On the theory of Jacobi forms. I

by

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Introduction

The functions studied in this paper are a cross between elliptic functions and modular forms in one variable. Specifically, we define a Jacobi form on $SL_2(\mathbb{Z})$ to be a holomorphic function

$$\phi: \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C} \qquad (\mathcal{H} = \text{upper half-plane})$$

satisfying the two transformation equations

$$(1) \qquad \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}} \phi(\tau,z) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\right),$$

$$(2) \qquad \phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi im(\lambda^2\tau+2\lambda z)} \phi(\tau,z) \qquad ((\lambda\ \mu) \in \mathbb{Z}^2)$$

and having a Fourier expansion of the form

$$(3) \qquad \phi(\tau,z) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2 \le 4nm}} c(n,r)\, e^{2\pi i(n\tau+rz)} \quad .$$

Here k and m are natural numbers, called the weight and index of ϕ, respectively. Note that the function $\phi(\tau,0)$ is an ordinary modular form of weight k, while for fixed τ the function $z \to \phi(\tau,z)$ is a function of the type normally used to embed the elliptic curve $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$ into a projective space.

If $m=0$, then ϕ is independent of z and the definition reduces to the usual notion of modular forms in one variable. We give three other examples of situations where functions satisfying (1) - (3) arise classically:

1. Theta series. Let $Q: \mathbb{Z}^N \longrightarrow \mathbb{Z}$ be a positive definite integer-valued quadratic form and B the associated bilinear form. Then for any vector $x_0 \in \mathbb{Z}^N$ the theta series

$$(4) \qquad \Theta_{x_0}(\tau,z) = \sum_{x\in\mathbb{Z}^N} e^{2\pi i(Q(x)\tau + B(x,x_0)z)}$$

is a Jacobi form (in general on a congruence subgroup of $SL_2(\mathbb{Z})$) of weight $N/2$ and index $Q(x_0)$; the condition $r^2 \le 4nm$ in (3) arises from the fact that the restriction of Q to $\mathbb{Z}x+\mathbb{Z}x_0$ is a positive definite binary quadratic form. Such theta series (for $N=1$) were first studied by Jacobi [10], whence our general name for functions satisfying (1) and (2).

2. Fourier coefficients of Siegel modular forms. Let $F(Z)$ be a Siegel modular form of weight k and degree 2. Then we can write Z as $\begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}$ with $z\in\mathbb{C}$, $\tau,\tau'\in H$ (and $\mathrm{Im}(z)^2<\mathrm{Im}(\tau)\mathrm{Im}(\tau')$), and the function F is periodic in each variable τ, z and τ'. Write its Fourier expansion with respect to τ' as

$$(5)\qquad F(Z) = \sum_{m=0}^{\infty} \phi_m(\tau,z)\, e^{2\pi i m\tau'} \quad ;$$

then for each m the function ϕ_m is a Jacobi form of weight k and index m, the condition $4nm\geq r^2$ in (3) now coming from the fact that F has a Fourier development of the form $\sum c(T)\, e^{2\pi i\,\mathrm{Tr}(TZ)}$ where T ranges over positive semi-definite symmetric 2×2 matrices. The expansion (5) (and generalizations to other groups) was first studied by Piatetski-Shapiro [26], who referred to it as the Fourier-Jacobi expansion of F and to the coefficients ϕ_m as Jacobi functions, a word which we will reserve for (meromorphic) quotients of Jacobi forms of the same weight and index, in accordance with the usual terminology for modular forms and functions.

3. The Weierstrass $\wp$-function. The function

$$(6)\qquad \wp(\tau,z) = z^{-2} + \sum_{\substack{\omega\in\mathbb{Z}+\mathbb{Z}\tau\\ \omega\neq 0}} \left((z+\omega)^{-2} - \omega^{-2}\right)$$

is a meromorphic Jacobi form of weight 2 and index 0; we will see later how to express it as a quotient of holomorphic Jacobi forms (of index 1 and weights 12 and 10).

Despite the importance of these examples, however, no systematic theory of Jacobi forms along the lines of Hecke's theory of modular forms seems to have been attempted previously*. The authors' interest in constructing such a theory arose form their attempts to understand and extend Maass' beautiful work on the "Saito-Kurokawa conjecture". This conjecture, formulated independently by Saito and by Kurokawa [15] on the basis of numerical calculations of eigenvalues of Hecke operators for the (full) Siegel modular group, asserted the existence of a "lifting" from ordinary modular forms of weight $2k-2$ (and level one) to Siegel modular forms of weight k (and also level one); in a more precise version, it said that this lifting should land in a specific subspace of the space of Siegel modular forms (the so-called Maass "Spezialschar", defined by certain identities among Fourier coefficients) and should in fact be an isomorphism from $M_{2k-2}(SL_2(\mathbb{Z}))$ onto this space, mapping Eisenstein series to Eisenstein series, cusp forms to cusp forms, and Hecke eigenforms to Hecke eigenforms. Most of this conjecture was proved by Maass [21,22,23], another part by Andrianov [2], and the remaining part by one of the authors [40]. It turns out that the conjectured correspondence is the composition of three isomorphisms

$$\begin{array}{c} \text{Maass "Spezialschar"} \subset M_k(Sp_4(\mathbb{Z})) \\ \downarrow\wr \\ \text{Jacobi forms of weight } k \text{ and index } 1 \\ \uparrow\wr \\ \text{Kohnen's "+"-space ([11])} \subset M_{k-\frac{1}{2}}(\Gamma_0(4)) \\ \downarrow\wr \\ M_{2k-2}(SL_2(\mathbb{Z})) \end{array} \tag{7}$$

*Shimura [31,32] has studied the same functions and also their higher-dimensional generalizations. By multiplication by appropriate elementary factors they become modular functions in τ and elliptic (resp. Abelian) functions in z, although non-analytic ones. Shimura used them for a new foundation of complex multiplication of Abelian functions. Because of the different aims Shimura's work does not overlap with ours. We also mention the work of R. Berndt [3,4], who studied the quotient field (field of Jacobi functions) from both an algebraic-geometrical and arithmetical point of view. Here, too, the overlap is slight since the field of Jacobi functions for $SL_2(\mathbb{Z})$ is easily determined (it is generated over $\mathbb{C}$ up to the modular invariant $j(\tau)$ and the Weierstrass $\wp$-function $\wp(\tau,z)$); Berndt's papers concern Jacobi functions of higher level. Finally, the very recent paper of Feingold and Frenkel (Math. Ann. 263, 1983) on Kac-Moody algebras uses functions equivalent to our Jacobi forms, though with a very different motivation; here there is some overlap of their results and our §9 (in particular, our Theorem 9.2 seems to be equivalent to their Corollary 7.11).

the first map associates to each F the function ϕ_1 defined by (5), the second is given by

$$\sum_{n\geq 0} c(n)\, e^{2\pi i n\tau} \longrightarrow \sum_{n\geq 0} \sum_{r^2\leq 4n} c(4n-r^2)\, e^{2\pi i(n\tau+rz)} \quad ,$$

and the third is the Shimura correspondence [29,30] between modular forms of integral and half-integral weight, as sharpened by Kohnen [11] for the case of forms of level 1 .

One of the main purposes of this paper will be to explain diagram (7) in more detail and to discuss the extent to which it generalizes to Jacobi forms of higher index. This will be carried out in Chapters I and II, in which other basic elements of the theory (Eisenstein series, Hecke operators,...) are also developed. In Chapter III we will study the bigraded ring of all Jacobi forms on $SL_2(\mathbb{Z})$. This is much more complicated than the usual situation because, in contrast with the classical isomorphism $M_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_4,E_6]$, the ring $J_{*,*} = \bigoplus_{k,m} J_{k,m}$ ($J_{k,m}$ = Jacobi forms of weight k and index m) is not finitely generated. Nevertheless, we will be able to obtain considerable information about the structure of $J_{*,*}$. In particular, we will find upper and lower bounds for $\dim J_{k,m}$ which agree for k sufficiently large ($k\geq m$), will prove that $J_{*,m} = \bigoplus_k J_{k,m}$ is a free module of rank $2m$ over the ring $M_*(SL_2(\mathbb{Z}))$, and will describe explicit algorithms for finding bases of $J_{k,m}$ as a vector space over $\mathbb{C}$ and of $J_{*,m}$ as a module over $M_*(SL_2(\mathbb{Z}))$. The dimension formula obtained has the form

$$(8) \qquad \dim J_{k,m} = \sum_{r=0}^{m} \dim M_{k+2r} - N(m)$$

for k even (and sufficiently large), where $N(m)$ is given by

$$N(m) = \sum_{r=0}^{m} \left\lceil \frac{r^2}{4m} \right\rceil \qquad (\lceil x\rceil = \text{smallest integer} \geq x) \; .$$

We will show that $N(m)$ can be expressed in terms of class numbers of imaginary quadratic fields and that (8) is equivalent to the formula

$$(9) \qquad \dim J_{k,m}^{\text{new}} = \dim M_{2k-2}^{\text{new}}(\Gamma_0(m))^+ ,$$

where $M_{2k-2}^{\text{new}}(\Gamma_0(m))^+$ is the space of new forms of weight $2k-2$ on $\Gamma_0(m)$ which are invariant under the Atkin-Lehner (or Fricke) involution $f(\tau) \to m^{-k+1} \tau^{-2k+2} f(-1/m\tau)$ and $J_{k,m}^{\text{new}}$ a suitably defined space of "new" Jacobi forms.

Chapter IV, which will be published as the second part of this paper, goes more deeply into the Hecke theory of Jacobi forms. In particular, it is shown with the aid of a trace formula that the equality of dimensions (9) actually comes from an isomorphism of the corresponding spaces as modules over the ring of Hecke operators.

Another topic which will be treated in a later paper (by B. Gross, W. Kohnen and the second author) is the relationship of Jacobi forms to Heegner points. These are specific points on the modular curve $X_0(m) = H/\Gamma_0(m) \cup \{\text{cusps}\}$ (namely, those satisfying a quadratic equation with leading coefficient divisible by m). It turns out that for each n and r with $r^2 < 4nm$ one can define in a natural way a class $P(n,r) \in \text{Jac}(X_0(m))(\mathbb{Q})$ as a combination of Heegner points and cusps and that the sum $\sum_{n,r} P(n,r)\, q^n \zeta^r$ is an element of $\text{Jac}(X_0(m))(\mathbb{Q}) \otimes_{\mathbb{Q}} J_{2,m}$.

One final remark. Since this is the first paper on the theory of Jacobi forms, we have tried to give as elementary and understandable an exposition as possible. This means in particular that we have always preferred a more classical to a more modern approach (for instance, Jacobi forms are defined by transformation equations in $H \times \mathbb{C}$ rather than as sections of line bundles over a surface or in terms of the representation theory of Weil's metaplectic group), that we have often given two proofs of the same result if the shorter one seemed to be too uninformative or to depend too heavily on special properties of the full modular group, and that we have included a good many numerical examples. Presumably the theory will be developed at a later time from a more sophisticated point of view.

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This work originated from a much shorter paper by the first author, submitted for publication early in 1980. In this the Saito-Kurokawa conjecture was proved for modular (Siegel and elliptic) forms on $\Gamma_0(N)$ with arbitrary level N . However, the exact level of the forms in the bottom of diagram (7) was left open. The procedure was about the same as here in §§4-6. The second author persuaded the first to withdraw his paper and undertake a joint study in a much broader frame. Sections 2 and 8-10 are principally due to the second author, while sections 1, 3-7 and 11 are joint work.

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Notations

We use $\mathbb{N}$ to denote the set of natural numbers, $\mathbb{N}_0$ for $\mathbb{N}\cup\{0\}$. We use Knuth's notation $\lfloor x\rfloor$ (rather than the usual $[x]$) for the greatest-integer function $\max\{n\in\mathbb{Z} \mid n\le x\}$ and similarly $\lceil x\rceil = \min\{n\in\mathbb{Z} \mid n\ge x\} = -\lfloor -x\rfloor$. The symbol $\square$ denotes any square number. By $d\|n$ we mean $d|n$ and $(d,\frac{n}{d})=1$. In sums of the form $\sum\limits_{d|n}$ or $\sum\limits_{ad=\ell}$ it is understood that the summation is over positive divisors only. The function $\sum\limits_{d|n} d^{\nu}$ $(d\in\mathbb{N})$ is denoted $\sigma_\nu(n)$.

The symbol $e(x)$ denotes $e^{2\pi ix}$, while $e^m(x)$ and $e_m(x)$ $(m\in\mathbb{N})$ denote $e(mx)$ and $e(x/m)$, respectively. In $e(x)$ and $e^m(x)$, x is a complex variable, but in $e_m(x)$ it is to be taken in $\mathbb{Z}/m\mathbb{Z}$; thus $e_m(ab^{-1})$ means $e_m(n)$ with $bn\equiv a \pmod m$, and not $e(a/bm)$.

We use M^t and I_n for the transpose of a matrix and for the $n\times n$ identity matrix, respectively. The symbol $[a,b,c]$ denotes the quadratic form $ax^2+bxy+cy^2$.

H denotes the upper half-plane $\{\tau\in\mathbb{C} \mid \mathrm{Im}(\tau)>0\}$. The letters τ and z will always be reserved for variables in H and $\mathbb{C}$, respectively, with $\tau = u+iv$, $z = x+iy$, $q = e(\tau)$, $\zeta = e(z)$. The group $SL_2(\mathbb{Z})$ will often be denoted by Γ_1 and the space of modular (resp. cusp) forms of weight k on Γ_1 by M_k (resp. S_k). The normalized Eisenstein series $E_k\in M_k$ $(k\ge 4$ even) are defined in the usual way; in particular one has $M_* := \bigoplus_k M_k = \mathbb{C}[E_4,E_6]$ with $E_4 = 1+240\sum\sigma_3(n)\,q^n$, $E_6 = 1-504\sum\sigma_5(n)\,q^n$.

The symbol ":=" means that the expression on the right is the definition of that on the left.

Chapter I. Basic Properties

§ 1. Jacobi forms and the Jacobi group

The definition of Jacobi forms for the full modular group $\Gamma_1 = SL_2(\mathbb{Z})$ was already given in the introduction. In order to treat subgroups $\Gamma \subset \Gamma_1$ with more than one cusp, we have to rewrite the definition in terms of an action of the groups $SL_2(\mathbb{Z})$ and $\mathbb{Z}^2$ on functions $\phi: \mathcal{H} \times \mathbb{C} \to \mathbb{C}$. This action, analogous to the action

$$(1) \qquad (f|_k M)(\tau) := (c\tau+d)^{-k} f(\tfrac{a\tau+b}{c\tau+d}) \qquad (M=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1)$$

in the usual theory of modular forms, will be important for several later constructions (Eisenstein series, Hecke operators). We fix integers k and m and define

$$(2) \qquad (\phi|_{k,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix})(\tau,z) := (c\tau+d)^{-k} e^m(\tfrac{-cz^2}{c\tau+d})\, \phi(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}) \qquad (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1)$$

and

$$(3) \qquad (\phi|_m [\lambda\ \mu])(\tau,z) := e^m(\lambda^2\tau + 2\lambda z)\ \phi(\tau, z + \lambda\tau + \mu) \qquad ((\lambda\ \mu) \in \mathbb{Z}^2) ,$$

where $e^m(x) = e^{2\pi i m x}$ (see "Notations"). Thus the two basic transformation laws of Jacobi forms can be written

$$\phi|_{k,m} M = \phi \quad (M \in \Gamma_1), \quad \phi|_m X = \phi \quad (X \in \mathbb{Z}^2),$$

where we have dropped the square brackets around M or X to lighten the notation. One easily checks the relations

$$(4) \qquad \begin{aligned} &(\phi|_{k,m} M)\,|_{k,m} M' = \phi|_{k,m}(MM'), \quad (\phi|_m X)|_m X' = \phi|_m (X + X') , \\ &(\phi|_{k,m} M)|_m XM = (\phi|_m X)|_{k,m} M \qquad (M,M' \in \Gamma_1,\ X,X' \in \mathbb{Z}^2) . \end{aligned}$$

They show that (2) and (3) jointly define an action of the semi-direct product $\Gamma_1^J := \Gamma_1 \ltimes \mathbb{Z}^2$ (= set of products (M,X) with $M \in \Gamma_1$, $X \in \mathbb{Z}^2$ and group law $(M,X)(M',X') = (MM', XM' + X')$; notice that we are writing our vectors as row vectors, so Γ_1 acts on the right), the (full) Jacobi group. We will discuss this action in more detail at the end of this section.

We can now give the general definition of Jacobi forms.

Definition. A Jacobi form of weight k and index m $(k,m \in \mathbb{N})$ on a subgroup $\Gamma \subset \Gamma_1$ of finite index is a holomorphic function $\phi: H \times \mathbb{C} \to \mathbb{C}$ satisfying

i) $\phi|_{k,m} M = \phi \quad (M \in \Gamma)$

ii) $\phi|_m X = \phi \quad (X \in \mathbb{Z}^2)$

iii) for each $M \in \Gamma_1$, $\phi|_{k,m} M$ has a Fourier development of the form $\sum c(n,r) q^n \zeta^r$ $(q = e(\tau), \zeta = e(z))$ with $c(n,r) = 0$ unless $n \geq r^2/4m$. If ϕ satisfies the stronger condition $c(n,r) \neq 0 \Rightarrow n > r^2/4m$, it is called a cusp form.

The vector space of all such functions ϕ is denoted $J_{k,m}(\Gamma)$; if $\Gamma = \Gamma_1$ we write simply $J_{k,m}$ for $J_{k,m}(\Gamma_1)$.

Remarks: We could also define Jacobi forms with character, $J_{k,m}(\Gamma,\chi)$, by inserting a factor $\chi(M)$ in i) in the usual way. Also, we could replace $\mathbb{Z}^2$ by some other lattice invariant under Γ (e.g. by imposing congruence conditions modulo N if $\Gamma = \Gamma(N)$);if we did this, then the exponents n and r in iii) would in general be rational numbers but we would still require $4nm \geq r^2$ as the condition of holomorphy at the cusps . It would therefore be more proper to refer to functions satisfying i) - iii) as Jacobi forms on the Jacobi group $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$ (rather than on Γ). However, we will not worry about this since most of the time we will be concerned only with the full Jacobi group.

Our first main result is

Theorem 1.1. The space $J_{k,m}(\Gamma)$ is finite-dimensional.

This will follow from two other results, both of independent interest :

Theorem 1.2. Let ϕ be a Jacobi form of index m. Then for fixed $\tau \in H$, the function $z \mapsto \phi(\tau,z)$, if not identically zero, has exactly $2m$ zeros (counting multiplicity) in any fundamental domain for the action of the lattice $\mathbb{Z}\tau + \mathbb{Z}$ on $\mathbb{C}$.

Proof: It follows easily from the transformation law ii) that

$$\frac{1}{2\pi i}\oint_{\partial F} \frac{\phi_z(\tau,z)}{\phi(\tau,z)}\, dz = 2m \qquad \left(\phi_z = \frac{\partial\phi}{\partial z}\,,\ F = \text{fundamental domain for } \mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}\right)$$

(the expression $\frac{1}{2\pi i}\frac{\phi_z}{\phi}$ is invariant under $z \to z+1$ and changes by $2m$ when one replaces z by $z+\tau$), and this is equivalent to the statement of the theorem. Notice that the same proof works for ϕ meromorphic (with "number of zeros" replaced by "number of zeros minus number of poles") and any $m \in \mathbb{Z}$. A consequence is that there are no holomorphic Jacobi forms of negative index, and that a holomorphic Jacobi form of index is independent of z (and hence simply an ordinary modular form of weight k in τ).

Theorem 1.3. Let ϕ be a Jacobi form on Γ of weight k and index m and λ,μ rational numbers. Then the function $f(\tau) = e^m(\lambda^2\tau)\ \phi(\tau,\lambda\tau+\mu)$ is a modular form (of weight k and on some subgroup of Γ_1 of finite index depending only on Γ and on λ,μ).

For $\lambda = \mu = 0$ it is clear that $\tau \to \phi(\tau,0)$ is a modular form of weight k on Γ. We will prove the general case later on in this section, when we have developed the formalism of the action of the Jacobi group further. Note that the Fourier development of $f(\tau)$ at infinity is

$$\sum_{n,r} c(n,r)\ e((m\lambda^2+r\lambda+n)\tau) \quad ,$$

so that the conditions $n \geq 0$, $r^2 \leq 4mn$ in the definition of Jacobi forms are exactly what is required to ensure the holomorphicity of f at ∞ in the usual sense.

To deduce 1.1, we pick any $2m$ pairs of rational numbers $(\lambda_i,\mu_i) \in \mathbb{Q}^2$ with $(\lambda_i,\mu_i) \not\equiv (\lambda_j,\mu_j) \pmod{\mathbb{Z}^2}$ for $i \neq j$. Then the functions $f_i(\tau) = e^m(\lambda_i^2\tau)\ \ \phi(\tau,\lambda_i\tau + \mu_i)$ lie in $M_k(\Gamma_i)$ for some subgroups Γ_i of Γ, and the map $\phi \to \{f_i\}_i$ is injective by Theorem 1.2. Therefore $\dim J_{k,m}(\Gamma) \leq \sum_i \dim M_k(\Gamma_i)$; this proves Theorem 1.1 and also shows that $J_{k,m}(\Gamma)$ is 0 for $k \leq 0$ unless $k = m = 0$, in which case it reduces to the constants.

To prove Theorem 1.3, we would like to apply (3) to $(\lambda,\mu) \in \mathbb{Q}^2$. However, we find that formula (3) no longer defines a group action if we allow non-integral λ and μ , since

$$\begin{aligned}((\phi|_m[\lambda\ \mu])|_m[\lambda'\ \mu'])(\tau,z) &= \\ &= e^m(\lambda'^2\tau + 2\lambda' z + \lambda^2\tau + 2\lambda(z+\lambda'\tau+\mu'))\quad \phi(\tau,z+\lambda'\tau+\mu'+\lambda z+\mu) \\ &= e(2m\lambda\mu')\ \ (\phi|_m[\lambda+\lambda'\ \ \mu+\mu'])(\tau,z)\end{aligned}$$

and $e(2m\lambda'\mu)$ will not in general be equal to 1 . Similarly, the third equation of (4) breaks down if X is not in $\mathbb{Z}^2$. Hence if we want to extend our actions to $SL_2(\mathbb{Q})$ (or $SL_2(\mathbb{R})$) and $\mathbb{Q}^2$ (or $\mathbb{R}^2$), we must modify the definition of the group action.

The verification of the third equation in (4) depends on the two elementary identities

$$\frac{z}{c\tau+d} + \lambda\frac{a\tau+b}{c\tau+d} + \mu = \frac{z+\lambda_1\tau+\mu_1}{c\tau+d} \quad ,$$

$$\lambda^2\frac{a\tau+b}{c\tau+d} + 2\lambda\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} + \lambda\mu = \lambda_1^2\tau + 2\lambda_1 z - \frac{c(z+\lambda_1\tau+\mu_1)^2}{c\tau+d} + \lambda_1\mu_1 \ ,$$

where $(\lambda_1\,\mu_1) = (\lambda\,\mu)\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Thus to make this equation hold for arbitrary $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{R})$ and $X = (\lambda\,\mu) \in \mathbb{R}^2$ we should replace (3) by

(5) $$(\phi|_m[\lambda\,\mu])(\tau,z) := e^m(\lambda^2\tau+2\lambda z+\lambda\mu)\ \phi(\tau,z+\lambda\tau+\mu)$$
$$((\lambda\,\mu)\in\mathbb{R}^2)\ ;$$

this is compatible with (3) because $e^m(\lambda\mu) = 1$ for $\lambda,\mu \in \mathbb{Z}$. Unfortunately, (5) still does not define a group action; we now find

(6) $$(\phi|_m X)|_m X' = e^m(\lambda\mu'-\lambda'\mu)\ \phi|_m(X+X') \qquad (X = (\lambda\,\mu),\ X' = (\lambda'\,\mu')\in\mathbb{R}^2)$$

To absorb the extra factor, we must introduce a scalar action of the group $\mathbb{R}$ by

(7) $$(\phi|_m[\kappa])(\tau,z) := e(m\kappa)\ \phi(\tau,z) \qquad (\kappa\in\mathbb{R})$$

and then make a central extension of $\mathbb{R}^2$ by this group $\mathbb{R}$, i.e. replace $\mathbb{R}^2$ by the Heisenberg group

$$H_{\mathbb{R}} := \{[(\lambda\ \mu),\kappa] \ \ |\ \ (\lambda,\mu)\in\mathbb{R}^2,\ \kappa\in\mathbb{R}\}\ ,$$

$$[(\lambda\ \mu),\kappa][(\lambda'\ \mu'),\kappa'] = [(\lambda+\lambda'\ \ \mu+\mu'),\ \kappa+\kappa'+\lambda\mu'-\lambda'\mu].$$

(This group is isomorphic to the group of upper triangular unipotent 3×3 matrices via

$$[(\lambda\ \mu),\kappa] \quad \mapsto \quad \begin{pmatrix} 1 & \lambda & \frac{1}{2}(\kappa+\lambda\mu) \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix}.)$$

The subgroup $C_{\mathbb{R}} := \{[(0\ 0),\kappa],\ \kappa\in\mathbb{R}\}$ is the center of $H_{\mathbb{R}}$ and $H_{\mathbb{R}}/C_{\mathbb{R}} \cong \mathbb{R}^2$. We can now combine (5) and (7) into an action of $H_{\mathbb{R}}$ by setting

$$(\phi|[(\lambda\ \mu),\kappa])(\tau,z) = e^{m}(\lambda^2\tau+2\lambda z+\lambda\mu+\kappa) \quad \phi(\tau,z+\lambda\tau+\mu) \ ,$$

and this now <u>is</u> a group action because the extra factor $e^{m}(\lambda'\mu-\lambda\mu')$ in (6) is compensated by the twisted group law in $H_{\mathbb{R}}$. Because this twist involves $\lambda\mu'-\lambda'\mu = \det\begin{pmatrix}\lambda & \mu \\ \lambda' & \mu'\end{pmatrix}$ and the determinant is preserved by SL_2, the group $SL_2(\mathbb{R})$ acts on $H_{\mathbb{R}}$ on the right by

$$[X,\kappa]M = [XM,\kappa] \quad (X\in\mathbb{R}^2,\ \kappa\in\mathbb{R},\ M\in SL_2(\mathbb{R}));$$

the above calculations then show that all three identities (4) remain true if we now take $M,M'\in SL_2(\mathbb{R})$ and $X,X'\in H_{\mathbb{R}}$ and hence that equations (2), (5) and (7) together define an action of the semidirect product $SL_2(\mathbb{R})\ltimes H_{\mathbb{R}}$.

