y_{n-l}) \in W_n,\ l = 0, 1, \dots, n \Big\}.$$

The assignment $w = (\overline{z}_1, \dots, \overline{z}_l; y_1, \dots, y_{n-l}) \mapsto I = (z_1, \dots, z_l)$ establishes a bijection between the poset $W^{(n)}$ and the poset of all strict partitions contained in ρ_n. One has divided differences $\partial_w : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$ $(w \in W)$ i.e. operators of degree $-l(w)$, whose definition has been explained in Section 5. ∂_w induces an operator on $A^*(Sp(V)/B)$ which will be denoted by the same symbol, for brevity. (We specialize (x_i) to the Chern roots of the tautological subbundle on $LG_n V$.) It will be clear from the context in which ring ∂_w actually acts.

Let V be an $2n$-dimensional vector space endowed with a nondegenerate symplectic form. Let B be a Borel subgroup in $Sp(V)$ and B^- its opposite. Then with every

$w \in W_n$ one associates the Schubert cycle $X_w = [B^- wB/B]$ in $A^*(Sp(V)/B)$. Note that $Sp(V)/B \simeq LFl(V)$ in the previous notation. The latter ring is isomorphic to $\mathbb{Z}[X_n]$ modulo the ideal $\mathcal{I}$ generated by $e_i(x_1^2, \ldots, x_n^2)$, $1 \leqslant i \leqslant n$ (see [B]). We have

$$A^*(LG_n V) = A^*(Sp(V)/B)^{W_n} \subset A^*(Sp(V)/B),$$

and, denoting by w_I the element of $W^{(n)}$ that corresponds to a strict partition $I \subset \rho_n$, these are precisely X_{w_I}, I–strict $\subset \rho_n$, that, among all X_w's belong to $A^*(LG_n V)$. The following fact comes from comparison of the results from [B-G-G] and [D2] with [P, Theorem 6.17] recalled in Theorem 2.1(i) (see also a recent work of Billey and Haiman [B-H] for an alternative proof).

Theorem 8.1. *For every strict partition $I \subset \rho_n$, one has in $A^*(LG_n V)$,*

$$X_{w_I} = \widetilde{Q}_I R^\vee = \partial_{w_I^{-1} w_0} X_{w_0}.$$

where R is the tautological subbundle on $LG_n V$.

Algebraically, this means that applying the operator $\partial_{w_I^{-1} w_0}$ to a representative of X_{w_0} in $\mathbb{Z}[X_n]$, one gets (modulo $\mathcal{I}$) the polynomial $\widetilde{Q}_I(X_n)$.

Note that if we replace V by a vector bundle $V \to B$, then the right hand side equality in Theorem 8.1 holds with X_{w_0} replaced through a generator of the top degree component of $\mathbb{Z}[X_n]/\mathcal{I}$, e.g., the one equal to $x_1^{2n+1} x_2^{2n-1} \ldots x_n$.

Fix now an integer $0 < k < n$ and denote:

$$w^{(k)} := (\overline{n}, \overline{n-1}, \ldots, \overline{k+1}; 1, 2, \ldots, k).$$

Observe first that for a strict partition $I \subset \rho_n$ of length $l(I)$, $\partial_{w^{(k)}} \widetilde{Q}_I(X_n) \neq 0$ only if $l(I) \geqslant n - k$. (This is because $\partial_{w^{(k)}}$ decreases the degree by $l(w^{(k)}) = n + (n-1) + \ldots + (k+1)$.) More precisely, we have:

Proposition 8.2. *For a strict partition I of length$\geqslant n-k$, $\partial_{w^{(k)}} \widetilde{Q}_I(X_n) \neq 0$ iff $I \supset (n, n-1, \ldots, k+1)$. In the latter case, writing $I = (n, n-1, \ldots, k+1, j_1, \ldots, j_l)$, where $j_l > 0$ and $l \leqslant k$, one has*

$$\partial_{w^{(k)}} \widetilde{Q}_I(X_n) \equiv \widetilde{Q}_{j_1, \ldots, j_l}(X_n) \ \ (mod\ \mathcal{J}).$$

(This is a congruence in $SPol(X_n)$; recall that $\mathcal{J} = \mathcal{I} \cap SPol(X_n)$.)

Proof. Let I be a strict partition of length $h \geqslant n-k$. Let

$$w_I = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_h; w_{h+1}, \ldots, w_n)$$

be the element of $W^{(n)}$ corresponding to I. Then taking into account that

$$(w^{(k)})^{-1} = (n-k+1, n-k+2, \ldots, n; \overline{n-k}, \overline{n-k-1}, \ldots, \overline{1}),$$

we get

$$w_I \circ (w^{(k)})^{-1} =$$

$$\left(\overline{w}_{n-k+1} > \overline{w}_{n-k+2} > \ldots > \overline{w}_h, w_{h+1} < w_{h+2} < \ldots < w_n, w_{n-k} < w_{n-k-1} < \ldots < w_1\right).$$

