THE COMBINATORICS OF BERNSTEIN FUNCTIONS

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Abstract. A construction of Bernstein associates to each cocharacter of a split p-adic group an element in the center of the Iwahori-Hecke algebra, which we refer to as a Bernstein function. A recent conjecture of Kottwitz predicts that Bernstein functions play an important role in the theory of bad reduction of a certain class of Shimura varieties (parahoric type). It is therefore of interest to calculate the Bernstein functions explicitly in as many cases as possible, with a view towards testing Kottwitz' conjecture. In this paper we prove a characterization of the Bernstein function associated to a *minuscule* cocharacter (the case of interest for Shimura varieties). This is used to write down the Bernstein functions explicitly for some minuscule cocharacters of Gl_n; one example can be used to verify Kottwitz' conjecture for a special class of Shimura varieties (the "Drinfeld case"). In addition we prove some general facts concerning the support of Bernstein functions, and concerning an important set called the "μ-admissible" set. These facts are compatible with a conjecture of Kottwitz-Rapoport on the shape of the special fiber of a Shimura variety with parahoric type bad reduction.

1. Introduction and Statement of Main Results

Let $\mathcal{H}$ denote the Iwahori-Hecke algebra of an almost simple, split connected reductive group G over a p-adic field F. More concretely, if $I \subset G(F)$ is an Iwahori subgroup, then $\mathcal{H}$ is the convolution algebra of compactly supported I-bi-invariant functions on $G(F)$, where convolution is defined using the Haar measure on $G(F)$ which gives I volume 1. Let q denote the size of the residue field of F, and let $\mathbb{Z}' = \mathbb{Z}[q^{1/2}, q^{-1/2}]$. Fix an F-split maximal torus T of G and a Borel subgroup B containing T. Let W denote the Weyl group of G and denote the cocharacter lattice by $X_*(T)$. Bernstein has constructed a $\mathbb{Z}'$-algebra isomorphism between the Weyl group invariant elements of the group algebra of $X_*(T)$ and the center of $\mathcal{H}$:

$$\mathbb{Z}'[X_*(T)]^W \xrightarrow{\sim} Z(\mathcal{H}).$$

This is achieved by constructing, for each dominant cocharacter $\mu \in X_*(T)$, an element z_μ in the center of $\mathcal{H}$ (see §2 for the definition), and then by showing these elements form a $\mathbb{Z}'$-basis for $Z(\mathcal{H})$, as μ runs over all dominant cocharacters. We call the elements z_μ the *Bernstein functions.*

The main aim of this paper is to study the Bernstein functions from a combinatorial viewpoint, meaning that we seek an explicit expression for z_μ as a $\mathbb{Z}'$-linear combination of the standard generators T_w, where T_w is the characteristic function of the Iwahori double coset corresponding to the element w in the extended affine Weyl group $\widetilde{W}$ of G (according to the Iwahori decomposition, $\widetilde{W}$ indexes the set $I\backslash G(F)/I$). A second goal of this article is the study of the μ-*admissible* subset of the extended affine Weyl group (see §2 for definition).

The motivation for this work comes from two conjectures in the theory of Shimura varieties with parahoric type bad reduction, which we now briefly explain. Fix a rational prime p. A Shimura variety S_K is, roughly speaking, attached to a triple of group-theoretic data $(G, K, \{\mu\})$, where G is a connected reductive $\mathbb{Q}$-group, $K = K_pK^p \subset G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ is a compact open subgroup, and $\{\mu\}$ is the $G(\overline{\mathbb{Q}}_p)$-conjugacy class of a cocharacter $\mu : (\mathbb{G}_m)_{\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ which is *minuscule*, meaning that $\langle \alpha, \mu \rangle \in \{-1, 0, 1\}$, for every root α of G. It is known that S_K is defined over a number field E. We say S_K is of *parahoric type* (resp. *Iwahori type*) (at p) if K_p is a parahoric (resp. Iwahori) subgroup of $G(\mathbb{Q}_p)$, meaning that the "reduction modulo p" of K_p is a parabolic (resp. Borel) subgroup of $G(\mathbb{F}_p)$.

The motivating conjectures concern the "semi-simplified" local zeta function $Z^{ss}_{\mathfrak{p}}(S_K, s)$ of a parahoric type Shimura variety S_K at a prime ideal $\mathfrak{p}$ of E dividing p. This is considered by M. Rapoport in [11]. Assume that $E_{\mathfrak{p}}$ is an unramified extension of $\mathbb{Q}_p$. By reasoning that is analogous to the case of good reduction, Rapoport shows in loc. cit. that in order to understand $Z^{ss}_{\mathfrak{p}}(S_K, s)$ one should study the function on $S_K(\mathbb{F}_q)$ defined by

$$x_0 \mapsto tr(Fr_q\,;\, R\Psi^{\mathcal{I}}_{x_0}(\overline{\mathbb{Q}}_l)),$$

where $R\Psi$ is the sheaf of *nearby cycles* on $(S_K)_{\overline{\mathbb{F}}_q}$, $q = p^j$ is such that $\mathbb{Q}_{p^j}$ contains $E_{\mathfrak{p}}$, $x_0 \in S_K(\mathbb{F}_q)$, Fr_q is the geometric Frobenius on $(S_K)_{\overline{\mathbb{F}}_q}$, and we take invariants under $\mathcal{I}$, the inertia subgroup of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We need a purely group-theoretic interpretation of this function, if we are eventually going to use the Arthur-Selberg trace formula to express the local zeta function in terms of automorphic L-functions.

Such an interpretation has been conjectured by R. Kottwitz. To state Kottwitz' conjecture, fix an unramified extension $F = \mathbb{Q}_{p^j} \supset E_{\mathfrak{p}}$, and assume that G is split over F. For simplicity we also assume that K_p is an *Iwahori* subgroup. Then there is an Iwahori subgroup $I \subset G(F)$ such that $I \cap G(\mathbb{Q}_p) = K_p$. Write $p^j = q$, and $\mathcal{H} = C_c(I\backslash G(F)/I)$. Associated to the element $\{\mu\}$ in the Shimura datum is the Bernstein function $z_\mu \in Z(\mathcal{H})$.

Locally in the étale topology, the special fiber of S_K has a stratification indexed by certain elements of the extended affine Weyl group $\widetilde{W}$ of G. In fact S_K is étale locally isomorphic to a "local model" $M^{\rm loc}$ (see [11]) and the special fiber of $M^{\rm loc}$ is a set of Iwahori orbits in the affine flag variety for $G(\overline{\mathbb{F}}_p((t)))$. To x_0 we associate the element $x \in \widetilde{W}$ indexing the stratum containing x_0.

Conjecture 1 (*Kottwitz*) *In the case of Iwahori type reduction,*

$$tr(Fr_q\,;\,R\Psi^{\mathcal{I}}_{x_0}(\overline{\mathbb{Q}}_l)) = q^{dim(S_K)/2} z_\mu(x),$$

where q is the cardinality of the residue field of F, and $z_\mu(x)$ is the coefficient of T_x for the Bernstein function z_μ in $C_c(I\backslash G(F)/I)$ corresponding to the cocharacter μ.

Concerning the shape of the stratification of the "local model" we have the following conjecture.

Conjecture 2 (*Kottwitz-Rapoport*) *The strata of M^{loc} are indexed by elements of the μ-admissible set.*

If Conjecture 1 were true it would follow that $q^{\dim(S_K)/2} z_\mu$ is the correct "test function" which is "plugged into" the twisted orbital integrals which come into the computation of the semi-simplified local zeta function when one attempts to use the Arthur-Selberg trace formula. This is in contrast to the case of good reduction, where K_p is a hyperspecial maximal compact subgroup, and the test function is simply the characteristic function of the double coset $K_p(F)\mu(p^{-1})K_p(F)$ in the spherical Hecke algebra $C_c(K_p(F)\backslash G(F)/K_p(F))$ (see §16 of [9]).

In [11], Rapoport has given explicit formulae in the "Drinfeld case" for the trace of Frobenius on nearby cycles appearing in Conjecture 1 (see loc. cit. Thm 3.12). In order to compare these formulae with Kottwitz' group-theoretic prescription, one needs to compute explicitly the coefficients $z_\mu(x)$ of the Bernstein function z_μ with respect to the basis $\{T_x,\ (x \in \widetilde{W})\}$ of $\mathcal{H}$.

The main theorem of this paper is a characterization of z_μ for any *minuscule* cocharacter, which is Theorem 5.8 in §5:

Theorem 3 (Main Theorem) *Let μ be a minuscule dominant cocharacter. Then z_μ is the unique element of $\mathcal{H}$ such that the following three properties hold:*

1. $z_\mu \in Z(\mathcal{H})$,
2. $\mathrm{supp}(z_\mu)$ *is contained in the μ-admissible set*,
3. $z_\mu(\mu) = 1$.

We use the characterization to compute the coefficients $z_\mu(x)$ for Bernstein functions in certain important cases. The main application is the computation of the function $z_{(1,0,\ldots,0)}$ for the group Gl_n (Proposition. 7.1). This can be reformulated as follows (comp. notation in §2).

Proposition 4 *For x an element of the extended affine Weyl group of Gl_n, let $z_\mu(x)$ denote the coefficient of T_x in the expression for the element z_μ. Then for $\mu = (1, 0^{n-1})$, we have*

$$q^{\dim(S_K)/2} z_\mu(x) = \begin{cases} (1-q)^{l(\mu)-l(x)} & \text{if } x \text{ is } \mu\text{-admissible}, \\ 0 & \text{if } x \text{ is not } \mu\text{-admissible}. \end{cases}$$

Here we have used $\dim(S_K) = l(\mu)$, where $l(x)$ denotes the length of an element $x \in \widetilde{W}$ (see §2 for precise definition). Putting this formula for $z_\mu(x)$ together with Rapoport's explicit formulae for the trace of Frobenius on nearby cycles ([11]), one can prove that Conjecture 1 holds in the "Drinfeld case" (see [5] for further details and another proof of Proposition 4). A related Shimura variety associated to the data (Gl_4, $\mu = (1,1,0,0)$) has been studied by U. Görtz. He computed the trace of Frobenius on the sheaf of nearby cycles and, comparing the result with the formulae for the coefficients $z_{(1,1,0,0)}(x)$ given in §9, he verified that Conjecture 1 also holds in this case.

If Conjectures 1 and 2 were both true, one would expect the support of z_μ to lie in the μ-admissible set. This is in fact true even if μ is not minuscule (see Corollary 5.7):

Proposition 5 *For any dominant cocharacter μ, the support of z_μ is contained in the μ-admissible set.*

This result is the main ingredient in the proof of Theorem 3 above. The proof is inspired by a result of Deodhar ([4]) and uses a generalization of one of the definitions appearing in loc. cit.

It appears to be true that the support of z_μ is precisely the μ-admissible set. This is true in the case of Gl_2, as follows from the Corollary 10.4 of §10. Moreover, it also holds for any group if μ is minuscule (see [5]). We make the following conjecture:

Conjecture 6 *For any dominant cocharacter μ, $supp(z_\mu)$ is the μ-admissible set.*

We now outline the contents of the paper. In §2 we define notation and recall some important definitions. In the next three sections (§3 − 5) we establish general facts which will be required for the applications in the later sections. In particular in §3 we give simple necessary and sufficient conditions for an arbitrary element in the Iwahori-Hecke algebra to be central. In §4 we study the relationship between conjugacy and μ-admissibility. In §5 the support of z_μ is characterized, leading to the proof of Proposition 5 (Corollary 5.7) and Theorem 3 (Theorem 5.8) cited above.

The second half of the paper is devoted to applications and examples of the general facts proved in §3 − 5. In §6, some facts about μ-admissible sets for Gl_n and μ minuscule are proved; these are used in the proof of Proposition 4 (Proposition 7.1) mentioned above, which is given in §7. In §8 the complexity of the μ-admissible sets is illustrated by counting the μ-admissible set for some minuscule cocharacters of Gl_n and GSp_{2n}. In §9, further examples of Bernstein functions for minuscule cocharacters of Gl_n and GSp_{2n} are given, using the technical results from §6.

Finally, in §10, we present a proof that the Bernstein and Satake isomorphisms are compatible, using results of G. Lusztig ([7]) and S. Kato ([8]). This implies that for any dominant μ there is a unique function k_μ in $Z(\mathcal{H})$

whose image in the spherical Hecke algebra has Satake transform $q^{l(\mu)/2}\chi_\mu$, where χ_μ denotes the character of the highest weight module attached to μ. We calculate the function k_μ explicitly for any dominant cocharacter μ of the group Gl_2 (Proposition 10.3), and then use this to compute all the Bernstein functions for Gl_2 (Corollary 10.4).