In the situation of usual modular forms, we write $\mathcal{H}$ as G/K where $G = SL_2(\mathbb{R})$ contains Γ as a discrete subgroup with $\mathrm{Vol}(\Gamma\backslash G)$ finite and $K = SO(2)$ is a maximal compact subgroup of G. Here we would like to do the same. However, the group $SL_2(R)\ltimes H_{\mathbb{R}}$ contains $\Gamma^J = \Gamma\times\mathbb{Z}^2$ with infinite covolume (because of the extra $\mathbb{R}$ in $H_{\mathbb{R}}$) and its quotient by the maximal compact subgroup $SO(2)$ is $\mathcal{H}\times\mathbb{C}\times\mathbb{R}$ rather than $\mathcal{H}\times\mathbb{C}$. To correct this, we observe that the subgroup $\mathbb{Z}\subset\mathbb{R}$ acts trivially in (7) , so that (2),(5) and (7) actually define an action of the quotient group

$$G^J := SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}/C_{\mathbb{Z}} .$$

Here it does not matter on which side of $H_{\mathbb{R}}$ we write $C_{\mathbb{Z}}$, since C is central in H ; the quotient $H_{\mathbb{R}}/C_{\mathbb{Z}}$ is a central extension of $\mathbb{R}^2$ by $S^1=\{\zeta\in\mathbb{C} \mid |\zeta| = 1\}$ $(\zeta=e(\kappa))$ and will also be denoted $\mathbb{R}^2\cdot S^1$ . . Now Γ^J is a discrete subgroup of G with $\mathrm{Vol}(\Gamma^J\backslash G^J)<\infty$, and if we choose the maximal compact subgroup

$$K^J := SO(2)\times S^1 \subset G^J = SL_2(\mathbb{R}) \ltimes (\mathbb{R}^2\cdot S^1)$$

then G^J/K^J can be identified naturally with $\mathcal{H}\times\mathbb{C}$ via

$$\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix},(\lambda\ \mu),\zeta\right]K^J \mapsto \left(\frac{ai+b}{ci+d}, \frac{\lambda i+\mu}{ci+d}\right) .$$

The above discussion now gives

<u>Theorem 1.4.</u> <u>Let</u> G^J <u>be the set of triples</u> $[M,X,\zeta]$ $(M\in SL_2(\mathbb{R}),\ X\in\mathbb{R}^2,\ \zeta\in\mathbb{C},\ |\zeta|=1)$. <u>Then</u> G^J <u>is a group via</u>

$$[M,X,\zeta][M',X',\zeta'] = \left[MM',XM'+X',\zeta\zeta' e^m\left(\det\begin{pmatrix} XM' \\ X'\end{pmatrix}\right)\right]$$

<u>and the formula</u>

$$\left(\phi \mid \left[\begin{pmatrix} a & b \\ c & d\end{pmatrix},(\lambda\ \mu),\zeta\right]\right)(\tau,z)$$

$$= \zeta^m(c\tau+d)^{-k}\, e^m\!\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau+2\lambda z+\lambda\mu\right)\phi\!\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)$$

<u>defines an action of</u> G^J <u>on</u> $\{\phi : \mathcal{H}\times\mathbb{C}\to\mathbb{C}\}$. <u>The functions</u> ϕ <u>satisfying the transformation laws i) and ii) of Jacobi forms</u>

are precisely those invariant with respect to this action under the discrete subgroup $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$ of G^J, and the space of such ϕ can be identified via

$$F(g) := (\phi|g)(i,0)$$

with the set of functions $F : G^J \to \mathbb{C}$ left invariant under Γ^J and transforming on the right by the representation

$$F(g\cdot [(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}),(0\ 0),\zeta]\ ) = \zeta^m e^{ik\theta}F(g)$$

of the maximal compact subgroup $K^J = SO(2)\times S^1$ of G^J.

Thus the two integers k and m in the definition of Jacobi forms appear, as they should, as the parameters for the irreducible (and here one-dimensional) representations of a maximal compact subgroup of G^J.

As an application of all this formalism, we now give the proof of 1.3. The function $f(\tau)$ in that theorem is up to a constant (namely $e^m(\lambda\mu)$) equal to $\phi_X(\tau) := (\phi|X)(\tau,0)$, where $X = (\lambda\ \mu) \in \mathbb{Q}^2$ and $\phi|X$ is defined by (5) (from now on we often omit the indices k,m on the sign $|$). For $X' = (\lambda'\ \mu') \in \mathbb{Z}^2$ we have

$$\phi_{X+X'}(\tau) = e^m(\lambda\mu'-\lambda'\mu)\ \phi_X(\tau)$$

by (6), so ϕ_X depends up to a scalar factor only on $X \pmod{\mathbb{Z}^2}$ and ϕ_X itself depends only on $X \pmod{N\mathbb{Z}^2}$ if $X \in N^{-1}\mathbb{Z}^2$. For $M = (\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}) \in \Gamma$ we have

$$\begin{aligned}(c\tau+d)^{-k}\,\phi_X\left(\frac{a\tau+b}{c\tau+d}\right) &= (\phi|X|M)(\tau,0)\\ &= (\phi|M|(XM))(\tau,0)\\ &= (\phi|(XM))(\tau,0)\\ &= \phi_{XM}(\tau)\ ,\end{aligned}$$

so ϕ_X behaves like a modular form with respect to the congruence subgroup

$$\{M \in \Gamma \mid \; XM \equiv M \pmod{\mathbb{Z}^2} \;,\; m\cdot\det\binom{X}{XM} \in \mathbb{Z}\}$$

of Γ (this group can be written explicitly

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \; (a-1)\lambda+c\mu, \;\; b\lambda+(d-1)\mu, \; m(c\mu^2+(d-a)\;\lambda\mu-b\lambda^2) \in \mathbb{Z}$$

and hence contains $\Gamma \cap \Gamma(\frac{N^2}{(N,m)})$ if $NX \in \mathbb{Z}^2$). Finally, if M is any element of Γ_1 then

$$\begin{aligned}(\phi_X|_k M)(\tau) &= (\phi|M|XM)(\tau,0)\\ &= e^m(\lambda_1^2\tau+\lambda_1\mu_1)(\phi|M)(\tau,\lambda_1\tau+\mu_1)\end{aligned}$$

where $(\lambda_1\,\mu_1) = XM$, and since $\phi|M$ has a Fourier development containing $q^n\zeta^r$ only for $4nm \geq r^2$, this contains only non-negative powers of $e(\tau)$ by the same calculation as given for $M=Id$ after the statement of 1.3.

We end with one other simple, but basic, property of Jacobi forms.

<u>Theorem 1.5</u>. <u>The Jacobi forms form a bigraded ring</u>.

<u>Proof</u>: That the product of two Jacobi forms ϕ_1 and ϕ_2 of weight k_1 and k_2 and index m_1 and m_2, respectively, transforms like a Jacobi form of weight $k=k_1+k_2$ and index $m=m_1+m_2$ is clear; we have to check the condition at infinity. One way to see this is to use the converse of Theorem 1.3, i.e. to observe that the conditions at infinity for a Jacobi form $\phi(\tau,z)$ of index m is <u>equivalent</u> to the condition that $f(\tau) = e^m(\lambda^2\tau)\phi(\lambda\tau+\mu)$ be holomorphic at ∞ (in the usual sense) for all $\lambda,\, \mu \in \mathbb{Q}$; this condition is clearly satisfied for $\phi(\tau,z) = \phi_1(\tau,z)\phi_2(\tau,z)$ with $f(\tau) = f_1(\tau)f_2(\tau)$. A more direct proof is to write

the (n,r)-Fourier coefficient of ϕ as

$$c(n,r) = \sum_{\substack{n_1+n_2=n \\ r_1+r_2=r}} c_1(n_1,r_1)\, c_2(n_2,r_2) \quad ,$$

where the c_i are the Fourier coefficients of ϕ_i (the sum is finite since $n_i \le n$, $r_i^2 \le 4n_im_i$) and deduce the inequality $r^2 \le 4nm$ from the identity

$$n_1+n_2 - \frac{(r_1+r_2)^2}{4(m_1+m_2)} = (n_1 - \frac{r_1^2}{4m_1}) + (n_2 - \frac{r_2^2}{4m_2}) + \frac{(m_1r_2-m_2r_1)^2}{4m_1m_2(m_1+m_2)} \quad .$$

This identity also shows that (as for modular forms) the product $\phi_1\phi_2$ is a cusp form whenever either ϕ_1 or ϕ_2 is one but that (unlike the situation for modular forms) $\phi_1\phi_2$ can be a cusp form even if neither ϕ nor ϕ_2 is.

The ring $J_{*,*} = \bigoplus_{k,m} J_{k,m}$ of Jacobi forms will be the object of study of Chapter III.

§ 2. Eisenstein series and cusp forms

As in the usual theory of modular forms, we will obtain our first examples of Jacobi forms by constructing Eisenstein series. In the modular case one sets (for $k > 2$)

$$E_k(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} 1|_k = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (c\tau + d)^{-k} ,$$

where $\Gamma_\infty = \{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} | n \in \mathbb{Z}\}$ is the subgroup of Γ_1 of elements γ with $1|_k\gamma = 1$, where 1 denotes the constant function. Similarly, here we define

(1) $$E_{k,m}(\tau,z) := \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma_1^J} 1|_{k,m}\gamma ,$$

where

$$\Gamma^J = \{\gamma \in \Gamma_1^J | \ 1|\gamma = 1\}$$
$$= \{ [\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}, (0\ \mu)] \ | \ n,\mu \in \mathbb{Z}\}.$$

Explicitly, this is

(2) $$E_{k,m}(\tau,z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau+d)^{-k}\, e^m\Big(\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d}\Big)$$

where a,b are chosen so that $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_1$. As in the case of modular forms, the series converges absolutely for $k \geq 4$; if is zero if k is odd (replace c, d by $-c, -d$). The invariance of $E_{k,m}$ under Γ^J is clear from the definition and the absolute convergence. To check the cusp condition, and in order to have an explicit example of a form in $J_{k,m}$, we must calculate the Fourier development of $E_{k,m}$, which we now proceed to do.

As with E_k, we split the sum over c, d into two parts, according as c is 0 or not. If $c=0$, then $d=\pm 1$; these terms give a contribution

$$(3) \qquad \sum_{\lambda\in\mathbb{Z}} e^m(\lambda^2\tau + 2\lambda z) = \sum_{\lambda\in\mathbb{Z}} q^{m\lambda}\,\zeta^{2m\lambda}$$

$(q = e^{2\pi i\tau},\ \zeta = e^{2\pi i z})$. This is a linear combination of $q^n\zeta^r$ with $4nm = r^2$ and corresponds to the constant term of the usual Eisenstein series. If $c \neq 0$, we can assume $c > 0$ (since k is even); using the identity

$$\lambda^2\frac{a\tau+b}{c\tau+d} + 2\lambda\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} = -\frac{c(z-\lambda/c)^2}{c\tau+d} + \frac{a\lambda^2}{c} \quad (c\neq 0)$$

we can write these terms as

$$\sum_{c=1}^{\infty} c^{-k} \sum_{\substack{d\in\mathbb{Z}\\(d,c)=1}} \sum_{\lambda\in\mathbb{Z}} (\tau+\tfrac{d}{c})^{-k}\, e^m\Big(-\frac{(z-\lambda/c)^2}{\tau+d/c} + \frac{a\lambda^2}{c}\Big).$$

Note that $d \to d+c$ and $\lambda \to \lambda+c$ correspond to $z \to z+1$ and $\tau \to \tau+1$, so this part equals

$$(4) \qquad \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{d \,(\mathrm{mod}\, c)\\(d,c)=1}} \sum_{\lambda\,(\mathrm{mod}\, c)} e_c(md^{-1}\lambda^2)\, F_{k,m}(\tau+\tfrac{d}{c},\ z-\tfrac{\lambda}{c})$$

with e_c as in "Notations" and

$$F_{k,m}(\tau,z) := \sum_{p,\,q\in\mathbb{Z}} (\tau+p)^{-k}\, e^m\Big(-\frac{(z+q)^2}{\tau+p}\Big) \quad ;$$

the function $F_{k,m}$ is periodic in τ and z, so (4) makes sense. Now the usual Poisson summation formula gives

$$F_{k,m} = \sum_{n,\,r\in\mathbb{Z}} \gamma(n,r)\, q^n\zeta^r$$

with

$$\gamma(n,r) = \int_{\mathrm{Im}(\tau)=C_1} \tau^{-k} e(-n\tau) \int_{\mathrm{Im}(z)=C_2} e(-mz^2/\tau - rz)\, dz\, d\tau$$

$(C_1>0,\ C_2$ arbitrary). The inner integral is standard and equals

$(\tau/2im)^{\frac{1}{2}}\, e(r^2\tau/4m)$. Hence

$$\gamma(n,r) = \int_{\mathrm{Im}(\tau)=C_1} \tau^{-k}(\tau/2im)^{\frac{1}{2}}\; e\left(\frac{r^2-4nm}{4m}\tau\right)\, d\tau$$

$$= \begin{cases} 0 & \text{if} \quad r^2 \ge 4nm \\ \alpha_k m^{1-k}(4nm-r^2)^{k-3/2} & \text{if} \quad r^2 < 4nm \end{cases}$$

with

$$\alpha_k := \frac{(-1)^{k/2}\, \pi^{k-1/2}}{2^{k-2}\, \Gamma(k-\frac{1}{2})}$$

(if $r^2 \ge 4nm$, we can deform the path of integration to $+i\infty$, *so* $\gamma=0$; if $r^2 \ge 4nm$ we deform it to a path from $-i\infty$ to $-i\infty$ circling 0 once in a clockwise direction and obtain a standard integral representation of $1/\Gamma($ Substituting the Fourier development of $F_{k,m}$ into (4) gives the expressi

$$\sum_{\substack{n,r\in\mathbb{Z} \\ 4nm > r^2}} e_{k,m}(n,r)\; q^n\, \zeta^r$$

with

$$(5) \qquad e_{k,m}(n,r) = \frac{\alpha_k}{m^{k-1}}(4nm-r^2)^{k-\frac{3}{2}} \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{\lambda,d \,(\mathrm{mod}\ c) \\ (d,c)=1}} e_c(md^{-1}\lambda^2 - r\lambda + nd).$$

(for d^{-1} , see "Notations"). To calculate this, we first replace λ by $d\lambda$ in the inner double sum (since $(d,c)=1$, this simply permutes the summands); then the summand becomes $e_c(dQ(\lambda))$ with $Q(\lambda) := m\lambda^2+r\lambda+n$. We now use the well-known identity

$$\sum_{\substack{d \,(\mathrm{mod}\ c) \\ (d,c)=1}} e_c(dN) = \sum_{a\,|\,(c,N)} \mu\left(\frac{c}{a}\right) a ,$$

where μ is the Möbius function (so-called Ramanujan sum; see Hardy-Wright or most other number theory texts); then the inner double sum in (5) becomes

$$\sum_{a|c} \mu(\tfrac{c}{a})\, a \sum_{\substack{\lambda (\mathrm{mod}\ c) \\ Q(\lambda) \equiv 0 (\mathrm{mod}\ a)}} 1 .$$

Now the condition $Q(\lambda)\equiv 0(\mathrm{mod}\ a)$ depends only on $\lambda\ (\mathrm{mod}\ a)$, so the inner sum is $\frac{c}{a}$ times $N_a(Q)$, where

$$N_a(Q) := \#\{\lambda (\mathrm{mod}\ a) \mid Q(\lambda) \equiv 0 (\mathrm{mod}\ a)\} .$$

Hence the triple sum in (5) simplifies to

$$\sum_{c=1}^{\infty} c^{1-k} \sum_{a|c} \mu(\tfrac{c}{a})\, N_a(Q) = \zeta(k-1)^{-1} \sum_{a=1}^{\infty} \frac{N_a(Q)}{a^{k-1}}$$

(the last equality follows by writing $c=ab$ and using $\sum \mu(b) b^{-s} = \zeta(s)^{-1}$). To calculate the Dirichlet series, we first calculate $N_a(Q)$ for $(a,m)=1$; this will suffice completely if $m=1$ and (using the obvious multiplicativity of N_a) will give the Dirichlet series up to a finite Euler product involving the prime divisors of m in general. If $(a,m)=1$, then

$$\begin{aligned} N_a(Q) &= \#\{\lambda (\mathrm{mod}\ a) \mid m\lambda^2+r\lambda+n \equiv 0 (\mathrm{mod}\ a)\} \\ &= \#\{\lambda (\mathrm{mod}\ a) \mid (2m\lambda+r)^2 \equiv r^2-4mn (\mathrm{mod}\ 4a)\} \\ &= N_a(r^2-4nm) , \end{aligned}$$

where

$$N_a(D) := \#\{x (\mathrm{mod}\ 2a) \mid x^2 \equiv D\ (\mathrm{mod}\ 4a)\} .$$

It is a classical fact that

$$\sum_{a=1}^{\infty} N_a(D) a^{-s} = \frac{\zeta(s)}{\zeta(2s)} L_D(s) , \tag{6}$$

if $D=1$ or if D is the discriminant of a real quadratic field, where $L_D(s) = L(s,(\frac{D}{\cdot}))$ is the Dirichlet L-series associated to D . It was shown in [39,p.130] that the same formula holds for all $D \in \mathbb{Z}$ if $L_D(s)$ is defined by

$$L_D(s) = \begin{cases} 0 & \text{if } D \not\equiv 0,1 \pmod 4 \\ \zeta(2s-1) & \text{if } D=0\ , \\ L_{D_0}(s) \cdot \sum_{d|f} \mu(d)\left(\frac{D_0}{d}\right) d^{-s} \sigma_{1-2s}(f/d) & \text{if } D \equiv 0,1 \pmod 4 \end{cases}$$

where in the last line D has been written as $D_0 f^2$ with $f \in \mathbb{N}$ and D_0 = discriminant of $\mathbb{Q}(\sqrt{D})$ (the finite sum in this case can also be written as a finite Euler product over the prime divisors of f) . Inserting (6) into the preceding equations, we find that we have proved

$$e_{k,1}(n,r) = \alpha_k |D|^{k-\frac{3}{2}} \zeta(2k-2)^{-1} L_D(k-1)$$

if $m=1$ and $D=r^2-4n<0$, while for m arbitrary there is a similar formula (now with $D=r^2-4mn$) but multiplied by an Euler factor involving the prime divisors of m . Using the functional equations of $L_D(s)$ and $\zeta(s)$ we can rewrite this formula in the simpler form

$$e_{k,1}(n,r) = L_D(2-k)/\zeta(3-2k) \ ,$$

where now all numerical factors have disappeared. The values $L_D(2-k)$ ($D<0$, k even) are well-known to the rational and non-zero; they have been studied extensively by Cohen [6] , who denoted them $H(k-1,|D|)$. Summarizing, we have proved

<u>Theorem 2.1.</u> <u>The series</u> $E_{k,m}$ ($k\geq 4$ even) <u>converges and defines a non-zero element of</u> $J_{k,m}$. <u>The Fourier development of</u> $E_{k,m}$ <u>is given by</u>

$$E_{k,m}(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4nm\geq r^2}} e_{k,m}(n,r)\, q^n \zeta^r \ ,$$

<u>where</u> $e_{k,m}(n,r)$ <u>for</u> $4nm=r^2$ <u>equals</u> 1 <u>if</u> $r\equiv 0 \pmod{2m}$ <u>and</u> 0 <u>otherwise</u>, <u>while for</u> $4nm>r^2$ <u>we have</u>

$$e_{k,1}(n,r) = \frac{H(k-1,\ 4n-r^2)}{\zeta(3-2k)}$$

($H(k-1,N) = L_{-N}(2-k) =$ Cohen's function) <u>and</u>

$$e_{k,m}(n,r) = \frac{H(k-1,\ 4nm-r^2)}{\zeta(3-2k)} \cdot \prod_{p|m} (\text{elementary p-factor}) \ .$$

<u>In particular,</u> $e_{k,m}(n,r) \in \mathbb{Q}$.

One can in fact complete the calculation of $e_{k,m}$ in general with little extra work; the result for m square-free is

$$(7) \qquad e_{k,m}(n,r) = \frac{\sigma_{k-1}(m)}{\zeta(3-2k)} \sum_{d|(n,r,m)} d^{k-1} H\left(k-1, \frac{4nm-r^2}{d^2}\right) .$$

However, we do not bother to give the calculation since this result will follow from the properties of Hecke-type operators introduced in § 4 (Theorem 4.3).

For $m=1$ and the first few values of k we find, using the tables of $H(k-1,N)$ given in [6] , the expansions

$$E_{4,1} = 1 + (\zeta^2+56\zeta+126+56\zeta^{-1}+\zeta^{-2})q + (126\zeta^2 + 576\zeta+756+576\zeta^{-1}+126\zeta^{-2})q^2 + (56\zeta^3+756\zeta^2 + 1512\zeta+2072+1512\zeta^{-1}+756\zeta^{-2}+56\zeta^{-3})q^3 + \ldots ,$$

$$E_{6,1} = 1 + (\zeta^2-88\zeta-330-88\zeta^{-1}-\zeta^{-2})q + (-330\zeta^2 -4224\zeta-7524-4224\zeta^{-1}-330\zeta^{-2})q^2 + \ldots ,$$

$$E_{8,1} = 1 + (\zeta^2+56\zeta+366+56\zeta^{-1}+\zeta^{-2})q^2 + \ldots$$

We will give tables of the coefficients of these and other Jacobi forms of index 1 at the end of § 3.

In the formula for the Fourier coefficients of $E_{k,1}$, it is striking that $e_{k,1}(n,r)$ depends only on $4n-r^2$. We now show that this is true for any Jacobi form of index 1 ; more generally, we have

Theorem 2.2. Let ϕ be a Jacobi form of index m with Fourier development $\sum c(n,r)q^n\zeta^r$. Then $c(n,r)$ depends only on $4nm-r^2$ and on $r \pmod{2m}$. If k is even and $m=1$ or m is prime, then $c(n,r)$ depends only on $4nm-r^2$. If $m=1$ and k is odd, then ϕ is identically zero.

Proof. This is essentially a restatement of the second transformation law of Jacobi forms: we have

$$\begin{aligned}\sum c(n,r)q^n\zeta^r &= \phi(\tau,z) = e^m(\lambda^2\tau+2\lambda z)\ \phi(\tau,z+\lambda\tau+\mu)\\ &= q^{m\lambda^2}\zeta^{2m}\sum c(n,r)\ q^n(\zeta q^\lambda)^r\\ &= \sum c(n,r)\ q^{n+r\lambda+m\lambda^2}\ \zeta^{r+2m\lambda}\end{aligned}$$

and hence

$$c(n,r) = c(n+r\lambda+m\lambda^2\ ,\ r+2m\lambda)\quad ,$$

i.e. $c(n,r) = c(n',r')$ whenever $r' \equiv r \pmod{2m}$ and $4n'm-r'^2 = 4nm-r^2$, as stated in the theorem. If k is even, then we also have $c(n,-r) = c(n,r)$ (because applying the first transformation law of Jacobi forms to $-I_2 \in \Gamma_1$ gives $\phi(\tau,-z) = (-1)^k\ \phi(\tau,z)$), so if m is 1 or a prime then

$$4n'm-r'^2 = 4nm-r^2 \Rightarrow \pm r' \equiv r \pmod{2m} \Rightarrow c(n,r) = c(n',r')\ .$$

Finally, if $m=1$ and k is odd then $\phi \equiv 0$ because $c(n,-r) = -c(n,r)$ but $4nm-(-r)^2 = 4nm-r^2$ and $-r \equiv r \pmod{2m}$ in this case.

Remark: Theorem 2.2 is the basis of the relationship between Jacobi forms and modular forms of half-integral weight (cf. § 5).

In the definition of Jacobi cusp forms, there were apparently infinitely many conditions to check, namely $c(n,r)=0$ for all n, r with $4nm=r^2$. Theorem 2.2 tells us in particular that we in fact need only check this for a set of representatives of $r \pmod{2m}$. The number of residue classes $r \pmod{2m}$ with $r^2 \equiv 0 \pmod{4m}$ is b, where b^2 is the largest square dividing m (namely if $m=ab^2$ with a square-free then $4m|r^2 \Leftrightarrow 2ab|r$). Thus for $\phi \in J_{k,m}$ we have

ϕ a cusp form $\Longleftrightarrow$ $c(as^2, 2abs)=0$ for $s=0,1,\dots,b-1$;

in particular, the codimension of $J_{k,m}^{cusp}$ in $J_{k,m}$ is at most b. Using $c(n,-r) = (-1)^k c(n,r)$ we see that in fact it suffices to check the condition $c(as^2, 2abs)=0$ for $s=0,1,\dots,\lfloor\frac{b}{2}\rfloor$ if k is even and $s=1,2,\dots,\lfloor\frac{b-1}{2}\rfloor$ if k is odd. Hence we have

Theorem 2.3. *The codimension of $J_{k,m}^{cusp}$ in $J_{k,m}$ is at most $\lfloor\frac{b}{2}\rfloor+1$ if k is even (resp. $\lfloor\frac{b-1}{2}\rfloor$ if k is odd), where b is the largest integer such that $b^2|m$.*

On the other hand, if $k>2$ then for each integer s we can construct an Eisenstein series

$$(8) \qquad E_{k,m,s}(\tau,z) := \sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} q^{as^2}\zeta^{2abs}|\gamma \quad .$$

($m=ab^2$ as above), where the summation is the same as in the definition of $E_{k,m} = E_{k,m,0}$. Then repeating the beginning of the proof of Theorem 2.1 we find that

$$(9) \qquad E_{k,m,s} = \frac{1}{2}\sum_{\substack{r\in\mathbb{Z}\\ r\equiv 2abs \,(\mathrm{mod}\ 2m)}} q^{r^2/4m}(\zeta^r + (-1)^k\zeta^{-r}) + \dots ,$$

where "..." (the contribution from all terms in the sum with $c\neq 0$) has a Fourier development consisting only of terms $q^n\zeta^r$ with $4nm-r^2>0$. It is then clear that $E_{k,m,s}$ depends only on $s \pmod b$, that $E_{k,m,-s}=(-1)^k E_{k,m,s}$, and that the series $E_{k,m,s}$ with $0\le s\le\frac{b}{2}$ (k even) or $0<s<\frac{b}{2}$ (k odd) are linearly independent. Comparing this with 2.3, we see that the bound given there is sharp and that we have proved:

Theorem 2.4. *If* $k>2$, *then* $J_{k,m} = J_{k,m}^{cusp} \oplus J_{k,m}^{Eis}$, *where* $J_{k,m}^{cusp}$ *is the space of cusp forms in* $J_{k,m}$ *and* $J_{k,m}^{Eis}$ *the space spanned by the functions* $E_{k,m,s}$. *The functions* $E_{k,m,s}$ *with* $0 \le s \le \frac{b}{2}$ (k even) *or* $0 < s < \frac{b}{2}$ (k odd) *form a basis of* $J_{k,m}^{Eis}$.

We will not give the entire calculation of the Fourier developments of the functions $E_{k,m,s}$ here, since it is tedious and we do not need the result. However, we make some remarks. In §4 we will introduce certain operators U_ℓ and V_ℓ which map Jacobi forms to Jacobi forms of higher index. These will act in a simple way on Fourier developments and will send Eisenstein series to Eisenstein series. Hence certain combinations of the $E_{k,m,s}$ ("old forms") have Fourier coefficients which can be given in a simple way in terms of the Fourier coefficients of Eisenstein series of lower index (compare equation (7), where the coefficients of $E_{k,m}$ are simple linear combinations of those of $E_{k,1}$), and we need only consider the remaining, "new," forms. A convenient basis for these is the set of forms

$$(10) \qquad E_{k,m}^{(\chi)} := \sum_{s \pmod f} \chi(s)\, E_{k,m,s} \qquad (m = f^2)$$

of index f^2, where χ is a primitive Dirichlet character (mod f) with $\chi(-1)=(-1)^k$. Then a calculation analogous to the proof of Theorem 2.1 for the case $m=1$ shows that the coefficient of $q^n\zeta^r$ in $E_{k,m,\chi}^{(\chi)}$ is given by

$$(11) \qquad e_{k,m}^{(\chi)}(n,r) = e(\chi)\ \chi(r)\ L_{r^2-4nm}(2-k,\bar{\chi})$$

if $(r,f)=1$, where $L_D(s,\chi)$ is the convolution of $L_D(s)$ and $L(s,\chi)$ and $e(\chi)$ a simple constant (essentially a quotient of Gauss sums attached to χ and χ^2 divided by $L(3-2k,\chi^{-2})$); in particular, the coefficients are algebraic (in $\mathbb{Q}(\chi)$) and non-zero. If $(r,f)>1$, then $e_{k,m}^{(\chi)}(n,r)$ is given by a formula like (11) with the right-hand side multiplied by a finite Euler product extending over the common prime factors of r and f.