We have $l(w_I) = w_1 + \ldots + w_h$, $l(w^{(k)}) = n + (n-1) + \ldots + (k+1)$, and

$$l\left(w_I \circ (w^{(k)})^{-1}\right) = w_{n-k+1} + w_{n-k+2} + \ldots + w_h + \sum_{j=1}^{n-h} card\{1 \leqslant p \leqslant n-k \mid w_p < w_{h+j}\}$$

by Lemma 5.1. Thus, denoting the above sum $\sum\limits_{j=1}^{n-h}(\ldots)$ by $\sum$, we get:

$$l(w_I) - l(w^{(k)}) - l\left(w_I \circ (w^{(k)})^{-1}\right) =$$

$$= w_1 + \ldots + w_{n-k} - \left(n + (n-1) + \ldots + (k+1)\right) - \textstyle\sum.$$

Now, a necessary condition for $\partial_{w^{(k)}} \widetilde{Q}_I(X_n) \neq 0$ is:

$$w_1 + \ldots + w_{n-k} - \left(n + (n-1) + \ldots + (k+1)\right) - \sum = 0,$$

which implies $(w_1, \ldots, w_{n-k}) = (n, n-1, \ldots, k+1)$ and $\sum = 0$, i.e., $w_n < w_{n-k}$. Assume this and write $l = h - (n-k)$, $j_p = w_{n-k+p}$ $(p = 1, \ldots, l)$. Since we have

$$w_{j_1, \ldots, j_l} = \left(\overline{w}_{n-k+1}, \ldots, \overline{w}_h; w_{h+1}, \ldots, w_n, w_{n-k}, \ldots, w_1\right) \in W^{(n)},$$

we conclude that $\partial_{w^{(k)}}\left(\widetilde{Q}_I(X_n)\right) \equiv \widetilde{Q}_{j_1, \ldots, j_l}(X_n)$ $(mod\ \mathcal{J})$, as desired. □

We now pass to a geometric interpretation of the proposition. The setup and the notation is the same as in the proof of Proposition 3.1: $V \to B$ - rank $2n$ vector bundle endowed with a nondegenerate symplectic form, $X = LG_nV$, V_n denotes here the tautological subbundle on X and $\pi : \mathcal{F} \to X$ is the composition (see Section 1):

$$LG_{n-k}(C^{\perp}/C) \xrightarrow{\pi_1} G_k(V_n) \xrightarrow{\pi_2} X,$$

where C is a tautological rank k bundle on $G_k(V_n)$. Let S be the tautological subbundle on $LG_{n-k}(C^{\perp}/C)$; hence $rank\ S = n-k$. We claim that from the point of view of Chern classes computations in Proposition 3.1, we can identify V_n and D, or equivalently V_n/C and S (recall that $S = D/C_{\mathcal{F}}$). By the splitting principle the sequence of the Chern roots of C is a subsequence of the sequence of the Chern roots of V_n. We have (see Lemma 1.1) that

$$[S] + [S^{\vee}] = [(V_n)_{\mathcal{F}}] + [(V_n^{\vee})_{\mathcal{F}}] - [C_{\mathcal{F}}] - [C_{\mathcal{F}}^{\vee}].$$

It follows that any symmetric polynomial in the squares of $n-k$ variables takes the same value when evaluated in the Chern roots of V_n/C and S. Since we know by Proposition

5.9 and Theorem 5.10 that the image of a polynomial in Chern classes of S via $(\pi_1)_*$ is a symmetric polynomial in the squares of Chern roots of V_n/C, we conclude that in the process of our calculation we can identify the Chern roots of V_n/C with those of S, the final effect of the calculation being the same in both instances. Denote now by $(q_1, \dots, q_n)$ the Chern roots of $V_n^\vee$. Therefore, without changing the effect of our calculation, we can identify the Chern roots of $D^\vee$ with $(q_1, \dots, q_n)$ (recall that D is the rank n tautological subbundle on $\mathcal{F}$). Having this identification in mind, we now give:

Another proof of Proposition 3.1.

In virtue of the previous proposition it suffices to show that for every polynomial f in n variables $\pi_*\big(f(q_1, \dots, q_n)\big) = \big(\partial_{w^{(k)}} f\big)(q_1, \dots, q_n)$, where $q_1, \dots, q_n$ are the above Chern roots. Let v and u be the following elements of W_n:

$$\begin{aligned} v &= \big(\overline{n-k}, \overline{n-k-1}, \dots, \overline{1}, n-k+1, n-k+2, \dots, n\big) \\ u &= \big(k+1, \dots, n, 1, 2, \dots, k\big). \end{aligned}$$

It follows from Proposition 5.8 that

$$(\pi_1)_*\big(f(q_1, \dots, q_n)\big) = \big(\partial_v f\big)(q_1, \dots, q_n).$$

On the other hand, as Lascoux showed to us several years ago, one has

$$(\pi_2)_*\big(f(q_1, \dots, q_n)\big) = \big(\partial_u f\big)(q_1, \dots, q_n).$$

(This can be proved using a reasoning similar to the one in the proof of Proposition 5.8 above.) Since $w^{(k)} = u \circ v$, we thus have

$$\big(\partial_{w^{(k)}} f\big)(q_1, \dots, q_n) = \Big((\partial_u \circ \partial_v) f\Big)(q_1, \dots, q_n) =$$

$$= (\pi_2 \circ \pi_1)_*\big(f(q_1, \dots, q_n)\big) = \tau_*\big(f(q_1, \dots, q_n)\big),$$

which is the desired assertion. □

In the odd orthogonal case, this way of arguing translates mutatis mutandis, thus giving another proof of the odd orthogonal analog of Proposition 3.1.