2. Further Notation

2.1. **Affine Weyl group.** We denote by Π the set of simple roots, and S the corresponding set of simple reflections in W. Let $\tilde{\alpha}$ denote the unique highest root, and let $s_0 = \check{\tilde{\alpha}} s_{\tilde{\alpha}} = s_{(-\tilde{\alpha},1)}$ denote the affine reflection about the affine hyperplane $\{x \in X_*(T^{ad}) \otimes \mathbb{R} \mid \langle \tilde{\alpha}, x\rangle = 1\}$,where T^{ad} is the image of T in the adjoint group G^{ad}. Let $S_a = S \cup \{s_0\}$. We denote by T^{sc} the inverse image of T under the map $G^{sc} \to G$; T^{sc} is a maximal torus in the simply connected cover G^{sc} of G^{der}. The affine Weyl group is by definition the group $W_a = X_*(T^{sc}) \rtimes W$. It is known that (W_a, S_a) is a Coxeter group; therefore there is a length function $l(x)$ and a Bruhat order $\leq$ defined on W_a. If Ω denotes the subgroup of $\widetilde{W}$ which preserves the set S_a under conjugation, then $\widetilde{W} = W_a \rtimes \Omega$. We extend the length function to $\widetilde{W}$ by declaring it to be trivial on ω, and we extend the Bruhat order by defining $x_1\omega_1 \leq x_2\omega_2$ if and only if $x_1 \leq x_2$ and $\omega_1 = \omega_2$.

Fix a dominant cocharacter μ.

Definition 2.1. We say an element $x \in \widetilde{W}$ is *μ-admissible* if $x \leq w(\mu)$ for some $w \in W$.

2.2. **Hecke algebra.** The *braid group* of $\widetilde{W}$ is the group generated by symbols T_w for $w \in \widetilde{W}$ subject to the relation $T_{ww'} = T_w T_{w'}$ whenever $l(ww') = l(w) + l(w')$. The Hecke algebra $\mathcal{H}$ is by definition the quotient of the group algebra (over $\mathbb{Z}'$) of the braid group of $\widetilde{W}$, by the two sided ideal generated by the elements

$$(T_s + 1)(T_s - q),$$

for $s \in S_a$. We continue to denote by T_w the image in $\mathcal{H}$ of the element T_w in the braid group. It is known that the elements T_w form a $\mathbb{Z}'$-basis for $\mathcal{H}$.

For $\psi = \sum_{x \in \widetilde{W}} a_x T_x \in \mathcal{H}$, we call $supp(\psi) = \{x \in \widetilde{W} \mid a_x \neq 0\}$ the *support* of ψ. We occasionally denote the coefficient a_x by $\psi(x)$.

For any $x \in X_*$, define an element $\Theta_x = q^{-(l(x_1)-l(x_2))/2} T_{x_1} T_{x_2}^{-1}$, where $x = x_1 - x_2$ and $x_i \in X_{dom}$, for $i = 1, 2$. This element is independent of the choice of x_1 and x_2, and moreover the elements $\Theta_x T_w$ (for $x \in X_*$ and $w \in W$) form a $\mathbb{Z}'$-basis for $\mathcal{H}$. We use these to define the Bernstein function attached to a dominant cocharacter μ:

Definition 2.2. $z_\mu = \sum_\lambda \Theta_\lambda$, where λ ranges over the W-orbit of μ.

The following theorem is due to Bernstein (see [7]):

Theorem 2.3 (*Bernstein*) *The center $Z(\mathcal{H})$ of the Hecke algebra is the free $\mathbb{Z}'$-module generated by the elements z_μ, where μ ranges over all dominant cocharacters of A.*

(Note: In [7] this is only stated for G a split semi-simple group, but it is easy to deduce the general statement for G any split quasi-simple connected reductive group.)

For any T_w, we define a renormalization $\tilde{T}_w = q^{-l(w)/2}T_w$. Define a new parameter $Q = q^{-1/2} - q^{1/2}$. Then the elements $\tilde{T}_w$ also form a basis for $\mathcal{H}$, and the usual relations can be written as

$$\widetilde{T}_s\widetilde{T}_w = \begin{cases} \widetilde{T}_{sw}, & \text{if } l(sw) = l(w)+1 \\ -Q\widetilde{T}_w + \widetilde{T}_{sw}, & \text{if } l(sw) = l(w)-1, \end{cases}$$

for $w \in \widetilde{W}$ and $s \in S_a$. There is also a right-handed version of this relation. Note in particular that $\tilde{T}_s^{-1} = \tilde{T}_s + Q$, for $s \in S_a$.

3. Conditions on central elements

The purpose of this section is to write down in a convenient form the conditions for an arbitrary element $\psi = \sum_w b_w\widetilde{T}_w$ to be in the center of $\mathcal{H}$. The results of this section are used later (§$6-7, 9$) to construct certain elements in the center of the Iwahori-Hecke algebra. We then show these are in fact equal to Bernstein functions by using the characterization of Bernstein functions given in Theorem 3.

First note that for any $s \in S_a$, $\tau \in \Omega$, and $w \in \widetilde{W}$, we have

$$\widetilde{T}_s\widetilde{T}_w = \begin{cases} \widetilde{T}_{sw}, & \text{if } l(sw) = l(w)+1 \\ -Q\widetilde{T}_w + \widetilde{T}_{sw}, & \text{if } l(sw) = l(w)-1, \end{cases}$$

$$\widetilde{T}_w\widetilde{T}_s = \begin{cases} \widetilde{T}_{ws}, & \text{if } l(ws) = l(w)+1 \\ -Q\widetilde{T}_w + \widetilde{T}_{ws}, & \text{if } l(ws) = l(w)-1, \end{cases}$$

$$\widetilde{T}_\tau\widetilde{T}_w = \widetilde{T}_{\tau w},$$

$$\widetilde{T}_w\widetilde{T}_\tau = \widetilde{T}_{w\tau}.$$

It follows that we can write, for $s \in S_a$, and $\tau \in \Omega$,

$$\begin{aligned} \widetilde{T}_s\psi &= \sum_{w\,:\,l(sw)=l(w)+1} b_w\widetilde{T}_{sw} + \sum_{w\,:\,l(sw)=l(w)-1} b_w(-Q\widetilde{T}_w + \widetilde{T}_{sw}), \\ \psi\widetilde{T}_s &= \sum_{w\,:\,l(ws)=l(w)+1} b_w\widetilde{T}_{ws} + \sum_{w\,:\,l(ws)=l(w)-1} b_w(-Q\widetilde{T}_w + \widetilde{T}_{ws}), \end{aligned}$$

$$\widetilde{T}_\tau\psi = \sum_w b_w\widetilde{T}_{\tau w},$$

$$\psi \widetilde{T}_\tau = \sum_w b_w \widetilde{T}_{w\tau}.$$

On equating the coefficients of $\widetilde{T}_x$ in the first two equations (and also in the second two equations), we deduce that ψ is in the center of the Iwahori-Hecke algebra if and only if the following five conditions hold, for every $x \in \widetilde{W}$, $s \in S_a$, and $\tau \in \Omega$:

- If $l(sx) = l(x) - 1$ and $l(xs) = l(x) - 1$, then $b_{sx} = b_{xs}$,
- If $l(sx) = l(x) - 1$ and $l(xs) = l(x) + 1$, then $b_{sx} - Qb_x = b_{xs}$,
- If $l(sx) = l(x) + 1$ and $l(xs) = l(x) - 1$, then $b_{sx} = b_{xs} - Qb_x$,
- If $l(sx) = l(x) + 1$ and $l(xs) = l(x) + 1$, then $b_{sx} = b_{xs}$,
- $b_{\tau x} = b_{x\tau}$.

Replacing x with xs, we see that the first and fourth conditions can be expressed as a single condition, and similarly the second and third conditions, in the presence of the first and fourth, can be expressed as a single condition. We have proved the following Lemma:

Lemma 3.1. *The element ψ is in the center of the Iwahori-Hecke algebra if and only if the following three conditions hold, for every $x \in \widetilde{W}, s \in S_a$ and $\tau \in \Omega$:*

1. *If $l(sxs) = l(x)$, then $b_{sxs} = b_x$,*
2. *If $l(sxs) = l(x) - 2$, then $b_x = b_{sxs} - Qb_{xs}$,*
3. *$b_{\tau x \tau^{-1}} = b_x$.*

4. Conjugacy and μ-admissibility

In this section we will establish some general facts concerning μ-admissibility and conjugacy classes. These facts play a role in the proof of the main computation (Proposition 4 in the introduction).

The first step is an easy lemma that applies to arbitrary Coxeter groups.

Lemma 4.1. *Let (W, S) be a Coxeter group. Let $w \in W$ and $s, t \in S$. Suppose that $l(swt) = l(w)$ and $l(wt) = l(sw)$. Then $sw = wt$.*

Proof. This can be found on p.148 of [6], but we repeat the proof for convenience. Write $w = s_1, \cdots s_r$ (reduced). There are two possibilities, either $l(sw) > l(w)$ or $l(sw) < l(w)$. Suppose first that $l(sw) > l(w)$. Then $l(swt) = l(w) < l(sw)$, so the Exchange Condition of Coxeter groups applies to the pair sw, t, meaning that $sw = w't$, where either $w' = ss_1 \cdots \hat{s}_i \cdots s_r$ or $w' = w$. The first alternative is impossible because it would imply that $w = s(sw) = s(w't) = s_1 \cdots \hat{s}_i \cdots s_r t$, forcing wt to be shorter than w, contrary to assumption. Therefore $w' = w$ and so $sw = wt$.

Now suppose that $l(sw) < l(w)$. The hypotheses of the first case are now satisfied with sw in place of w, so we can conclude that $w = swt$. $\square$

Remark. The lemma as stated applies to affine Weyl groups (W_a, S_a), but it easy to see that it remains true even if we allow w to be an element of an extended affine Weyl group $\widetilde{W}$.

We now recall three basic facts about the Bruhat order.

Lemma 4.2. *Let x, y be elements of a Coxeter group (W, S). Suppose that $x^{-1}y$ is a reflection and $l(x) < l(y)$. Then $x \leq y$ in the Bruhat order.*

Proof. This is Lemma 8.11 of [1]. $\square$

Lemma 4.3. *Let $x, y \in \widetilde{W}$, and let $s \in S_a$. Assume that $x \leq y$. Then*

1. *either $sx \leq y$ or $sx \leq sy$, and*
2. *either $x \leq sy$ or $sx \leq sy$, and*
3. *either $xs \leq y$ or $xs \leq ys$, and*
4. *either $x \leq ys$ or $xs \leq ys$.*

Proof. The first two statements, for ordinary or affine Weyl groups, are proved in Lemma 2.5 of [1]. It is easy to check that they remain valid for extended affine Weyl groups. The last two statements follow from the first two and the fact that, for any $x, y \in \widetilde{W}$, $x \leq y$ if and only if $x^{-1} \leq y^{-1}$. $\square$

Lemma 4.4. *Let $x, y \in \widetilde{W}$ and let $x = s_1 \cdots s_q$ be a reduced expression of x. Then $y \leq x$ if and only if there exists a strictly increasing sequence $i_1 < i_2 < \cdots i_k$ (possibly empty) of integers drawn from $\{1, 2, ...q\}$, such that $y = s_{i_1} \cdots s_{i_k}$.*

Proof. This is Proposition 2.8 of [1]. $\square$

For the next lemma let μ denote an arbitrary dominant cocharacter of the torus T, the maximal F-split torus in G we fixed above. Let x denote an element of the extended affine Weyl group $\widetilde{W}$, and $s \in S_a$ a simple reflection in W_a.