If $k=2$, then the Eisenstein series fail to converge; however, by the same type of methods as are used for ordinary modular forms ("Hecke's convergence trick") one can show that for χ non-principal there is an Eisenstein series $E_{2,m,\chi} \in J_{2,m}$ having a Fourier development given by the same formula as for $k>2$. Since χ must be even $(\chi(-1) = (-1)^k)$ and since there exists an even non-principal character (mod b) only if $b=5$ or $b\geq 7$, such series exist only for m divisible by 25, 49, 64 ... .

There is one more topic from the theory of cusp forms in the classical case which we want to generalize, namely the characterization of cusp forms in terms of the Petersson scalar product. We write

$$\tau = u+iv \ (v>0) \ , \ z = x+iy$$

and define a volume element dV on $\mathbb{H}\times\mathbb{C}$ by

(12) $$dV := v^{-3}\, dx\, dy\, du\, dv \ .$$

If is easily checked that this is invariant under the action of G^J on $\mathbb{H}\times\mathbb{C}$ defined in §1 and is the unique G^J-invariant measure up to a constant. (The form $v^{-2}\,du\,dv$ is the usual $SL_2(\mathbb{R})$-invariant volume form on $\mathbb{H}$; the form $v^{-1}\,dx\,dy$ is the translation-invariant volume form on $\mathbb{C}$, normalized so that the fibre $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$ has volume 1.) If ϕ and ψ transform like Jacobi forms of weight k and index m, then the expression

$$v^k\, e^{-4\pi m y^2/v}\, \phi(\tau,z)\, \overline{\psi(\tau,z)}$$

is easily checked to be invariant under Γ^J, so we can define the <u>Petersson scalar product</u> of ϕ and ψ by

(13) $$(\phi,\psi) := \int_{\Gamma^J\backslash \mathbb{H}\times\mathbb{C}} v^k\, e^{-4\pi m y^2/v}\, \phi(\tau,z)\, \overline{\psi(\tau,z)}\, dV \ .$$

Then we have

Theorem 2.5. *The scalar product (13) is well-defined and finite for $\phi, \psi \in J_{k,m}$ and at least one of ϕ and ψ a cusp form. It is positive-definite on $J_{k,m}^{cusp}$ and the orthogonal complement of $J_{k,m}^{cusp}$ with respect to (,) is $J_{k,m}^{Eis}$.*

This will follow from the results in §5 concerning the connection between Jacobi forms and modular forms of half-integral weight.

§ 3. Taylor expansions of Jacobi forms

The restriction of a Jacobi form $\phi(\tau,z)$ to $z=0$ gives a modular form of the same weight. In §1 we proved an analogous statement for the restriction to $z=\lambda\tau+\mu$ (λ,μ rational) and used it to show that $J_{k,m}(\Gamma)$ is finite-dimensional. Another and even more useful way to get modular forms is to consider the Taylor development of ϕ around $z=0$; by forming certain linear combinations of the coefficients one obtains a series of modular forms $D_\nu\phi$. (D_ν for "ν^{th} development coefficient") with $D_0\phi=\phi(\tau,0)$ and $D_\nu\phi$ a modular form of weight $k+2\nu$. The precise result is

Theorem 3.1. *For $\nu\in\mathbb{N}_0$, $k\in\mathbb{N}$ define a homogeneous polynomial $p_{2\nu}^{(k-1)}$ of two variables by*

(1) $$\frac{(k+\nu-2)!}{(2\nu)!(k-2)!}\;p_{2\nu}^{(k-1)}(r,n)=\text{coefficient of } t^{2\nu} \text{ in } (1-rt+nt^2)^{-k+1}$$

Then for $\phi\in J_{k,m}(\Gamma)$ a Jacobi form with Fourier development $\sum_{n,r} c(n,r)q^n\zeta^r$, the function

(2) $$D_{2\nu}\phi \quad := \sum_{n=0}^{\infty}\Big(\sum_r p_{2\nu}^{(k-1)}(r,nm)\;c(n,r)\Big)\,q^n$$

is a modular form of weight $k+2\nu$ on Γ . If $\nu>0$, it is a cusp form.

Explicitly, one has

$$D_0\phi=\sum_n\Big(\sum_r c(n,r)\Big)\,q^n \quad ,$$

$$D_2\phi=\sum_n\Big(\sum_r (kr^2-2nm)\;c(n,r)\Big)\,q^n \;,$$

$$D_4\phi=\sum_n\Big(\sum_r((k+1)(k+2)r^4-12(k+1)r^2nm+12n^2m^2)c(n,r)\Big)\,q^n \;.$$

Notice that the summation over r is finite since $c(n,r)\neq 0 \Rightarrow r^2\leq 4nm$. The polynomial $p_{2\nu}^{(k-1)}$ is given explicitly by

(3) $$p_{2\nu}^{(k-1)}(r,n) := \sum_{\mu=0}^{\nu} (-1)^{\mu} \frac{(2\nu)!}{\mu!(2\nu-2\mu)!} \frac{(k+2\nu-\mu-2)!}{(k+\nu-2)!} r^{2\nu-2\mu} n^{\mu} .$$

and is, up to a change of notation and normalization, the so-called Gegenbauer or "ultraspherical" polynomial, studied in any text on orthogonal polynomials); we have chosen the normalization given so as to make $p_{2\nu}^{(k-1)}$ a polynomial with integral coefficients in k,r,n in a minimal way (actually, $\frac{1}{\nu!}$ times $p_{2\nu}^{(k-1)}$ would still have integral coefficients as a function of r and n for fixed $k\in\mathbb{N}$). The characteristic property of the polynomial $p_{2\nu}^{(k-1)}$ is that the function $p_{2\nu}^{(k-1)}(B(x,y),Q(x)Q(y))$, where Q is a quadratic form in $2k$ variables and B the associated bilinear form, is a spherical function of x and y with respect to Q (Theorem 7.2).

There is a similar result involving odd polynomials and giving modular forms $D_1\phi$, $D_3\phi,\ldots$ weight $k+1, k+3, \ldots$ (simply take $\nu\in\frac{1}{2}+\mathbb{N}_0$ and replace $(k+\nu-2)!$ by $(k+\nu-\frac{3}{2})!$ in (1) and (3)), but, as we shall see, this can be reduced to the even case in a trivial way, so we content ourselves with stating the latter case.

As an example of Theorem 3.1 we apply it to the function $E_{k,1}$ studied in the last section; using the formula given there for the Fourier coefficients of $E_{k,1}$ we obtain

<u>Corollary</u> (Cohen [6, Th.6.2]). <u>Let</u> k <u>be even and</u> $H(k-1,N)$ ($N\in\mathbb{N}_0$) <u>be Cohen's function</u>

$$H(k-1,N) = \begin{cases} L_{-N}(2-k) & \text{if } N>0,\ N\equiv 0 \text{ or } 3 \pmod 4, \\ \zeta(3-2k) & \text{if } N=0, \\ 0 & \text{if } N\equiv 1 \text{ or } 2 \pmod 4. \end{cases}$$

<u>Then for each</u> $\nu \in \mathbb{N}_0$ <u>the function</u>

$$C_k^{(\nu)}(\tau) = \sum_{n\geq 0} \Big(\sum_{r^2 \leq 4n} p_{2\nu}^{(k-1)}(r,n)\ H(k-1,4n-r^2)\Big)\ q^n$$

<u>is a modular form of weight</u> $k+2\nu$ <u>on the full modular group</u> Γ_1 .
<u>If</u> $\nu>0$, <u>it is a cusp form</u>.

Cohen's proof of this result used modular forms of half-integral weight; the relation of this to Theorem 3.1 will be discussed in §5 . Yet another proof was given in [39] , where it was shown that $C_k^{(\nu)}$ h the property that its scalar product with a Hecke eigenform $f = \sum a(n)q^n \in S_{k+2\nu}$ is equal, up to a simple numerical factor, to the value of the Rankin series $\sum a(n)^2 n^{-s}$ at $s = 2k+2\nu-2$. This property characterizes the form $C_k^{(\nu)}$ and also shows (since the value of the Rankin series is non-zero) that it generates $S_{k+2\nu}$ (resp. M_k if $\nu=0$) as a module over the Hecke algebra; an application of this will be mentioned in §7 .

To prove Theorem 3.1, we first develop $\phi(\tau,z)$ in a Taylor expansion around $z=0$:

(4) $$\phi(\tau,z) = \sum_{\nu=0}^{\infty} \chi_\nu(\tau)\ z^\nu$$

and then apply the transformation equation

(5) $$\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^k\ e^m\Big(\frac{cz^2}{c\tau+d}\Big)\ \phi(\tau,z)$$

to get

(6) $$\chi_\nu\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{k+\nu}\Big(\chi_\nu(\tau) + \frac{2\pi imc}{c\tau+d}\ \chi_{\nu-2}(\tau) + \frac{1}{2!}\Big(\frac{2\pi imc}{c\tau+d}\Big)^2\ \chi_{\nu-4}(\tau$$

i.e. χ_ν transforms under Γ like a modular form of weight $k+\nu$ modul corrections coming from previous coefficients. The first three cases of (6) are

$$\chi_0\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \quad \chi_0(\tau)$$

$$\chi_1\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+1} \quad \chi_1(\tau)$$

$$\chi_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2} \quad \chi_2(\tau) + 2\pi imc(c\tau+d)^{k+1} \quad \chi_0(\tau) \; .$$

Differentiating the first of these equations gives

$$\chi_0'\left(\frac{a\tau+b}{c\tau+d}\right) = kc(c\tau+d)^{k+1} \quad \chi_0(\tau) + (c\tau+d)^{k+2} \quad \chi_0'(\tau) \; ,$$

and subtracting a multiple of this from the third equation gives

$$\xi_2 := \chi_2 - \frac{2\pi im}{k} \chi_0' \quad \in \quad M_{k+2}(\Gamma) \; .$$

Proceeding in this way, we find that for each ν the function

$$\xi_\nu(\tau) := \sum_{0\le\mu\le\frac{\nu}{2}} \frac{(-2\pi im)^\mu (k+\nu-\mu-2)!}{(k+\nu-2)! \; \mu!} \quad \chi_{\nu-2\mu}^{(\mu)}(\tau) \tag{7}$$

transforms like a modular form of weight $k+\nu$ on Γ . The algebraic manipulations required to obtain the appropriate coefficients in (7) directly (i.e. like what we just did for $\nu=2$) are not very difficult and can be made quite simple by a judicious use of generating series, but we will in fact prove the result in a slightly different way in a moment. If ϕ is periodic in z and has a Fourier development $\sum_{n,r} c(n,r) \, q^n \zeta^r$, then $\chi_\nu = \frac{1}{\nu!}\sum_n\left(\sum_r (2\pi ir)^\nu \; c(n,r)\right) q^n$ and hence

$$\xi_\nu(\tau) = (2\pi i)^\nu \sum_{n\ge0}\left(\sum_r\left(\sum_{0\le\mu\le\frac{\nu}{2}} \frac{(k+\nu-\mu-2)!}{(k+\nu-2)!} \; \frac{(-mn)^\mu r^{\nu-2\mu}}{\mu!(\nu-2\mu)!}\right) c(n,r)\right) q^n \; , \tag{8}$$

so

$$D_{2\nu}\phi(\tau) = (2\pi i)^{-2\nu} \frac{(k+2\nu-2)!\,(2\nu)!}{(k+\nu-2)!} \; \xi_{2\nu}(\tau) \; . \tag{9}$$

Thus Theorem 3.1 follows from the following more general result:

<u>Theorem 3.2. Let</u> $\phi(\tau,z)$ <u>be a formal power series in</u> z <u>as in (4) with coefficients</u> χ_ν <u>which satisfy (6) for all</u> $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ <u>and</u> <u>are holomorphic everywhere (including the cusps of</u> Γ <u>). Then the function</u> ξ_ν <u>defined by (7) is a modular form of weight</u> $k+\nu$ <u>on</u> Γ.

Proof: Let $M_{k,m}(\Gamma)$ denote the set of all functions ϕ satisfying the conditions of the theorem. (Note that $M_{k,m}(\Gamma)$ is isomorphic to $M_{k,1}(\Gamma)$ via $z \mapsto \sqrt{m}\,z$.) Since ξ_ν involves only $\chi_{\nu'}$, with $\nu' \equiv \nu \pmod 2$ we can split up $M_{k,m}(\Gamma)$ into odd and even power series, say $M_{k,m}(\Gamma) = M^{+}_{k,m}(\Gamma) \oplus M^{-}_{k,m}(\Gamma)$ and look at the two parts separately (this corresponds to adjoining $-I_2$ to Γ and looking at the action of $-I_2$ on ϕ; if Γ already contains $-I_2$, then $M_{k,m}(\Gamma) = M^{(-1)^k}_{k,m}(\Gamma)$). If $\phi \in M^{-}_{k,m}(\Gamma)$, then $\phi = z\phi_1$ with $\phi_1 \in M^{+}_{k+1,m}(\Gamma)$ and the functions χ_ν, ξ_ν for ϕ and ϕ_1 are the same except for the shift $\nu \to \nu - 1$, $k \to k+1$. Hence it suffices to look at $M^{+}_{k,m}(\Gamma)$. We now introduce the differential operators

$$L := 8\pi i m \frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}$$

(the <u>heat operator</u>) and

$$L_k := L - \frac{2k-1}{z}\,\frac{\partial}{\partial z} .$$

The operator L is natural in the context of Jacobi forms because it acts on monomials $q^n\zeta^r$ by multiplication by $(2\pi i)^2(4nm-r^2)$ and hence, in view of Theorem 2.2, preserves the second transformation law of Jacobi forms; this can also be seen directly by checking that

$$(10) \qquad L(\phi|_m X) = (L\phi)|_m X \qquad (X \in \mathbb{R}^2) \ .$$

If L satisfied a similar equation with respect to the operation of $SL_2(\mathbb{R})$, then it would map Jacobi forms to Jacobi forms. Unfortunately, this is not quite true; when we compute the difference between $L(\phi|_{k,m}M)$ and $(L\phi)|_{k+2,m}M$ we find that most of the terms cancel but there is one term $4\pi im(2k-1)\frac{c}{c\tau+d}(\phi|_{k,m}M)(\tau,z)$ left over (unless $k=\frac{1}{2}$, in which case L really does map Jacobi forms to Jacobi forms of weight $\frac{5}{2}$ and the same index m; examples are the Jacobi theta-series, which are annihilated by L). To correct this we replace L by L_k, which no longer satisfies (10) but does satisfy

$$(11) \qquad L_k(\phi|_{k,m}M) = (L_k\phi)|_{k+2,m}M \qquad (M \in SL_2(\mathbb{R})) \ ,$$

as one checks by direct computation. Because of the z in the denominator, L_k only acts on power series with no linear term; in particular it acts on $M^+_{k,m}(\Gamma)$ and (because of (11)) maps $M^+_{k,m}(\Gamma)$ to $M^+_{k+2,m}(\Gamma)$. Explicitly, we have

$$L_k: \sum_{\lambda\geq 0}\chi_\lambda\, z^{2\lambda} \mapsto \sum_{\lambda\geq 0}(8\pi im\chi_\lambda' - 4(\lambda+1)(\lambda+k)\chi_{\lambda+1})z^{2\lambda} \ .$$

Iterating this formula ν times, we find by induction on ν that the composite map

$$M^+_{k,m}(\Gamma) \xrightarrow{L_k} M^+_{k+2,m}(\Gamma) \xrightarrow{L_{k+2}} \cdots \xrightarrow{L_{k+2\nu-2}} M^+_{k+2\nu,m}(\Gamma)$$

maps $\sum\chi_\lambda z^{2\lambda}$ to

$$\sum_{\lambda\geq 0}\Big(\sum_{\mu=0}^{\nu}(-4)^{\nu-\mu}\,(8\pi im)^\mu\binom{\nu}{\mu}\,\frac{(\lambda+\nu-\mu)!}{\lambda!}\,\frac{(\lambda+k+2\nu-\mu-2)!}{(\lambda+k+\nu-2)!}\,\chi^{(\mu)}_{\lambda+\nu-\mu}(\tau)\Big)z^{2\lambda} \ ,$$

and composing this with the map

$$M^+_{k+2\nu,m}(\Gamma) \to M_{k+2\nu}(\Gamma) \qquad \big(\phi(\tau,z) \mapsto \phi(\tau,0)\big)$$

gives $\xi_{2\nu} \in M_{k+2\nu}(\Gamma)$. This proves Theorem 3.2 and hence also Theorem 3.1 except for the assertion about cusp forms. But the latter is clearly true, because the constant term of (2) is $p_{2\nu}^{(k-1)}(0,0)\ c(0,0)$, which is 0 for $\nu>0$, and the expansion of $D_\nu\phi$ at the other cusps is given by a similar formula applied to $\phi|_{k,m}M,\ M\in\Gamma_1$.

By mapping an even (resp. odd) function $\phi\in M_{k,m}(\Gamma)$ to $(\xi_0,\xi_2,\xi_4,\dots)$ (resp. to $(\xi_1,\xi_3,\dots)$), we obtain maps

$$M^+_{k,m}(\Gamma) \longrightarrow \prod_{\nu\geq 0} M_{k+2\nu}(\Gamma)\ ,\ M^-_{k,m}(\Gamma) \longrightarrow \prod_{\nu\geq 0} M_{k+2\nu+1}(\Gamma)\ .$$

It is clear that these maps are isomorphisms: one can express χ_ν in terms of ξ_ν by inverting (7) to get

$$(12)\qquad \chi_\nu(\Gamma) = \sum_{0\leq\mu\leq\frac{\nu}{2}} \frac{(2\pi i m)^\mu (k+\nu-2\mu-1)!}{(k+\nu-\mu-1)!\ \ \mu!}\ \xi^{(\mu)}_{\nu-2\mu}(\tau)\quad ,$$

and then the transformation equations (6) of the χ_ν follow from $\xi_\nu\in M_{k+\nu}(\Gamma)$. In particular, taking $\xi_0=f$, $\xi_\nu=0$ $(\nu>1)$ and $m=1$ we obtain the following result, due (independently of one another) to Kuznetsov and Cohen:

<u>Theorem 3.3</u> (Kuznetsov [16] , Cohen [7]): <u>Let</u> $f(\tau)$ <u>be a modular form of weight</u> k <u>on</u> Γ . <u>Then the function</u>

$$(13)\quad \tilde f(\tau,z) := \sum_{\nu=0}^{\infty} \frac{(2\pi i)^\nu (k-1)!}{\nu!\ (k+\nu-1)!}\ f^{(\nu)}(\tau)\ z^{2\nu}$$

<u>satisfies the transformation equation</u>

$$(14)\quad \tilde f\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) = (c\tau+d)^k\ e\Big(\frac{cz^2}{c\tau+d}\Big)\ \tilde f(\tau,z)\qquad \Big(\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma\Big)\ .$$

We mention a corollary which will be used later.

<u>Corollary</u> (Cohen [6 ,Th. 7.1]): <u>Let</u> f_1,f_2 <u>be modular forms on</u> Γ <u>of weight</u> k_1 <u>and</u> k_2 , <u>respectively</u>, $\nu\in\mathbb{N}_0$. <u>Then the function</u>

$$F_\nu(f_1,f_2) := (2\pi i)^{-\nu}\sum_{\mu=0}^{\nu}(-1)^{\nu-\mu}\binom{\nu}{\mu}\frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)}\frac{\Gamma(k_2+\nu)}{\Gamma(k_2+\mu-\nu)}\ f_1^{(\mu)}\ f_2^{(\nu-\mu)}$$

<u>is a modular form of weight</u> $k_1+k_2+2\nu$ <u>on</u> Γ <u>and is a cusp form if</u> $\nu>0$.

(We have modified Cohen's definition by a factor $(2\pi i)^{-\nu}$ to make the Fourier coefficients of $F_\nu(f_1,f_2)$ rational in those of f_1 and f_2.) The corollary follows by computing the coefficient of $z^{2\nu}$ in $\tilde{f}_1(\tau,z)\tilde{f}_2(\tau,iz)$, which by Theorem 3.3 transforms like a modular form of weight k_1+k_2 under Γ.

We observe that the known result 3.3 could also have been used to prove 3.1 and 3.2. (We preferred to give a direct proof in the context of the theory of Jacobi forms, especially as the use of the differential operators L_k makes the proof rather natural.) Indeed, let $M_{k,m}^{(\nu)}$ be the subspace of $M_{k,m}$ of functions ϕ which are $O(z^\nu)$, i.e. have a development $\chi_\nu(\tau)z^\nu+\chi_{\nu+1}(\tau)z^{\nu+1}+\dots$. From (6) it is clear that the leading coefficient χ_ν is then a modular form of weight $k+\nu$ and we get an exact sequence

(14) $0 \longrightarrow M_{k,m}^{(\nu+1)} \longrightarrow M_{k,m}^{(\nu)} \longrightarrow M_{k+\nu}$

in which the first arrow is the inclusion and the second is $\phi \longmapsto \chi_\nu$ On the other hand, $M_{k,m}^{(\nu)} \simeq M_{k+\nu,m}$ by division by z^ν (this was already used for $\nu=1$ when we reduced the study of $M_{k,m}^-$ to that of $M_{k,m}^+$) , and 3.3 gives a map $M_{k+\nu} \longrightarrow M_{k+\nu,m}$ by $f \longmapsto \tilde{f}(\tau,\sqrt{m}z)$; this shows that the last map in (15) is surjective and gives an explicit splitting. To get the sequence of modular forms $\xi_0,\xi_1,\dots$ associated to $\phi \in M_{k,m}$ we now proceed by induction: having found $\xi_0,\xi_1,\dots,\xi_{\nu-1}$ such that

$$\phi(\tau,z) - \sum_{\nu'<\nu} \tilde{\xi}_{\nu'}(\tau,\sqrt{m}z)z^{\nu'} \equiv 0 \pmod{z^\nu}$$

we define $\xi_\nu(\tau)$ as the leading coefficient (coefficient of z^ν) in the expression on the left-hand side; then $\phi = \sum_\nu \tilde{\xi}_\nu(\tau,\sqrt{m}z)\, z^\nu$ as a formal power series and this is equivalent to the series of

We have gone into the meaning of the development coefficients $D_\nu\phi$ fairly deeply because they play an important role in the study of Jacobi forms and because the relation with the identity (14) of Kuznetsov and Cohen concerning $\tilde{f}$ (which is not a Jacobi form) seemed striking. In particular, we should mention that (13) can be written

$$\tilde{f}(\tau,z) = a(0) + (k-1)!\sum_{n=1}^{\infty} a(n)\,\frac{J_{k-1}(4\pi\sqrt{mn}\,z)}{(2\pi\sqrt{mn}\,z)^{k-1}}\;q^n$$

if $f = \sum a(n)\,q^n$ (this is the form in which Kuznetsov gave the identity). To see where the Bessel functions come from, note that the function $h(z) = (k-1)!\,J_{k-1}(\pi z)/(2\pi z)^{k-1}$ satisfies the ordinary differential equation $h'' + \frac{2k-1}{z}h' + (\pi)^2 h = 0$ and is the only solution holomorphic at the origin and with $h(0) = 1$. By separation of variables we see that $\tilde{f}(\tau,z) = \sum_0^{\infty} a(n)\,h(\sqrt{n}\,z)\;e^{2\pi in\tau}$ is the unique solution of the parial differential equation $L_k\tilde{f} = 0$ satisfying the boundary conditions $\tilde{f}(\tau+1,z) = \tilde{f}(\tau,z)$ and $\tilde{f}(\tau,0) = f(\tau)$, and this uniqueness together with the fact that L_k commutes with the operation of $SL_2(\mathbb{R})$ (eq. (11)) immediately implies that $\tilde{f}$ has the property (14).

As a first application of the maps D_ν to Jacobi forms, we have a second proof and sharpening of Theorem 1.1:

<u>Theorem 3.4</u>: $$\dim J_{k,m}(\Gamma) \le \dim M_k(\Gamma) + \sum_{\nu=1}^{2m} \dim S_{k+\nu}(\Gamma)\ .$$

Indeed, $\xi_0 = \ldots = \xi_{2m} = 0$ implies $\chi_0 = \ldots = \chi_{2m} = 0$ or $\phi = O(z^{2m+1})$, so Theorem 1.2 implies that the map

$$D = \bigoplus_{\nu=0}^{2m} D_\nu : J_{k,m}(\Gamma) \to M_k(\Gamma) \oplus S_{k+1}(\Gamma) \oplus \ldots \oplus S_{k+2m}(\Gamma)$$

is injective. Note that half of the spaces $M_{k+\nu}(\Gamma)$ are 0 if $-I_2 \in \Gamma$; in particular, for $\Gamma = \Gamma_1$ we have

$$(15)\qquad \dim J_{k,m} \le \begin{cases} \dim M_k + \dim S_{k+2} + \ldots + \dim S_{k+2m} & (k \text{ even}) \\ \dim S_{k+1} + \dim S_{k+3} + \ldots + \dim S_{k+2m-1} & (k \text{ odd}) \end{cases}$$

Here the second estimate can even be strengthened to

(16) $\dim J_{k,m} \leq \dim S_{k+1} + \ldots + \dim S_{k+2m-3}$,

because an odd Jacobi form must vanish at the three 2-division points $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ and hence cannot have more than a $(2m-3)$-fold zero at $z=0$.

Application: Jacobi forms of index one

Theorem 3.4 is the basis for the analysis of the structure of $J_{*,*} = \bigoplus_{k,m} J_{k,m}$ as given in Chapter III, to which the reader may now skip if he so desires (the results of §§ 4-7 are not used there). As an example, we now treat the case $m=1$, which is particularly easy and will be used in Chapter II. Equations (15) and (16) (or Theorem 2.2) give

$$J_{k,1}=0 \text{ (k odd)} , \quad \dim J_{k,1} \leq \dim M_k + \dim S_{k+2} \quad \text{(k even)}.$$

On the other hand, the Fourier developments of $E_{4,1}$ and $E_{6,1}$, as given after Theorem 2.1, show that the quotient

$$\frac{E_{6,1}(\tau,z)}{E_{4,1}(\tau,z)} = 1 - (144\zeta + 456 + 144\zeta^{-1})q + \ldots$$

depends on z and hence is not a quotient of two modular forms, so the map

$$M_{k-4} \oplus M_{k-6} \longrightarrow J_{k,1}$$

$$(\ f\ ,\ g\ ) \longmapsto f(\tau)E_{4,1}(\tau,z) + g(\tau)E_{6,1}(\tau,z)$$

is injective. Since $\dim M_{k-4} + \dim M_{k-6} = \dim M_k + \dim S_{k+2}$ for all k (this follows from the well-known formula for $\dim M_k$), we deduce

__Theorem 3.5__. __The space of Jacobi forms of index__ 1 __on__ $SL_2(\mathbb{Z})$ __is a free module of rank__ 2 __over__ M_* , __with generators__ $E_{4,1}$ and $E_{6,1}$.

$$D_0 \oplus D_2 : J_{k,1} \to M_k \oplus S_{k+2}$$

(D_0 , D_2 as in Th. 3.1) <u>is an isomorphism</u>.