Finally, we pass to the even orthogonal case. In type D_n the Weyl group W_n is isomorphic to $S_n \ltimes \mathbb{Z}_2^{n-1}$ and is identified with the group of "even barred permutations". Consider a system S of generators of W_n consisting of $s_{\overline{1}} = (\overline{2}, \overline{1}, 3, \dots, n)$ and $s_i = (1, 2, \dots, i-1, i+1, i, i+2, \dots, n)$, $i = 1, 2, \dots, n-1$. (W_n, S) is a Coxeter system of type D_n and the length function w.r.t. S is

$$l(w) = \sum_{i=1}^{n} a_i + \sum_{\tau_j = -1} 2b_j,$$

where $a_i = card\{p \mid p > i \;\&\; w_p < w_i\}$ and $b_j = card\{p \mid p < j \;\&\; w_p < w_j\}$. The longest element w_0 in W_n is equal to $(\overline{1}, ..., \overline{n})$ if n is even and to $(1, \overline{2}, ..., \overline{n})$ if n is odd. Consider the poset $W^{(n)}$ of minimal length (w.r.t. S) left coset representatives of W_n modulo the subgroup generated by $\{s_i\}_{i=1,\dots,n-1}$. We have

$$W^{(n)} = \left\{(\overline{z}_1 > \overline{z}_2 > \ldots > \overline{z}_{2t}; y_1 < y_2 < \ldots < y_{n-2t}) \in W_n \mid t = 0, 1, \ldots, \left[\frac{n}{2}\right]\right\}.$$

Observe that for $w \in W^{(n)}$ we have $l(w) = \sum_{i=1}^{2t}(z_i - 1)$. The assignment

$$(\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_{2t}; y_1, y_2, \ldots, y_{n-2t}) \mapsto (z_1 - 1, z_2 - 1 \ldots, z_{2t} - 1)$$

establishes a bijection between $W^{(n)}$ and the poset of strict partitions contained in ρ_{n-1}. Given such a partition I, let w_I be the corresponding element of $W^{(n)}$. Following [B-G-G] and [D1,2] one defines the operators $\partial_w : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$ $\Big($resp. $\partial_w : A^*(SO(2n, K)/B) \to A^*(SO(2n, K)/B)\Big)$ for $w \in W_n$ mutatis mutandis ; here,

$$\partial_{\overline{1}} f = (f - f(-x_2, -x_1, x_3, ..., x_n))/(-x_1 - x_2).$$

Also, the definition of the Schubert cycles $X_w \in A^{l(w)}(SO(2n, K)/B)$, $w \in W_n$, is completely analogous to that in the Lagrangian case.

If $\mathcal{I}'$ is an ideal in $\mathbb{Z}[X_n]$ generated by $e_i(x_1^2, \ldots, x_n^2)$, $1 \leqslant i \leqslant n-1$, and $x_1 \cdot \ldots \cdot x_n$, then $A^*(SO(2n, K)/B)$ is isomorphic to $\mathbb{Z}[X_n]/\mathcal{I}'$ (see [B]). By comparing Theorem 2.1(iii) with [B-G-G] and [D2] we get, for strict I, in $A^*(OG'_n V)$:

$$X_{w_I} = \widetilde{P}_I R^{\vee} = \partial_{w_I^{-1} w_0}(X_{w_0}),$$

(see also [B-H] for an alternative proof).

Fix now an integer $0 < k < n$ such that $k \equiv n \ (mod\ 2)$ and denote:

$$w^{(k)} := (\overline{n}, \overline{n-1}, \ldots, \overline{k+1}; 1, 2, \ldots, k).$$

Note that $l(w^{(k)}) = (n-1) + (n-2) + \ldots + k$. Hence for a strict partition $I \subset \rho_{n-1}$, $\partial_{w^{(k)}} \widetilde{P}_I(X_n) \neq 0$ only if $l(I) \geqslant n - k$.

Proposition 8.3. *Let $k \equiv n \ (mod\ 2)$. For a strict partition $I \subset \rho_{n-1}$ of length $\geqslant n-k$, $\partial_{w^{(k)}}(X_{w_I}) \neq 0$ only if I is of the form $I = (n-1, n-2, \ldots, k, j_1, j_2, \ldots, j_l)$ for some $J = (j_1, \ldots, j_l)$ with $j_l > 0$ and $l \leqslant k-1$. In the latter case, $\partial_{w^{(k)}} \widetilde{P}_I(X_n) \equiv \widetilde{P}_J(X_n)$ (mod $\mathcal{I}'$).*

Proof. We imitate the proof of Proposition 8.2. Consider the element w_I of $W^{(n)}$,

$$w_I = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_h; w_{h+1}, \ldots, w_n),$$

with h-even and $h \geqslant n - k$, corresponding to I; so $I = (w_1 - 1, \ldots, w_h - 1)$. We have $l(w_I) = w_1 + \ldots + w_h - h$, $l(w^{(k)}) = n + (n-1) + \ldots + (k+1) - (n-k)$, and

$$l\big(w_I \circ (w^{(k)})^{-1}\big) = w_{n-k+1} + w_{n-k+2} + \ldots + w_h - h + (n-k) + \sum,$$

with $\sum$ as in the proof of Proposition 8.2. Hence the same proof as the imitated one yields the desired assertion. $\square$

Observe that for $k \equiv n \pmod 2$ there exists a completely analogous operator proof of Proposition 3.5 (with the same u and v) to that of Proposition 3.1 given in this Section. Invoking Remark 3.4, this leads to another proof of Proposition 3.5.