Lemma 4.5. *If $l(sxs) = l(x)$, then x and sxs are simultaneously μ-admissible.*

Proof. We must prove that if the lengths of x and sxs are the same and x is μ-admissible, then sxs is μ-admissible. Consider the four elements x, sx, xs, and sxs. Since x and sxs have the same length, there are only four possible configurations of these four elements in the Bruhat order. Two of them involve the cases where $l(sx) = l(xs)$ (which necessarily differs from $l(x) = l(sxs)$ by one). But in these two cases we know by Lemma 4.1 (and following Remark) that $x = sxs$, so we are done. In the remaining two cases, either xs is the least upper bound and sx is the greatest lower bound of the set $\{x, xs, sx, sxs\}$ (in the Bruhat order), or vice-versa. Suppose for instance that xs is the least upper bound, so that $x \leq xs$ and $sxs \leq xs$. If xs is μ-admissible, then so is sxs and we are done. So assume that xs is not μ-admissible. We can write $x \leq \mu'$ for some $\mu' \in W \cdot \mu$. By Lemma 4.3 (3), we know that either $xs \leq \mu'$ or $xs \leq \mu' s$. The first possibility does not occur (since xs is not μ-admissible), so the second one does. Applying Lemma 4.3 (2) to this relation we see that either $xs \leq s\mu' s = s(\mu')$ or $sxs \leq s(\mu')$. Since $s(\mu') \in W \cdot \mu$, only the second case occurs and sxs is μ-admissible. The case where sx is the least upper bound is similar. $\square$

Corollary 4.6. *(of Proof) Let $\mu' \in W(\mu)$ and $s \in S_a$. If $x \leq \mu'$ and either sx or xs is not μ-admissible, then $s(x) \leq s(\mu')$.*

5. The Support of z_μ and the Main Theorem

In this section we prove the main theoretical result of the paper, Theorem 5.8 (Theorem 3 in the introduction), which is a characterization of z_μ for μ minuscule. This is the principal tool used to produce the explicit expressions for z_μ given later (we write down some explicit expressions, and use the main theorem to show that they are in fact the desired Bernstein functions). To start the proof we give a characterization of the support of the function Θ_λ for any $\lambda \in X_*$ (see Proposition 5.4). Using this we then show that the support of Θ_λ is contained in the set $\{x \in \widetilde{W} \mid x \leq \lambda\}$ (Proposition 5.5). An immediate consequence is that $supp(z_\mu)$ is contained in the μ-admissible subset of $\widetilde{W}$, for any dominant $\mu \in X_*$ (Corollary 5.7). We then use these observations about the supports of arbitrary Bernstein functions to prove Theorem 5.8.

Suppose $\lambda = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are both dominant. Write $\lambda_i = w_i\sigma_i$ ($w_i \in W_a$ and $\sigma_i \in \Omega$) for $i = 1, 2$. Then $\Theta_\lambda = \tilde{T}_{w_1\sigma_1}\tilde{T}^{-1}_{w_2\sigma_2} = \tilde{T}_{w_1}\tilde{T}^{-1}_{\tau(w_2)}\tilde{T}_\tau$, where $\tau = \sigma_1\sigma_2^{-1}$. Note that for support questions it is enough to understand the first two factors, meaning that we have effectively replaced $\widetilde{W}$ with the Coxeter group (W_a, S_a). In fact the results of this section apply to any Coxeter group. Therefore fix a Coxeter group (W, S) and consider the Hecke algebra H_W with $\mathbb{Z}'$-basis $\{\tilde{T}_w\}$ $(w \in W)$ and having relations

$$\widetilde{T}_w\widetilde{T}_s = \begin{cases} \widetilde{T}_{ws}, & \text{if } l(ws) = l(w) + 1 \\ -Q\widetilde{T}_w + \widetilde{T}_{ws}, & \text{if } l(ws) = l(w) - 1, \end{cases}$$

for $w \in W$ and $s \in S$. Fix elements v and w in W. Denote by $\Theta(v, w)$ the element $\tilde{T}_v\tilde{T}^{-1}_{w^{-1}}$. We want to understand the support of the function $\Theta(v, w)$. This is simplified by the following positivity statement (where $\mathbb{Z}_+$ denotes the nonnegative integers):

Lemma 5.1. *Write $\Theta(v, w) = \sum_{x \in W} a_x(Q)\tilde{T}_x$. Then $a_x(Q) \in \mathbb{Z}_+[Q]$ for every $x \in W$.*

Proof. This is an easy induction on $l(w)$, using the identity $\tilde{T}_s^{-1} = \tilde{T}_s + Q$ and

$$\tilde{T}_y(\tilde{T}_s + Q) = \begin{cases} Q\tilde{T}_y + \tilde{T}_{ys}, & \text{if } l(ys) = l(y) + 1 \\ \tilde{T}_{ys}, & \text{if } l(ys) = l(y) - 1. \end{cases}$$

□

Now we need to recall a definition from [4]. Fix a reduced expression $w = s_1 s_2 \cdots s_r$.

Definition 5.2. We call a sequence $[\alpha_0, \alpha_1, \ldots, \alpha_r]$ a *subexpression for* w (with respect to $w = s_1 s_2 \cdots s_r$) if $\alpha_0 = 1$ and $\alpha_j \in \{\alpha_{j-1}, \alpha_{j-1}s_j\}$ for every $j \geq 1$.

The following definition generalizes Deodhar's notion of *distinguished subexpression* (see loc. cit.).

Definition 5.3. Let v and w be arbitrary elements of W. Fix for w a reduced expression $w = s_1 \cdots s_r$. We call a sequence $[\sigma_0, \sigma_1, \ldots, \sigma_r]$ v-*distinguished* (with respect to $w = s_1 s_2 \cdots s_r$) if

1. $\sigma_j \in \{\sigma_{j-1}, \sigma_{j-1}s_j\}$, for every $j \geq 1$,
2. $\sigma_j \leq \sigma_{j-1}s_j$, for every $j \geq 1$,
3. $\sigma_0 = v$.

Note that this last definition is actually a generalization of the one introduced by Deodhar in [4]: any *distinguished subexpression* of w is v-*distinguished* for $v = 1$.

Given $w = s_1 s_2 \cdots s_r$ as above and $v \in W$, define $D(v, w) \subset W$ to be the set of elements σ_r such that there exists a subexpression $[\alpha_0, \ldots, \alpha_r]$ of w such that $[v\alpha_0, \ldots, v\alpha_r]$ is v-distinguished and $v\alpha_r = \sigma_r$.

Proposition 5.4. $supp(\Theta(v, w)) = D(v, w)$.

Proof. The proof is by induction on r. If $r = 0$ the statement is evident. Assume the result holds for $r \geq 0$; we will show it holds for $r + 1$. Suppose therefore that we have a reduced expression $ws = s_1 s_2 \cdots s_r s$. Note that $\Theta(v, ws) = \Theta(v, w)(\tilde{T}_s + Q)$.

First we claim that $supp(\Theta(v, ws)) \subset D(v, ws)$. Any element in the left hand side must be in the support of $\tilde{T}_{\sigma_r}(\tilde{T}_s + Q)$ for some element $\sigma_r \in supp(\Theta(v, w))$. We must consider two cases.
Case 1: $\sigma_r s < \sigma_r$. Then $\tilde{T}_{\sigma_r}(\tilde{T}_s + Q) = \tilde{T}_{\sigma_r s}$ and we must show $\sigma_r s \in D(v, ws)$. But by the induction hypothesis there is a subexpression $[\alpha_0, \ldots, \alpha_r]$ of w such that $[v\alpha_0, \ldots, v\alpha_r]$ is distinguished, with $\sigma_r = v\alpha_r$. But then it is clear that $[\alpha_0, \ldots, \alpha_r, \alpha_r s]$ is a subexpression of ws and $[v\alpha_0, \ldots, v\alpha_r, v\alpha_r s]$ is distinguished and $\sigma_r s = v\alpha_r s$, implying that $\sigma_r s \in D(v, ws)$.
Case 2: $\sigma_r < \sigma_r s$. Then $\tilde{T}_{\sigma_r}(\tilde{T}_s + Q) = Q\tilde{T}_{\sigma_r} + \tilde{T}_{\sigma_r s}$ and we have to show that $\sigma_r s$ and σ_r are in $D(v, ws)$. The same argument as above shows that $\sigma_r s \in D(v, ws)$. The induction hypothesis implies the existence of a subexpression $[\alpha_0, \ldots, \alpha_r]$ of w such that $[v\alpha_0, \ldots, v\alpha_r]$ is distinguished and $v\alpha_r = \sigma_r$. Note that $[\alpha_0, \ldots, \alpha_r, \alpha_r]$ is a subexpression for ws and $[v\alpha_0, \ldots, v\alpha_r, v\alpha_r]$ is distinguished. This shows that $\sigma_r \in D(v, ws)$, and proves the claim.

Now we claim that $supp(\Theta(v, ws)) \supset D(v, ws)$. Suppose $[\alpha_0, \ldots, \alpha_{r+1}]$ is a subexpression of ws and $[v\alpha_0, \ldots, v\alpha_{r+1}]$ is distinguished. Write $\sigma_j = v\alpha_j$ for $j \geq 0$. We must show $\sigma_{r+1} \in supp(\Theta(v, ws))$. By induction we know that $\sigma_r \in supp(\Theta(v, w))$.

Case 1: $\sigma_{r+1} = \sigma_r$. Then since $[\sigma_0, \ldots, \sigma_r, \sigma_r]$ is distinguished, we have $\sigma_r < \sigma_r s$ and so $\tilde{T}_{\sigma_r}(\tilde{T}_s + Q) = Q\tilde{T}_{\sigma_r} + \tilde{T}_{\sigma_r s}$. It follows that $\sigma_{r+1} = \sigma_r \in supp(\Theta(v, ws))$.
Case 2: $\sigma_{r+1} = \sigma_r s$. If $\sigma_r < \sigma_r s$, then $\tilde{T}_{\sigma_r}(\tilde{T}_s + Q) = Q\tilde{T}_{\sigma_r} + \tilde{T}_{\sigma_r s}$ and so $\sigma_{r+1} = \sigma_r s \in supp(\Theta(v, ws))$. If $\sigma_r s < \sigma_r$, then $\tilde{T}_{\sigma_r}(\tilde{T}_s + Q) = \tilde{T}_{\sigma_r s}$ and so $\sigma_{r+1} = \sigma_r s \in supp(\Theta(v, ws))$. This completes the proof.

□

Proposition 5.5. *Suppose $[\alpha_0, \ldots, \alpha_r]$ is a subexpression of $w = s_1 \cdots s_r$ and $[v, v\alpha_1, \ldots, v\alpha_r]$ is v-distinguished. Then $v\alpha_r \leq vw$.*

Proof. We use induction on r. If $r = 0$ the statement is evident. We assume the statement holds for $r \geq 0$ and deduce it for $r+1$. Suppose therefore that $ws_{r+1} = s_1 \cdots s_r s_{r+1}$ is a reduced expression, that $[\alpha_0, \ldots, \alpha_{r+1}]$ is a subexpression of ws_{r+1} and $[v, v\alpha_1, \ldots, v\alpha_{r+1}]$ is v-distinguished with respect to $ws_{r+1} = s_1 \cdots s_r s_{r+1}$. We want to prove that $v\alpha_{r+1} \leq vws_{r+1}$. By Lemma 4.4 we can write $\alpha_{r+1} = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_q} \cdots s_{r+1}$, for $1 \leq i_1 < \cdots < i_q \leq r+1$ and where the ˆ symbol means we omit that term.

If $q = 0$ then $v\alpha_{r+1} = vws_{r+1}$ and so we are done. Therefore we can assume $q > 0$. We consider two cases.
Case 1: $\alpha_1 = s_1$. This means that $i_1 > 1$ and our sequence is of the form

$$[v, vs_1, \ldots, v\alpha_{r+1}].$$

The last $r + 1$ terms of this sequence form a vs_1-distinguished sequence with respect to $s_2 \cdots s_{r+1}$. Therefore by the induction hypothesis we have $vs_1(s_2 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_q} \cdots s_{r+1}) \leq vs_1(s_2 \cdots s_{r+1})$, i.e., $v\alpha_{r+1} \leq vws_{r+1}$.
Case 2: $\alpha_1 = \alpha_0 = 1$. Then $i_1 = 1$ and our sequence is of the form

$$[v, v, \ldots, v\alpha_{r+1}].$$

The last $r+1$ terms of this sequence form a v-distinguished sequence with respect to $s_2 \cdots s_{r+1}$ and so by the induction hypothesis we have $v\alpha_{r+1} \leq v(s_2 \cdots s_{r+1})$. Since the sequence is distinguished we have $v < vs_1$ and so the desired conclusion follows from the following lemma (taking $n = r+1$). □

Lemma 5.6. *Let $s_1 \cdots s_n$ be reduced. Suppose $v < vs_1$. Then $vs_2 \cdots s_n < vs_1 \cdots s_n$.*

Proof. Use induction on n. If $n = 1$, then the statement is evident. Suppose that it holds for $n \geq 1$; we show it holds for $n+1$. We assume $v < vs_1$. By the induction hypothesis we have $vs_2 \cdots s_n < vs_1 \cdots s_n$. We want to show $vs_2 \cdots s_{n+1} < vs_1 \cdots s_{n+1}$. Call the first term x and the second y. Note that $x^{-1}y = s_{n+1} \cdots s_2 s_1 s_2 \cdots s_{n+1}$ is a reflection. By Lemma 4.2, we need only show $l(x) < l(y)$. Assume this is not the case. Then because $x^{-1}y$ is a reflection it is impossible to have $l(x) = l(y)$. Therefore $l(y) < l(x)$ and hence $x \not\leq y$. The induction hypothesis asserts that $xs_{n+1} < ys_{n+1}$, so by Lemma 4.3 we deduce $x \leq ys_{n+1}$ and $xs_{n+1} \leq y$. Equality is impossible in each of these cases, for otherwise $s_1 \cdots s_{n+1} = s_2 \cdots s_n$. Hence we have strict inequalities

$$l(xs_{n+1}) < l(y) < l(x),$$

which is a contradiction because the numbers on the left and right must differ by 1. Therefore we conclude that $l(x) < l(y)$ and so $x \leq y$. □

Corollary 5.7. *Let $\lambda \in X_*$ and $\mu \in X_{dom}$. Then the following statements hold:*

1. *The support of Θ_λ is contained in $\{x \mid x \leq \lambda\}$.*
2. *The support of z_μ is contained in the μ-admissible set.*

Proof. Using the notation at the beginning of this section we see that $\lambda = \lambda_1\lambda_2^{-1} = w_1 \, {}^\tau(w_2)^{-1}\tau$, where $\tau = \sigma_1\sigma_2^{-1}$ and $\Theta_\lambda = \tilde{T}_{w_1}\tilde{T}^{-1}_{{}^\tau(w_2)}\tilde{T}_\tau$. Therefore we need to show that $supp(\Theta(w_1, \, {}^\tau(w_2^{-1}))$ is contained in the set $\{x \mid x \leq w_1 \, {}^\tau(w_2^{-1})\}$. This follows immediately from the previous two propositions. Thus the first statement is proved, and the second is a direct consequence of the first and the definition of z_μ.