In particular, we find that the space $J_{8,1}$ is one-dimensional, with generator $E_{8,1} = E_4 \cdot E_{4,1}$ ($E_4 = 1 + 240\,q + \ldots$ the Eisenstein series in M_4), while the first cusp forms of index 1 are the forms

$$(17) \qquad \phi_{10,1} = \frac{1}{144}(E_6\,E_{4,1} - E_4\,E_{6,1})\ , \quad \phi_{12,1} = \frac{1}{144}(E_4^2\,E_{4,1} - E_6\,E_{6,1})$$

of weight 10 and 12 , respectively (the factor 144 has been inserted to make the coefficients of $\phi_{10,1}$ and $\phi_{12,1}$ integral and coprime). We have tabulated the first coefficients $e_{k,1}$ of $E_{k,1}$ ($k = 4,6,8$) and $c_{k,1}$ of $\phi_{k,1}$ ($k = 10,12$) in Table 1; notice that it suffices to give a single sequence of coefficients $c(N)$ ($N \geq 0$, $N \equiv 0,3 \pmod 4$) since by 2.2 any Jacobi form of index 1 has Fourier coefficients of the form $c(n,r) = c(4n-r^2)$ for some $\{c(N)\}$. To compute the $c(N)$, we can use either assertion of Th. 3.5, e.g. for $\phi_{10,1}$, $\phi_{12,1}$ we can either use (17) and the known Fourier expansions of E_k and $E_{k,m}$ or else (what is quicker) use the expansions

$$(18) \qquad D_0\,\phi_{10,1} = 0\ , \quad D_2\,\phi_{10,1} = 20\Delta\ , \quad D_0\,\phi_{12,1} = 12\Delta\ , \quad D_2\,\phi_{12,1} = 0$$

($\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)\,q^n$) to obtain the identities

$$\sum_{|r|<2\sqrt{n}} c_{10,1}(4n-r^2) = 0\ , \qquad \sum_{0<r<2\sqrt{n}} r^2\,c_{10,1}(4n-r^2) = \tau(n)\ ,$$

$$\sum_{|r|<2\sqrt{n}} c_{12,1}(4n-r^2) = 12\tau(n)\ , \qquad \sum_{0<r<2\sqrt{n}} r^2\,c_{12,1}(4n-r^2) = n\tau(n)$$

and then solve these recursively for the $c_{k,1}(N)$.

The functions

$$\phi_{10,1} = (\zeta-2+\zeta^{-1})q + (2\zeta^2-16\zeta+36-16\zeta^{-1}+2\zeta^{-2})q^2 + \ldots$$

$$\phi_{12,1} = (\zeta+10+\zeta^{-1})q + (10\zeta^2-88\zeta-132-88\zeta^{-1}+10\zeta^{-2})q^2 + \ldots$$

n	$e_4(n)$	$e_6(n)$	$e_8(n)$	$c_{10}(n)$	$c_{12}(n)$
0	1	1	1	0	0
3	56	-88	56	1	1
4	126	-330	366	-2	10
7	576	-4224	14016	-16	-88
8	756	-7524	33156	36	-132
11	1512	-30600	260712	99	1275
12	2072	-46552	462392	-272	736
15	4032	-130944	1987392	-240	-8040
16	4158	-169290	2998638	1056	-2880
19	5544	-355080	9090984	-253	24035
20	7560	-464904		-1800	13080
23	12096	-899712	.	2736	-14136
24	11592	-1052040	.	-1464	-54120
27	13664	-1732192	.	-4284	-128844
28	16704	-2099328		12544	115456
31	24192	-3421440		-6816	389520
32	24948	-3859812		-19008	38016
35	27216	-5593104		27270	-256410
36	31878	-6522450		-4554	-697950
39	44352	-9651840		-6864	-806520
40	39816	-10433544		39880	963160
43	41832	-14002824		-66013	1892363
44	55944	-16187400		-26928	938400
47	72576	-22429440		44064	-1227600
48	66584	-23836120		12544	-2309120
51	67536	-30320400		108102	-813450
52	76104	-33965448		-93704	-2813096
55	100800	-45141888		-22000	2311640
56	99792	-47828880		80784	5549040
59	101304	-58659480		-281943	-3336015
60	116928	-65079168		188160	10548480
63	145728	-83487360		-36432	6141960
64	133182	-86676810		-295424	-20142080
67	126504	-103023624		659651	-11654893
68	160272	-114521616		193392	-10887888
71	205632	-143637120		-84816	5100360
72	177660	-147492972		-390420	24801876
75	176456	-171930088		-635225	31406575
76	205128	-187837320		68816	17689760
79	249984	-230334720		-109088	-47059760
80	249480	-238495752		950400	-3767040
83	234360	-272322072		-22455	-3738471
84	265104	-295334160		-484368	-64883280
87	326592	-356805504		1050768	-5321448
88	281736	-362360328		143176	26020696
91	277200	-408875280		195910	66711190
92	350784	-447156864		-2145024	18546432
95	423360	-532388736		-370800	96031320
96	382536	-539696520		772992	15586560
99	355320	-599851800		-1073655	-239563575
100	390726	-644325330		2832950	118753250

Table 1. <u>**Coefficients of Jacobi forms of index 1**</u>

have several beautiful properties and will play a role in the structure theory developed in Chapter III. Here we mention only the following:

Theorem 3.6. The quotient

$$\frac{\phi_{12,1}(\tau,z)}{\phi_{10,1}(\tau,z)} = \frac{\zeta+10+\zeta^{-1}}{\zeta-2+\zeta^{-1}} + 12(\zeta-2+\zeta^{-1})q + \dots$$

is $-3/\pi^2$ times the Weierstrass $\wp$-function $\wp(\tau,z)$.

Indeed, since $\phi_{10,1}$ vanishes doubly at $z=0$ and (by Theorem 1.2) nowhere else in $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$, and since by (18)

$$(19) \qquad \phi_{10,1} = (2\pi i)^2\,\Delta(\tau)\,z^2 + O(z^4)\ ,\quad \phi_{12,1} = 12\,\Delta(\tau) + O(z^2)\ ,$$

the quotient in question is a doubly periodic function of z with a double pol with principal part $\frac{12}{(2\pi i)^2}z^{-2}$ at $z=0$ and no other poles in a period parallelogr so must equal $\frac{12}{(2\pi i)^2}\,\wp(\tau,z)$.

Finally, we note that, just as the two Eisenstein series $E_{4,1}$ and $E_{6,1}$ form a free basis of $J_{*,1}$ over M_* , the two cusp forms $\phi_{10,1}$ and $\phi_{12,1}$ form a basis of $J_{*,1}^{cusp}$ over M_* , i.e. we have an isomorphism

$$\begin{aligned} M_{k-10} \oplus M_{k-12} &\to J_{*,1}^{cusp} \\ (f\ ,\ g) &\mapsto f(\tau)\phi_{10,1}(\tau,z) + g(\tau)\phi_{12,1}(\tau,z)\ . \end{aligned}$$

Thus the Jacobi forms

$$E_4(\tau)^a\,E_6(\tau)^b\,\phi_{j,1}(\tau,z) \qquad (a,b\ge 0,\ j\in\{10,12\},\ 4a+6b+j=k)$$

form an additive basis of the space of Jacobi cusp forms of weight k and index 1 . Each of them has a Fourier expansion of the form $\sum c(4n-r^2)q^n\zeta^r$; the coefficients $c(N)$ for $N\le 20$ and all weights $k\le 50$ are given in Table 2.

k	a	b	j	c(3)	c(4)	c(7)	c(8)	c(11)	c(12)	c(15)	c(16)	c(19)	c(20)
10	0	0	10	1	-2	-16	36	99	-272	-240	1056	-253	-1800
12	0	0	12	1	10	-88	-132	1275	736	-3040	-2380	24035	13080
14	1	0	10	1	-2	224	-444	-1581	4046	-4320	96	65987	-129000
16	0	1	10	1	-2	-520	1044	-8469	14848	93000	-214656	-90973	628200
16	1	0	12	1	10	152	2268	-17685	-9344	114600	-44160	274595	199800
18	2	0	10	1	-2	464	-924	54339	-105832	105800	-864	-2962973	5619000
18	0	1	12	1	10	-592	-5172	28995	-99056	690000	591840	-6840445	129000
20	1	1	10	1	-2	-280	564	-131109	261088	-3056040	5590464	459107	-11836920
20	2	0	12	1	10	392	4668	20955	556576	-3794760	2679360	-9382045	-15165480
22	3	0	10	1	-2	704	-1404	167959	-332912	14157120	-27649824	143276867	-231589800
22	0	2	10	1	-2	-1024	2052	236979	-478064	12687040	-24815904	157306227	-265458600
22	1	1	12	1	10	-352	-2772	-110925	-1318736	6375800	-34285920	217428035	-106371000
24	2	1	10	1	-2	-40	84	-196149	392128	-35120280	69456384	-1018050013	1897579560
24	3	0	12	1	10	632	7068	117195	1698496	2087880	146407680	-872209885	1861629240
24	0	2	12	1	10	-1096	-10212	310731	2341312	-4200312	135307068	-764576221	1980045624
26	4	0	10	1	-2	944	-1884	338979	-674192	55970640	-110594784	3908309507	-7596107400
26	1	2	10	1	-2	-784	1572	-6621	10096	67556880	-135132384	3755208707	-7240139400
26	2	1	12	1	10	-112	-372	-193245	-1962416	-20996800	-356702880	1505914115	-11201914200
28	3	1	10	1	-2	200	-396	-203589	407968	-82275720	163735104	-9870850333	19414637640
28	0	3	10	1	-2	-1528	3060	736443	-1479008	-89642184	182245440	-10153799965	19941636168
28	4	0	12	1	10	872	9468	271035	3416416	31586520	569380800	-113712925	40715895960
28	1	2	12	1	10	-856	-7812	49851	-87968	68014488	675231168	-1108819741	39442512024
30	5	0	10	1	-2	1184	-2364	567699	-1130672	139371360	-276483744	18079918947	-35607805800
30	2	2	10	1	-2	-544	1092	-192621	383056	64281120	-129327264	19949307587	-39639575400
30	3	1	12	1	10	128	2028	-217965	-2030096	-67612800	-828419040	-3951942205	-101051748600
30	0	3	12	1	10	-1600	-15252	846483	7321840	-142703040	-876098016	-3681447997	-103904883960
32	4	1	10	1	-2	440	-876	-153429	308608	-130698360	260778624	-30055413853	59589577320
32	1	3	10	1	-2	-1288	2580	371883	-748928	83810376	-166120320	-30087457885	60506412648
32	5	0	12	1	10	1112	11868	482475	5710336	98525160	1409838720	8058364835	184810546680
32	2	2	12	1	10	-616	-5412	-153429	-1941248	78136488	637312128	15316600739	201255660024
34	6	0	10	1	-2	1424	-2844	854019	-1702352	278183280	-552964704	52763149187	-104422077000
34	3	2	10	1	-2	-304	612	-321021	640816	16883760	-35048544	34957076867	-69843414600
34	0	4	10	1	-2	-2032	4068	1489923	-2987984	-435518736	877017504	22964715395	-47686445640
34	4	1	12	1	10	368	4428	-185085	-1521776	-119641200	-1311194400	-20648940925	-304243522200
34	1	3	12	1	10	-1360	-12852	464643	3682960	57003600	848266464	-36112508797	-298455551640
36	5	1	10	1	-2	680	-1356	-45669	94048	-166564200	332938944	-61751452573	122837118600
36	2	3	10	1	-2	-1048	2100	64923	-134048	170286936	-340303680	-9178338205	19037153928
36	6	0	12	1	10	1352	14268	751515	8580256	216727800	2806021440	32754039395	535586093400
36	3	2	12	1	10	-376	-3012	-299109	-3218528	39989688	159789888	33733829219	349981281624
36	0	4	12	1	10	-2104	-20292	1636251	14842528	-542842248	-4313863872	54358407779	217741975896

Table 2. Coefficients of Jacobi cusp forms of index 1

k	a	b	j	c(3)	c(4)	c(7)	c(8)	c(11)	c(12)	c(15)	c(16)	c(19)	c(20)
38	7	0	10	1	-2	1664	-3324	1197939	-2389232	486230400	-967685664	121381404227	-240829833000
38	4	2	10	1	-2	-64	132	-391821	783376	-60811200	120055776	38313748547	-76866825000
38	1	4	10	1	-2	-1792	3588	1004403	-2015984	-82319616	168674784	-78355185085	156371011800
38	5	1	12	1	10	608	6828	-94605	-437456	-163260000	-1666788960	-49760122045	-622187283000
38	2	3	12	1	10	-1120	-10452	140403	620080	165587040	1704483744	-21437137597	-87002596920
40	6	1	10	1	-2	920	-1836	119691	-235712	-176049240	352568064	-101820918493	202936461480
40	3	3	10	1	-2	-808	1620	-184437	365632	183611496	-367952640	31823735075	-62911195992
40	0	5	10	1	-2	-2536	5076	2497419	-5004992	-1152766680	2315548416	217935113507	-440506318680
40	7	0	12	1	10	1592	16668	1078155	12026176	400018440	4896168960	86401086755	1227660648120
40	4	2	12	1	10	-136	-612	-387189	-3919808	-32601912	-619095552	42682769699	381358768824
40	1	4	12	1	10	-1864	-17892	1133451	9994048	-154679928	-795420672	-72403550941	-785661679944
42	8	0	10	1	-2	1904	-3804	1599459	-3191312	777336720	-1548294624	240675448067	-478257505800
42	5	2	10	1	-2	176	-348	-405021	810736	-154979760	308337696	22872314627	-46360494600
42	2	4	10	1	-2	-1552	3108	576483	-1159184	154873104	-307424736	-95954407165	192522510840
42	6	1	12	1	10	848	9228	53475	1222864	-184645200	-1756962720	-89142765565	-1023115479000
42	3	3	12	1	10	-880	-8052	-126237	-1866800	196871280	1830793824	18599513603	323342812200
42	0	5	12	1	10	-2608	-25332	2680035	24903376	-1332642000	-11458231200	300994982915	2147558543400
44	7	1	10	1	-2	1160	-2316	342651	-680672	-145329480	292017984	-143808003613	287031285960
44	4	3	10	1	-2	-568	1140	-376197	750112	137608056	-276715200	75486697955	-150419213112
44	1	5	10	1	-2	-2296	4596	1890939	-3791072	-558857160	1125301056	-53351489053	104448594120
44	8	0	12	1	10	1832	19068	1462395	16048096	662221080	7818521280	184745042915	2428829922840
44	5	2	12	1	10	104	1788	-417669	-4045088	-125814312	-1561104192	34021086179	224305113624
44	2	4	12	1	10	-1624	-15492	688251	5721568	113328792	1564571328	-107090988061	-955095556584
46	9	0	10	1	-2	2144	-4284	2058579	-4108592	1165326240	-2322439584	430703904707	-856767047400
46	6	2	10	1	-2	416	-828	-360621	722896	-251797920	502149216	-15196472893	29389368600
46	3	4	10	1	-2	-1312	2628	206163	-417584	289883424	-578929056	-57550070845	116257587480
46	0	6	10	1	-2	-3040	6084	3758931	-7530032	-2369410080	4753886304	757703781827	-1524922854120
46	7	1	12	1	10	1038	11628	259155	3459184	-169972800	-1443475680	-133336391485	-1442082958200
46	4	3	12	1	10	-640	-5652	-335277	-3777680	164680320	1365436704	65570052803	758647107720
46	1	5	12	1	10	-2368	-22932	2056275	19845296	-695060160	-5536070880	-13067729725	-548795708280
48	8	1	10	1	-2	1400	-2796	623211	-1240832	-60580920	123640704	-177939139933	355629752040
48	5	3	10	1	-2	-328	660	-510357	1019392	46100616	-94239360	107696246435	-215202993432
48	2	5	10	1	-2	-2056	4116	1342059	-2692352	-109984440	225357696	-183408190813	366362982120
48	9	0	12	1	10	2072	21468	1904235	20646016	1017159720	11711318400	346849203875	4340067229560
48	6	2	12	1	10	344	4188	-390549	-3594368	-225823512	-2527996032	2924202659	-159085091976
48	3	4	12	1	10	-1384	-13092	300651	2025088	275007912	2904352128	-78416351581	-567343782024
48	0	6	12	1	10	-3112	-30372	3977835	37504384	-2640126360	-23589440640	928392397475	7511424019320
50	10	0	10	1	-2	2384	-4764	2575299	-5141072	1664022960	-3317768544	714843158147	-1423055929800
50	7	2	10	1	-2	656	-1308	-258621	519856	-337441680	673842336	-76404102013	151461036600
50	4	4	10	1	-2	-1072	2148	-106557	208816	336535344	-673486176	12458463875	-23569742280
50	1	6	10	1	-2	-2800	5604	3031491	-6074192	-1473826320	2959806624	197144242307	-400214160840
50	8	1	12	1	10	1328	14028	522435	6271504	-105418800	-588087840	-173562759805	-1780966968600
50	5	3	12	1	10	-400	-3252	-486717	-5112560	82838160	446652384	104364848003	1078154321640
50	2	5	12	1	10	-2128	-20532	1490115	13363216	-206662320	-1062665760	-175456509565	-1836900807960

Table 2. <u>Coefficients of Jacobi cusp forms of index 1</u> (contd.)

§ 4. Hecke operators.

We define operators U_ℓ, V_ℓ, T_ℓ $(\ell>0)$ on functions $\phi: H\times\mathbb{C}\to\mathbb{C}$ by

(1) $$(\phi|_{k,m}U_\ell)(\tau,z) = \phi(\tau,\ell z),$$

(2) $$(\phi|_{k,m}V_\ell)(\tau,z) = \ell^{k-1} \sum_{\substack{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_1\backslash M_2(\mathbb{Z}) \\ ad-bc=\ell}} (c\tau+d)^{-k}\, e^{m\ell}\left(\frac{-cz^2}{c\tau+d}\right) \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{\ell z}{c\tau+d}\right),$$

(3) $$(\phi|_{k,m}T_\ell)(\tau,z) = \ell^{k-4} \sum_{\substack{M\in\Gamma_1\backslash M_2(\mathbb{Z}) \\ \det M=\ell^2 \\ \text{g.c.d}(M)=\square}} \sum_{X\in\mathbb{Z}^2/\ell\mathbb{Z}^2} \phi|_{k,m} M \,|_m X ,$$

where the symbols $|_{k,m} M$, $|_m X$ have the same meanings as in §1 (except that for $M\in GL_2^+(\mathbb{R})$ one first replaces M by $(\det M)^{-1/2}M \in SL_2(\mathbb{R})$) and $\text{g.c.d.}(M)=\square$ means that the greatest common divisor of the entries of M is a square. Then we have

<u>Theorem 4.1.</u> <u>The operators</u> U_ℓ, V_ℓ, T_ℓ <u>are well-defined</u> (i.e. independent of the choice of representatives) <u>on</u> $J_{k,m}$ <u>and map</u> $J_{k,m}$ <u>to</u> $J_{k,m\ell^2}$, $J_{k,m\ell}$ <u>and</u> $J_{k,m}$, <u>respectively</u>.

<u>Proof:</u> The well-definedness and the fact that $\phi|V_\ell$, $\phi|V_\ell$, $\phi|T_\ell$ transform correctly follow by straightforward calculations from the properties of the Jacobi group given in §1 ; the conditions at infinity will follow from the explicit Fourier expansions given below.

Before proceeding to give the properties of the operators U,V,T, we explain the motivation for the definitions given. The operator U_ℓ is an obvious one to introduce, corresponding to the endomorphism "multiplication by ℓ " on the elliptic curve $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$. As to the other two, we would like to define Hecke-operators as in the theory of modular forms by replacing ϕ by $\phi|M$, where M runs over a system of representatives of matrices of determinent ℓ modulo left multiplication by elements of Γ_1 . Doing this would produce a new function transforming like a Jacobi form with respect to Γ_1 . However, since $\phi|[M]$ is a multiple of

$$(4) \qquad \phi\left(\frac{a\tau+b}{c\tau+d}\ ,\ \frac{\sqrt{\ell}\ z}{c\tau+d}\right)$$

($M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\ell = \det M$) , and $\sqrt{\ell}$ is in general irrational, the function $\phi|[M]$ transforms in z for translations by a lattice incommensurable with $\mathbb{Z}\tau + \mathbb{Z}$, and there is no way to make a Jacobi form of the same index out of it. However, replacing $\sqrt{\ell}$ by ℓ in (4) restores the rationality; since this is formally the operator $U_{\sqrt{\ell}}$, and U_ℓ multiplies the index by ℓ^2 , we obtain in this way an operator which multiplies indices by ℓ . This explains the definition (2). Finally, if ℓ is a square then the function (4) transforms like a Jacobi form with respect to translations in the sublattice $\sqrt{\ell}(\mathbb{Z}\tau+\mathbb{Z})$ of $\mathbb{Z}\tau + \mathbb{Z}$, so we get a function with the right translations properties by averaging over the quotient lattice. This explains (3) except for the condition on g.c.d.(M) , which was introduced for later purposes: If we define T_ℓ^0 by the same formula as (3) but with the condition "M primitive" (i.e. g.c.d.(M) = 1), then T_ℓ and T_ℓ^0 are related by

$$(5) \qquad T_\ell = \sum_{d^2|\ell} d^{2k-4}\ T^0_{\ell/d^2} \quad .$$

Eventually we want to show that the Jacobi-Hecke operators T_ℓ correspond to the usual Hecke operators T_ℓ on modular forms of weight $2k-2$, and equation (5) is precisely the relation between these Hecke operators and the corresponding operators defined with primitive matrices.

Our main goal is to describe the action of our operators on Fourier coefficients and to give their commutation relations. We start with U_ℓ and V_ℓ since they are much easier to treat.

Theorem 4.2. i) Let $\phi \in J_{k,m}$, $\phi = \sum c(n,r)\, q^n \zeta^r$. Then

(6) $$\phi|U_\ell = \sum_{n,r} c(n,r/\ell)\, q^n\, \zeta^r$$

(with the convention $c(n,r/\ell) = 0$ if $\ell \nmid r$) and

(7) $$\phi|V_\ell = \sum_{n,r} \Big(\sum_{a|(n,r,\ell)} a^{k-1}\, c\big(\tfrac{n\ell}{a^2}, \tfrac{r}{a}\big)\Big)\, q^n\, \zeta^r .$$

ii) The operators U_ℓ, V_ℓ satisfy the relations

(8) $$U_\ell \circ U_{\ell'} = U_{\ell\ell'} ,$$

(9) $$U_\ell \circ V_{\ell'} = V_{\ell'} \circ U_\ell ,$$

(10) $$V_\ell \circ V_{\ell'} = \sum_{d|(\ell,\ell')} d^{k-1}\, U_d \circ V_{\ell\ell'/d^2} .$$

In particular, all of these operators commute.

Remark. Formula (7) nearly makes sense for $\ell = 0$ and suggests the definition

$$\phi|V_0 = c(0,0)\, [\, c_k + \sum_{n\geq 1} \sigma_{k-1}(n)\, q^n \,]$$

with some constant c_k. Since $\phi|V_0$ should belong to $J_{k,0} = M_k$, we take $c_k = -\frac{2k}{B_{2k}}$ so that $\phi|V_0$ is a multiple of the Eisenstein series of weight k. This definition will be used later.

Proof. Equation (6) is obvious. For (7) take the standard set of representatives

(11) $$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} , \quad a,d>0 , \quad b \pmod d , \quad ad = \ell$$

for the matrices in (2). Then

$$(\phi|V_\ell)(\tau,z) = \ell^{k-1} \sum_{ad=\ell} \sum_{b \pmod d} d^{-k}\, \phi\left(\frac{a\tau+b}{d}, az\right)$$

$$= \ell^{k-1} \sum_{ad=\ell} d^{-k} \sum_{b \pmod d} \sum_{n,r} c(n,r)\, q^{\frac{an}{d}} \zeta^{ar} e_d(nb)$$

$$= \ell^{k-1} \sum_{ad=\ell} d^{1-k} \sum_{\substack{n,r \\ n\equiv 0 \pmod d}} c(n,r)\, q^{\frac{an}{d}} \zeta^{ar}$$

$$= \sum_{a|\ell} a^{k-1} \sum_{n,r} c\left(\frac{n\ell}{a}, r\right) q^{an} \zeta^{ar} ,$$

which is equivalent to (7) . Eq. (8) is obvious. By (6) and (7), we have

$$\text{coefficient of } q^n\zeta^r \text{ in } (\phi|U_\ell)|V_{\ell'}$$

$$= \sum_{\substack{a|(n,r,\ell') \\ \ell|\frac{r}{a}}} a^{k-1} c\left(\frac{n\ell'}{a^2}, \frac{r}{a\ell}\right)$$

$$= \begin{cases} 0 & \text{if } \ell\nmid r \\ \displaystyle\sum_{a|(n,\frac{r}{\ell},\ell')} a^{k-1} c\left(\frac{n\ell'}{a^2}, \frac{r/\ell}{a}\right) & \text{if } \ell|r \end{cases}$$

$$= \text{coefficient of } q^n \zeta^r \text{ in } (\phi|V_{\ell'})\, U_\ell .$$

Finally, using (7) we find

$$\text{coefficient of } q^n \zeta^r \text{ in } (\phi|V_\ell)|V_{\ell'}$$

$$= \sum_{a|(n,r,\ell')} a^{k-1} \sum_{b|(\frac{n\ell'}{a^2}, \frac{r}{a}, \ell)} b^{k-1} c\left(\frac{n\ell\ell'}{a^2b^2}, \frac{r}{ab}\right)$$

$$= \sum_e N(e)\, e^{k-1} c\left(\frac{n\ell\ell'}{e^2}, \frac{r}{e}\right)$$

where $N(e)$ is the number of ways of writing e as $a\cdot b$ in the preceding sum. If such a decomposition exists then $e \mid \ell a$ and hence $a = \frac{e}{(e,\ell)}\delta$ for some integer δ ; writing down the conditions on a and $b=e/a$ we find the formula

$$N(e) = \text{number of divisors } \delta \text{ of } \left(n,\ell,\ell',e,\frac{n\ell}{e},\frac{n\ell'}{e},\frac{\ell\ell'}{e},\frac{n\ell\ell'}{e^2}\right)$$

($=0$ unless $e \mid (n\ell, n\ell', \ell\ell')$, $e^2 \mid n\ell\ell'$). On the other hand, using the calculation just given for $U_\ell \cdot V_{\ell'}$ we find

$$\text{coefficient of } q^n \zeta^r \text{ in } \sum_{d \mid (\ell,\ell')} d^{k-1} (\phi \mid U_d) \mid V_{\ell\ell'/d^2}$$

$$= \sum_{d \mid (\ell,\ell')} d^{k-1} \sum_{a \mid (n,\frac{r}{d},\frac{\ell\ell'}{d^2})} a^{k-1} c\left(\frac{n\ell\ell'}{a^2 d^2}, \frac{r}{ad}\right)$$

$$= \sum_e N'(e)\, e^{k-1} c\left(\frac{n\ell\ell'}{e^2}, \frac{r}{e}\right) ,$$

where now $N'(e)$ consits the decompositions of e as $a \cdot d$ satisfying the conditions in the sum; from $e \mid nd$ we find $\frac{e}{(n,e)}$ divides d , and writing d as $\frac{e}{(n,e)}\delta$ we obtain for $N'(e)$ the same formula as for $N(e)$. This proves (10) (another proof could be obtained by combining the Corollary below with the multiplicative properties of ordinary Hecke operators) and completes the proof of Theorem 4.2.