9. Main results in the generic case

Let V be a rank $2n$ vector bundle over a smooth equidimensional scheme X endowed with a nondegenerate symplectic form. Let E and $F_\cdot : F_1 \subset F_2 \subset \ldots \subset F_n = F$ be Lagrangian subbundles of V with $rank\ F_i = i$ and $rank\ E = n$. For a given sequence $a_\cdot = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$, we are interested in a locus

$$D(a_\cdot) := \Big\{ \ x \in X |\ dim(E \cap F_{a_p})_x \geqslant p,\ p = 1, \ldots, k \Big\}.$$

Let $G = LG_nV$ and let $R \subset V_G$ be the tautological rank n subbundle on G. By a well known universality property of Grassmannians there exists a morphism $s : X \to G$ such that $E = s^*R$. Therefore (in the set-theoretic sense) we have:

$$D(a_\cdot) = s^{-1}(\Omega(a_\cdot; F_\cdot)),$$

where

$$\Omega(a_\cdot; F_\cdot) = \Big\{ \ g \in G|\ dim(R \cap F_{a_p})_g \geqslant p,\ p = 1, \ldots, k \ \Big\}.$$

We take this equality as the definition of a scheme structure on $D(a_\cdot)$, i.e., $D(a_\cdot)$ is defined in X by the inverse image ideal sheaf (see [Ha, p.163]): $s^{-1}\mathcal{I}(\Omega(a_\cdot; F_\cdot))\cdot\mathcal{O}_X$ where $\mathcal{I}(\Omega(a_\cdot; F_\cdot)$ is the ideal sheaf defining Ω in G. It follows from the main theorem of [DC-L] that $\Omega(a_\cdot; F_\cdot)$ is a Cohen-Macaulay scheme. Hence, by [K-L, Lemma 9] we get $[D(a_\cdot)] = s^*[\Omega(a_\cdot; F_\cdot)]$ provided $D(a_\cdot)$ is either empty or equidimensional of codimension equal to the codimension of $\Omega(a_\cdot; F_\cdot)$ in G. Therefore, having a formula for the fundamental class of $\Omega(a_\cdot; F_\cdot)$ given by a polynomial P in $c_\cdot(R)$ and $c_\cdot(F_{a_p})_G$, $p = 1, \ldots, k$, the formula for $D(a_\cdot)$ becomes $P\big(c_\cdot(E), c_\cdot(F_{a_p})_{p=1,\ldots,k}\big)$. Moreover, by using the Chow groups for singular schemes and a technique from [F] one can prove the following refinement of the above. If X is an equidimensional Cohen-Macaulay scheme and $D(a_\cdot)$ is either empty or of codimension equal to the codimension of $\Omega(a_\cdot; F_\cdot)$ in G then the class of $D(a_\cdot)$ in the Chow group of X equals $P\big(c_\cdot(E), c_\cdot(F_{a_p})_{p=1,\ldots,k}\big)\cap[X]$. In particular, for $a_\cdot = (n-k+1, \ldots, n)$ we have by Proposition 3.2:

Theorem 9.1. *If X is an equidimensional Cohen-Macaulay scheme and*

$$D^k = \big\{x \in X |\ dim(E \cap F)_x \geqslant k\big\}$$

is either empty or an equidimensional subscheme of codimension $k(k+1)/2$, then the class of D^k (endowed with the above scheme structure) in the Chow group of X equals

$$[D^k] = \Big(\sum \widetilde{Q}_I E^\vee \cdot \widetilde{Q}_{\rho_k \setminus I} F^\vee\Big) \cap [X],$$

where the sum is over all strict partitions $I \subset \rho_k$.

Example 9.2. The expressions giving the classes for successive k are:

k=1 $\widetilde{Q}_1 E^\vee + \widetilde{Q}_1 F^\vee$;

k=2 $\widetilde{Q}_{21} E^\vee + \widetilde{Q}_2 E^\vee \cdot \widetilde{Q}_1 F^\vee + \widetilde{Q}_1 E^\vee \cdot \widetilde{Q}_2 F^\vee + \widetilde{Q}_{21} F^\vee$;

k=3 $\widetilde{Q}_{321} E^\vee + \widetilde{Q}_{32} E^\vee \cdot \widetilde{Q}_1 F^\vee + \widetilde{Q}_{31} E^\vee \cdot \widetilde{Q}_2 F^\vee + \widetilde{Q}_{21} E^\vee \cdot \widetilde{Q}_3 F^\vee + \widetilde{Q}_3 E^\vee \cdot \widetilde{Q}_{21} F^\vee + \widetilde{Q}_2 E^\vee \cdot \widetilde{Q}_{31} F^\vee + \widetilde{Q}_1 E^\vee \cdot \widetilde{Q}_{32} F^\vee + \widetilde{Q}_{321} F^\vee$.

For $a. = (n+1-i)$ we get:

Theorem 9.3. *Let X be an equidimensional Cohen-Macaulay scheme and assume that $S^i = \{x \in X \mid dim(E \cap F_{n+1-i})_x \geqslant 1\}$ is either empty or equidimensional of codimension i in X. Then*

$$[S^i] = \Big(\sum_{p=0}^{i} c_p E^\vee \cdot s_{i-p} F^\vee_{n+1-i}\Big) \cap [X].$$

Example 9.4. The expressions giving the classes for successive i are:

i=1 $c_1 E^\vee + s_1 F^\vee$;

i=2 $c_2 E^\vee + c_1 E^\vee s_1 F^\vee_{n-1} + s_2 F^\vee_{n-1}$;

i=3 $c_3 E^\vee + c_2 E^\vee s_1 F^\vee_{n-2} + c_1 E^\vee s_2 F^\vee_{n-2} + s_3 F^\vee_{n-2}$.