□

Now we can prove the main theorem.

Theorem 5.8. *Let μ be a minuscule dominant cocharacter. Then z_μ is the unique element of $\mathcal{H}$ such that the following three properties hold:*

1. *$z_\mu \in Z(\mathcal{H})$,*
2. *$supp(z_\mu)$ is contained in the μ-admissible set,*
3. *$z_\mu(\mu) = 1$.*

Proof. It is obvious that z_μ satisfies conditions (1) and (3). Corollary 5.7 above shows it also satisfies condition (2).

We now want to prove that if f satisfies these three conditions, then it must be z_μ. First we need a lemma.

Lemma 5.9. *Suppose $\lambda \in X_{dom}$ is μ-admissible. Then $\lambda = \mu$.*

Proof. For any cocharacter x let w_x denote the unique element of greatest length in the double coset WxW. It is well known that for dominant cocharacters λ and μ we have

$$w_\lambda \leq w_\mu \ \text{ if and only if } \ \lambda \preceq \mu,$$

where $\lambda \preceq \mu$ means $\mu - \lambda$ is 0 or a sum of simple coroots. Now λ being μ-admissible implies $\lambda \leq w\mu w^{-1}$ for some $w \in W$. It follows easily (using Lemma 4.3 repeatedly) that $w_\lambda \leq w_\mu$ and thus $\lambda \preceq \mu$. Since μ (being minuscule) is a minimal nonzero element of X_{dom} with respect to $\preceq$, we conclude that $\lambda = \mu$.

□

Now suppose $f \in \mathcal{H}$ satisfies the three conditions. Because $f \in Z(\mathcal{H})$, Bernstein's theorem (Theorem 2.3) implies that we can write f as a $\mathbb{Z}'$-linear combination of elements z_λ, for $\lambda \in X_{dom} : f = \sum_\lambda a_\lambda z_\lambda$. Among those λ

such that $a_\lambda \neq 0$, choose one of maximal length; say $l(\lambda) = r$. Then we can write

$$f = \sum_{\lambda' : l(\lambda')=r} a_{\lambda'} z_{\lambda'} \; + \sum_{\lambda'' : l(\lambda'')<r} a_{\lambda''} z_{\lambda''}.$$

Since $supp(z_{\lambda''})$ is contained in the λ''-admissible set for each λ'' and similarly for each λ' (Cor. 5.7), and the W-orbits of the elements λ' are disjoint, it follows that $\lambda \in supp(f)$. By (2), this means λ is μ-admissible. Hence by the lemma above, $\lambda = \mu$. Since λ was chosen arbitrarily (among the λ''s), we can now write

$$f = a_\mu z_\mu \; + \sum_{\lambda'' : l(\lambda'')<r} a_{\lambda''} z_{\lambda''}.$$

Suppose that the sum on the right hand side is nonempty. Then among the elements λ'' such that $a_{\lambda''} \neq 0$, choose one of maximal length, say λ_1. Then again by the argument used above we see that $\lambda_1 \in supp(\sum_{\lambda''} a_{\lambda''} z_{\lambda''})$. We now reach a contradiction: If $\lambda_1 \in supp(f)$, then by (2) and the lemma above, $\lambda_1 = \mu$, which can't happen (consider lengths). On the other hand, if $\lambda_1 \notin supp(f)$ then we must have $\lambda_1 \in supp(z_\mu)$, which again by the lemma leads to $\lambda_1 = \mu$. This contradiction shows that the sum is in fact empty, so that

$$f = a_\mu z_\mu.$$

Finally condition (3) implies that $a_\mu = 1$. □

6. Applications to Gl_n

The goal of this section is to extend some results in [10] and then use these new results together with Theorem 5.8 to give an explicit formula for the Bernstein function attached to Gl_n and $\mu = (1, 0, \ldots, 0)$. Some further examples for Gl_n will also be discussed.

6.1. **Minuscule alcoves for Gl_n.** We will need some results and definitions pertaining to the case $G = Gl_n$. All the terminology and notation, and some of the theorems, will be taken from [10].

The extended affine Weyl group $\widetilde{W}$ for Gl_n is the semidirect product of the symmetric group S_n and the group $\mathbb{Z}^n$. We view $\widetilde{W}$ as a group of affine linear transformations of $\mathbb{R}^n$, with S_n acting by permutations of the coordinates and $\mathbb{Z}^n$ acting by translations. We order the affine roots in such a way that the simple affine roots are the functions $(x_1, \ldots, x_n) \mapsto x_i - x_{i+1}$ $(1 \leq i \leq n-1)$ together with the affine linear function $(x_1, \ldots, x_n) \mapsto x_n - x_1 + 1$. The corresponding simple reflections S_a are the transpositions $s_i = (i, i+1)$ in S_n together with the affine linear tranformation s_0 given by $(x_1, \ldots, x_n) \mapsto (x_n + 1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_1 - 1)$. The affine Weyl group W_a is the semidirect product of S_n with the subgroup of $\mathbb{Z}^n$ consisting of elements $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $x_1 + \cdots + x_n = 0$. We define the length function $l(\cdot)$ and the Bruhat order on the groups W_a and $\widetilde{W}$ using the simple affine reflections S_a. (Note that these are the usual simple reflections defined by using the maximal torus T of diagonal matrices and the usual Borel subgroup B of upper triangular matrices.)

For a vector $v \in \mathbb{Z}^n$ we denote by $v(m)$ the m-th entry of v; thus $v = (v(1), \dots, v(n))$. Also we write $\sum(v)$ for the sum $v(1) + \cdots + v(n)$. Given two vectors $u, v \in \mathbb{Z}^n$, we write $u \leq v$ if $u(m) \leq v(m)$ for all m such that $1 \leq m \leq n$.

Definition 6.1. An *alcove* for Gl_n is a sequence $\mathbf{v}$ consisting of vectors $v_0, v_1, \dots, v_{n-1}, v_n = v_0 + (1, \dots, 1)$ in $\mathbb{Z}^n$ satisfying the following two conditions:

- $v_0 \leq v_1 \leq \cdots \leq v_{n-1} \leq v_n$,
- $\sum(v_i) = \sum(v_{i-1}) + 1$, for $1 \leq i \leq n$.

We will call a vector $v = (v(1), \dots, v(n)) \in \mathbb{Z}^n$ *minuscule* if $0 \leq v(m) \leq 1$ for all m such that $1 \leq m \leq n$.

For each r such that $0 \leq r \leq n$, let $\mathbf{e}_r$ denote the r-th standard basis vector in $\mathbb{R}^n$, with the convention $\mathbf{e}_0 = (0, \dots, 0)$. Also we will let ω_r denote the minuscule vector $(1^r, 0^{n-r})$.

Definition 6.2. We say that an alcove $\mathbf{v}$ consisting of the vectors $v_0, \dots v_n$ is *minuscule* if

$$\omega_i \leq v_i \leq \omega_i + (1, \dots, 1),$$

for all i in the range $0 \leq i \leq n-1$.

We refer to the minuscule alcove $\omega_0, \dots, \omega_n$ as the *standard alcove*. The group $\widetilde{W}$ acts on the set of alcoves in the obvious way. It is easy to see that this action is simply transitive. Therefore by taking the standard alcove as base point, we may identify $\widetilde{W}$ with the set of all alcoves. If $\mathbf{v}$ is an alcove and x is the corresponding element of $\widetilde{W}$, we write $\mathbf{v} \longleftrightarrow x$. If $w \in \widetilde{W}$ and $w\,\mathbf{v}$ is the alcove attained by the action of w on $\mathbf{v}$, then we have $w\,\mathbf{v} \longleftrightarrow wx$.

Definition 6.3. We call an element $x \in \widetilde{W}$ *minuscule* if there exists an r $(0 \leq r \leq n)$ such that $x \leq v$ in the Bruhat order on $\widetilde{W}$, for some permutation v of ω_r.

In other words, x is minuscule if and only if it is μ-admissible for some minuscule dominant cocharacter μ (all of which are of the form ω_r for some r such that $0 \leq r \leq n$). For the rest of this paper we will tacitly assume the following theorem.

Theorem 6.4. *(Kottwitz-Rapoport) Suppose the alcove* $\mathbf{v}$ *corresponds to* $x \in \widetilde{W}$. *Then* $\mathbf{v}$ *is minuscule if and only if* x *is minuscule.*

Proof. This is a direct consequence of Theorem 3.5 of [10]. □

If $v_0, \dots, v_n$ is a minuscule alcove , we associate to it a sequence of minuscule vectors $\mu_0, \dots, \mu_{n-1}$, where

$$\mu_k(j) = v_k(j) - \omega_k(j).$$

Also we define, for $1 \leq j \leq n$, the subset K_j of $\{0, 1, 2, \ldots, n-1\}$ by putting

$$K_j = \{k \mid 0 \leq k \leq n-1 \text{ and } \mu_k(j) = 1\}.$$

Note that the set of vectors $\{\mu_0, \ldots, \mu_{n-1}\}$, or equivalently, the collection of sets $\{K_1, \ldots, K_n\}$, determines the minuscule alcove $v_0, \ldots, v_{n-1}$ uniquely.

Definition 6.5. For any alcove $\mathbf{v}$ and any integer l in the range $0 \leq l \leq n-1$, we call the number $\mu_l(1) + \cdots + \mu_l(n)$ (which is independent of l) the *size* of $\mathbf{v}$.

Let $1 \leq i < j \leq n$ and $d \in \mathbb{Z}$. We will consider the decomposition of the affine reflection $w = w_{i,j;d} = d(\mathbf{e}_i - \mathbf{e}_j)s$ corresponding to the affine linear function $\alpha_{i,j;d} = x_i - x_j - d$ on $\mathbb{R}^n$. In our expression for w here the term $d(\mathbf{e}_i - \mathbf{e}_j)$ denotes translation by this vector, and the term s is the factor in the finite Weyl group; it is the transposition (ij). Now suppose $\mathbf{v}$ is the alcove $v_0, \ldots, v_{n-1}$ and x is the corresponding element of the extended affine Weyl group. Then we have a similar decomposition of x:

$$x = (v_0(1), \ldots, v_0(n))\pi,$$

where the first term is simply translation by the vector $v_0 \in \mathbb{Z}^n$ and the second term is a permutation $\pi \in W$. We can also describe the alcove $\mathbf{v}$ by specifying the initial vector v_0 and the ordered set of vectors $(v_1 - v_0, v_2 - v_1, \ldots, v_n - v_{n-1})$, which can also be written as $(\mathbf{e}_{\pi(1)}, \ldots, \mathbf{e}_{\pi(n)})$.

If $K \subset \mathbb{Z}/n\mathbb{Z}$, we denote the complement of K in $\mathbb{Z}/n\mathbb{Z}$ by K^c. The characteristic function of such a subset will be denoted by ξ_K.

The following are some facts which will be needed later.