As a consequence of the formula for the action of V_ℓ on Fourier coefficients we have the following

Corollary. For $\phi \in J_{k,m}$ and $\nu \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, one has

$$D_\nu(\phi|_{k,m} V_\ell) = (D_\nu \phi)|_{k+\nu} T_\ell ,$$

where T_ℓ on the right denotes the usual Hecke operator on modular forms.

Notice that this property characterizes V_ℓ completely, since we have

$$\begin{array}{ccccccc}
J_{k,m} & \xrightarrow{D} & M_k & \oplus\ S_{k+2} & \oplus \cdots \oplus & S_{k+N} \\
\Big\downarrow V_\ell & & \Big\downarrow T_\ell & \Big\downarrow T_\ell & \cdots & \Big\downarrow T_\ell \\
J_{k,m\ell} & \xrightarrow{D} & M_k & \oplus\ S_{k+2} & \oplus \cdots \oplus & S_{k+N}
\end{array}$$

and the horizontal maps are injective for $N \geq 2m\ell$ (cf. Theorem 3.4). To prove it, we calculate

$$c_n\left[(D_\nu \phi)\big|_{k+\nu} T_\ell\right] = \sum_{d|n,\ell} d^{k+\nu-1} \, c_{\frac{n\ell}{d^2}}(D_\nu \phi)$$

$$= \sum_{d|n,\ell} d^{k+\nu-1} \sum_r p_\nu^{(k-1)}\left(r, \tfrac{n\ell}{d^2} m\right) \; c\left(\tfrac{n\ell}{d^2}, r\right)$$

where c_n denotes "coefficient of q^n in" and $c(n,r)$ is the coefficie[nt] of $q^n \zeta^r$ in ϕ. Replacing r by r/d in this sum and using the homogeneity of $p_\nu^{(k-1)}$ we find

$$c_n\left[(D_\nu \phi)\big|_{k+\nu} T_\ell\right] = \sum_r p_\nu^{(k-1)}(r, n\ell m) \sum_{d|(n,r,\ell)} d^{k-1} \, c\left(\tfrac{n\ell}{d^2}, \tfrac{r}{d}\right)$$

$$= c_n\left[D_\nu(\phi|_{k,m} V_\ell)\right]$$

as was to be shown. Another proof comes from observing that the map $L_k : M^+_{k,m} \longrightarrow M^+_{k+2,m}$ used to construct $D_{2\nu}$ (cf. proof of Theorem 3.2) is equivariant with respect to the action of V_ℓ on $M^+_{*,*}$ (this follows from (11) of §3 since V_ℓ maps ϕ to $\sum_M (\phi|_k M)(\tau, \sqrt{\ell} z)$) and that V_ℓ acts as T_ℓ on the constant term $\phi(\tau,0)$ of a form ϕ in $M^+_{*,*}$.

As another application of Theorem 4.2 we have

<u>Theorem 4.3. Let $E_{k,m}$ denote the Eisenstein series of §2. Then for m square free we have</u>

(12) $\qquad E_{k,1}|V_m = \sigma_{k-1}(m)\, E_{k,m}$.

<u>Thus the Fourier coefficients of $E_{k,m}$ are as in eq. (7) of §2.</u>

Indeed, we have

$$E_{k,1}|V_m = m^{k-1} \sum_{M=\binom{a\ b}{c\ d}} (c\tau+d)^{-k} e^m\left(\frac{-cz^2}{c\tau+d}\right) \sum_{\binom{\alpha\ \beta}{\gamma\ \delta}} \sum_{\lambda\in\mathbb{Z}} (\gamma M\tau+\delta)^{-k}$$

$$\times\ e\left[\lambda^2 M'M\tau+2\lambda\,\frac{mz(c\tau+d)}{\gamma M\tau+\delta} - \frac{\gamma m^2z^2/(c\tau+d)^2}{\gamma M\tau+\delta}\right]$$

where M runs over $\Gamma_1\backslash\{M\in M_2(\mathbb{Z})\mid \det M = m\}$ and M' over $\Gamma_\infty\backslash\Gamma_1$. Writing $\binom{a\ b}{c\ d}$ for the product MM' gives

$$E_{k,1}|V_m = m^{k-1} \sum_{\binom{a\ b}{c\ d}} (c\tau+d)^{-k} \sum_{\lambda\in\mathbb{Z}} e\left[\lambda^2\,\frac{a\tau+b}{c\tau+d} + 2\lambda\,\frac{mz}{c\tau+d} - \frac{mcz^2}{c\tau+d}\right],$$

where now $\binom{a\ b}{c\ d}$ runs over $\Gamma_\infty\backslash\{M\mid\det M = m\}$. Now suppose m is square-free. Then $(c,d)=\delta$ divides m and $\delta'=\frac{m}{\delta}$ is prime to δ. It follows that $\binom{a\ b}{c\ d}$ can be multiplied on the left by an element of Γ_∞ so as to make $a\equiv b\equiv 0 \pmod{\delta'}$, so

$$\Gamma_\infty\backslash\{\binom{a\ b}{c\ d},\ ad-bc=m,\ (c,d)=\delta\} = \begin{pmatrix}1 & \delta'\mathbb{Z}\\ 0 & 1\end{pmatrix}\backslash\left\{\begin{pmatrix}\delta'a & \delta'b\\ \delta c' & \delta d'\end{pmatrix}\middle|\ a'd'-b'c'=1\right\}$$

$$\leftrightarrow \begin{pmatrix}1 & \delta\mathbb{Z}\\ 0 & 1\end{pmatrix}\backslash SL_2(\mathbb{Z}),$$

and hence

$$E_{k,1}|V_m = m^{k-1}\sum_{\delta|m}\delta^{-k} \sum_{\binom{a\ b}{c\ d}\in\binom{1\ \delta\mathbb{Z}}{0\ \ 1}\backslash SL_2(\mathbb{Z})} (c\tau+d)^{-k} \sum_{\lambda\in\mathbb{Z}} e\left[\lambda^2\,\frac{\delta'}{\delta}\,\frac{a\tau+b}{c\tau+d} + 2\lambda\frac{\delta' z}{c\tau+d} - \frac{cz^2 m}{c\tau+d}\right].$$

Replacing $\binom{a\ b}{c\ d}$ by $\binom{1\ 1}{0\ 1}\binom{a\ b}{c\ d}$ multiplies the final exponential by $e(\lambda^2\delta'/\delta)$, and this is 1 only if $\delta|\lambda$ (since $(\delta,\delta')=1$ and δ is square-free). Therefore the terms with $\delta\nmid\lambda$ can be omitted, so

$$E_{k,1}|V_m = m^{k-1}\sum_{\delta|m}\delta^{1-k} \sum_{\Gamma_\infty\backslash\Gamma_1} (c\tau+d)^{-k} \sum_{\lambda\in\delta\mathbb{Z}} e^m\left[\frac{\lambda^2}{\delta^2}\,\frac{a\tau+b}{c\tau+d} + 2\,\frac{\lambda}{\delta}\,\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d}\right]$$

$$= m^{k-1}\sum_{\delta|m}\delta^{1-k}\ E_{k,m}(\tau,z) = \sigma_{k-1}(m)\,E_{k,m}(\tau,z).$$

This proves (12); the formula for the Fourier coefficients of $E_{k,m}$ now follows directly from Theorem 4.2.

For m not square-free we find after a similar calculation the more general result

$$(13)\qquad E_{k,m} = m^{-k+1} \prod_{p|m} (1+p^{-k+1})^{-1} \cdot \sum_{d^2|m} \mu(d)\, E_{k,1} \big| U_d \cdot V_{\frac{m}{d^2}} \; ;$$

yet more generally, we see that the space $J^{Eis}_{k,m}$ defined in Theorem 2.4 is mapped by U_ℓ and V_ℓ to the corresponding spaces with index $m\ell^2$ and $m\ell$, respectively, the precise formulas for the images of $E_{k,m,s}$ under U_ℓ and V_ℓ being easily ascertained by looking at the "constant term" (terms $q^n\zeta^r$ with $4nm=r^2$). One then sees by induction that all Jacobi Eisenstein series can be written uniquely as a linear combination of $E^{(\chi)}_{k,m}|U_\ell \cdot V_{\ell'}$, ($\ell,\ell' \in \mathbb{N}$, χ a primitive Dirichlet character modulo f with $\chi(-1)=(-1)^k$, $m=f^2$, $E^{(\chi)}_{k,m}$ as in (10) of §2). Combining these remarks with the commutation relations of Theorem 4.2, we obtain:

Theorem 4.4. Let

$$J^{old}_{k,m} = \sum_{\substack{d|m\\ d>1}} \sum_{\ell^2|d} J_{k,\frac{m}{d}} \Big| U_\ell \cdot V_{\frac{d}{\ell^2}}$$

and similarly for $J^{cusp,old}_{k,m}$ and $J^{Eis,old}_{k,m}$. Define $J^{cusp,new}_{k,m}$ as the orthogonal complement of $J^{old}_{k,m}$ in $J^{cusp}_{k,m}$ (with respect to the Petersson scalar product) and $J^{Eis,new}_{k,m}$ as the span of the functions $E^{(\chi)}_{k,m}$ (χ a primitive Dirichlet character (mod f)) if $m=f^2$ and 0 otherwise. Th

$$J_{k,m}^{Eis} = \bigoplus_{\substack{\ell,\ell' \\ \ell^2\ell'|m}} J_{k,m/\ell'\ell^2}^{Eis,new} | U_\ell V_{\ell'}$$

and

$$J_{k,m}^{cusp} = \sum_{\substack{\ell,\ell' \\ \ell^2\ell'|m}} J_{k,m/\ell'\ell^2}^{cusp,new} | U_\ell V_{\ell'} .$$

Eventually, we will show that the latter decomposition is also a direct sum, but this will be harder and will depend on using a trace formula.

It remains to describe the action of the Hecke operators T_ℓ on the Fourier coefficients of Jacobi forms. We do this only for $(\ell,m)=1$. The formula obtained involves the Legendre symbol $(\frac{\cdot}{\ell})$ if ℓ is an odd prime; in the general case it involves a slightly generalized symbol $\varepsilon_D(n)$, $(D,n\in\mathbb{Z})$, which we now define.

If D is congruent to 2 or 3 (mod 4) we set $\varepsilon_D(n)=0$. For $D=0$ we set

$$\varepsilon_0(n) = \begin{cases} r & \text{if } n=r^2,\ r\geq 0 , \\ 0 & \text{if } n\neq\square . \end{cases}$$

Otherwise D can be written uniquely as $D_0 f^2$ where $f\geq 1$ and D_0 is the discriminant of $\mathbb{Q}(\sqrt{D})$. Then let χ be the primitive Dirichlet character $(\text{mod}\, D_0)$ corresponding to $\mathbb{Q}(\sqrt{D})$, i.e. the multiplicative function with

$$\chi(p) = (\frac{D_0}{p}) \quad (p \text{ odd}), \quad \chi(2) = \begin{cases} 1 & D_0\equiv 1\ (8) \\ -1 & D_0\equiv 5\ (8) \\ 0 & D_0\equiv 0\ (4) \end{cases} , \quad \chi(-1) = \operatorname{sign} D$$

and set

$$\varepsilon_D(n) = \begin{cases} \chi(n_0)\, g & \text{if } n=n_0 g^2,\ g|f ,\ (\frac{f}{g},n_0)=1 , \\ 0 & \text{if } (n,f^2)\neq\square . \end{cases}$$

The function ε_D, which reduces to χ if $D = D_0$, occurs naturally whenever one studies non-fundamental discriminants; it has the properties

$$\varepsilon_D(n+D) = \varepsilon_D(n)$$

and (as shown in [13], p.188)

$$\sum_{n=1}^{\infty} \frac{\varepsilon_D(n)}{n^s} = L_D(s) \quad ,$$

where $L_D(s)$ is the L-series already introduced in connection with Jacobi-Eisenstein series (eq.(6) of §2 and the following formula). We can now state the formula for the Fourier coefficients of $\phi|T_\ell$.

<u>Theorem 4.5. Let</u> $\phi = \sum c(n,r)\, q^n \zeta^r$ <u>be a Jacobi form of weight</u> k <u>and index</u> m <u>and</u> ℓ <u>a positive integer prime to</u> m. <u>Then</u> $\phi|T_\ell = \sum c^*(n,r)\, q^n \zeta^r$ <u>with</u>

(14) $$c^*(n,r) = \sum_a \varepsilon_{r^2-4nm}(a)\, a^{k-2}\, c(n',r') \quad ,$$

<u>where the sum is over</u> a <u>with</u> $a|\ell^2$, $a^2|\ell^2(r^2-4nm)$, $a^{-2}\ell^2(r^2-4nm) \equiv 0, 1 \pmod 4$

(15) $$r'^2 - 4n'm = \ell^2(r^2-4nm)/a^2 \quad , \quad ar' \equiv \ell r \pmod{2m} \quad .$$

<u>In particular, if</u> r^2-4nm <u>is a fundamental discriminant and</u> χ <u>the corresponding quadratic character, then</u>

(16) $$c^*(n,r) = \sum_{d|\ell} \chi(d)\, d^{k-2}\, c\left(\frac{\ell^2}{d^2}n, \frac{\ell}{d}r\right) \quad .$$

To see that (15) makes sense, note that $a|\ell^2$ and $(\ell,m)=1$ imply $(a,m)=1$, so the second equation in (15) defines r' uniquely modulo $2m$ if m is even and modulo m if m is odd; in the latter case, r' is determined (mod 2) by the mod 4 reduction of the first of equations (15). Thus in both cases r' is uniquely defined modulo $2m$ and satisfies

$$r'^2 \equiv \ell^2(r^2-4nm)/a^2 \pmod{4m} \quad ,$$

so the number n' defined by the first of equations (15) is integral; that $c(n',r')$ is independent of the choices made then follows from Theorem 2.2.

Before proving Theorem 4.5 we state two consequences.

Corollary 1. For ℓ and ℓ' both prime to m one has

$$T_\ell \cdot T_{\ell'} = \sum_{d|(\ell,\ell')} d^{2k-3} T_{\ell\ell'/d^2} .$$

In particular, the operators T_ℓ $((\ell,m)=1)$ all commute. Furthermore, the operators T_ℓ commute with $V_{\ell'}$ and $U_{\ell'}$ for $(\ell,\ell'm)=1$.

Proof. Exercise.

Corollary 2. The spaces $J_{k,m}$ and $J_{k,m}^{new}$ are spanned by common eigenforms of the Hecke operators T_ℓ ($(\ell,m)=1$).

Proof. This follows by a standard argument from Corollary 1 and the easily-proved fact that the Hecke operators in question are hermitian with respect to the Petersson scalar product defined in §2.

Proof of Theorem 4.5: We write the definition of T_ℓ as

$$\phi|T_\ell = \ell^{k-2} \sum_M \phi|_{k,m} M \,|_m \underline{A} ,$$

where the sum is over all $M \in \Gamma_1 \backslash M_2(\mathbb{Z})$ with $\det(M)=\ell^2$ and g.c.d.(M) a square and $|_m \underline{A}$ is the "averaging operator" which replaces a function $\psi(\tau,z)$ that transforms like a Jacobi form with respect to some sublattice $L \subseteq \mathbb{Z}^2$ by

$$\psi|_m \underline{A} := \frac{1}{[\mathbb{Z}^2:L]} \sum_{X \in \mathbb{Z}^2/L} \psi|_m X .$$

(Note that this is independent of the choice of L and projects L-invariant functions to $\mathbb{Z}^2$-invariant functions; in (3) we took $L = \ell\mathbb{Z}^2$.)

As in the usual computation of the action of Hecke operators on Fourier coefficients we choose upper triangular representatives M for the left Γ_1-cosets; then $\phi|T_\ell = \phi_1|\underline{A}$ where

$$\phi_1 = \ell^{k-2} \sum_{ad=\ell^2} \sum_{\substack{b \,(\mathrm{mod}\, d) \\ (a,b,d)=\square}} \phi|_{k,m} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$= \ell^{-2} \sum_{ad=\ell^2} a^k \sum_{\substack{b \pmod d \\ (a,b,d)=\square}} \phi\left(\frac{a\tau+b}{d}, \frac{\ell z}{d}\right) .$$

To get rid of the condition " $(a,b,d) = \square$ " we use the identity

$$\sum_{\delta | n} \lambda(\delta) = \begin{cases} 1 & (n = \square) \\ 0 & (n \neq \square) \end{cases}$$

with $n = (a,b,d)$, where λ is Liouville's function (the multiplicative function with $\lambda(p^\nu) = (-1)^\nu$). This gives

$$\phi_1(\tau,z) = \ell^{-2} \sum_{ad=\ell^2} a^k \sum_{\delta | (a,d)} \lambda(\delta) \sum_{\substack{b \pmod d \\ b \equiv 0 \pmod \delta}} \phi\left(\frac{a\tau+b}{d}, \frac{\ell z}{d}\right)$$

$$= \sum_{ad=\ell^2} a^{k-1} \sum_{\delta | (a,d)} \lambda(\delta) \sum_{\substack{n,r \\ n \equiv 0 \pmod{d/\delta}}} c(n,r)\ e\left(\frac{an}{d}\tau + \frac{r\ell}{d} z\right)$$

$$= \sum_{ad=\ell^2} a^{k-1} \sum_{\substack{n,r \\ n \equiv 0 \pmod{d/(a,d)}}} \Lambda\left((a,d), \frac{n}{d/(a,d)}\right) c(n,r) e\left(\frac{an}{d}\tau + \frac{r\ell}{d} z\right) ,$$

where

$$\Lambda(\alpha,\beta) := \sum_{\delta | \alpha,\ \alpha\delta^{-1} | \beta} \lambda(\delta)/\delta = \frac{1}{\alpha} \sum_{\gamma | (\alpha,\beta)} \lambda\left(\frac{\alpha}{\gamma}\right) \gamma \qquad (\alpha , \beta \in \mathbb{N}) .$$

Replacing n and r by $\frac{dn}{a}$ and $\frac{dr}{\ell}$ gives

(17) $$\phi_1(\tau,z) = \sum_{ad=\ell^2} a^{k-1} \sum_{\substack{n \in a(a,d)^{-1}\mathbb{Z} \\ r \in \ell d^{-1}\mathbb{Z}}} \Lambda\left((a,d), \frac{n}{a/(a,d)}\right) c\left(\frac{dn}{a}, \frac{dr}{\ell}\right) e(n\tau + rz) .$$

This gives the Fourier development of ϕ_1 (which, notice, involves non-integral powers of ζ). We still must apply the averaging operator $\underline{A}$. We can factor $\underline{A}$ as $\underline{A}_1 \circ \underline{A}_2$ where $\underline{A}_1$ is the averaging operator with respect to $0 \times \mathbb{Z}$, i.e.

$$\psi|_m \underline{A}_1 \;:=\; \frac{1}{[\mathbb{Z}:L']} \sum_{X \in 0\times\mathbb{Z}/L'} \psi|_m X$$

for any sublattice $L'\subset\mathbb{Z}$ such that ψ is invariant with respect to $0\times L'$, and $\underline{A}_2$ (for a function invariant under a lattice $L\subset\mathbb{Z}^2$ containing $0\times\mathbb{Z}$) is the averaging operator with respect to $\mathbb{Z}^2/0\times\mathbb{Z}$. It is easily seen that the effect of $\underline{A}_1$ is to leave any term $q^n\zeta^r$ with n and r in $\mathbb{Z}$ unchanged and to replace all terms $q^n\zeta^r$ with $r\notin\mathbb{Z}$ by 0 . Hence we have $\phi|T_\ell = \phi_2|\underline{A}_2$ where $\phi_2 = \phi_1|\underline{A}_1$ is obtained from ϕ_1 by omitting all terms in (17) for which $r\notin\mathbb{Z}$, i.e. by replacing the condition $r\in\frac{\ell}{d}\mathbb{Z}$ by $r\in\frac{\ell}{(\ell,d)}\mathbb{Z}$. Thus

$$\phi_2(\tau,z) \;=\; \sum_{ad=\ell^2} a^{k-1}\,\phi_{2,a}(\tau,z)$$

with

$$(18)\quad \phi_{2,a}(\tau,z) \;=\; \sum_{\substack{n,\,r\in\mathbb{Z}\\ dn\equiv 0 \;(\mathrm{mod}\; a)\\ dr\equiv 0 \;(\mathrm{mod}\; \ell)}} A\Big((a,d),\frac{n}{a/(a,d)}\Big)\, c\Big(\frac{dn}{a},\frac{dr}{\ell}\Big)\, q^n\,\zeta^r \quad .$$

To compute the coefficients $c^*(n,r)$ of $\phi|T_\ell$, we must still determine the operation of the operator $\underline{A}_2$ on each $\phi_{2,a}$. The operator $\underline{A}_2$ acts by

$$(\psi|_m A_2)(\tau,z) \;=\; \frac{1}{N}\sum_{\lambda\;(\mathrm{mod}\;N)} e^m(\lambda^2\tau + 2\lambda z)\,\psi(\tau\,,\,z+\lambda\tau) \qquad ,$$

where N is any integer such that $\psi|X=\psi$ for $X\in N\mathbb{Z}\times\mathbb{Z}$, i.e. such that the coefficient of $q^n\zeta^r$ in ψ depends only on $4nm-r^2$ and on $r\ (\mathrm{mod}\ 2mN)$. (For $\psi=\phi_{2,a}$ one checks that $N=(a,\ell)$ works.) Letting $C(r,r^2-4nm)$ denote the coefficient of $q^n\zeta^r$ in ψ (so that $C(r,\Delta)$ depends only on $\Delta\in-\mathbb{N}$ and on $r\in\mathbb{Z}/2mN\mathbb{Z}$) , we easily deduce that the coefficient of $q^n\zeta^r$ in $\psi|\underline{A}_2$ is $C^*(r\,,\,r^2-4nm)$ with

$$(19) \qquad C^*(r,\Delta) = \frac{1}{N} \sum_{\substack{R \ (\mathrm{mod}\ 2Nm) \\ R\equiv r \ (\mathrm{mod}\ 2m)}} C(R,\Delta)$$

(which depends, as it should, only on Δ and on $r \pmod{2m}$). Applying this to $\psi=\phi_{2,a}$ we find that the coefficient of $q^n\zeta^r$ in $\phi|T_\ell$ is given by

$$(20) \qquad c^*(n,r) = \sum_{ad=\ell^2} a^{k-1}\, C_a^*(r,r^2-4nm)$$

with C_a^* related as in (19) to the coefficient $C_a(R,\Delta)$ of $q^{(R^2-\Delta)/4m}\zeta^R$ in $\phi_{2,a}$. From eq. (18) we see that this coefficient is given by

$$C_a(R,\Delta) = \Lambda\left((a,d),\frac{R^2-\Delta}{4ma/(a,d)}\right)\, c\left(d\frac{R^2-\Delta}{4ma},\frac{dR}{\ell}\right) ,$$

with the convention that $\Lambda(\alpha,\beta)$ and $c(n,r)$ are 0 unless α, β, n and r are integral. Set $\alpha=(a,d)$; then $ad=\ell^2$ implies that

$$a=x^2\alpha , \quad d=y^2\alpha , \quad \ell=xy\alpha , \quad \frac{\ell}{(\ell,a)}=y , \quad \frac{\ell}{(\ell,d)}=x$$

for some $x,y\in\mathbb{N}$ with $(x,y)=1$. Then

$$C_a(R,\Delta)\neq 0 \implies x=\frac{\ell}{(\ell,d)}\Big|R , \quad x^2=\frac{a}{(a,d)}\Big|\frac{R^2-\Delta}{4m}$$

$$\implies x|R , \quad x^2|\Delta , \quad \frac{\Delta}{x^2}\equiv\left(\frac{R}{x}\right)^2 \pmod 4 .$$

Conversely, if $R=xr_0$, $\Delta=x^2\Delta_0$, $\Delta_0\equiv r_0^2 \pmod 4$, then

$$C_a(R,\Delta) = \Lambda\left(\alpha,\frac{r_0^2-\Delta_0}{4m}\right)\, c(r_0^2y^2-\Delta_0y^2 , r_0y) .$$

Hence (taking $N=(a,\ell)=x\alpha$ in (19))

$$C_a^*(r,\Delta) = \frac{1}{x\alpha} \sum_{\substack{r_0 \ (\mathrm{mod}\ 2\alpha m) \\ xr_0\equiv r \ (\mathrm{mod}\ 2m) \\ r_0^2\equiv\Delta_0 \ (\mathrm{mod}\ 4)}} \Lambda\left(\alpha,\frac{r_0^2-\Delta_0}{4m}\right)\, c(r_0^2y^2-\Delta_0y^2,r_0y)$$

if $\Delta=x^2\Delta_0$ and $C_a^*(r,\Delta)=0$ if $x^2\nmid\Delta$ (or if $x^{-2}\Delta\equiv 2$ or $3 \pmod 4$). Note that, since m is prime to $\ell^2=ad$ by hypothesis, the conditions

$r^2 \equiv \Delta \pmod{4m}$ and $r_0^2 \equiv \Delta_0 \pmod 4$ already assure that $r_0^2 \equiv \Delta_0 \pmod{4m}$. The fact that m is prime to ℓ also implies that the two numbers r and $r' := r_0 y = ryx^{-1} = r\ell/d$ determine one another (modulo 2m) and that the number $c(r_0^2y^2 - \Delta_0 y^2, r_0 y)$ in the last formula is the same as the number $c(n',r')$ in (14) (with $\Delta = r^2 - 4nm$). Moreover, we have

$$(21) \qquad \sum_{\substack{r_0 \pmod{2\alpha m} \\ xr_0 \equiv r \pmod{2m} \\ r_0^2 \equiv \Delta_0 \pmod 4}} \Lambda\left(\alpha, \frac{r_0^2 - \Delta_0}{4m}\right) = \varepsilon_{\Delta_0}(\alpha) ,$$

as we will show in a moment, and substituting this into the last equation and into (20) gives the desired formula (14), if we observe that
$\frac{1}{x\alpha} \varepsilon_{\Delta_0}(\alpha) = \frac{1}{x^2\alpha} \varepsilon_{\Delta_0 x^2}(\alpha x^2) = \frac{1}{a} \varepsilon_\Delta(a)$.

It remains to prove equation (21). From the definition of $\Lambda(\alpha,\beta)$ and the fact that α and m are relatively prime, we find that the left-hand side of (21) equals

$$\sum_{\gamma|\alpha} \lambda\left(\frac{\alpha}{\gamma}\right) \frac{\gamma}{\alpha} \sum_{\substack{r_0 \pmod{2\alpha m} \\ xr_0 \equiv r \pmod{2m} \\ r_0^2 \equiv \Delta_0 \pmod{4\gamma}}} 1 .$$

The inner sum equals $\frac{\alpha}{\gamma} N_\gamma(\Delta_0)$ with $N_\gamma(\Delta_0)$ as in the formula preceding equation (6) of §2. That equation gives

$$\frac{\zeta(2s)}{\zeta(s)} \sum_{\gamma=1}^{\infty} N_\gamma(\Delta_0)\, \gamma^{-s} = L_{\Delta_0}(s) = \sum_{\alpha=1}^{\infty} \varepsilon_{\Delta_0}(\alpha)\, \alpha^{-s}$$

or (since the coefficient of n^{-s} in $\zeta(2s)/\zeta(s)$ is $\lambda(n)$)

$$\sum_{\gamma|\alpha} \lambda\left(\frac{\alpha}{\gamma}\right) N_\gamma(\Delta_0) = \varepsilon_{\Delta_0}(\alpha) .$$

This completes the proof.