In a similar way one can interpret other formulas proved earlier for Schubert subschemes in Lagrangian Grassmannian bundles.

In the odd orthogonal case, the setup is the same as above. Repeating mutatis mutandis the above definitions and arguments, one gets the following analog of Theorem 9.1.:

Theorem 9.5. *If X is an equidimensional Cohen-Macaulay scheme over a field of characteristic different from 2 and*

$$D^k = \big\{x \in X \mid dim(E \cap F)_x \geqslant k\big\}$$

is either empty or an equidimensional subscheme of codimension $k(k+1)/2$, then the class of D^k in the Chow group of X equals

$$\Big(\sum \widetilde{P}_I E^\vee \cdot \widetilde{P}_{\rho_k \setminus I} F^\vee\Big) \cap [X],$$

where the sum is over all strict partitions $I \subset \rho_k$.

An analog of Theorem 9.3 in this case is left to the (interested in) reader.

Let now V be a rank $2n$ vector bundle over a connected equidimensional scheme X endowed with a nondegenerate orthogonal form. Let E and $F. : F_1 \subset F_2 \subset \ldots \subset F_n = F$ be isotropic subbundles of V with $rank\ F_i = i$ and $rank\ E = n$. One should be careful here with the definition of $D(a.)$. For a given sequence $a_. = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$, where k is such that $dim(E \cap F)_x \equiv k(mod\ 2)$ if $a_k = n$, we are interested in the locus

$$D(a_.) = \Big\{x \in X \mid dim(E \cap F_{a_p})_x \geqslant p\ ,\ p = 1, \ldots , k\Big\}.$$

There is a morphism $s = (s', s'') : X \to OG'_nV \cup OG''_nV$ such that $s^*R = E$ where R is the tautological rank n subbundle on $OG'_nV \cup OG''_nV$. We have (in the scheme – theoretic sense) that if $k \equiv n\ (mod\ 2)$ then

$$D(a_.) = (s')^{-1}\Omega(a_.; (F_.)_{OG'_nV});$$

and if $k \equiv n + 1(mod\ 2)$ then

$$D(a_.) = (s'')^{-1}\Omega(a_.; (F_.)_{OG''_nV}).$$

Hence, arguing as above we have the following analog of Theorem 9.1 :

Theorem 9.6. *If X is a connected equidimensional Cohen-Macaulay scheme over a field of characteristic different from 2 and the locus*

$$D^k = \big\{x \in X \mid dim(E \cap F)_x \geqslant k\big\},$$

defined for k such that $k \equiv dim(E \cap F)_x\ (mod\ 2)$ where $x \in X$, is either empty or is an equidimensional subscheme of codimension $k(k-1)/2$ in X, then the class of D^k in the Chow group of X equals

$$\Big(\sum \widetilde{P}_I E^\vee \cdot \widetilde{P}_{\rho_{k-1}\setminus I} F^\vee\Big) \cap [X],$$

where the sum is over all strict partitions $I \subset \rho_{k-1}$.

Example 9.7. The expressions giving the classes for successive k are:

k=1 $\quad 1$;

k=2 $\quad \widetilde{P}_1E^\vee + \widetilde{P}_1F^\vee$;

k=3 $\quad \widetilde{P}_{21}E^\vee + \widetilde{P}_2E^\vee \cdot \widetilde{P}_1F^\vee + \widetilde{P}_1E^\vee \cdot \widetilde{P}_2F^\vee + \widetilde{P}_{21}F^\vee$;

k=4 $\quad \widetilde{P}_{321}E^\vee + \widetilde{P}_{32}E^\vee \cdot \widetilde{P}_1F^\vee + \widetilde{P}_{31}E^\vee \cdot \widetilde{P}_2F^\vee + \widetilde{P}_{21}E^\vee \cdot \widetilde{P}_3F^\vee + \widetilde{P}_3E^\vee \cdot \widetilde{P}_{21}F^\vee + \widetilde{P}_2E^\vee \cdot \widetilde{P}_{31}F^\vee + \widetilde{P}_1E^\vee \cdot \widetilde{P}_{32}F^\vee + \widetilde{P}_{321}F^\vee$.

An analog of Theorem 9.3 in this case is left to the (interested in) reader.

Remark 9.8. All the formulas stated in this Section in the Chow groups have their direct analogs in topology. Maybe the simplest version is the following. Assume that X is a compact complex manifold, the bundles E, F_i are holomorphic and the morphism s from X to LG_nV above is transverse to the smooth locus of the Schubert variety $\Omega(a.; F.)$. Then the cohomology fundamental classes of $D(a.)$ are evaluated by the corresponding (given above) expressions in the Chern classes of E and F_i. The same applies to the orthogonal case.