Lemma 6.6. *(Kottwitz-Rapoport) Let $\mathbf{v}$ be a minuscule alcove, and let w be as above.*

1. *For each $1 \leq j \leq n$, K_j is either $\mathbb{Z}/n\mathbb{Z}$, $\emptyset$, or an interval of the form $[?, j)$.*
2. *Suppose that $w\,\mathbf{v}$ is minuscule. Then $d \in \{0, 1\}$.*
3. *If $d = 0$, then $w\,\mathbf{v}$ is minuscule if and only if $[i, j) \subset K_i^c \cap K_j$.*
4. *If $d = 1$, then $w\,\mathbf{v}$ is minuscule if and only if $[i, j)^c \subset K_j^c \cap K_i$.*
5. *Consider the statement: Either $d = 0$ and $[i, j) = K_i^c \cap K_j$ or else $d = 1$ and $[i, j)^c = K_j^c \cap K_i$. Then this statement holds if and only if $\mathbf{v} \leq w\,\mathbf{v}$ and $w\,\mathbf{v}$ is minuscule.*

Proof. These are contained in §5 of [10]. ☐

Now fix x, $\mathbf{v}$, and w as before. We denote by $\mathbf{v}', \mathbf{v}''$, and $\mathbf{v}'''$, the alcoves corresponding to the group elements xw, wx, and wxw, respectively. If these alcoves are minuscule, we also denote the corresponding collections of sets with the appropriate number of primes, that is, $\mathbf{v}'$ corresponds to the collection of sets $\{K_j'\}$, etc. We will follow the same convention when discussing the sets of minuscule vectors μ_k, that is, $\mathbf{v}'$ will correspond to the set $\{\mu'_j\}$, etc. It is easy to see that

$$\begin{aligned} x &= v_0\pi, \\ xw &= (v_0 + d(\mathbf{e}_{\pi(i)} - \mathbf{e}_{\pi(j)}))\pi s, \\ wx &= (s(v_0) + d(\mathbf{e}_i - \mathbf{e}_j))s\pi, \\ wxw &= (s(v_0) + d(\mathbf{e}_i - \mathbf{e}_j) + d(\mathbf{e}_{s\pi(i)} - \mathbf{e}_{s\pi(j)}))s\pi s. \end{aligned}$$

If we consider the case where $d = 0$, the expressions above can be simplified somewhat. In particular we can write $w = s = (ij)$. In this case the associated alcoves may be specified by the following data:

$$\begin{array}{lll} \mathbf{v} & \longleftrightarrow & (v_0(1), \dots, v_0(n));\ \mathbf{e}_{\pi(1)}, \dots, \mathbf{e}_{\pi(n)}, \\ \mathbf{v}' & \longleftrightarrow & (v_0(1), \dots, v_0(n));\ \mathbf{e}_{\pi s(1)}, \dots, \mathbf{e}_{\pi s(s)}, \\ \mathbf{v}'' & \longleftrightarrow & (v_0(s(1)), \dots, v_0(s(n)));\ \mathbf{e}_{s\pi(1)}, \dots \mathbf{e}_{s\pi(n)}, \\ \mathbf{v}''' & \longleftrightarrow & (v_0(s(1)), \dots, v_0(s(n)));\ \mathbf{e}_{s\pi s(1)}, \dots \mathbf{e}_{s\pi s(n)}. \end{array}$$

Next we want to describe the conditions on the element $x = v_0\pi$ that ensure that $\mathbf{v}$ is a minuscule alcove:

Lemma 6.7. $\mathbf{v}$ *is a minuscule alcove if and only if* v_0 *is a minuscule vector and the following condition holds, for every* $1 \le i \le n$*:*

$$v_0(i) = 0 \Rightarrow \pi^{-1}(i) \le i, \quad \textit{and} \quad v_0(i) = 1 \Rightarrow \pi^{-1}(i) \ge i.$$

Proof. This is an easy exercise using the definitions. □

For x and w as above, define elements i' and j' such that $1 \le i' < j' \le n$ and $\pi(\{i', j'\}) = \{i, j\}$. As an application of the previous lemma and the explicit expressions above we get the following

Lemma 6.8. *Suppose* x*,* xw*,* wx*, and* wxw *are all minuscule elements of* $\widetilde{W}(Gl_n)$*. Then* $\{i', j'\} \cap \{i, j\} = \emptyset$*.*

Proof. We know from Lemma 6.6 (2) that $d \in \{0, 1\}$. First we consider the case $d = 1$. In this case we will prove something stronger: if the translation parts of the four elements x, xw, wx, and wxw are minuscule vectors, then $\{i', j'\} \cap \{i, j\} = \emptyset$. Indeed, if the translation parts of the first and second elements are minuscule, we must have $v_0(\pi(i)) = 0$ and $v_0(\pi(j)) = 1$. On the other hand if the translation parts of the first and third elements are minuscule, then $s(v_0)(i) = 0$ and $s(v_0)(j) = 1$, so $v_0(j) = 0$ and $v_0(i) = 1$. This implies that $\pi^{-1}(i) \ne i$ and $\pi^{-1}(j) \ne j$. Finally if the translation parts of the first and fourth elements are minuscule, then we have $i \ne s\pi(i)$ and $j \ne s\pi(j)$, that is, $\pi^{-1}(j) \ne i$ and $\pi^{-1}(i) \ne j$. This proves the lemma in the case $d = 1$.

Now consider the case $d = 0$. Let $\tau = (1, 0, \dots, 0)c \in \Omega \subset \widetilde{W}(Gl_n)$, where $c = (12 \cdots n) \in S_n = W(Gl_n)$. Conjugating w by an appropriate power of τ yields an element $w_{l,k;1}$. Indeed, a direct computation shows that for any

$1 \leq i < j < n$ we have

$$\begin{aligned} \mathrm{Int}(\tau)(w_{i,j;0}) &= w_{i+1,j+1;0} \\ \mathrm{Int}(\tau)(w_{i,n;0}) &= w_{1,i+1;1}. \end{aligned}$$

Moreover conjugating any element of $\widetilde{W}(Gl_n)$ by τ preserves the property of being minuscule, since $\mathrm{Int}(\tau)$ preserves the Bruhat order and the W-orbit of any minuscule dominant cocharacter μ. Therefore we reduce this case to the case $d = 1$ if we show that the cardinality of the set $\{i', j'\} \cap \{i, j\}$ does not change when x and w are replaced with ${}^\tau x$ and ${}^\tau w$. The permutation factor of ${}^\tau w$ is ${}^c s = (i+1, j+1)$. The permutation factor of ${}^\tau x$ is ${}^c\pi$. Define $i'' < j''$ by ${}^c\pi(\{i'', j''\}) = \{i+1, j+1\}$. Then we must show that $|\{i'', j''\} \cap \{i+1, j+1\}| = |\{i', j'\} \cap \{i, j\}|$. But it is clear from the definition that $\{i'', j''\} = \{i'+1, j'+1\}$, so the result follows.

□

6.2. **A technical lemma.**

Lemma 6.9. *Suppose $w = w_{i,j;d}$ is any affine reflection for Gl_n and x is an element of $\widetilde{W}$. Suppose that x, xw, wx and wxw are all minuscule. Then the following statements hold:*

1. $K_i = K_i' = [\pi^{-1}(i), i)$ *and* $K_i'' = K_i''' = [\pi^{-1}(j), i)$,
2. $K_j = K_j' = [\pi^{-1}(j), j)$ *and* $K_j'' = K_j''' = [\pi^{-1}(i), j)$.

In particular, all the K_r-sets above are proper, for $r \in \{i, j\}$.

Proof. In light of Lemma 6.8, the last statement is a consequence of the explicit forms given for the K-sets. We need to prove those explicit forms are correct. By Lemma 6.6 (2) we know that $d \in \{0, 1\}$. Assume first that $d = 0$ and write $w = s = (ij)$.

The descriptions of the alcoves $\mathbf{v}, \mathbf{v}', \mathbf{v}'', \mathbf{v}'''$ at the end of the previous subsection imply the following formulas, for every $0 \leq k \leq n-1$ and $1 \leq r \leq n$:

$$\mu_k(r) = v_0(r) + \sum_{l=1}^{k} [\mathbf{e}_{\pi(l)}(r) - \mathbf{e}_l(r)], \tag{1}$$

$$\mu_k'(r) = v_0(r) + \sum_{l=1}^{k} [\mathbf{e}_{\pi s(l)}(r) - \mathbf{e}_l(r)], \tag{2}$$

$$\mu_k''(r) = v_0(s(r)) + \sum_{l=1}^{k} [\mathbf{e}_{s\pi(l)}(r) - \mathbf{e}_l(r)], \tag{3}$$

$$\mu_k'''(r) = v_0(s(r)) + \sum_{l=1}^{k} [\mathbf{e}_{s\pi s(l)}(r) - \mathbf{e}_l(r)], \tag{4}$$

where the sums are interpreted to be 0 if $k = 0$. Now we need a definition to simplify the notation.

Definition 6.10.

$$\Delta(s\pi s,\pi,k)(r) = \sum_{l=1}^{k}[\mathbf{e}_{s\pi s(l)}(r) - \mathbf{e}_{\pi(l)}(r)],$$
$$\Delta(s\pi,\pi s,k)(r) = \sum_{l=1}^{k}[\mathbf{e}_{s\pi(l)}(r) - \mathbf{e}_{\pi s(l)}(r)].$$

We then deduce the following two formulas:

$$\mu'''_k(r) = \mu_k(r) + [v_0(s(r)) - v_0(r)] + \Delta(s\pi s,\pi,k)(r), \tag{5}$$
$$\mu''_k(r) = \mu'_k(r) + [v_0(s(r)) - v_0(r)] + \Delta(s\pi,\pi s,k)(r). \tag{6}$$

There are six possible configurations of the four integers i, j, i', j' (e.g. $i < i' < j < j'$). Although Lemma 6.8 implies that $s(i') = i'$ and $s(j') = j'$ we retain the s in the notation in much of what follows, as it clarifies some calculations (in particular the calculation of the values in the table below). A property that the six configurations of i', j', i, j have in common which we use below is $s(i') < j'$ and $i' < s(j')$. We note that the four intervals $[i', s(j')), [s(i'), j'), [j', s(i')), [s(j'), i')$ (which are really only two distinct intervals here) are all *proper* intervals in $\mathbb{Z}/n\mathbb{Z}$, in the sense of Lemma 6.6.

There are some divisions into cases that are necessary, namely

- Case(1): $\pi(i') = i$ and $\pi(j') = j$,
- Case(2): $\pi(i') = j$ and $\pi(j') = i$.

and

- Case(A): $(v_0(i), v_0(j)) = (0, 1)$,
- Case(B): $(v_0(i), v_0(j)) = (1, 0)$,
- Case(C): $(v_0(i), v_0(j)) = (0, 0)$ or $(1, 1)$.

When discussing these cases we will refer to them as Case(A.1), etc. There are a total of six cases, and it is necessary to handle each of them separately. While carrying out this task it is necessary to make frequent reference to the following table giving the values of the functions $\Delta(s\pi s,\pi,k)(r)$ and $\Delta(s\pi,\pi s,k)(r)$ for $r = i, j$.

The Δ functions for $r = i, j$.

Case \ Δ	$\Delta(s\pi s,\pi,k)(i)$	$\Delta(s\pi s,\pi,k)(j)$	$\Delta(s\pi,\pi s,k)(i)$	$\Delta(s\pi,\pi s,k)(j)$
1	$-\xi_{[i',s(j'))}(k)$	$\xi_{[s(i'),j')}(k)$	$-\xi_{[s(i'),j')}(k)$	$\xi_{[i',s(j'))}(k)$
2	$\xi_{[s(i'),j')}(k)$	$-\xi_{[i',s(j'))}(k)$	$\xi_{[i',s(j'))}(k)$	$-\xi_{[s(i'),j')}(k)$

The first two columns are computed directly from the definitions. The third and fourth columns can be derived from the first two, using the relations

$$\Delta(s\pi,\pi s,k)(i) = -\Delta(s\pi s,\pi,k)(j),$$
$$\Delta(s\pi,\pi s,k)(j) = -\Delta(s\pi s,\pi,k)(i).$$

Now we begin the process of deducing the explicit descriptions of the K-sets asserted in this lemma. We have six cases (A.1, A.2, B.1, B.2, C.1, C.2) to consider. We claim that in fact cases A.2 and B.1 do not occur. Take for example Case(A.2). Taking $r = i$ in (5) and looking at the table we get

$$\mu_k'''(i) = \mu_k(i) + 1 + \xi_{[s(i'),j')}(k),$$

from which it follows (since the μ-functions take values in $\{0, 1\}$) that $[s(i'), j') = \emptyset$, a contradiction. A similar and equally easy contradiction arises in Case(B.1).