Chapter II. Relations with other types of modular forms

§ 5. Jacobi forms and modular forms of half-integral weight

In §2 we showed that the coefficients $c(n,r)$ of a Jacobi form of index m depend only on the "discriminant" r^2-4nm and on the value of $r \pmod{2m}$, i.e.

(1) $$c(n,r) = c_r(4nm-r^2)\,, \quad c_{r'}(N) = c_r(N) \quad \text{for } r' \equiv r \pmod{2m}$$

From this it follows very easily, as we will now see, that the space of Jacobi forms of weight k and index m is isomorphic to a certain space of (vector-valued) modular forms of weight $k-\frac{1}{2}$ in one variable; the rest of this section will then be devoted to identifying this space with more familiar spaces of modular forms of half-integral weight and studying the correspondence more closely.

Equation (1) gives us coefficients $c_\mu(N)$ for all $\mu \in \mathbb{Z}/2m\mathbb{Z}$ and all integers $N \geq 0$ satisfying $N \equiv -\mu^2 \pmod{4m}$ (notice that μ^2 is well-defined modulo $4m$ if μ is given modulo $2m$), namely

(2) $$c_\mu(N) := c\left(\frac{N+r^2}{4m}, r\right) \qquad (\text{any } r \in \mathbb{Z},\ r \equiv \mu \pmod{2m})$$

(since μ is a residue class, one should more property write $r \in \mu$ rather than $r \equiv \mu \pmod{2m}$; we permit ourselves the slight abuse of notation). We extend the definition to all N by setting $c_\mu(N)=0$ if $N \not\equiv -\mu^2 \pmod{4m}$ and set

(3) $$h_\mu(\tau) := \sum_{N=0}^{\infty} c_\mu(N)\, q^{N/4m} \qquad (\mu \in \mathbb{Z}/2m\mathbb{Z})$$

and

(4) $$\vartheta_{m,\mu}(\tau,z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2m}}} q^{r^2/4m}\, \zeta^r$$

(The $\vartheta_{m,\mu}$ are independent of the function ϕ.) Then

$$\phi(\tau,z) = \sum_{\mu \pmod{2m}} \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu\,(2m)}} \sum_{n \geq r^2/4m} c_\mu(4nm-r^2)\, q^n \zeta^r$$

$$= \sum_{\mu \pmod{2m}} \sum_{r \equiv \mu\,(2m)} \sum_{N \geq 0} c_\mu(N)\, q^{\frac{N+r^2}{4m}}\, \zeta^r$$

Thus knowing the (2m)-tuple $(h_\mu)_{\mu(\text{mod } 2m)}$ of functions of one variable is equivalent to knowing ϕ. Reversing the above calculation, we see that given any functions h_μ as in (3) with $c_\mu(N)=0$ for $N \not\equiv -\mu^2 \pmod{4m}$, equation (5) defines a function ϕ (with Fourier coefficients as in (1)) which transforms like a Jacobi form with respect to $z \mapsto z+\lambda\tau+\mu$ $(\lambda,\mu \in \mathbb{Z})$ and satisfies the right conditions at infinity. In order for ϕ to be a Jacobi form, we still need a transformation law with respect to $SL_2(\mathbb{Z})$. Since the theta-series (4) have weight $\frac{1}{2}$ and index m, while ϕ has weight k and index m, we see from (5) that the h_μ must be modular forms of weight $k-\frac{1}{2}$. To specify their precise transformation law, it suffices to consider the generators $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ of Γ_1. For the first we have

(6) $$\vartheta_{m,\mu}(\tau+1,z) = e_{4m}(\mu^2)\ \vartheta_{m,\mu}(\tau,z)$$

and

(7) $$h_\mu(\tau+1) = e_{4m}(-\mu^2)\ h_\mu(\tau)\ ,$$

as one sees either from the invariance of the sum (5) under $\tau \longmapsto \tau+1$ or from the congruence $N \equiv -\mu^2 \pmod{4m}$ in (3). For the second we have as an easy consequence of the Poisson summation formula the identity

(8) $$\vartheta_{m,\mu}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \sqrt{\tau/2mi}\ e^{2\pi i m z^2/\tau} \sum_{\nu \pmod{2m}} e_{2m}(-\mu\nu)\ \vartheta_{m,\nu}(\tau,z)\ ,$$

so (5) and the transformation law of ϕ under $(\tau,z) \mapsto \left(-\frac{1}{\tau},\frac{z}{\tau}\right)$ give

(9) $$h_\mu\left(-\frac{1}{\tau}\right) = \frac{\tau^k}{\sqrt{2m\tau/i}} \sum_{\nu \pmod{2m}} e_{2m}(\mu\nu)\ h_\nu(\tau)\ .$$

We have proved

<u>Theorem 5.1.</u>: <u>Equation</u> (5) <u>gives an isomorphism between</u> $J_{k,m}$ <u>and the space of vector valued modular forms</u> $(h_\mu)_{\mu(\text{mod } 2m)}$ <u>on</u> $SL_2(\mathbb{Z})$ <u>satisfying the transformation laws (7) and (9) and bounded as</u> $\text{Im}(\tau) \to \infty$.

When we speak of "vector-valued" forms in Theorem 5.1, we mean that the vector $\vec{h}(\tau) = (h_\mu)_{\mu (\mathrm{mod}\ 2m)}$ satisfies

$$\vec{h}(M\tau) = (c\tau+d)^{k-\frac{1}{2}}\, U(M)\, \vec{h}(\tau) \qquad \left(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1\right) \tag{10}$$

where $U(M) = \bigl(U_{\mu\nu}(M)\bigr)$ is a certain $2m \times 2m$ matrix (the map $U: \Gamma_1 \to GL_{2m}(\mathbb{C})$ is not quite a homomorphism because of the ambiguities arising from the choice of square-root in (10); to get a homomorphism one must replace Γ_1 by a double cover). The result 5.1 would be more pleasing if we could identify $J_{k,m}$ with a space of ordinary (i.e. scalar) modular forms of weight $k-\frac{1}{2}$ on some congruence subgroup of Γ_1. We will do this below in the cases $m=1$ and m prime, k even, and also discuss the general case a little. First, however, we look at some immediate consequences of Theorem 5.1.

First of all, by combining (5) with the equations $\vartheta_{m,-\mu}(\tau,z) = \vartheta_{m,\mu}(\tau,$ and $\phi(\tau,-z) = (-1)^k \phi(\tau,z)$ we deduce the symmetry property

$$h_{-\mu} = (-1)^k\, h_\mu \qquad (\mu \in \mathbb{Z}/2m\mathbb{Z}) \tag{11}$$

(this can also be proved by applying (9) twice), so that in fact (h_μ) reduces to an $(m+1)$-tuple of forms $\bigl(h_\mu + h_{-\mu}\bigr)_{0 \le \mu \le m}$ if k is even and to an $(m-1)$-tuple $\bigl(h_\mu - h_{-\mu}\bigr)_{0<\mu<m}$ if k is odd. However, we can introduce a finer splitting if m is composite. For each divisor m' of m with $(m',m/m')=1$ (there are 2^t such divisors, where t is the number of distinct prime factors of m) choose an integer $\xi = \xi_{m'}$ satisfying

$$\xi \equiv 1 \ (\mathrm{mod}\ 2m/m') \quad , \quad \xi \equiv -1 \ (\mathrm{mod}\ 2m') \ ; \tag{12}$$

such a ξ clearly exists and is unique (mod $2m$), and the set of $\xi_{m'}$ for all $m' \| m$ is precisely $\{\xi \ (\mathrm{mod}\ 2m) \mid \xi^2 \equiv 1 \ (\mathrm{mod}\ 4m)\}$. Now map the collection of $(2m)$-tuples $(h_\mu)_\mu$ into itself by the per-

(13) $\qquad (h_\mu)_{\mu\,(\mathrm{mod}\ 2m)} \longmapsto (h_{\xi\mu})_{\mu\,(\mathrm{mod}\ 2m)}$.

Because $\xi^2 \equiv 1\,(\mathrm{mod}\ 4m)$, if is clear that equations (7) and (9) are preserved. Hence we deduce

Theorem 5.2. For each divisor m' of m with $(m', m/m')=1$ there is an operator $W_{m'}$ from $J_{k,m}$ to itself such that the coefficient of $q^n \zeta^r$ in $\phi|W_{m'}$ is $c(n',r')$ where $r' \equiv -r\,(\mathrm{mod}\ 2m')$, $r' \equiv r\,(\mathrm{mod}\ 2m/m')$, $4n'm - r'^2 = 4nm - r^2$. These operators are all involutions and together form a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^t$ and generated by the $W_{p_i^{\nu_i}}$ $(m = \prod_{i=1}^{t} p_i^{\nu_i})$.

Next, we relate the expansion (5) to the Petersson product introduced in §2.

Theorem 5.3. Let

$$\phi = \sum_\mu h_\mu \vartheta_{m,\mu} \quad , \quad \psi = \sum_\mu g_\mu \vartheta_{m,\mu}$$

be two Jacobi forms in $J_{k,m}$. Then

$$(\phi,\psi) = \frac{1}{\sqrt{2m}} \int_{\Gamma_1\backslash \mathcal{H}} \sum_{\mu\,(\mathrm{mod}\ 2m)} h_\mu(\tau)\,\overline{g_\mu(\tau)}\; v^{k-\frac{5}{2}}\, du\, dv \ .$$

In other words, the Petersson scalar product of ϕ and ψ as defined in §2 is equal (up to a constant) to the Petersson product in the usual sense of the vector-valued modular forms $(h_\mu)_\mu$, $(g_\mu)_\mu$ of weight $k - \frac{1}{2}$. The assertions of Theorem 2.5 (that (ϕ,ψ) is well-defined and is finite if ϕ or ψ is cuspidal) now follow from the corresponding statements for modular forms in one variable.

Proof: We first compute the scalar product of $\vartheta_{m,\mu}$ and $\vartheta_{m,\nu}$ in a fixed fiber ($\tau \in \mathcal{H}$ fixed):

$$\int_{\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}} \vartheta_{m,\mu}(\tau,z)\ \overline{\vartheta_{m,\nu}(\tau,z)}\ e^{-4\pi m y^2/v}\ dx\, dy$$

$$= \int_{\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}} \sum_{\substack{r\equiv\mu(2m)\\ s\equiv\nu(2m)}} e(rz-s\bar{z})\ e((r^2\tau-s^2\bar{\tau})/4m)\ dx\, dy\ .$$

Using

$$\int_{\mathbb{R}/\mathbb{Z}} e(rz-s\bar{z})\ dx = \delta_{rs}\ e^{-4\pi r y}$$

we find that this equals

$$\delta_{\mu\nu} \int_{\mathbb{R}/v\mathbb{Z}} \sum_{r\equiv\mu\,(\mathrm{mod}\ 2m)} e^{-\frac{4\pi m}{v}\left(y+\frac{rv}{2m}\right)^2}\ dy$$

$$= \delta_{\mu\nu} \int_{-\infty}^{\infty} e^{-4\pi m y^2/v}\ dy$$

$$= \sqrt{v/4m}\ \delta_{\mu\nu}$$

(here δ_{rs} is the Kronecker delta of r, s and $\delta_{\mu\nu}$ of μ and ν modulo $2m$). It immediately follows that

$$(\phi,\psi) = \frac{1}{\sqrt{4m}} \int_{\Gamma_1\backslash H} \sum_{\mu\,(\mathrm{mod}\ 2m)} h_\mu(\tau)\ \overline{g_\mu(\tau)}\ v^{k-\frac{1}{2}}\ \frac{du\, dv}{v^2}$$

as claimed.

Since $W_{m'}$ simply permutes the h_μ, it follows from Theorem 5.3 that $W_{m'}$ is Hermitian. From Theorem 4.5 it is clear that the $W_{m'}$ commute with all T_ℓ $((\ell,m)=1)$. Hence we deduce

<u>Corollary:</u> $J_{k,m}$ <u>has a basis of simultaneous eigenforms for all</u> T_ℓ $((\ell,m)=1)$ <u>and</u> $W_{m'}$ $(m'\,\|\,m)$.

Theorem 5.2 gives a splitting of $J_{k,m}$ as $\bigoplus_{\pm\ldots\pm} J_{k,m}^{\pm\ldots\pm}$, where the sum is over all t-tuples of signs with product $(-1)^k$; Theorem 5.3 shows that this splitting is orthogonal and that each summand has a basis consisting of Hecke eigenforms.

We now discuss the connection between Jacobi forms and scalar-valued modular forms of weight $k-\frac{1}{2}$. We recall that modular forms of half-integral weight are defined like forms of integral weight, except that the automorphy factor describing the action of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ involvs the Legendre symbol $(\frac{c}{d})$; the easiest way to specify the automorphy factor exactly is to say that for a modular form $h(\tau)$ on $\Gamma_0(4m)$ the quotient h/θ^{2k-1}, where $\theta(\tau) = \sum q^{n^2}$, is invariant under $\Gamma_0(4m)$. We denote the space of such forms by $M_{k-\frac{1}{2}}(\Gamma_0(4m))$. Shimura developed an extensive theory of such forms in [29], [30]. In particular, he showed that one can define Hecke operators T_p on $M_{k-\frac{1}{2}}(\Gamma_0(4m))$ for all primes $p \nmid 4m$, that $M_{k-\frac{1}{2}}(\Gamma_0(4m))$ is spanned by simultaneous eigenforms of these operators, and that the set of eigenvalues of an eigenform is the same as the set of eigenvalues of a certain Hecke eigenform of weight $2k-2$. (Shimura used the notation $T(p^2)$ for the Hecke operators in half-integral weight because they are defined using matrices of determinant p^2, but we prefer to write T_p since these are the only naturally definable operators and correspond to the operators T_p in weight $2k-2$.) His conjecture that the eigenforms of integral weight obtained in this way have level $2m$ was proved by Niwa [24]. For the case $m=1$ (and later for the case of odd, square-free m [12]), Kohnen [11] showed how one could get all the way down to level m by passing to the subspace

$$M^+_{k-\frac{1}{2}}(4m) = \{ h \in M_{k-\frac{1}{2}}(\Gamma_0(4m)) \mid h = \sum_{\substack{N=0 \\ (-1)^{k-1}N \equiv 0,\,1 \ (\mathrm{mod}\ 4)}} c(N)\, q^N \}$$

of forms in $M_{k-\frac{1}{2}}(\Gamma_0(4m))$ whose N-th Fourier coefficient vanishes for all N with $(-1)^{k-1}N$ congruent to 2 or 3 (mod 4). Following Kohnen's notation in [12], we shall write simply $M_{k-\frac{1}{2}}(m)$ for $M^+_{k-\frac{1}{2}}(4m)$ and $M_{k-\frac{1}{2}}$ instead of $M_{k-\frac{1}{2}}(1)$. Then Kohnen's main result for $m=1$

says that one can define commuting and hermitian Hecke operators T_p on $M_{k-\frac{1}{2}}$ for all p (agreeing with Shimura's operators if $p \neq 2$) and that $M_{k-\frac{1}{2}}$ then becomes isomorphic to M_{2k-2} as a module over the ring of Hecke operators, i.e. there is a 1-1 correspondence between eigenforms $h \in M_{k-\frac{1}{2}}$ and $\tilde{h} \in M_{2k-2}$ such that the eigenvalues of h and $\tilde{h}$ under T_p agree for all p. Explicitly, $T_p\colon M_{k-\frac{1}{2}} \longrightarrow M_{k-\frac{1}{2}}$ (k even) is given by

$$(14)\quad T_p\colon \sum_{\substack{N\geq 0\\ N\equiv 0 \text{ or } 3 \pmod 4}} c(N)\,q^N \longmapsto \sum_{\substack{N\geq 0\\ N\equiv 0 \text{ or } 3 \pmod 4}} \Big(c(Np^2)+\big(\tfrac{-N}{p}\big)p^{k-2}c(N)+p^{2k-3}c\big(\tfrac{N}{p^2}\big)\Big)\,q^N .$$

Observe also that $M_{*-\frac{1}{2}} := \bigoplus_k M_{k-\frac{1}{2}}$ is a module over M_* by $h(\tau) \longmapsto f(4\tau)\,h(\tau)$ $(h \in M_{*-\frac{1}{2}},\ f \in M_*)$.

We can now state:

<u>Theorem 5.4</u>. <u>The correspondence</u>

$$(15)\qquad \sum_{\substack{N\geq 0\\ N\equiv 0,\,3 \pmod 4}} c(N)\,q^N \longmapsto \sum_{\substack{n,\,r\in\mathbb{Z}\\ 4n\geq r^2}} c(4n-r^2)\,q^N\,\zeta^r$$

<u>gives an isomorphism between</u> $M_{k-\frac{1}{2}}$ <u>and</u> $J_{k,1}$ (k <u>even</u>). <u>This isomorphism is compatible with the Petersson scalar products, with the actions of Hecke operators, and with the structures of</u> $M_{2*-\frac{1}{2}}$ <u>and</u> $J_{*,1}$ <u>as bundles over</u> M_*.

<u>Proof</u>: Denote the functions defined by the two Fourier series in (15) by $h(\tau)$ and $\phi(\tau,z)$ and set $h_\mu(\tau) = \sum c_\mu(N)\,q^N$ with

$$c_\mu(N) = \begin{cases} c(N) & N \equiv -\mu^2 \pmod 4 \\ 0 & \text{otherwise} \end{cases},$$

so that $c(N) = c_0(N) + c_1(N)$ $(\forall N)$ and $h(\tau) = h_0(4\tau) + h_1(4\tau)$. By Theorem 5.1, ϕ is a Jacobi form if and only if h_0 and h_1 satisfy the transformation laws (7) and (8), which now become

$$(16)\quad \begin{aligned} h_0(\tau+1) &= h_0(\tau)\ , \quad h_0(-\tfrac{1}{\tau}) = \tfrac{1+i}{2}\,\tau^{k-\frac12}(h_0(\tau)+h_1(\tau)) \ , \\ h_1(\tau+1) &= -ih_1(\tau)\ , \quad h_1(-\tfrac{1}{\tau}) = \tfrac{1+i}{2}\,\tau^{k-\frac12}(h_0(\tau)-h_1(\tau)) \ . \end{aligned}$$

These easily imply

$$h(\tau+1) = h(\tau)\ , \quad h(\tfrac{\tau}{4\tau+1}) = (4\tau+1)^{k-\frac12}h(\tau)$$

(for the three-line calculation, see [40], p.385), and since the matrices $\binom{1\ 1}{0\ 1}$ and $\binom{1\ 0}{4\ 1}$ generate $\Gamma_0(4)$, it follows that $h \in M_{k-\frac12}(\Gamma_0(4))$ and hence $h \in M_{k-\frac12}$. Conversely, if $h \in M_{k-\frac12}$, then reversing the same calculation shows that h_0 and h_1 satisfy (16) and hence by Theorem 5.1 that $\phi(\tau,z)$ is in $J_{k,1}$. This establishes the isomorphism claimed. To see that it preserves the Petersson scalar product (up to a constant), we combine Theorem 5.3 with the fact that $(h,h) = \text{const}[(h_0,h_0) + (h_1,h_1)]$ for h and h_μ related as above. This fact follows easily from Rankin's method, which expresses (h,h) as a multiple of $\text{Res}_{s=k-\frac12}(\sum_{N>0} |c(N)|^2 N^{-s})$ and similarly for h_0 and h_1 (for more explicit formulas, look at the proof of Corollary 5 in [13], pp.189-191, and take residues at $s=1$ in the identities proved there). The compatibility with Hecke operators is clear from (14) and Theorem 4.5. The compatibility with the structures of $M_{2*-\frac12}$ and $J_{*,1}$ as modules over M_* is also clear.

Theorem 5.4 tells us that the spaces $J_{*,1}$ and $M_{2*-\frac12}$ are in some sense identical: they are related by a canonical isomorphism preserving their Hilbert space structures, their structures as modules over the Hecke algebra and their structures as modules over the ring of modular forms of integral weight, and such that the Fourier coefficients of corresponding forms are the same (up to permutation). Combining this with results on Jacobi forms proved earlier, we can obtain various (previously known) results about forms of half-integral weight, in particular the following two:

Corollary 1. Let $k \geq 4$ be even and $H(k-1,N) = L_{-N}(2-k)$ as in Theorem 2.1.

Then the function

$$\underline{H}_{k-1}(\tau) = \sum_{N=0}^{\infty} H(k-1,N)\, q^N$$

lies in $M_{k-\frac{1}{2}}$.

This was proved by H. Cohen for both odd and even k in [6]; it was Cohen's discovery of the existence of Eisenstein series half of whose Fourier coefficients vanish which led to Kohnen's definition of the space $M^{+}_{k-\frac{1}{2}}(4)$. Cohen's calculation of the coefficients of the two Eisenstein series in $M_{k-\frac{1}{2}}(\Gamma_0(4))$ is of the same order of complexity as the calculation leading to Theorem 2.1, but our proof makes it clearer where the condition $-N \equiv \square \pmod 4$ comes from. On the other hand, it does not apply at all to the case of odd k.

Corollary 2. $M_{2*-\frac{1}{2}}$ is a free module of rank 2 over M_*, with generators $\underline{H}_3$ and $\underline{H}_5$.

This was proved by Kohnen [11, Prop.1], who also gave the corresponding result for $M_{2*+\frac{1}{2}}$ (it is also free, with generators θ and $\underline{H}_2$); our proof, which is a restatement of Theorem 3.5, works only for even k.

Conversely, by combining Theorem 5.4 with Kohnen's results as quoted above, we obtain

Corollary 3. If $\phi \in J_{k,1}$ is an eigenvalue of all Hecke operators T_ℓ, then there is a Hecke eigenform in M_{2k-2} with the same eigenvalues. The correspondence

$$\{\text{eigenforms in } J_{k,1}\}/\text{scalars} \longleftrightarrow \{\text{normalized eigenforms in } M_{2k-2}\}$$

is bijective. More precisely, for fixed n_0 and r_0 with $4n_0 \geq r_0^2$, there is a map $S_{n_0,r_0} : J_{k-1} \to M_{2k-2}$ given by

$$\sum c(n,r)\, q^n \zeta^r \longmapsto \sum_{\ell=0} \left(\sum_{\substack{d \mid \ell^2 \\ d^2 \mid \ell^2(4n_0-r_0^2)}} \varepsilon(d)\, d^{k-2}\, c\!\left(\frac{n_0\ell^2}{d^2}, \frac{r_0\ell}{d}\right) \right) q^\ell ,$$

<u>all of these maps are compatible with Hecke operators, and some combination of them is an isomorphism.</u>

The second statement follows from the explicit description of the correspondence $M_{k-\frac{1}{2}} \to M_{2k-2}$ and from the corresponding result for half-integral weight, namely that the map

$$S_N : \sum_{n\geq 0} c(n)\, q^n \to \frac{H(k-1,N)}{2} c(0) + \sum_{\ell\geq 1} \Bigg(\sum_{\substack{d \mid \ell^2 \\ d^2 \mid \ell^2 N}} \varepsilon_{-N}(d)\, d^{k-2}\, c\Big(\frac{\ell^2 N}{d^2}\Big)\Bigg)\, q^{\ell}$$

sends $M_{k-\frac{1}{2}}$ to M_{2k-2} for all $N>0$ and that some linear combination of the S_N is an isomorphism ([11], Theorem 1, iii; Kohnen states the result only for $-N$ a fundamental discriminant). However, it also follows from the first statement of the Corollary and from Theorem 4.5, by noting that the map S_{n_0,r_0} is simply

$$\phi \longmapsto \sum_{\ell=0}^{\infty} (\text{coefficient of } q^{n_0}\zeta^{r_0} \text{ in } \phi|T_\ell)\, q^{\ell} \quad .$$

Finally, by combining Theorem 5.4 with the main theorem of [13], which is a refinement for modular forms of level one of a theorem of Waldspurger [36,37] one gets:

<u>Corollary 4.</u> <u>Let</u> $\phi \in J_{k,1}^{cusp}$ <u>be an eigenform of all</u> T_ℓ <u>and</u> $f \in S_{2k-2}$ <u>the corresponding normalized eigenform as in Corollary 3.</u> <u>Then for all</u> n, r <u>with</u> $r^2 < 4n$

$$|c(n,r)|^2 = \frac{(k-2)!}{\pi^{k-1}} (4n-r^2)^{k-\frac{3}{2}} \frac{(\phi,\phi)}{(f,f)} L(f,\varepsilon_{r^2-4n},k-1) \quad ,$$

<u>where</u> $L(f,\varepsilon_D,s)$ <u>denotes the twisted L-series</u> $\sum \varepsilon_D(n)\, a(n)\, n^{-s}$ $(f = \sum a(n)\, q^n)$.

We mention one more result about the correspondences $J_{k,1} \to M_{k-\frac{1}{2}}$.

<u>Theorem 5.5.</u> <u>Let</u> $\phi \in J_{k,1}$ <u>and</u> $h \in M_{k-\frac{1}{2}}$ <u>be forms corresponding to one</u>

another as in Theorem 5.4 and $\nu \geq 0$ an integer. Then

$$D_\nu(\phi) = F_\nu(\theta,h)|U_4 ,$$

where D_ν is the Taylor development operator of §3, F_ν is Cohen's operator as defined in the Corollary to Theorem 3.3, $\theta(\tau) = \sum q^{n^2}$, and U_4 is the operator $\sum c(n)\,q^n \longmapsto \sum c(4n)\,q^n$.

This can be checked easily by direct computation. The special case $\phi = E_{k,1}$, $h = \underline{H}_{k-1}$ gives Cohen's function $C_k^{(\nu)}$ as in the Corollary to Theorem 3.1. (Indeed, Cohen proved that $C_k^{(\nu)}$ belongs to $M_{k+2\nu}$ by defining it as $F_\nu(\theta,\underline{H}_{k-1})\,|\,U_4$.)

We now turn to the case of general m , where, however, we will not be able to give such precise results as for $m=1$. By Theorem 5.2 we have a splitting

$$(17) \qquad J_{k,m} = \bigoplus_{\varepsilon} J^{\varepsilon}_{k,m} ,$$

where ε runs over all characters of the group

$$\Xi = \{\ \xi \pmod{2m}\ |\ \xi^2 \equiv 1 \pmod{4m}\ \} \cong (\mathbb{Z}/2\mathbb{Z})^t$$

with $\varepsilon(-1)=(-1)^k$ and $J^{\varepsilon}_{k,m}$ (which was denoted $J^{\pm,\dots,\pm}_{k,m}$ in the remark following the Corollary to Theorem 5.3) is the corresponding eigenspace, i.e. if $(h_\mu)_{\mu \pmod{2m}}$ are the modular forms of weight $k-\frac{1}{2}$ associated to $\phi \in J^{\varepsilon}_{k,m}$ then $h_{\xi\mu} = \varepsilon(\xi)\,h_\mu$ for all $\xi \in \Xi$. In particular, the map

$$(18) \qquad \phi(\tau,z) \mapsto h(\tau) = \sum_{\mu \pmod{2m}} h_\mu(4m\tau)$$

(which generalizes the map $\phi \to h_0(4\tau)+h_1(4\tau)$ used in the proof of Theorem 5.4) annihilates all $J^{\varepsilon}_{k,m}$ with $\varepsilon \neq 1$. For this map we can prove the analogue of Theorem 5.4:

Theorem 5.6. The function $h(\tau)$ defined by (18) lies in $M_{k-\frac{1}{2}}(m)$. If m is prime and k even, then the map (18) defines an isomorphism between $J_{k,m}$ and the space $M^{+}_{k-\frac{1}{2}}(m)$ of modular forms in $M_{k-\frac{1}{2}}(m)$ whose N th Fourier coefficient is zero for all N with $(\frac{-N}{m}) = -1$.