Appendix: Quaternionic Schubert calculus

Let $\mathbb{H}$ denote the (skew) field of quaternions. Let $\mathbb{P}^n_{\mathbb{H}}$ be the projective space that is identified with $(\mathbb{H}^{n+1} \setminus \{0\})/\sim$, where $(h_1, \ldots, h_{n+1}) \sim (h'_1, \ldots, h'_{n+1})$ iff there is $0 \neq h \in \mathbb{H}$ such that $h_i = h \cdot h'_i$ for every i. It is a compact, oriented manifold over $\mathbb{R}$ of dimension $4n$. Let us recall after Hirzebruch [H1], that, in general, this real manifold does not admit a structure of a complex analytic manifold.

Let $G_k(\mathbb{H}^n)$ be the set of all k-dimensional subspaces[2] of $\mathbb{H}^n$. $G_k(\mathbb{H}^n)$ has a natural structure of $4k(n-k)$-dimensional, compact, oriented manifold over $\mathbb{R}$. Of course $G_1(\mathbb{H}^{n+1}) = \mathbb{P}^n_{\mathbb{H}}$.

Let $Fl_{k_1,\ldots,k_r}(\mathbb{H}^n)$ be the set of all flags of subspaces of consecutive dimensions $(k_1, \ldots, k_r)$ over $\mathbb{H}$. It is also a compact, oriented manifold over $\mathbb{R}$. One has (see [B], [Sl]), $Fl_{k_1,\ldots,k_r}(\mathbb{H}^n) \doteq Sp(n)/\prod_{i=0}^r Sp(k_{i+1} - k_i)$ (here, $k_0 = 0$ and $k_{r+1} = n$). Of course $Fl_{k_1}(\mathbb{H}^n) = G_{k_1}(\mathbb{H}^n)$.

10.1. ([B, 31.1 p.202]) *Let $y_1, \ldots, y_n$ be a sequence of independent variables with $deg\ y_i = 4$. Then*

$$H^*\Big(Fl_{k_1,\ldots,k_r}(\mathbb{H}^n), \mathbb{Z}\Big) \simeq SPol(y_1, \ldots, y_n)/I_{k_1,\ldots,k_r},$$

where $I_{k_1,\ldots,k_r}$ is the ideal generated by polynomials symmetric in each of the sets $\{y_{k_i+1}, \ldots, y_{k_{i+1}}\}$, $i = 0, 1, \ldots, r$, separately ($k_0 = 0$, $k_{r+1} = n$).

For instance (all cohomology groups are taken with coefficients in $\mathbb{Z}$),

$$H^*(\mathbb{P}^n_{\mathbb{H}}) = \mathbb{Z}[y]/(y^{n+1}),\ deg\ y = 4;$$

$$H^*\Big(G_k(\mathbb{H}^n)\Big) = SPol(y_1, \ldots, y_n)/I_k,\ deg\ y_i = 4.$$

We see that these cohomology rings are double-degree isomorphic with the cohomology rings of their complex analogues.

Fix now a flag $V.\ :\ V_1 \subset V_2 \subset \ldots \subset V_n$ of subspaces of $\mathbb{H}^n$ with $dim_{\mathbb{H}} V_i = i$. For every partition $I \subset (n-k)^k$ we set

$$\mathring{\sigma}(I) = \Big\{L \in G_k(\mathbb{H}^n) \mid dim_{\mathbb{H}}(L \cap V_{n-k+p-i_p}) = p\ ,\ p = 1, \ldots, k\Big\}.$$

The so defined $\mathring{\sigma}(I)$ $\big(I \subset (n-k)^k\big)$ give a cellular decomposition of $G_k(\mathbb{H}^n)$ and the codimension of $\mathring{\sigma}(I)$ is $4|I|$. Now define

$$\sigma(I) = \sigma(I, V.) = \Big\{L \in G_k(\mathbb{H}^n) \mid dim_{\mathbb{H}}\big(L \cap V_{n-k+p-i_p}\big) \geqslant p\ ,\ p = 1, \ldots, k\Big\}.$$

The cohomology classes of $\sigma(I, V.)$, in fact, do not depend on the flag V. chosen and will be denoted by the same symbol $\sigma(I)$. We record:

[2] the word "(sub)space" means always a "left $\mathbb{H}$-(sub)space".

10.2. *(Pieri-type formula) In $H^*(G_k(\mathbb{H}^n)$ one has*

$$\sigma(I) \cdot \sigma(r) = \sum \sigma(J),$$

where the sum is over J such that $i_p \leqslant j_p \leqslant i_{p-1}$ and $|J| = |I| + r$.

Not all proofs of the Pieri formula for Complex Grassmannians can be extended to the quaternionic case. However, the proof in [G-H, pp.198-204] has this advantage. As a matter of fact, $G_k(\mathbb{H}^n)$ is an oriented compact manifold and thus its cohomology ring is endowed with the Poincaré duality. Moreover, one checks by direct examination that

$$\sigma(I) \cdot \sigma(n-k-i_k, \dots, n-k-i_1) = \sigma\big((n-k)^k\big) = [pt].$$

Then the proof in loc.cit. goes through mutatis mutandis also in the quaternionic case.