The remaining four cases are A.1, B.2, C.1, and C.2. We will discuss B.2 as an example; the others are similar. Using (5) and (6) and the table we get

$$\mu_k'''(i) = \mu_k(i) - 1 + \xi_{[s(i'),j')}(k), \tag{7}$$

$$\mu_k'''(j) = \mu_k(j) + 1 - \xi_{[i',s(j'))}(k), \tag{8}$$

$$\mu_k''(i) = \mu_k'(i) - 1 + \xi_{[i',s(j'))}(k), \tag{9}$$

$$\mu_k''(j) = \mu_k'(j) + 1 - \xi_{[s(i'),j')}(k). \tag{10}$$

These imply the first four lines of the following relations between the associated sets, the last two being consequences of Lemma 6.6 (3):

(11) $K_i''' = K_i \cap [s(i'), j')$ $\qquad$ $K_i^c \cap [s(i'), j')^c = \emptyset$,

(12) $K_j''' = K_j \amalg [i', s(j'))^c$,

(13) $K_i'' = K_i' \cap [i', s(j'))$ $\qquad$ $(K_i')^c \cap [i', s(j'))^c = \emptyset$,

(14) $K_j'' = K_j' \amalg [s(i'), j')^c$,

(15) $[i, j) \subset K_i^c \cap K_j$ $\qquad$ $[i, j) \subset (K_i''')^c \cap K_j'''$,

(16) $[i, j) \subset (K_i')^c \cap K_j'$ $\qquad$ $[i, j) \subset (K_i'')^c \cap K_j''$.

The next step is to use these properties to deduce the precise descriptions of the sets K_i, K_j, K_i', K_j', etc. Note that (11),(12), and (15) are consequences only of (7), (8), and Lemma 6.6.

Claim 6.11. The relations (11), (12), and (15) imply that the sets K_i, K_j, K_i''' and K_j''' are proper.

Proof. It is obvious from (15) that $K_i \neq \mathbb{Z}/n\mathbb{Z}$, and $K_i''' \neq \mathbb{Z}/n\mathbb{Z}$. If $K_i = \emptyset$, then (11) implies that $[s(i'), j')^c = \emptyset$, a contradiction. Therefore K_i is proper. If $K_i''' = \emptyset$, then (11) implies that $K_i = [j', s(i'))$, and thus by Lemma 6.6 (1), $i = s(i')$ and $j = i'$, contrary to Lemma 6.8. Therefore K_i''' is proper. Similar arguments using only (11),(12), and (15) show that K_j and K_j''' are both proper. $\square$

Now using the claim and Lemma 6.6 we can describe explicitly the sets K_i, K_i''', K_j, K_j'''. In fact, since K_i and K_i''' are both half-open intervals with upper endpoint i, we see from (11) that $K_i = [j', i)$ and $K_i''' = [s(i'), i)$.

Moreover, since K_j and K_j''' are both intervals with upper endpoint j, we see from (12) that $K_j = [i', j)$ and $K_j''' = [s(j'), j)$.

We can now argue similarly to determine the remaining four K-sets explicitly. Rather than repeating the arguments, note that we deduced the first four sets formally from (7), (8) and Lemma 6.6. Now (9) and (10) are formally the result of altering (7) and (8) in the following way: in the μ-functions replace $'''$ with $''$ and "no primes" with $'$, and replace i' with $s(i')$ and j' with $s(j')$. Applying the same rule to any formal consequence of (7), (8) and Lemma 6.6 gives us a formal consequence of (9), (10) and Lemma 6.6, so we can simply read off the explicit descriptions of the remaining K-sets by applying the rule to the four sets already determined. We have therefore proved the following:

$$\begin{aligned} K_i &= [j', i) & K_i''' &= [s(i'), i), \\ K_j &= [i', j) & K_j''' &= [s(j'), j), \\ K_i' &= [s(j'), i) & K_i'' &= [i', i), \\ K_j' &= [s(i'), j) & K_j'' &= [j', j). \end{aligned}$$

Now recalling that $s(i') = i'$ and $s(j') = j'$ and that we are in Case(2) yields the desired result. This completes the proof in Case(B.2). The other three cases are similar and will be omitted. This completes the proof of the lemma when $d = 0$.

When $d = 1$ we proceed as above. We give only a sketch of how the argument differs from that above. Recall the explicit expressions for the four elements x, xw, wx, and wxw given in the previous section, with $d = 1$:

$$\begin{aligned} x &= v_0\pi, \\ xw &= (v_0 + (\mathbf{e}_{\pi(i)} - \mathbf{e}_{\pi(j)}))\pi s, \\ wx &= (s(v_0) + (\mathbf{e}_i - \mathbf{e}_j))s\pi, \\ wxw &= (s(v_0) + (\mathbf{e}_i - \mathbf{e}_j) + (\mathbf{e}_{s\pi(i)} - \mathbf{e}_{s\pi(j)}))s\pi s. \end{aligned}$$

Here $s = (ij)$ and $w = (\mathbf{e}_i - \mathbf{e}_j)s$. Because the translation parts of these four elements are minuscule vectors, we see as in the proof of Lemma 6.8 that $v_0(i) = 1$, $v_0(j) = 0$, $v_0(\pi(i)) = 0$, and $v_0(\pi(j)) = 1$. This means that we are automatically in Case(A), so we have only two cases (A.1 and A.2) to consider here rather than the six cases of the $d = 0$ situation. As before we can use the explicit expressions for x,xw, wx, and wxw above to write down expressions for the μ-functions, as in equations (1) - (4). Then one can compute the analogs of equations (5) and (6); using the result of Lemma 6.8 and the explicit values for $v_0(r)$ and $v_0(\pi(r))$ for $r = i, j$ given above, one finds

$$\begin{aligned} \mu_k'''(r) &= \mu_k(r) + \Delta(s\pi s, \pi, k)(r), \\ \mu_k''(r) &= \mu_k'(r) + \Delta(s\pi, \pi s, k)(r), \end{aligned}$$

for $r \in \{i, j\}$. This combined with the table is enough information to proceed as before, imitating the steps in the Case(B.2) described above. The only difference is that where recourse to Lemma 6.6 (3) was made in the $d = 0$ case, here we use Lemma 6.6 (4) instead.

□

As a consequence we get:

Corollary 6.12. *Let μ be the minuscule cocharacter $(1, 0, \ldots, 0)$ or $(1, \ldots, 1, 0)$ of Gl_n, let $x \in \widetilde{W}$, and let w be a reflection. Then x, xw, wx, and wxw cannot all be μ-admissible.*

Proof. Suppose $\mu = (1, 0, \ldots, 0)$. Let $\mathbf{v}$ be the alcove corresponding to x. Because x is μ-admissible, its K-sets have the following property: $K_{j_1} \cap K_{j_2} = \emptyset$, for $j_1 \neq j_2$. For suppose k were an element of the intersection. We recall from [10] the notion of *size* of an alcove: $\mathrm{size}(\mathbf{v}) = \mu_l(1) + \cdots + \mu_l(n)$ (this is independent of choice of l). By definition of the K-sets, we have $\mu_k(j_1) = 1$ and $\mu_k(j_2) = 1$, so that $\mathrm{size}(\mathbf{v}) \geq 2$, contrary to $\mathrm{size}(\mathbf{v}) = \mathrm{size}(\mu) = 1$ (the equality $\mathrm{size}(\mathbf{v}) = \mathrm{size}(\mu)$ holds because $\mathbf{v}$ is μ-admissible, as one sees from the definition of the Bruhat order on $\widetilde{W}$). Now assume that all four elements x, xw, wx, and wxw are μ-admissible, so that this comment applies to each of their K-sets. Write $w = w_{i,j;d}$. Using the explicit forms of the K-sets given in Lemma 6.9 we see

$$[\pi^{-1}(i), i) \cap [\pi^{-1}(j), j) = \emptyset \quad \text{and} \quad [\pi^{-1}(j), i) \cap [\pi^{-1}(i), j) = \emptyset.$$

But an easy case-by-case analysis of the (twelve) possible configurations of the numbers i, j, $\pi^{-1}(i)$, and $\pi^{-1}(j)$ shows that these intersections cannot simultaneously be empty. This contradiction yields the desired claim.
A similar argument (replacing $\emptyset$ with $\mathbb{Z}/n\mathbb{Z}$ and $\cap$ with $\cup$) works for the case $\mu = (1, \ldots, 1, 0)$. □

Remark 6.13. It is essential for the Corollary that we assume $G = Gl_n$ and μ as prescribed. The conclusion is false for instance in the case $G = Gl_3$ and $\mu = (2, 0, 0)$, the case $G = Gl_4$ and $\mu = (1, 1, 0, 0)$, and in the case $G = GSp_4$ and $\mu = (1, 1, 0, 0)$.

7. The Bernstein function for Gl_n, $\mu = (1, 0, \ldots, 0)$

Here we prove the main application in the paper.

Proposition 7.1. *Let $G = Gl_n$, let μ be one of the minuscule coweights $(1, 0, \ldots, 0)$ or $(1, \ldots, 1, 0)$ of Gl_n, and let $\phi = \sum_{w \in \widetilde{W}} a_w \widetilde{T}_w$ be the element of the Iwahori-Hecke algebra given by*

$$a_w = \begin{cases} 0 & \text{if } w \text{ is not } \mu\text{-admissible}, \\ Q^{l(\mu) - l(w)} & \text{if } w \text{ is } \mu\text{-admissible}. \end{cases}$$

Then $\phi = z_\mu$.

Proof. Note that $supp(\phi)$ is contained in the μ-admissible set and $\phi(\mu) = 1$ by construction. Therefore by the characterization proved in Theorem 5.8, it suffices to show that $\phi \in Z(\mathcal{H})$. This we do by verifying the three conditions of Lemma 3.1.

The fact that ϕ satisfies the first condition is an immediate consequence of Lemma 4.5.

It is easy to see that ϕ satisfies the third condition. This is an immediate consequence of two observations. Let $x \in \widetilde{W}$ and $\tau \in \Omega$ and write $\mathrm{Int}(\tau)(x) = \tau x \tau^{-1}$. Then

1. x is μ-admissible if and only if $\mathrm{Int}(\tau)(x)$ is μ-admissible.
2. $l(\mathrm{Int}(\tau)(x)) = l(x)$.

The second observation follows from the fact that subgroup Ω preserves the set S_a under conjugation. The first observation is also an easy consequence of the definitions, using this same property of Ω along with Lemma 4.4 above and the fact that $W \cdot \mu$ is stable under conjugation by τ (or even any element of $\widetilde{W}$), as is easily checked.

It remains to prove that ϕ satisfies the second condition of Lemma 3.1. So assume that $l(sxs) = l(x) - 2$. Note that in this case $l(xs) = l(sx)$ and $s(sx)s = xs$, so by Lemma 4.5, sx and xs are simultaneously μ-admissible. There are therefore four cases to consider:

1. x, xs, sx , and sxs are not μ-admissible,
2. sxs is μ-admissible, but the other three are not,
3. sxs, xs, and sx are μ-admissible, but x is not,
4. sxs, xs, sx, and x are all μ-admissible.

Now in case (1), the second condition of Lemma 3.1 is clearly satisfied. Case (2) does not arise, because the fact that sxs is μ-admissible and sx is not implies by Corollary 4.6 (with sxs instead of x) that x is μ-admissible, a contradiction. In case (3), the conclusion of the second condition of Lemma 3.1 holds, as is easily checked by the definition of ϕ. Finally, Corollary 6.12 above implies that case (4) does not arise. This completes the proof. □

Remark 7.2. After this paper was written, the author learned that J.-L. Waldspurger, in a letter to Rapoport ([13]), had earlier proved that the function ϕ defined in Proposition 7.1 lies in $Z(\mathcal{H})$.

8. Counting μ-admissible Elements

When trying to make explicit computations of the Bernstein function z_μ, it is helpful to know the number of μ-admissible elements of $\widetilde{W}$ (see Conjecture 6). In this section we present a few formulas for the cardinalities of μ-admissible sets for some cocharacters of the groups Gl_n and GSp_{2n}. They illustrate the combinatorial complexity of the μ-admissible sets.

We will only consider here μ-admissible sets for some *minuscule* cocharacters μ. It turns out that in the minuscule case the support of z_μ is precisely the μ-admissible set (comp. Conjecture 6), as is shown in a forthcoming paper by the author [5]. So one can also interpret the formulas below as counting the number of Iwahori double cosets needed to express the Bernstein function z_μ (as a linear combination of characteristic functions of Iwahori double cosets).

For the groups Gl_n or GSp_{2n} and the case where μ is minuscule, one can enumerate the μ-admissible subset of $\widetilde{W}$ by counting instead the more concrete set of *minuscule alcoves* of given size, thanks to work of Kottwitz and Rapoport (see Theorem 6.4). For the group $G = GSp_{2n}$ there is a notion of *G-alcove* and a theorem in [10] analogous to Theorem 6.4, which we now briefly discuss.

Let Θ denote the automorphism of $\mathbb{R}^{2n}$ defined by $\Theta : (x_1, \dots, x_{2n}) \mapsto (-x_{2n}, -x_{2n-1}, \dots, -x_2, -x_1)$.