More generally, the image of (18) is contained in the subspace $M^{+,\dots,+}_{k-\frac{1}{2}}(m)$ consisting of all modular forms of weight $k-\frac{1}{2}$ on $\Gamma_0(4m)$ whose N^{th} Fourier coefficient vanishes unless $-N$ is a square modulo $4m$, and for m squarefree the map (18) gives an isomorphism between $J^1_{k,m}$ and $M^{+,\dots,+}_{k-\frac{1}{2}}(m)$ (note that $J_{k,m} = J^1_{k,m}$ for m prime and k even). The spaces $M^{+,\dots,+}_{k-\frac{1}{2}}(m)$, and more generably certain spaces $M^{\pm,\dots,\pm}_{k-\frac{1}{2}}(m) \subsetneq M_{k-\frac{1}{2}}(m)$ defined as eigenspaces of appropriate operators, were studied (for m odd and squarefree) by Kohnen [12], who showed that $M_{k-\frac{1}{2}}(m)$ decomposes in a Hecke-invariant way into the sum of these spaces. It is tempting to assume that this decomposition corresponds to the splitting (17) of $J_{k,m}$, but this is not the case. We discuss this in more detail below after proving Theorem 5.6.

Proof (Sketch): It follows from equation (9) that $\tau^{-k+\frac{1}{2}}\, h(-\frac{1}{4m\tau})$ is a multiple of $h_0(4m\tau)$ and hence invariant under $\tau \mapsto \tau+1$; this and the invariance of h itself under $\tau \mapsto \tau+1$ suffice to show that $h \in M_{k-\frac{1}{2}}(\Gamma_0(4m))$. We have $h = \sum c(N)\, q^N$ with $c(N) = \sum_{\mu} c_{\mu}(N)$; since $c_{\mu}(N)$ is non-zero only for $\mu^2 \equiv -N \pmod{4m}$, this shows that h lies in $M^{+,\dots,+}_{k-\frac{1}{2}}$. The map (18) is clearly zero on all $J^{\varepsilon}_{k,m}$ with $\varepsilon \neq 1$, but for m squarefree it is injective on $J^1_{k,m}$. Indeed, for $\phi \in J^1_{k,m}$ we have $c_{\xi\mu}(N) = c_{\mu}(N)$ for all $\xi \in \Xi$, and for m squarefree this implies that $c_{\mu}(N)$ is independent of the choice of the square-root μ of $-N \pmod{4m}$; hence $c(N) = 2^t c_{\mu}(N)$ and therefore $h=0 \Rightarrow \phi=0$. Finally, the surjectivity claimed in the theorem results from the formula

$$(19) \qquad \dim J_{k,m} = \dim M_{2k-2}(\Gamma^*(m)) \qquad (\text{ m prime, k even })$$

which will be proved in §§9-10 and the isomorphism $M^{+}_{k-\frac{1}{2}}(m) \cong M_{2k-2}(\Gamma^*(m))$ proved by Kohnen [12] , where $\Gamma^*(m)$ denotes the normalizer of $\Gamma_0(m)$ in $SL_2(\mathbb{R})$. (Actually, (19) will be proved in §§9-10 only for k sufficiently large, but the inequality with = replaced by ≥ will be proved for all k, and that is sufficient for the application here; it then actually follows from Theorem 5.6 and Kohnen's work that one has equality in (19) for all k.) This completes the proof.

Theorem 5.6 and the remarks following it describe the relation between $J^1_{k,m}$ and forms of half-integral weight. As we said, the situation for the other eigenspaces $J^\varepsilon_{k,m}$ is more complicated; the authors are indebted to N. Skoruppa for the following remarks which clarify it somewhat.

Suppose that m is odd and square-free and let $\varepsilon\colon \Xi \to \{\pm 1\}$ be a character as above. Write $\varepsilon = \prod_{p|m} \varepsilon_p$ with ε_p a character on $\{\xi \pmod p \mid \xi^2 \equiv 1 \pmod p\} \cong \mathbb{Z}/2\mathbb{Z}$ and let f be the product of those p for which ε_p is non-trivial. We extend ε to a character $\tilde{\varepsilon} \pmod f$ by taking $\tilde{\varepsilon}$ to be the product of arbitrary odd characters $\tilde{\varepsilon}_p \pmod p$ for $p|f$ and define

$$h_{\tilde{\varepsilon}}(\tau) = \sum_{\mu \pmod{2m}} \tilde{\varepsilon}(\mu)\, h_\mu(4m\tau)$$

(thus $h_{\tilde{\varepsilon}}$ is our old h for $\varepsilon = 1$). Then

i) $h_{\tilde{\varepsilon}}$ lies in $M_{k-\frac{1}{2}}(\Gamma_0(4mf), \chi)$, where $\chi = \left(\frac{-4}{\cdot}\right)^k \tilde{\varepsilon}$;

ii) the map $\phi \mapsto h_{\tilde{\varepsilon}}$ is 0 on all $J^{\varepsilon'}_{k,m}$ for $\varepsilon' \neq \varepsilon$ and is injective on $J^\varepsilon_{k,m}$;

iii) the image of this map is the set of h in $M_{k-\frac{1}{2}}(\Gamma_0(4mf),\chi)$ such that the representation of $\widetilde{SL_2(\mathbb{Z})}$ (= double cover of $SL_2(\mathbb{Z})$) on $\sum_{\gamma \in SL_2(\mathbb{Z})} \mathbb{C}\cdot h|\gamma$ is isomorphic to C_ε (or to 0), where C_ε is a certainly explicitly known irreducible representation of $SL_2(\mathbb{Z})/\Gamma(4m)$. This set can be characterized in terms of Fourier expansions and its dimension (resp. traces of Hecke operators) explicitly computed; it is contained in but in general not equal to the space of modular forms in $M_{k-\frac{1}{2}}(\Gamma_0(4mf),\chi)$ with Fourier expansions of the form $\sum_{-N \equiv \square \pmod{4m}} c(n)\, q^N$.

Notice that there is a choice involved in extending ε to $\tilde{\varepsilon}$ but that the different functions $h_{\tilde{\varepsilon}}$ which are obtained in this way are twists of one another. Skoruppa also has corresponding results for m arbitrary (if m is not square-free, then one must restrict to the U-new part, i.e. the complement of the space $\bigoplus_{d>1} V_{k,m/d^2}|U_d$; if m is even, then we must take $\tilde{\varepsilon}_2$ to be $(\frac{-4}{\cdot})$ of conductor 4 rather than 2 and thus f to be twice as big as above; the level of $h_{\tilde{\varepsilon}}$ in general is 4 times the smallest common multiple of m and f^2) and has given a formula for $\dim J^{\varepsilon}_{k,m}$ for arbitrary m and ε [34].

Finally, the reader might want to see some numerical examples illustrating the correspondence between Jacobi forms and modular forms of half-integral weight. Examples of Jacobi forms of index 1 were given at the end of §3. According to Theorem 5.4, these should correspond to modular forms in $M_{k-\frac{1}{2}}$, so that the coefficients given in the Table in §3 should be the Fourier coefficients of the Eisenstein series in $M_{k-\frac{1}{2}}$ $(k=4,6,8)$ and cusp forms in $M_{k-\frac{1}{2}}$ $(k=10,12)$; one can check that this is indeed the case by comparing these coefficients with the table in Cohen [7]. Numerical examples for higher m illustrating Theorem 5.6 and the following discussion can be found at the end of §9.

§ 6. Fourier-Jacobi expansions of Siegel modular forms and the Saito-Kurokawa conjecture

We recall the definition of Siegel modular forms. The Siegel upper half-space of degree n is defined as the set H_n of complex symmetric $n\times n$ matrices Z with positive-definite imaginary part. The group

$$Sp_{2n}(\mathbb{R}) = \{M \in M_{2n}(\mathbb{R}) \mid MJ_{2n}M^t = J_{2n}\}, \quad J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

$$= \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A,B,C,D \in M_n(\mathbb{R}),\ AB^t=BA^t,\ CD^t=DC^t,\ AD^t-BC^t=I_n\}$$

acts on H_n by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\cdot Z = (AZ+B)(CZ+D)^{-1} ;$$

a Siegel modular form of degree n and weight k with respect to the full Siegel modular group $\Gamma_n = Sp_{2n}(\mathbb{Z})$ is a holomorphic function $F\colon H_n \longrightarrow \mathbb{C}$ satisfying

(1) $\qquad F(M\cdot Z) = \det(CZ+D)^k\, F(Z)$

for all $Z \in H_n$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. If $n>1$, such a function will automatically possess a Fourier development of the form

(2) $$F(Z) = \sum_{T\geq 0} A(T)\ e(\operatorname{tr} TZ) ,$$

where the summation is over positive semidefinite semi-integral (i.e. $2t_{ij}$, $t_{ii} \in \mathbb{Z}$) $n\times n$ matrices T ; if $n=1$, of course, this must be posed as an extra requirement.

If $n=2$ we can write Z as $\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau,\tau' \in H$, $z \in \mathbb{C}$, $\operatorname{Im}(z)^2 < \operatorname{Im}(\tau)\operatorname{Im}$ and we write $F(\tau,z,\tau')$ instead of $F(Z)$; similarly we have $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ with $n,r,m \in \mathbb{Z}$, $n,m\geq 0$, $r^2\leq 4nm$ and we write $A(n,r,m)$ for $A(T)$, so the Fourier development of F becomes

$$(3) \qquad F(\tau,z,\tau') = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,m,4nm-r^2 \geq 0}} A(n,r,m)\, e(n\tau+rz+n\tau') .$$

The relation to Jacobi forms is given by the following result, which, as mentioned in the Introduction, is contained in Piatetski-Shapiro's work [26] .

Theorem 6.1. Let F be a Siegel modular form of weight k and degree 2 and write the Fourier development of F in the form

$$(4) \qquad F(\tau,z,\tau') = \sum_{m=0}^{\infty} \phi_m(\tau,z) \; e(m\tau') .$$

Then $\phi_m(\tau,z)$ is a Jacobi form of weight k and index m .

Proof: For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ and $(\lambda\ \mu) \in \mathbb{Z}^2$ the matrices

$$(5) \qquad \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

belong to Γ_2 and act on H_2 by

$$(\tau, z, \tau') \mapsto \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, \tau' - \frac{cz^2}{c\tau+d}\right), \quad (\tau, z, \tau') \mapsto (\tau, z+\lambda\tau+\mu, \tau'+2\lambda z+\lambda^2\tau) ,$$

respectively. Applying (1), we deduce the two transformation laws of Jacobi forms for the Fourier coefficients ϕ_m ; the condition at infinity follows directly from (3). Following Piatetski-Shapiro, we call (4) the Fourier-Jacobi expansion of the Siegel modular form F .

Note that the proof of Theorem 6.1 still applies if Γ_2 is replaced by a congruence subgroup. Note, too, that the first collection of matrices in (5) from a group (isomorphic to $SL_2\mathbb{Z}$)

but that the second do not; the group they generate has the form

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; \lambda,\mu,\kappa \in \mathbb{Z} \right\}$$

so that we again see the necessity of replacing $\mathbb{Z}^2$ by the Heisenberg group $H_{\mathbb{Z}}$. The complete embedding at $SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}$ into $Sp_4(\mathbb{R})$ is given by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda\ \mu), \kappa\right] \mapsto \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(\lambda'\ \mu') = (\lambda\ \mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. However, we shall make no further use of this.

The real interest of the relation between Jacobi and Siegel modular forms, and the way to the proof of the Saito-Kurokawa conjecture, is the following result, due essentially to Maass [21].

<u>Theorem 6.2</u>: <u>Let</u> ϕ <u>be a Jacobi form of weight</u> k <u>and index</u> 1. <u>Then the functions</u> $\phi|V_m$ $(m\geq 0)$ <u>defined in §4 are the Fourier-Jacobi coefficients of a Siegel modular form</u> $V\phi$ <u>of weight</u> k <u>and degree</u> 2.

<u>Proof</u>: Reversing the proof of Theorem 6.1, we see that the function defined by (4), where ϕ_m $(m\geq 0)$ are <u>any</u> Jacobi forms of weight k and index m, transforms like a Siegel modular form under the action of the matrices (5). In particular, this holds for the function

$$V\phi(\tau,z,\tau') := \sum_{m\geq 0} (\phi|V_m)(\tau,z)\ e(m\tau').$$

On the other hand, Th. 4.2 gives the formula

$$(6) \qquad A(n,r,m) = \sum_{d|(n,r,m)} d^{k-1}\ c\left(\frac{4nm-r^2}{d^2}\right) \qquad ((n,r,m) \neq (0,0,0))$$

for the Fourier coefficients (defined as in (3)) of $V\phi$, where

$\sum_{n\geq r^2/4m} c(n,r)\, q^n \zeta^r$ is the Fourier expansion of ϕ, and since this is symmetric in n and m we deduce that $V\phi$ is symmetric in τ and τ', i.e. transforms like a Siegel modular form with respect to the matrix

$$(7) \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

as well as the matrices (5). Since these matrices are known to generate Γ_2, it follows that $V\phi$ is indeed a Siegel modular form.

The same proof works for the standard congruence subgroups of Γ_2, since these are known to be generated by the matrices (5) (now with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the corresponding subgroup of Γ_1) and (7); nevertheless, it would be nice to have a "real" proof of Theorem 6.2, i.e. a direct verification of the transformation law of $V\phi$ with respect to all elements of Γ_2.

Theorems 6.1 and 6.2 give an injective map

$$FJ : M_k(\Gamma_2) \longrightarrow J_{k,0} \times J_{k,1} \times J_{k,2} \times \cdots$$

and a map

$$V : J_{k,1} \longrightarrow M_k(\Gamma_2)$$

such that the composite

$$J_{k,1} \xrightarrow{V} M_k(\Gamma_2) \xrightarrow{FJ} \prod_{m\geq 0} J_{k,m} \xrightarrow{pr} J_{k,1}$$

is the identity. Thus V is injective and its image is exactly the set of F with $F = V(FJ(F))$, i.e. of Siegel modular forms whose Fourier-Jacobi expansion (5) has the property $\phi_m = \phi_1|V_m$ $(\forall\, m)$. This means that the Fourier coefficients $A(n,r,m)$ (defined by (3)) are given by the formula (6) , where

$$(8) \qquad c(N) = \begin{cases} A(n,0,1) & \text{if } N = 4n \\ A(n,1,1) & \text{if } N = 4n-1 \end{cases}$$

(the last statement could be omitted, since the validity of (6) for any sequence of numbers $\{c(N)\}$ forces the $c(N)$ to be given by this formula). Equivalently, we can characterize these functions by the Fourier coefficient identity

$$(9) \qquad A(n,r,m) = \sum_{d|(n,r,m)} d^{k-1} A(\tfrac{nm}{d^2}, \tfrac{r}{d}, 1) \qquad (\forall\, n,r,m) \ .$$

The space $M_k^*(\Gamma_2)$ of functions satisfying (9) was studied by Maass ([20], [21]; see also [17]) after the existence of Siegel forms satisfying these identities had been discovered experimentally by Resnikoff and Saldaña [27] (whose Tables IV and V should be compared to the tables in §3); he called it the "Spezialschar." We thus have established inverse isomorphisms

$$M_k^*(\Gamma_2) \underset{V}{\overset{FJ_1}{\rightleftarrows}} J_{k,1} \ ;$$

combining these with the isomorphism $J_{k,1} \xrightarrow{\sim} M_{k-\frac{1}{2}}$ given in §5 we obtain

Corollary 1. If $h(\tau) = \sum_{\substack{N\geq 0 \\ N\equiv 0,3 (\mathrm{mod}\ 4)}} c(N)\, q^N$ is a modular form in Kohnen's "plus-space" $M_{k-\frac{1}{2}}$, then the numbers $A(n,r,m)$ defined by eq. (6) are the coefficients of a modular form F in Maass' "Spezialschar" $M_k^*(\Gamma_2)$. The map $h \mapsto F$ is an isomorphism from $M_{k-\frac{1}{2}}$ to $M_k^*(\Gamma_2)$, the inverse map being given in terms of Fourier coefficients by eq. (8).

From Theorem 3.5 (or Corollary 2 to Theorem 5.4) we also deduce:

Corollary 2. The "Spezialschar" $M_*^*(\Gamma_2) = \bigoplus_{k \text{ even}} M_k^*(\Gamma_2)$ is free over $M_*(SL_2(\mathbb{Z}))$ on two generators, of weights 4 and 6 .

This result was proved by Maass [21], [22].

Finally, we must check that the map V is compatible with the action of Hecke opeartors in $J_{n,1}$ and $M_k(\Gamma_2)$, i.e. that there is an algebra map $\iota\colon \mathbf{T}_S \to \mathbf{T}_J$ from the Hecke algebra for Siegel modular forms of weight k and degree 2 to the Hecke algebra for Jacobi forms of weight k and index 1 such that

$$(10) \qquad V(\phi)|T = V(\phi|\iota(T)) \qquad \forall\, T \in \mathbf{T}_S .$$

This will imply in particular that V maps Hecke eigenforms to Hecke eigenforms. It is well known (cf. Andrianov [1]) that $\mathbf{T}_S$ is generated by the operators $T_S(p)$ and $T_S(p^2)$ with p prime (until the end of the chapter we write $T_S(\ell)$ and $T_J(\ell)$ for the Hecke operators in $\mathbf{T}_S$ and $\mathbf{T}_J$). We use the somewhat more convenient generators $T_S(p)$ and $T_S'(p) = T_S(p)^2 - T_S(p^2)$.

Theorem 6.3. The map $V\colon J_{k,1} \to M_k(\Gamma_2)$ is Hecke-equivariant (in the sense of (10)) with respect to the homomorphism of Hecke algebras $\iota\colon \mathbf{T}_S \to \mathbf{T}_J$ defined on generators by

$$\iota(T_S(p)) = T_J(p) + p^{k-1} + p^{k-2} ,$$
$$\iota(T_S'(p)) = (p^{k-1}+p^{k-2})\, T_J(p) + 2p^{2k-3} + p^{2k-4} .$$

Proof: In [2], Andrianov proves that Maass' "Spezialschar" is invariant under the Hecke algebra by calculating the Fourier coefficients of the two functions

$$F_1 = F|T_S(p) \quad , \qquad F_2 = F|T_S'(p)$$

for $F \in M_k^*$ and checking that they satisfy the identity (9). Since the Spezialschar is just the image of V , this says that for $\phi \in J_{k,1}$:

$$V\phi|T_S(p) = V\phi_1 \quad , \quad V\phi|T_S'(p) = V\phi_2$$

for some ϕ_1 , $\phi_2 \in J_{k,1}$ with explicitly given coefficients, so to prove the theorem we need only check that

$$(11) \qquad \begin{aligned} \phi_1 &= \phi|(T_J(p) + p^{k-1} + p^{k-2}) \quad , \\ \phi_2 &= \phi|((p^{k-1} + p^{k-2})\, T_J(p) + 2p^{2k-3} + p^{2k-4}) \quad . \end{aligned}$$

The coefficients of a Siegel modular form in the Spezialschar are determined by a single function $c(N)$ $(N\equiv 0, 3 \pmod 4)$ as in (6), (8). If $c(N)$, $c_1(N)$ and $c_2(N)$ are the coefficients corresponding in this way to ϕ , ϕ_1 and ϕ_2 , then equations (13)-(16) of [2] (with $D=-N$, $t=1$, $d=0$) give

$$c_1(N) = c(p^2N) + p^{k-1}c(N) + p^{k-2}(1+(\tfrac{D}{p}))c(N)$$

if $p^2\nmid N$,

$$c_1(N) = c(p^2N) + p^{k-1}c(N) + p^{k-2}(c(N) + p^{k-1}c(N/p^2))$$

if $p^2|N$,

$$\begin{aligned} c_2(N) = {} & p^{2k-4}(1+(\tfrac{D}{p}))c(N) + p^{2k-3}c(N) + p^{k-2}(1+(\tfrac{D}{p}))(c(p^2N) + p^{k-1}c(N)) \\ & + p^{k-2}(p-(\tfrac{D}{p}))c(p^2N) \end{aligned}$$

if $p^2\nmid N$, and

$$\begin{aligned} c_2(N) = {} & p^{2k-4}c(N) + p^{2k-3}c(N) + p^{3k-5}c(N/p^2) \\ & + p^{k-2}(c(p^2N) + p^{k-1}c(N) + p^{2k-2}c(N/p^2)) + p^{k-1}c(p^2N) \end{aligned}$$

if $p^2|N$. In a unified notation, this can be written

$$\begin{aligned} c_1(N) &= c(p^2N) + (p^{k-1} + p^{k-2} + p^{k-2}(\tfrac{D}{p}))c(N) + p^{2k-3}c(N/p^2) \quad , \\ c_2(N) &= (p^{k-1} + p^{k-2})c(p^2N) + (2p^{2k-3} + p^{2k-4} + (p^{2k-3} + p^{2k-4})(\tfrac{D}{p}))c(N) \\ &\quad + (p^{3k-4} + p^{3k-5})c(N/p^2) \quad , \end{aligned}$$

with the usual convention $c(N/p^2) = 0$ if $p^2\nmid N$. On the other hand, from Theorem 4.5 with $m=1$, $\ell=p$ we find that the Nth coefficient for $\phi|T_J(p)$ is given by

$$c(p^2N) + p^{k-2}(\tfrac{D}{p})c(N) + p^{2k-3}c(N/p^2)$$

(cf. equation (14) of §5), and comparing this with the formulas for c_1 and c_2 gives the desired identities (11).

In [1], §1.3, Andrianov associates to any Hecke eigenform $F \in S_k(\Gamma_2)$ the Euler product

$$Z_F(s) = \prod_p (1-\gamma_p p^{-s} + (\gamma_p' - p^{2k-4})p^{-2s} - \gamma_p p^{2k-3-3s} + p^{4k-6-4s})^{-1}$$

where $F|T_S(p) = \gamma_p F$, $F|T_S'(p) = \gamma_p' F$. If $F = V\phi$ with $\phi|T_J(p) = \lambda_p\phi$, then it follows from Theorem 6.3 that

$$\gamma_p = \lambda_p + p^{k-1} + p^{k-2} , \quad \gamma_p' = (p^{k-1} + p^{k-2})\, p + 2p^{2k-3} + p^{2k-4}$$

and hence

$$1-\gamma_p t + (\gamma_p' - 2p^{2k-4})t^2 - \gamma_p p^{2k-3}t^3 + p^{4k-6}t^4$$
$$= (1-p^{k-1}t)(1-p^{k-2}t)(1-\lambda_p t + p^{2k-3}t^2) \quad .$$

By Corollary 3 of Theorem 5.4, there is a 1-1 correspondence between eigenforms in M_{2k-2} and $J_{k,1}$, the eigenvalues being the same. We deduce:

Corollary 1 (Saito-Kurokawa conjecture). The space $S_k^*(\Gamma_2)$ is spanned by Hecke eigenforms. These are in 1-1 correspondence with normalized Hecke eigenforms $f \in S_{2k-2}$, the correspondence being such that

(10) $\quad Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)\, L(f,s)$

As stated in the Introduction, most of the Saito-Kurokawa conjecture was proved by Maass [21,22,23]. In particular, he found the bijection between functions in the Spezialschar and pairs of functions (h_0,h_1) satisfying (16) of §5, showed that $\dim M_k^*(\Gamma_2) = \dim M_{2k-2}$, and established the existence of a lifting from the Spezialschar to M_{2k-2} satisfying (10), without, however, being able to show that it was an isomorphism. The statement that the Spezialschar is spanned by eigenforms was proved by Andrianov [2] in the paper cited in the proof of Theorem 6.3, and the bijectivity of Maass' lifting by one of the authors [40]. See also Kojima [14]. A detailed exposition of the proof of the conjecture (more or less along the same lines as the one given here) can be found in [40].

A further consequence of Theorem 6.3 is

<u>Corollary 2.</u> <u>The Fourier coefficients of the Eisenstein series</u>

$$E_k^{(2)}(Z) = \sum_{\{C,D\}} \det(CZ+D)^{-k}$$

(sum over non-associated pairs of coprime symmetric matrices $C, D \in M_2(\mathbb{Z})$) <u>in</u> $M_k(\Gamma_2)$ <u>are given by</u>

$$A\left(\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right) = \sum_{d|(n,r,m)} d^{k-1}\, H\left(k-1, \frac{4nm-r^2}{d^2}\right) .$$

Indeed, since $E_k^{(2)}$ is the unique eigenform of all Hecke operators with $A\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1$, it must equal $V(E_{k,1})$, and the result follows. The formula for the coefficients $A(T)$ of Siegel Eisenstein series of degree 2 was proved by Maass ([18], correction in [19]), but his proof involved much more work, especially in the case of non-primitive T.

§ 7. Jacobi theta series and a theorem of Waldspurger

Perhaps the most important modular forms for applications to arithmetic are the theta series

(1) $$\Theta_Q(\tau) = \sum_{x \in \Lambda} q^{Q(x)} ,$$

where Q is a positive-definite rational-valued quadratic form on a lattice Λ of finite rank. This series is a modular form of weight $\frac{1}{2}\,\mathrm{rk}(\Lambda)$ and some level; in the simplest case of unimodular Q (i.e. Q is integer-valued and can be written with respect to some basis of Λ as $\frac{1}{2}\,x^tAx$ with A a symmetric matrix with integer entries and determinant 1) it is a modular form on the full modular group. As is well-known, the theta-series (1) should really be considered as the restriction to $z=0$ ("Thetanullwert") of a function $\Theta(\tau,z)$ which satisfies a transformation law for $z \mapsto z+\lambda\tau+\mu$ and is, in fact, a Jacobi form. (This fact, which goes back to Jacobi, is the primary motivation for the definition of Jacobi forms.) We give a precise formulation in the case of unimodular Q.

Theorem 7.1. Let $Q(x)$ be a unimodular positive definite quadratic form on a lattice Λ of rank $2k$ and $B(x,y)$ the associated bilinear form with $Q(x) = \frac{1}{2}\,B(x,x)$. Then for fixed $y \in \Lambda$ the series

(2) $$\Theta_{Q,y}(\tau,z) := \sum_{x \in \Lambda} q^{Q(x)}\ \zeta^{B(x,y)}$$

is a Jacobi form of weight k and index $m = Q(y)$ on $SL_2(\mathbb{Z})$.