We can restate these information about the multiplicative structure in $H^*\big(G_k(\mathbb{H}^n)\big)$ as follows:

10.3. *Let $Y = (y_1, \dots, y_k)$ be independent variables of degree 4. The assignment $s_I(y_1, \dots, y_k) \mapsto \sigma(I)$ for $I \subset (n-k)^k$, and 0 -otherwise, is a ring homomorphism, and allows one to identify $H^*\big(G_k(\mathbb{H}^n)\big)$ with a quotient of $SPol(Y)$ modulo the ideal $\oplus \mathbb{Z} s_I(Y)$, the sum over $I \not\subset (n-k)^k$.*

This result has a number of useful consequences. For example, it implies immediately that the signature of the Complex Grassmannian (see [H, p.163] and [H-S, Formula (23) p.336] is the same as the one of the Quaternionic Grassmannian - a result proved originally in [Sl] using different methods.

We now describe a certain fibration which makes the Quaternionic Grassmannians useful in study of the Grassmannians of non-maximal Lagrangian subspaces (which are not Hermitian symmetric spaces).

Let $V = \mathbb{C}^{2n}$ be endowed with a nondegenerate symplectic form Φ given by the matrix

$$A = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

where I_n is the $(n \times n)$-identity matrix.

Having in mind the standard notation associated with $\mathbb{H}$ we endow V with a structure of $\mathbb{H}$-space setting $\mathbf{j} \cdot v = A\overline{v}$, where " $^-$ " denotes the complex conjugation (note that $A^2 = -id_V$).

10.4. *If $U \subset V$ is $k-$dimensional Lagrangian $\mathbb{C}$-subspace of V then $dim_{\mathbb{H}}(\mathbb{H} \cdot U) = k$. Moreover, the restriction of the symplectic form Φ to any $\mathbb{H}$-subspace of V, is nondegenerate.*

To show this consider the standard Hermitian scalar product $<,>$ on $V = \mathbb{C}^{2n}$. Now given U, we pick up its $\mathbb{C}-$basis $u_1, \dots, u_k$ such that $< u_p, u_q >= \delta_{p,q}$. We claim that

$u_1, \ldots, u_k, \mathbf{j}u_1, \ldots, \mathbf{j}u_k$ are linearly independent over $\mathbb{C}$ (which implies $dim_{\mathbb{H}}(\mathbb{H}\cdot U) = k$). This claim follows immediately from $\Phi(u_p, u_q) = 0 = \Phi(\mathbf{j}u_p, \mathbf{j}u_q)$ and $\Phi(u_p, \mathbf{j}u_q) = u_p^t A(A\overline{u}_q) = -< u_p, u_q >= -\delta_{p,q}$.

Suppose now a $\mathbb{H}$–subspace $W \subset V$ is given with $dim_{\mathbb{H}} W = k$, say. We can always find $\mathbb{C}$–linearly independent vectors $w_1, \ldots, w_k \in W$ such that $\Phi(w_p, w_q) = 0$ and $< w_p, w_q >= \delta_{p,q}$. Then $\mathbf{j}w_1, \ldots, \mathbf{j}w_k$ also belong to W. It follows from $\Phi(w_p, w_q) = 0 = \Phi(\mathbf{j}w_p, \mathbf{j}w_q)$ and $\Phi(w_p, \mathbf{j}w_q) = -\delta_{p,q}$ that $w_1, \ldots, w_k, \mathbf{j}w_1, \ldots, \mathbf{j}w_k$ form a $\mathbb{C}$–basis of W and the form Φ restricted to W is nondegenerate.

We infer from the above

10.5. *The assignment $U \mapsto \mathbb{H} \cdot U$, defines a locally trivial fibration of $LG_k(\mathbb{C}^{2n})$ over $G_k(\mathbb{H}^n)$ with the fiber $LG_k(\mathbb{C}^{2k})$.*

In other words, denoting by S the tautological (sub)bundle over $G_k(\mathbb{H}^n)$, $rank_{\mathbb{H}} S = k$, we have an identification $LG_k(\mathbb{C}^{2n}) \simeq LG_k(S)$, where the latter symbol denotes (the total space of) the corresponding Grassmannian bundle.

This identification can be used in reduction of some problems about Grassmannians of non-maximal Lagrangian subspaces to the problems about the Grassmannians of maximal ones. For example, we get from 10.5 the following identity of Poincaré series:

$$P_{LG_k(\mathbb{C}^{2n})}(t) = P_{G_k(\mathbb{H}^n)}(t) \cdot P_{LG_k(\mathbb{C}^{2k})}(t),$$

thus reproving the result from [P-R2, Corollary 1.7].

Similar fibrations exist for Flag varieties. Let $LFl_{k_1,\ldots,k_r}(\mathbb{C}^{2n})$ be the variety of Lagrangian (w.r.t. Φ) flags of dimensions $(k_1, \ldots, k_r)$ in $\mathbb{C}^{2n}$.

10.6. *The assignment $(dim_{\mathbb{C}} U_i = k_i,\ i = 1, \ldots, r)$:*

$$\Big(U_1 \subset U_2 \subset \ldots \subset U_r\Big) \mapsto \Big(\mathbb{H} \cdot U_1 \subset \mathbb{H} \cdot U_2 \subset \ldots \subset \mathbb{H} \cdot U_r\Big)$$

is a locally trivial fibration of $LFl_{k_1,\ldots,k_r}(\mathbb{C}^{2n})$ over $Fl_{k_1,\ldots,k_r}(\mathbb{H}^n)$. If $\mathbb{C}^{2k_1} \subset \mathbb{C}^{2k_2} \subset \ldots \subset \mathbb{C}^{2k_r}$ is a (part of) the standard flag, then the fiber of this fibration is the variety of Lagrangian flags $W_1 \subset W_2 \subset \ldots \subset W_r$ such that $W_i \subset \mathbb{C}^{2k_i}$ and $dim_{\mathbb{C}} W_i = k_i$, $i = 1, \ldots, r$.