The Weyl group W_n of Sp_{2n} is the group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \subset S_{2n}$. Here is a concrete realization of W_n: choose a coordinate system $(x_1, \dots, x_n, y_n, \dots, y_1)$ for $\mathbb{Z}^{2n}$. Then $\mathbf{e}_i \in (\mathbb{Z}/2\mathbb{Z})^n$ is the "switch" $x_i \longleftrightarrow y_i$ and an element $\sigma \in S_n$ acts on the vector by simultaneously permuting the x_i's and the y_i's.

The affine Weyl group for Sp_{2n} is the semidirect product of the Weyl group W_n and the lattice in $\mathbb{Z}^{2n}$ consisting of the elements fixed by Θ:

$$\{(x_1, \dots, x_n, -x_n, \dots, -x_1) \,|\, (x_1, \dots, x_n) \in \mathbb{Z}^n\}.$$

The extended affine Weyl group $\widetilde{W}(G)$ of GSp_{2n} is the semidirect product of W_n and the lattice of translations consisting of the elements $(x_1, \dots, x_{2n}) \in \mathbb{Z}^{2n}$ such that there exists a $c \in \mathbb{Z}$ such that

$$c = x_1 + x_{2n} = x_2 + x_{2n-1} = \cdots = x_n + x_{n+1}.$$

One can define a set of simple affine reflections S_a for G analogous to the case of Gl_n, and (W_a, S_a) is a Coxeter group; thus there is a length function and a Bruhat order on W_a and on $\widetilde{W}(G)$.

A *G-alcove* is a sequence of vectors v_0, ... v_{2n-1}, $v_{2n} = v_0 + (1, \dots, 1)$ of $\mathbb{Z}^{2n}$ which is an alcove for Gl_{2n} and for which there exists a $d \in \mathbb{Z}$ such that

$$v_{2n-i} = \mathbf{d} + \Theta(v_i),$$

for $1 \le i \le 2n$, where $\mathbf{d}$ denotes the vector $(d, \dots, d)$. We call the G-alcove *minuscule* if it is minuscule as an alcove for Gl_{2n}.

The group $\widetilde{W}(G)$ acts simply transitively on the set of all G-alcoves; we identify $\widetilde{W}(G)$ with the set of all G-alcoves using as base point the minuscule G-alcove $\omega_0, \dots, \omega_{2n}$.

The following theorem is the analogue for the symplectic group of Theorem 6.4.

Theorem 8.1. *(Kottwitz-Rapoport) Let $G = GSp_{2n}$, and let μ denote the minuscule vector $(1^n, 0^n)$. Suppose $\mathbf{v}$ is a G-alcove and x is the corresponding element of $\widetilde{W}(G)$. Then $\mathbf{v}$ is minuscule of size n if and only if x is μ-admissible.*

Proof. This is Theorem 4.5 (3) of [10]. $\square$

For a group G and a dominant cocharacter μ, let $N(G,\mu)$ denote the number of μ-admissible elements of $\widetilde{W}(G)$.

Proposition 8.2. *Let n be a positive integer. Then*
1. $N(Gl_n, (1,0,\dots,0)) = 2^n - 1$,
2. $N(Gl_n, (1,1,0,\dots,0)) = 3^n - 2^n - n2^{n-1}$,
3. $N(GSp_{2n}, (1^n, 0^n)) = \sum_{i=0}^n 2^{n-i}n!/i!$.

Proof. Since the idea is the same for each of these formulas, we present only the proof of the last one as an example. By Theorem 8.1, we need to count the set of minuscule G-alcoves $\mathbf{v}$ of size n.

One can prove that the analogue of Lemma 6.7 holds: For a G-alcove $\mathbf{v}$, $\mathbf{v}$ is minuscule if and only if v_0 is a minuscule vector and the following condition holds:
(*) For every i with $1 \leq i \leq 2n$,

$$v_0(i) = 0 \Rightarrow \pi^{-1}(i) \leq i \quad \text{and} \quad v_0(i) = 1 \Rightarrow \pi^{-1}(i) \geq i.$$

Moreover, a minuscule G-alcove $\mathbf{v}$ is of size n if and only if v_0 is in the W_n-orbit of $(1^n, 0^n)$.

Each G-alcove $\mathbf{v}$ is completely determined by the pair (v_0, π), where $x = v_0\pi \in X_*(T) \rtimes W_n = \widetilde{W}(G)$ is the element corresponding to $\mathbf{v}$. Therefore we need to count the pairs (v_0, π) such that
- v_0 is of size n (has precisely n 1's and n 0's),
- $\pi \in W_n$,
- (v_0, π) satisfies (*) above.

We count, for each π, the v_0's such that (v_0, π) satisfies the above conditions. The π's are partioned according to the subset of the x_i's they move. Fix $d : 0 \leq d \leq n$, and a subset $S = \{j_1 < j_2 < \dots < j_d\}$ of $\{1, 2, \dots n\}$. Suppose π moves precisely the x_{j_l}'s. It may be considered then as an element of $(\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d$ in the obvious way. We may simplify by writing the subscript l in place of j_l, and write $\pi = \phi \circ \sigma$, where $\phi \in (\mathbb{Z}/2\mathbb{Z})^d$ and $\sigma \in S_d$. These π's which move precisely S may be further partitioned according to the subset $T = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, d\}$ which σ moves, where $0 \leq k \leq d$, $k \neq 1$. For each T there are precisely a_k possible σ's, where a_k is the number of "derangements" (permutations with no fixed points) on k letters, and each derangement can be paired with 2^k "switches" ϕ such that the product $\phi\sigma$ still moves precisely the set S. The number of sets T (inside S) with cardinality k is $\binom{d}{k}$. Therefore the number of π's which move precisely the set S is

$$\sum_{k=0}^{d} \binom{d}{k} 2^k a_k,$$

(noting that $a_1 = 0$). Now it is easy to see that any π which fixes precisely the set S can be paired with 2^{n-d} possible vectors v_0 such that (v_0, π) satisfies (*). Indeed, any v_0 has exactly n 1's and n 0's, and moreover for each i we have $(v_0(i), v_0(2n+1-i))$ is either $(1,0)$ or $(0,1)$. If i is such that π moves x_i, then only one of these possibilites can occur, by (*), but if π fixes x_i, then both can occur, giving a degree of freedom of 2 in the determination of v_0. Thus the total number of allowed pairs (v_0, π) such that π fixed precisely S is 2^{n-d} times the number above.

Finally the number of subsets S is $\binom{n}{d}$, so we see on summing over d that

$$N(GSp_{2n},(1^n,0^n)) = \sum_{d=0}^{n} \binom{n}{d} 2^{n-d} \sum_{k=0}^{d} \binom{d}{k} 2^k a_k.$$

The right hand side can be simplified, using a generating function coming from the power series expansion of $e^x/(1-2x)$ (this was pointed out by Gerd Mersmann), and we obtain

$$\sum_{i=0}^{n} 2^{n-i} n!/i!.$$

□

Remark 8.3. 1. The formula $N(Gl_n,(1,0,\dots,0)) = 2^n - 1$ was known earlier to Rapoport ([12]).
2. Helena Verrill has found a recursive procedure to calculate the numbers $N(Gl_n,(1^r,0^{n-r}))$.
3. Ilan Vardi has pointed out that the formula (3) can be rewritten as

$$N(GSp_{2n},(1^n,0^n)) = [2^n n! \sqrt{e}],$$

where $[x]$ denotes the greatest integer less than or equal to x.

9. More Examples of Bernstein Functions

In this section we give the coefficients $a_x(Q)$ for the expression of z_μ in terms of the *normalized* basis elements $\tilde{T}_x$ (comp. notation in §2):

$$z_\mu = \sum_{x \text{ is } \mu\text{-admissible}} a_x(Q)\tilde{T}_x.$$

In the examples below we will only specify $a_x(Q)$ for x a *μ-admissible* element, since all other coefficients are automatically 0.

For each example we give the number N of μ-admissible elements and the length of μ (however we will not explicitly describe the μ-admissible sets). We will always let τ denote the unique element of Ω such that $\mu\tau^{-1} \in W_a$. We use a short-hand notation for elements in the extended affine Weyl group: for example we write $s_{321}\tau$ instead of $s_3 s_2 s_1 \tau$. We use the following system to number simple reflections. In the Gl_n case, let $s_i = (i, i+1) \in S_n$ for $1 \le i \le n-1$. Let $s_0 = (1,0,\dots,0,-1)(1,n)$ be the simple affine reflection. For GSp_{2n}, let $s_i = (i,i+1)(2n+1-i, 2n-i) \in W_n$ for $1 \le i \le n-1$. Let $s_n = (n, n+1) \in W_n$. Let $s_0 = (1,0,\dots,0,-1)(1,2n)$ be the simple affine reflection.

9.1. Gl_n. Examples (1) and (2) below are for *nonminuscule* cocharacters. They were computed directly from the definition. Examples (3) and (4) were deduced from Theorem 5.8 with the help of Lemma 6.6 and Lemma 6.9. We will explain the method for Gl_4 and $\mu = (1,1,0,0)$ below. The case of Gl_5 and $\mu = (1,1,0,0,0)$ follows the same general pattern, but is considerably more complicated. The details of that case will be omitted.

1. $n = 3$, $\mu = (2,0,0)$. Then $N = 19$, $l(\mu) = 4$, and

$$a_x(Q) = \begin{cases} Q^2 + Q^4, & \text{if } l(x) = 0, \\ Q^{l(\mu)-l(x)}, & \text{if } l(x) > 0. \end{cases}$$

2. $n = 3$, $\mu = (2,1,0)$. Then $N = 25$, $l(\mu) = 4$, and

$$a_x(Q) = \begin{cases} 3Q^2 + Q^4, & \text{if } l(x) = 0, \\ Q + Q^3, & \text{if } l(x) = 1, \\ Q^{l(\mu)-l(x)}, & \text{if } l(x) > 1. \end{cases}$$

3. $n = 4$, $\mu = (1,1,0,0)$. Then $N = 33$, $l(\mu) = 4$, and

$$a_x(Q) = \begin{cases} Q^2 + Q^4, & \text{if } l(x) = 0, \\ Q^{l(\mu)-l(x)}, & \text{if } l(x) > 0. \end{cases}$$

4. $n = 5$, $\mu = (1,1,0,0,0)$. Then $N = 131$, $l(\mu) = 6$, and

$$a_x(Q) = \begin{cases} 2Q^4 + Q^6, & \text{if } l(x) = 0, \\ Q^3 + Q^5, & \text{if } l(x) = 1, \\ Q^2 + Q^4, & \text{if } x \in S_1, \\ Q^4, & \text{if } x \in S_2, \\ Q^{l(\mu)-l(x)}, & \text{if } l(x) > 2. \end{cases}$$

Here S_1 and S_2 are the following subsets of μ-admissible elements of length two:

$$S_1 = \{s_{01}\tau, s_{34}\tau, s_{23}\tau, s_{12}\tau, s_{40}\tau\}$$
$$S_2 = \{s_{43}\tau, s_{41}\tau, s_{42}\tau, s_{21}\tau, s_{31}\tau, s_{32}\tau, s_{20}\tau, s_{30}\tau, s_{10}\tau, s_{04}\tau\}.$$

9.2. GSp_{2n}. The calculation for GSp_4 below was done using the definition, jointly with Robert Kottwitz. The calculation for GSp_6 was done using a method that will appear in a forthcoming paper ([5]).

1. $n = 2$, $\mu = (1,1,0,0)$. Then $N = 13$, $l(\mu) = 3$, and

$$a_x(Q) = \begin{cases} Q + Q^3, & \text{if } l(x) = 0, \\ Q^{l(\mu)-l(x)}, & \text{if } l(x) > 0. \end{cases}$$

2. $n = 3$, $\mu = (1,1,1,0,0,0)$. Then $N = 79$, $l(\mu) = 6$, and

$$a_x(Q) = \begin{cases} Q^2 + 3Q^4 + Q^6, & \text{if } l(x) = 0, \\ 2Q^3 + Q^5, & \text{if } l(x) = 1, \\ Q^2 + Q^4, & \text{if } l(x) = 2, \\ Q + Q^3, & \text{if } x \in T_1, \\ Q^3, & \text{if } x \in T_2, \\ Q^{l(\mu)-l(x)}, & \text{if } l(x) > 3. \end{cases}$$

Here T_1 and T_2 are the following sets of μ-admissible elements of length three:

$$\begin{aligned} T_1 &= \{s_{321}\tau, s_{232}\tau, s_{123}\tau, s_{210}\tau, s_{101}\tau, s_{012}\tau\}, \\ T_2 &= \{s_{323}\tau, s_{312}\tau, s_{212}\tau, s_{213}\tau, s_{230}\tau, s_{310}\tau, s_{320}\tau, s_{120}\tau, s_{301}\tau, s_{201}\tau, s_{101}\tau\}. \end{aligned}$$

9.3. **Explanation of the case: Gl_4, $\mu = (1,1,0,0)$.** The strategy is the following:
Step 1. Use Lemma 6.6 and Lemma 6.9 to find all elements x for which there exists a simple reflection s such that $x < sx$, $xs < sxs$ and sxs is μ-admissible. The proof of Proposition 7.1 shows that these elements are such that $a_x(Q) \neq Q^{l(\mu)-l(x)}$. Such elements may or may not exist (none exist for example in the situation of Proposition 7.1) but when they exist they lie at the center of the μ-admissible set, i.e. the element τ (the base alcove) plus perhaps a few other elements of small length.
Step 2. Use the answer to Step 1 to write down a candidate function ϕ such that its support is in the admissible set, $\phi(\mu) = 1$, and which is hoped to be in the center. For the x's not belonging to a pair (x, s) as above, set $a_x(Q) = Q^{l(\mu)-l(x)}$.
Step 3. Use Lemma 3.1 to verify that ϕ is in $Z(\mathcal{H})$.
Step 4. By Theorem 5.8 we conclude $\phi = z_\mu$ and therefore we have computed z_μ.