Therefore the fiber is a composition of Lagrangian Grassmannian bundles of maximal subspaces. In particular, we obtain the following formula for the Poincaré series of $LFl_{k_1,\ldots,k_r}(\mathbb{C}^{2n})$:

$$P_{LFl_{k_1,\ldots,k_r}(\mathbb{C}^{2n})}(t) = P_{Fl_{k_1,\ldots,k_r}(\mathbb{H}^n)}(t) \cdot \prod_{i=1}^{r} P_{LG_{k_i - k_{i-1}}(\mathbb{C}^{2(k_i - k_{i-1})})}(t),$$

where $k_0 = 0$. Since explicit expressions for the factors on the R.H.S. are known (see (10.1)), this gives an explicit formula for $P_{LFl_{k_1,\dots,k_r}(\mathbb{C}^{2n})}(t)$.

10.7. Finally, we show an algebro-topological interpretation (as well as another proof) of the identity:

$$s_I(x_1^2,\dots,x_n^2)\cdot s_{\rho_n}(x_1,\dots,x_n) = s_{2I+\rho_n}(x_1,\dots,x_n)$$

from Section 5. To this end we show two different ways of constructing $LFl := LFl(\mathbb{C}^{2n})$. The first way is given by taking the total space of the Flag bundle $Fl(R) \to LG_n(\mathbb{C}^{2n})$ where R is the tautological vector bundle on $LG_n(\mathbb{C}^{2n})$. The second way relies on the following observation: LFl can be interpreted as the variety of flags $W_1 \subset W_2 \subset \dots \subset W_{2n}$ such that $dim_{\mathbb{C}} W_j = j$ and each W_{2j} is a $\mathbb{H}$-subspace. This realization is given by the assignment:

$$\Big(V_1 \subset V_2 \subset \dots \subset V_n\Big) \mapsto \Big(V_1 \subset \mathbb{H}\cdot V_1 \subset \mathbb{H}\cdot V_1 + V_2 \subset \mathbb{H}\cdot V_1 + \mathbb{H}\cdot V_2 \subset \dots\Big)$$

Equivalently, using the tautological sequence $S_1 \subset S_2 \subset \dots \subset S_n$, $rank_{\mathbb{H}} S_i = i$, on $Fl_{\mathbb{H}}$, this corresponds to taking the total space of the product of Projective bundles

$$\mathbb{P} := \mathbb{P}(S_2/S_1) \times_{Fl_{\mathbb{H}}} \dots \times_{Fl_{\mathbb{H}}} \mathbb{P}(S_n/S_{n-1}) \to Fl_{\mathbb{H}}$$

where S_{i+1}/S_i, $i = 1,\dots,n$, are considered as rank 2 complex bundles.

The same holds in the relative situation, i.e. given a rank $2n$ vector bundle $V \to X$ endowed with a symplectic form we get a commutative diagram

$$\begin{array}{ccccc}
\mathbb{P} & = & LFl(V) & = & Fl(R) \\
{\scriptstyle \tau_1}\big\downarrow & & & & {\scriptstyle \pi_1}\big\downarrow \\
Fl_{\mathbb{H}}(V) & \xrightarrow{\tau_2} & X & \xleftarrow{\pi_2} & LG_n V
\end{array}$$

where $Fl_{\mathbb{H}}(V)$ is the Quaternionic (complete) Flag bundle. Let $x_1,\dots,x_n$ be the sequence of the Chern roots of the tautological quotient bundle on $LG_n V$. By Corollary 5.6(i) we know that if there exists an even i_p, then $(\pi_2 \circ \pi_1)_*(x_1^{i_1}\cdot\ldots\cdot x_n^{i_n}) = 0$. (Calculating the other way arround, this follows easily from the projection formula.) On the other hand, iff all i_p are odd, then (see Proposition 5.5)

$$s_{\rho_n}(x_1,\dots,x_n)\cdot(\pi_2\circ\pi_1)_*(x_1^{i_1}\cdot\ldots\cdot x_n^{i_n}) = s_{I-\rho_{n-1}}(x_1,\dots,x_n).$$

Putting $i_p = 2j_p + 1$ and calculating the other way around, we get

$$
\begin{aligned}
(\tau_2\tau_1)_*\Big(x_1^{2j_1+1}x_2^{2j_2+1}\ldots x_n^{2j_n+1}\Big) &= \\
&= (\tau_2)_*\Big((x_1^2)^{j_1}\cdot(x_2^2)^{j_2}\cdot\ldots\cdot(x_n^2)^{j_n}\Big) \\
&= s_{J-\rho_{n-1}}(x_1^2,\ldots,x_n^2).
\end{aligned}
$$

Indeed, recalling the notation from 10.1 we have $y_p = x_p^2$, $p = 1,\ldots,n$ (see [B, 31.1]), and we use the fact that $(\tau_2)_*$ is induced by the Jacobi symmetrizer (recalled in the proof of Corollary 5.6(ii) and that of Lemma 5.7(ii)) this time applied to $y_1,\ldots,y_n$. The latter statement follows from 10.1 by exactly the same reasoning as that used in the proof of Lemma 2.4 in [P1]. Comparison of the results of both computations, yields the desired identity.

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