To carry this out in our concrete case, note that the element $\tau_1 = (1,0,0,0)(1234) \in \Omega$ acts transitively on the set of simple reflections $s \in S_a$, so in order to find all pairs (x, s) that satisfy the condition in Step 1 it suffices to fix $s = s_1 = (12)$ and find all the x's that can be paired with *this* s (and then apply all powers of $\mathrm{Int}(\tau_1)$ to the pairs found). Once s is fixed, the condition that $x < sx$, $xs < sxs$ are all μ-admissible can be interpreted, using Lemmas 6.6 and 6.9, as conditions on π, where $x = v_0\pi$. In this case it is possible to determine the possible π's very quickly. Namely,

$x < sx$ implies by Lemma 6.6 and Lemma 6.9 that

$$[1,2) = K_1^c \cap K_2 = [\pi^{-1}(1), 1)^c \cap [\pi^{-1}(2), 2).$$

On the other hand Lemma 6.8 implies that $\{\pi^{-1}(1), \pi^{-1}(2)\} = \{3,4\}$. The only possibility compatible with all this is $\pi^{-1}(1) = 3$ and $\pi^{-1}(2) = 4$, so that $\pi = (13)(24)$ or $\pi = (1423)$. In either of these cases, Lemma 6.7 shows that the only possible v_0 is $(1,1,0,0)$. In the first case for π we get $x = (1,1,0,0)(13)(24) = \tau$, i.e., the base alcove. The other element is in fact $(1,1,0,0)(1423) = (1,1,0,0)(13)(24)(12) = \tau s$; therefore the second element does not occur, because $(\tau s)s < \tau s$. Thus we have proved that for $s = (12)$, the only x such that $x < sx$, $xs < sxs$ are all admissible is $x = \tau$. Conjugating by τ_1 produces no new such x's, so the answer to Step 1 is: τ.

Next we follow the recipe in Step 2 and write

$$\phi = (Q^2 + Q^4)\tilde{T}_\tau + \sum_{x\,:\,l(x)>0} Q^{l(\mu)-l(x)}\tilde{T}_x,$$

where in the sum on the right hand side x ranges only over elements in the μ-admissible set. It is easy to verify that ϕ is central, using Lemma 3.1 along the lines of the proof of Proposition 7.1. Therefore by Theorem 5.8 we conclude $z_\mu = \phi$, as desired.

10. Compatibility of Bernstein and Satake Isomorphisms

This section has two goals. First we deduce the compatibility of the Satake and Bernstein isomorphisms from work of Lusztig and Kato. Then we compute, rather indirectly, the Bernstein function for an arbitrary dominant cocharacter of Gl_2.

10.1. **Bernstein and Satake Isomorphisms.** Let W denote the finite Weyl group of a split p-adic group G over a p-adic field F. Let q denote the size of the residue field of F. Let T be an maximal F-split torus in G. Let $K \subset G(F)$ be a special maximal compact subgroup. Let $I \subset K$ be an Iwahori subgroup which is in good position, meaning that we have the decomposition $\cup_{w\in W} IwI = K$. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{C}[X_*(T)]^W \arrow[d, equal] & \mathrm{C}_c(K\backslash G(F)/K) \arrow[l, "b"', "\sim"] \\
\mathbb{C}[X_*(T)]^W \arrow[r, "B", "\sim"'] & Z(\mathrm{C}_c(I\backslash G(F)/I)) \arrow[u, "e_K * -"']
\end{tikzcd}$$

The convolution products in $\mathcal{H}_K = C_c(K\backslash G(F)/K)$ and $\mathcal{H} = C_c(I\backslash G(F)/I))$ are both defined using the Haar measure on $G(F)$ for which $\mathrm{vol}(I) = 1$. The right vertical map is convolution by $e_K = |K : I|^{-1}\mathrm{char}(K)$. The Satake isomorphism b is defined using the normalizations implicit in [7] and [8]. Note that this b differs from the usual definition of the Satake isomorphism, which is given for instance in [2]. If we let b' denote the Satake isomorphism as defined in [2], then we have the relation $b = |K : I|b'$. The difference comes from different choices of Haar measure on $G(F)$: in [2] the measure is chosen such that K has volume 1, whereas implicit in [7] and [8] is the measure giving I volume 1.

For each dominant cocharacter $\mu \in X_*(T)$, define an element $m_\mu = |W_\mu|^{-1}\sum_{w\in W} e^{w(\mu)} \in \mathbb{C}[X_*(T)]^W$, where W_μ denotes the stabilizer of μ in W. By definition the map B sends m_μ to the element $z_\mu \in Z(\mathcal{H})$. By Theorem 2.3, B is an algebra isomorphism. We call it the *Bernstein isomorphism.*

The next proposition seems to be well-known to the experts. It implies that the natural map $e_K * - : Z(C_c(I\backslash G(F)/I) \to C_c(K\backslash G(F)/K)$ is an isomorphism. A proof can be found in a paper of J.-F. Dat ([3]). It can also be deduced easily from known results of Lusztig ([7]) and S. Kato ([8]), as is explained below.

Proposition 10.1. *The Satake and Bernstein isomorphisms are compatible, i.e., the diagram above is commutative.*

Proof. Following Lusztig's notation ([7]), define a function in $\mathcal{H}$ by $\phi_0 = W(q)^{-1}\sum_{w\in W} T_w$, where $W(q) = \sum_{w\in W} q^{l(w)}$ is the Poincaré polynomial of W. Using $K = \cup_{w\in W} IwI$ one sees that ϕ_0 is just another way to write the function e_K above. Again following Lusztig, define for each dominant cocharacter $\mu \in X_{dom}$ the following elements of $\mathcal{H}$:

$$f_\mu = W(q)^{-1}\sum_{w\in W\mu W} T_w,$$

$$k'_\mu = \sum_{\substack{\lambda\preceq\mu \\ \lambda\in X_{dom}}} P_{w_0\lambda,w_0\mu}(1) z_\lambda,$$

$$c'_\mu = q^{-l(\mu)/2}\sum_{\substack{\lambda\preceq\mu \\ \lambda\in X_{dom}}} P_{w_0\lambda,w_0\mu}(q) f_\lambda.$$

where w_0 is the longest element of W and $P_{x,y}(q)$ are the Kazhdan-Lusztig polynomials for the extended affine Weyl group of G.

Note that f_μ is just another way to write the spherical function $|K : I|^{-1}\text{char}(K\mu K)$, so the functions f_μ form a basis of $\mathcal{H}_K$ as μ ranges over dominant cocharacters (by the Cartan decomposition). Furthermore, since the matrix of Kazhdan-Lusztig polynomials is invertible, Theorem 2.3 implies that $\{k'_\mu,\ \mu \in X_{dom}\}$ forms a basis for $Z(\mathcal{H})$. To show the commutativity of the diagram it is sufficient to show that $b^{-1}B^{-1}$ and $\phi_0 * -$ both map k'_μ to c'_μ.

We have $B^{-1}(k'_\mu) = \sum_{\lambda\preceq\mu} P_{w_0\lambda,w_0\mu}(1) m_\lambda = \chi_\mu$, the character of the module with highest weight μ (the first equality is trivial, and the second is a theorem of Lusztig ([7])). On the other hand by a result of Kato (see

Theorem 1.8 of [8]) we have

$$\chi_\mu = q^{-l(\mu)/2} \sum_{\lambda \preceq \mu} P_{w_0\lambda, w_0\mu}(q) b(f_\lambda).$$

It follows that $b^{-1}B^{-1}(k'_\mu) = c'_\mu$. It remains to prove

$$c'_\mu = \phi_0 * k'_\mu.$$

But this is precisely Proposition 8.6 of [7]. □

Definition 10.2. We call the function $k_\mu = q^{l(\mu)/2} k'_\mu$ the *test function* attached to the dominant cocharacter μ.

10.2. **Test functions and Bernstein functions for Gl_2.** Let $\mu = (n, m)$, $n \geq m$ be an arbitrary dominant coweight for Gl_2. Let s_0 denote the unique affine reflection $(1, -1)s_1$ where s_1 the unique reflection in the finite Weyl group $W = \{1, (1, 2)\}$.

By the discussion above, k_μ is the element in the Iwahori-Hecke algebra characterized by the following property: $q^{-l(\mu)/2} k_\mu$ is the unique element in the center of the Iwahori-Hecke algebra whose image under $\phi_0 * - \; : Z(\mathcal{H}_I) \xrightarrow{\sim} \mathcal{H}_K$ has Satake transform equal to χ_μ, the character of the module with highest weight μ. Here ϕ_0 is the element $W(q)^{-1} \sum_{w \in W} T_w$, which in the case of Gl_2 takes the form $(1+q)^{-1}(1 + T_{s_1})$. We write

$$k_\mu = \sum_{x \in \widetilde{W}} k_\mu(x) T_x.$$

Proposition 10.3.

$$k_\mu(x) = \begin{cases} 0, & \textit{if } x \textit{ in not } \mu\textit{-admissible}, \\ 1 - q + q^2 - \cdots + (-q)^{l(\mu) - l(x)}, & \textit{if } x \textit{ is } \mu\textit{-admissible}. \end{cases}$$

Proof. (Sketch)
Define a function $\tilde{k}_\mu$ with coefficients given as in the Proposition.
Step 1: Use Lemma 3.1 of to show that $\tilde{k}_\mu$ is in the center of the Iwahori-Hecke algebra.
Step 2: Write down explicitly the function in the spherical Hecke algebra whose Satake transform is χ_μ. This is

$$c'_\mu = q^{-l(\mu)/2} \sum_{\lambda \preceq \mu} f_\lambda,$$

where

$$f_\lambda = (1+q)^{-1} \sum_{x \in W\lambda W} T_x.$$

Note that this can be written out more explicitly, using that $W\lambda W = \{\lambda, s_1\lambda, \lambda^-, s_1\lambda^-\}$, where λ^- is the antidominant coweight in the W-orbit of λ.
Step 3: Verify by direct computation that $(\phi_0) * (q^{-l(\mu)/2} \tilde{k}_\mu) = c'_\mu$.
Conclude: $\tilde{k}_\mu = k_\mu$. □

On the other hand, we have

$$k_\mu = q^{l(\mu)/2} \sum_{\lambda \preceq \mu} P_{w_0\lambda, w_0\mu}(1) z_\lambda = q^{l(\mu)/2} \sum_{\lambda \preceq \mu} z_\lambda,$$

the second equality holding because the Kazhdan-Lusztig polynomials are trivial for the affine Weyl group of Gl_2. As a result we can invert to calculate the Bernstein functions. We find

$$q^{l(\mu)/2} z_\mu = \begin{cases} k_\mu - q k_{\mu - \check{\alpha}}, & \text{if } \mu \text{ is not minuscule}, \\ k_\mu, & \text{if } \mu \text{ is minuscule}. \end{cases}$$

Here $\check{\alpha} = (1, -1)$ is the unique simple coroot of Gl_2. Write

$$z_\mu = \sum_{x \in \widetilde{W}} z_\mu(x) T_x.$$

Using the explicit expression for k_μ given above, we find

Corollary 10.4.

$$q^{l(\mu)/2} z_\mu(x) = \begin{cases} 0, & \textit{if } x \textit{ is not } \mu\textit{-admissible}, \\ 1, & \textit{if } l(x) = l(\mu), \\ 1 - q, & \textit{if } l(x) = l(\mu) - 1, \\ 1 - 2q + 2q^2 - \cdots + 2(-q)^{r-1} + (-q)^r, & \textit{if } r = l(\mu) - l(x) \geq 2. \end{cases}$$

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