<u>Proof:</u> The transformation law with respect to $\begin{pmatrix}1&1\\0&1\end{pmatrix} \in SL_2(\mathbb{Z})$ is obvious and that for $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ an immediate consequence of the Poisson summation formula, so that the modular properties of $\theta_{Q,x}$ are particularly easy in this case. The transformation law with respect to $z \mapsto z+\lambda\tau+\mu$ is equally clear:

$$e^{m}(\lambda^2\tau+2\lambda z) \quad \theta_{Q,y}(\tau,z+\lambda\tau+\mu) = q^{\lambda^2 Q(y)} \; \zeta^{2\lambda Q(y)} \sum_{x\in\Lambda} q^{Q(x)} \; \zeta^{B(x,y)} \; q^{\lambda B}$$

$$= \sum_{x\in\Lambda} q^{Q(x+\lambda y)} \; \zeta^{B(x+\lambda y,y)} = \theta_{Q,y}(\tau,z) \; .$$

Finally, as pointed out in the introduction, the conditions at infinity are simply $Q(x)\geq 0$, $Q(y)\geq 0$, $B(x,y)^2 \leq 4Q(x)\,Q(y)$, which express the fact that the restriction of Q to the (possibly degenerate) sublattic $\mathbb{Z}x + \mathbb{Z}y \subset \Lambda$ is positive (semi-)definite.

If Q is not unimodular, then $\theta_{Q,x}$ will have a level and character which can be determined in a well-known manner.

We recall two generalizations of the theta-series (1) and explain their relation to the Jacobi theta-series (2) . First of all, one can insert a spherical polynomial $P(x)$ in front of the exponential in (1) (recall that "spherical" means $\Delta P=0$, where Δ is the standard Laplacian with respect to a basis of $\Lambda\otimes\mathbb{R}$ for which Q is $\sum x_i^2$); if P is homogeneous of degree 2ν, then the series

$$(3) \qquad \theta_{Q,P}(\tau) = \sum_{x\in\Lambda} P(x)\; q^{Q(x)}$$

is a modular form of weight $k+2\nu$ $(2k=\mathrm{rk}\ \Lambda)$ and the same level as θ_Q , and a cusp form if $\nu > 0$ (see for example Ogg [25]). We then have

Theorem 7.2. Let Λ, Q, B, y be as in Theorem 7.1, $\nu \in \mathbb{N}_0$. Then the polynomial

(4) $$P_{\nu,y}(x) = p_{2\nu}^{(k-1)}(B(x,y),\ m\,Q(x))$$

($p_{2\nu}^{(k-1)}$ as in Theorem 3.1) is a spherical polynomial of degree 2ν and

$$D_{2\nu}(\theta_{Q,y}) = \theta_{Q,P_{\nu,y}}$$

Proof. The final formula is clear from the definition of $D_{2\nu}$, and this makes it morally certain that $P_{\nu,y}$ is spherical with respect to Q, since (3) is never modular unless P is spherical. To check that this is really so, we use eq. (3) of §3 to get

$$P_{\nu,y}(x) = \text{const.} \times \text{coefficient of } t^{2\nu} \text{ in } Q(y-tx)^{-k+1}$$

(we may assume $y \neq 0$, so $m \neq 0$), from which the assertion follows easily (choose a basis so $Q = \sum x_i^2$ and compute $\left(\sum \frac{\partial^2}{\partial x_i^2}\right) Q(y-tx)^{-k+1}$, recalling that $k = \frac{1}{2}\,\mathrm{rk}\,\Lambda$).

The other generalization of (1) is the Siegel theta-series

(5) $$\theta_Q^{(2)}(Z) := \sum_{T \geq 0} r_Q(T)\ e(\mathrm{tr}\ TZ),$$

where $r_Q(T)$ is the number of representations of the binary quadratic form T by Q (just as the coefficient of q^n in (1) is the number of representations of n, or of the unary form nx^2, by Q). Explicitly, we have

$$r_Q\left(\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right) = \#\{(x,y) \in \Lambda \times \Lambda \mid Q(x)=n,\ Q(y)=m,\ B(x,y)=r\},$$

$$\theta_Q^{(2)}\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} = \sum_{x,y \in \Lambda} e(Q(x)\tau + B(x,y)z + Q(y)\tau').$$

Then the following is obvious.

Theorem 7.3. Let Q be as in Theorem 7.1. Then the m^{th} Fourier-Jacobi coefficient of $\theta_Q^{(2)}$ equals $\sum_{\substack{y\in\Lambda \\ Q(y)=m}} \theta_{Q,y}$.

Theorems 7.2 and 7.3 show how the Jacobi theta-series fit into the theory developed in §3 and §6, respectively. Their relation to §5 is also easily described: the modular form of weight $k-\frac{1}{2}$ associated to $\theta_{Q,y}$ is essentially the theta-series associated to the quadratic form of rank $2k-1$ obtained by restricting Q to the orthogonal complement of y. We illustrate this in the simplest example, with $k=4$ and $m=1$. Then (Λ,Q) is the E_8 lattice (cf. Serre [28,5,1.4.3]), i.e.

$$\Lambda = \{x\in \mathbb{Z}^8 \cup \mathbb{Z}^8+(\tfrac{1}{2},\ldots,\tfrac{1}{2}) \mid \sum_{i=1}^{8} x_i \in 2\mathbb{Z}\}\ ,\quad Q=\sum_{i=1}^{8} x_i^{\,2}\ .$$

The form θ_Q is E_4 (since $\dim M_4=1$); in particular, there are 240 choices of y with $Q(y)=1$, but since they are all equivalent under $\mathrm{Aut}(Q)$ we can choose $y=(\frac{1}{2},\ldots,\frac{1}{2})$. Then

$$\theta_{Q,y} = \sum_{x\in\Lambda} q^{\frac{1}{2}(x_1^2+\ldots+x_8^2)}\ \zeta^{\frac{1}{2}(x_1+\ldots+x_8)}\ ,$$

and since $J_{4,1}$ is one-dimensional this must equal $E_{4,1}$. Hence the formula for the coefficient of $q^n\zeta^r$ in $E_{k,1}$ (Th. 2.1) implies

$$\#\{x\in\Lambda \mid x_1^2+\ldots+x_8^2=2n\ ,\ x_1+\ldots+x_8=2r\} = \frac{H(3,4n-r^2)}{\zeta(-5)}$$

or more explicitly (replacing x_i by $\frac{1}{2}x_i$)

(6) $$\#\{x\in\mathbb{Z}^8 \mid x_1\equiv\ldots\equiv x_8 \pmod 2,\ \textstyle\sum x_i^2=8n,\ \sum x_i=4r\} = -252\ H(3,4n-r^2)$$

This first seems like an infinite family of identities, parametrized by $r\in\mathbb{Z}$, but in fact the identities depend only on $r \pmod 2$. Indeed, if r

is even then replacing x_i by $x_i + \frac{1}{2}r$ replaces the left-hand side of (6) by the same expression with n replaced by $n - \frac{1}{4}r^2$ and r by 0, so (6) for any even r is equivalent to the theta-series identity

$$(7) \qquad \sum_{\substack{x \in \mathbb{Z}^8,\, x_1+\ldots+x_8=0 \\ x_1 \equiv \ldots \equiv x_i \pmod 2}} q^{\frac{1}{8}(x_1^2+\ldots+x_8^2)} = 1 - 252 \sum_{n=1}^{\infty} H(3,4n)\, q^n$$

$$(= 1+126q + 756q^2 + 2072q^3 + \ldots) .$$

Similar remarks hold for r odd except that now replacing x_i by $x_i + \frac{1}{2}r$ replaces "$x \in \mathbb{Z}^8$" by "$x \in (\frac{1}{2},\ldots,\frac{1}{2})+\mathbb{Z}^8$"; doubling the x_i then turns this condition into "$x \in \mathbb{Z}^8$, all x_i odd, $x_1 \equiv \ldots \equiv x_8 \pmod 4$", so (6) for odd r is equivalent to

$$(8) \qquad \sum_{\substack{x \in \mathbb{Z}^8,\, x_1+\ldots+x_8=0 \\ x_i \text{ odd},\, x_1 \equiv \ldots \equiv x_8 \pmod 4}} q^{\frac{1}{8}(x_1^2+\ldots+x_8^2)} = -252 \sum_{n=1}^{\infty} H(3,4n-1)\, q^{4n-1}$$

$$(= 56q^3 + 576q^7 + 1512q^{11} + \ldots).$$

Thus the Jacobi form identity $\theta_{Q,y} = E_{4,1}$ is equivalent to the modular-form-of-half-integral-weight identity $\theta_{Q'} = H_3$, where Q' is the integer-valued quadratic form $\frac{1}{8}\sum x_i^2$ on the 7-dimensional lattice $\{x \in \mathbb{Z}^8 \mid \sum x_i = 0 ,\ x_1 \equiv \ldots \equiv x_8 \pmod 4\}$.

We end this section by combining various of the results of Chapters I and II to obtain a proof (related in content but different in presentation from the original one) of a beautiful theorem of Waldspurger's [35], generalizing Siegel's famous theorem on theta-series. Siegel's theorem [33], in the simplest case of forms of level 1, says

$$(9)\qquad \sum_{i=1}^{h} \frac{1}{w_i}\ \theta_{Q_i} = \Big(\sum_{i=1}^{h} \frac{1}{w_i}\Big)\ E_k \qquad (k\in 4\mathbb{Z}) .$$

Here (Λ_i, Q_i) $(1\le i\le h)$ denote the inequivalent unimodular positive-definite quadratic forms of rank $2k$ and w_i the number of automorphisms of Q_i . Explicitly, (9) says

$$(10)\qquad \sum_{i=1}^{h} \varepsilon_i\ r_{Q_i}(n) = \frac{2}{\zeta(1-k)}\ \sigma_{k-1}(n) \qquad (n>0)$$

$(\varepsilon_i := w_i^{-1}/(w_1^{-1}+\ldots+w_h^{-1}))$, i.e. it gives a formula for the average number (with appropriate weights) of representations of an integer n by the forms Q_i . The number of representations by a single form, however, remains mysterious. Waldspurger's result in this case is

$$(11)\qquad \sum_{i=1}^{h} \varepsilon_i\ r_{Q_i}(n)\ \theta_{Q_i} = \frac{2}{\zeta(1-k)}\ C_k|T_n \qquad (n\in\mathbb{N}_0)$$

where $C_k(\tau) = (H_{k-1}(\tau)\ \theta(\tau))|U_4$ is Cohen's function. Thus one has explicit evaluations of weighted linear combinations of theta-series with variable weights; in fact, since the $C_k|T_n$ are known to span M_k (cf. discussion after the Corollary to Theorem 3.1), one gets <u>all</u> modular forms in this way. Equation (9) follows from (11) by taking $n=0$ (with the convention $f|T_n = 2\zeta(1-k)^{-1}a(0)\ E_k$ for $f=\sum a(n)q^n\in M_k$) or by computing the constant term of both sides; taking the coefficient of q^m on both sides gives

$$(12)\qquad \sum_{i=1}^{h} \varepsilon_i\ r_{Q_i}(n)\ r_{Q_i}(m) = \frac{2}{\zeta(1-k)} \sum_{d|(n,m)} d^{k-1} \sum_{r^2\le 4nm/d^2} H(k-1,\frac{4nm}{d^2}-$$

Finally, Waldspurger has a generalization of (11) involving theta series with spherical polynomials.

To prove all of these results, we start with Siegel's own generalization of (9) to Siegel modular forms of degree 2:

$$(13) \qquad \sum_{i=1}^{h} \varepsilon_i \quad \theta_{Q_i}(Z) = E_k^{(2)}(Z) \quad .$$

Here $E_k^{(2)}$ is the Eisenstein series of degree 2 and weight k and θ_{Q_i} the theta-series (5). We then compute the m^{th} Fourier-Jacobi coefficient of both sides of (13); by Theorem 7.3 and (the proof of) Corollary 1 of Theorem 6.3, this gives

$$(14) \qquad \sum_{i=1}^{h} \varepsilon_i \sum_{\substack{y \in \Lambda_i \\ Q_i(y)=m}} \theta_{Q_i,y}(\tau,z) = (E_{k,1}|V_m)(\tau,z) \ ,$$

an identity of Jacobi forms of weight k and index m. We now apply the development map $D_{2\nu}$ of §3 to both sides of (14). By Theorem 7.2, the left-hand side of the resulting identity is

$$\sum_{i=1}^{h} \varepsilon_i \quad \theta_{Q_i,P_{i,m}^{\nu}}(\tau) \quad ,$$

where $P_{i,m}^{\nu}$ is given by

$$(15) \qquad P_{i,m}^{\nu}(x) := \sum_{\substack{y \in \Lambda_i \\ Q_i(y)=m}} p_{2\nu}^{(k-1)}(B_i(x,y),\ m\ Q_i(x))$$

and is a spherical polynomial of degree 2ν with respect to Q_i . By the Corollary to Theorem 4.2, the right-hand side is

$$D_{2\nu}(E_{k,1}) \mid_{k+2\nu} T_m \quad .$$

But $D_{2\nu}(E_{k,1})$ is Cohen's function $C_k^{(\nu)}$ as defined in §3 (Corollary to Theorem 3.1). Hence we have the following identity, of which (11) is the special case $\nu=0$.

Theorem 7.4. (Waldspurger [35]). Let Q_i (i=1,...,h) be the inequivalent unimodular quadratic forms of rank $2k$ ($k>0$, $k\equiv 0 \pmod 4$) and $P^{\nu}_{i,m}$ ($\nu\geq 0$) the polynomials (15), $p^{(k-1)}_{2\nu}$ as in (1) of §3. Then

$$\sum_{i=1}^{h} \frac{1}{|\mathrm{Aut}\ Q_i|} \ \theta_{Q_i, P^{\nu}_{i,m}} = \left(\sum_{i=1}^{h} \frac{1}{|\mathrm{Aut}\ Q_i|}\right) C^{(\nu)}_k | T_m$$

where $C^{(\nu)}_k$ is Cohen's function (as in the Corollary to Theorem 3.1) and T_m the m^{th} Hecke operator in $M_{k+2\nu}$.

Since, as mentioned in §3, the $C^{(\nu)}|T_m$ are known to span $S_{k+2\nu}$, one obtains

Corollary: For fixed k and ν, the functions $\theta_{Q_i,P}$ (i=1,...,h , P spherical of degree 2ν with respect to Q_i) span $S_{k+2\nu}$ (resp. M_k if $\nu=0$).

In particular, the theta-series associated to the E_8-lattice and to spherical polynomials of degree $k-4$ span S_k for every k .

Chapter III. The ring of Jacobi forms

§8. Basic structure theorems

The object of this and the following section is to obtain as much information as possible about the algebraic structure of the set of Jacobi forms, in particular about

i) the dimension of $J_{k,m}$ (k, m fixed), i.e. the structure of this space as a vector space over $\mathbb{C}$;

ii) the additive structure of $J_{*,m} = \bigoplus_k J_{k,m}$ (m fixed) as a module over the graded ring $M_* = \bigoplus_k M_k$ of ordinary modular forms;

iii) the multiplicative structure of the bigraded ring $J_{*,*} = \bigoplus_{k,m} J_{k,m}$ of all Jacobi forms.

We will study only the case of forms on the full Jacobi group Γ_1^J (and usually only the case of forms of even weight), but many of the considerations could be extended to arbitrary Γ.

The simplest properties of the space of Jacobi forms were already given in Chapter I. There we showed that $J_{k,m}$ is finite-dimensional for all k and m and zero if k or m is negative (Theorem 1.1 and its proof) and obtained the explicit dimension estimate

$$(1) \qquad \dim J_{k,m} \le \begin{cases} \dim M_k + \sum_{\nu=1}^{m} \dim S_{k+2\nu} & (k \text{ even }) , \\ \sum_{\nu=1}^{m-1} \dim S_{k+2\nu-1} & (k \text{ odd }) \end{cases}$$

(Theorem 3.4 and the following remarks). We also proved that $J_{*,m}$ reduces to M_* if $m = 0$ and is free over M_* on two generators $E_{4,1}$, $E_{6,1}$ for $m = 1$. The very precise result for $J_{*,1}$ was obtained by comparing the upper bound (1) with the lower bound coming from the linear independence of the two special modular forms $E_{4,1}$ and $E_{6,1}$. Similarly, we will get information for higher m by combining (1) with the following result.

Theorem 8.1. The forms $E_{4,1}$ and $E_{6,1}$ are algebraically independent over M_* .

Proof: Clearly the theorem is equivalent to the algebraic independence over M_* of the two cusp forms $\phi_{10,1}$ and $\phi_{12,1}$ defined in §3, (17). Suppose that these forms are dependent. Since both have index 1 and any relation can be assumed to be homogeneous, the relation between them has the form

$$(2) \qquad \sum_{j=0}^{m} g_j(\tau)\, \phi_{10,1}(\tau,z)^j\, \phi_{12,1}(\tau,z)^{m-j} = 0$$

for some m , where the g_j are modular forms, not all zero. Let j_0 be the smallest j for which g_j is not identically zero. Substituting into (2) the Taylor expansions given in §3, (19), we find that the left-hand side of (2) equals

$$(\text{const.})\ \Delta(\tau)^m\, g_{j_0}(\tau)^{2j_0} z^{2j_0} + O(z^{2j_0+2})$$

and hence cannot vanish identically. This proves the theorem.

Since the two functions $E_{4,1}$ and $E_{6,1}$ will play a basic role in our analysis of $J_{*,*}$, we introduce the abbreviations A , B to denote them. Thus $A \in J_{4,1}$, $B \in J_{6,1}$ and the theorem just proved says that the map $M_*[X,Y] \to J_{*,*}$ sending X to A and Y to B is injective. The (k,m)-graded component of this statement is that the map

$$M_{k-4m} \times M_{k-4m+2} \times \dots \times M_{k-6m} \longrightarrow J_{k,m}$$
$$(f_0, \quad f_1, \quad \dots, f_m) \longmapsto f_0A^m + f_1A^{m-1}B + \dots + f_mB^m$$

is injective. This implies:

Corollary 1. $\dim J_{k,m} \geq \sum_{j=0}^{m} \dim M_{k-4m-2j}$.

We now show how this estimate can be combined with (1) to obtain algebraic information about the ring of Jacobi forms.

Corollary 2. Fix an integer $m \ge 0$. Then the space $J_{2*,m}$ of Jacobi forms of index m and even weight is a module of rank $m+1$ over M_*.

Proof: The linear independence of the monomials $A^j B^{m-j}$ $(0 \le j \le m)$ over M_* implies that the rank is at least $m+1$. Using the two facts $\dim M_k \to \infty$ and $\dim M_{k+O(1)} = \dim M_k + O(1)$ (we do not need the more precise formula $\dim M_k = \frac{k}{12} + O(1)$), we can write Corollary 1 in the weakened form

$$\dim J_{k,m} \ge (m+1) \dim M_k + O(1) \qquad (k \to \infty, \ k \text{ even}) .$$

If there were $m+2$ Jacobi forms of index m linearly independent over M_*, then the same argument used to prove Corollary 1 would show that the factor $m+1$ in this inequality could be replaced by $m+2$, contradicting the upper bound

$$\dim J_{k,m} \le (m+1) \dim M_k + O(1)$$

coming from (1). Hence the rank is exactly $m+1$.

Corollary 3. Every Jacobi form can be expressed uniquely as a polynomial in and B with coefficients which are meromorphic modular forms (quotients of holomorphic modular forms).

Proof: If $\phi \in J_{*,m}$, then the forms ϕ, A^m, $A^{m-1}B, \ldots, B^m$ must be linearly dependent over M_* by Corollary 2, and this linear relation must involve ϕ by the Theorem, i.e. we have a formula

$$(3) \qquad f(\tau)\,\phi(\tau,z) = \sum_{j=0}^{m} f_j(\tau)\, E_{4,1}(\tau,z)^{m-j}\, E_{6,1}(\tau,z)^j$$

with $f, f_j \in M_*$ and $f \ne 0$; dividing by f gives the assertion of the corollary (the uniqueness follows at once from the algebraic independence of A and B).

Looking more carefully at the proof just given, we can obtain an estimate of the minimal weight of the form f in (3). Indeed, the proof of Corollary 2 depended on the fact that the upper and lower bounds we had obtained for $\dim J_{k,m}$ (m fixed) differ by a bounded amount, i.e.

$$(\dim M_k + \sum_{\nu=1}^{m} \dim S_{k+2\nu}) - (\sum_{\nu=0}^{m} \dim M_{k-4m-2\nu}) \leq C$$

for some constant C depending only on m ; using the explicit formulas for $\dim M_k$ we see that this holds with $C = \frac{m(m-1)}{2}$ and with equality for all even k. Hence the codimension of $M_*[A,B] \cap J_{k,m}$ in $J_{k,m}$ is bounded by C for all k. Now if $\phi \in J_{k,m}$ and there is no relation of the form (3) with f of weight h, then the subspaces $\phi \cdot M_h$ and $M_*[A,B] \cap J_{k+h,m}$ of $J_{k+h,m}$ are disjoint and hence the dimension of M_h is $\leq C$. Therefore there is a relation of type (3) at latest in weight $h = 12\,C = 6m(m-1)$. (Later we shall obtain a much better bound.)

Corollary 3 says that $J_{*,*} \otimes K_*$, where $K_* = \mathbb{C}(E_4,E_6)$ is the quotient field of M_*, is a free polynomial algebra $K_*[A,B]$ over K_*. In particular, the quotient field of $J_{*,*}$ is $\mathbb{C}(E_4,E_6,A,B)$. In view of Theorem 3.6, this is equivalent to the statement that the field of Jacobi functions (= meromorphic Jacobi forms of weight 0 and index 0) for $SL_2(\mathbb{Z})$ is $\mathbb{C}(j(\tau),\wp(\tau,z))$, a fact which is more or less obvious from the definition of Jacobi functions and the fact that every even elliptic function on $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$ is a rational function of $\wp(\tau,z)$.

Before proceeding with the theory we would like to discuss the case of forms of index 2 in some detail; this will both motivate and illustrate our results. Here, of course, we do not need Theorem 8.1, since we can check the linear independence of A^2, AB and B^2 (or of the monomials $A^m, \dots, B^m$ for any fixed m) directly by looking at the first few terms of their Fourier expansions, as was done in the case $m=1$ in §3. Thus we obtain the lower bound of Corollary 1 "by hand." This bound and the upper bound are given for small k by the table

k	2	4	6	8	10	12
$\dim M_k + \dim S_{k+2} + \dim S_{k+4}$	0	1	1	2	2	3
$\dim M_{k-8} + \dim M_{k-10} + \dim M_{k-12}$	0	0	0	1	1	2

Thus the upper and lower bounds no longer agree, as they did for $m=1$, but now always differ by 1. We will see that the upper bound is in fact always the correct one. In §2 we showed that $J_{k,m} \neq 0$ for all even $k \geq 4$ and all $m \geq 1$. Hence there exist non-zero forms $X \in J_{4,2}$, $Y \in J_{6,2}$. By Corollary 2, there must be two linear relations over M_* among the five Jacobi forms X, Y, A^2, AB and B^2 of index 2. To find them, we could calculate the leading Fourier coefficients of X and Y (we didn't give complete formulas for the coefficients of Eisenstein series of index >1 in §2, but from §4 we know that $E_{4,2}$ and $E_{6,2}$ are proportional to $E_{4,1}|V_2$ and $E_{6,1}|V_2$, from which the Fourier coefficients can be obtained painlessly). However, we prefer a different method which illustrates the use of the Taylor development coefficients $D_\nu\phi$ of §3. Since $S_6 = S_8 = S_{10} = \{0\}$, we must have $D_\nu X = D_\nu Y = 0$ for $\nu = 1, 2$; we normalize X and Y by assuming $D_0 X = E_4$, $D_0 Y = -E_6$ (thus $X = E_{4,2}$, $Y = -E_{6,2}$). Then equation (12) of §3 shows that the beginnings of the Taylor expansions of X, Y are

$$X = E_4 + \pi i\, E_4'\, z^2 - \frac{2\pi^2}{5} E_4''\, z^4 + O(z^6),$$
$$Y = -E_6 - \frac{2\pi i}{3} E_6'\, z^2 + \frac{4\pi^2}{21} E_6''\, z^4 + O(z^6);$$

we do not need to go beyond $O(z^6)$ since a Jacobi form of index 2 is determined by its Taylor development up to z^4 (by Theorem 1.2). For convenience we introduce the abbreviations Q and R for E_4 and E_6 (just as we already are using A, B for $E_{4,1}$ and $E_{6,1}$), as well as P for the near-Eisenstein series $1 - 24\sum \sigma_1(n)\, q^n$ (the notations P, Q, R are those of Ramanujan). Then one has the well-known identities

$$P' = \frac{2\pi i}{12}(P^2 - Q), \quad Q' = \frac{2\pi i}{3}(PQ - R), \quad R' = \frac{2\pi i}{2}(PR - Q^2)$$

for the derivatives of P, Q and R, so the above expansions become

$$X = Q - \frac{2\pi^2}{3}(PQ-R)\, z^2 + \frac{2\pi^4}{9}(P^2Q+Q^2-2PR)\, z^4 + O(z^6),$$
$$Y = -R + \frac{2\pi^2}{3}(PR-Q^2)\, z^2 - \frac{2\pi^4}{9}(P^2R+QR-2PQ^2)\, z^4 + O(z^6).$$

Similarly we find the expansions

$$A = Q - \frac{\pi^2}{3}(PQ-R)\,z^2 + \frac{\pi^4}{18}(P^2Q+Q^2-2PR)\,z^4 + O(z^6)\ ,$$
$$B = R - \frac{\pi^2}{3}(PR-Q^2)\,z^2 + \frac{\pi^4}{18}(P^2R+QE-2PQ^2)\,z^4 + O(z^6)$$

for the two basic Eisenstein series of index 1 . Hence the five forms $\phi = X, Y, A^2, AB, B^2$ have Taylor expansions $\phi = \chi_0 + \chi_2 z^2 + \chi_4 z^4 + O(z^6)$ with χ_ν given by the table

ϕ	χ_0	$\frac{3}{2\pi^2}\chi_2$	$\frac{9}{2\pi^4}\chi_4$
X	Q	$R-PQ$	P^2Q+Q^2-2PR
Y	$-R$	$PR-Q^2$	$-P^2R-QR+2PQ^2$
A^2	Q^2	$QR-PQ^2$	$\frac{1}{2}(Q^3+R^2)+P^2Q^2-2PQR$
AB	QR	$\frac{1}{2}(Q^3+R^2)-PQR$	$P^2QR-PR^2-PQ^3+Q^2R$
B^2	R^2	Q^2R-PR^2	$\frac{1}{2}Q(Q^3+R^2)+P^2R^2-2PQ^2R$

If any linear combination of the rows gives 0 in the three columns on the right, then the corresponding form ϕ is $O(z^6)$ and hence identically zero. Therefore by linear algebra we find the formulas

$$X = \frac{Q^2A^2 - 2R\,AB + Q\,B^2}{Q^3 - R^2}\ , \qquad Y = \frac{QR\,A^2 - 2Q^2\,AB + R\,B^2}{Q^3 - R^2}$$

expressing X and Y as polynomials in A and B as in Corollary 3. At this point we can discard AB and B^2 from our collection X, Y, A^2, AB, B^2 since

$$AB = RX - QY\ , \qquad B^2 = Q^2X - RY - QA^2\ ;$$

we can also replace A^2 by the more convenient basis element

$$Z = QX - A^2 = \frac{R^2A^2 - 2QRAB + Q^2B^2}{Q^3 - R^2}$$

(Z is a cusp form and is equal to $12\,\phi_{10,1}^2/\Delta$, where $\phi_{10,1}$ is the cusp form of index 1 defined in §3). Then X, Y, Z are Jacobi forms of index 2 and weights 4, 6 and 8 , respectively, and since they are clearly linearly independent over M_* we can improve our lower bound to

$$\dim J_{k,2} \geq \dim M_{k-4} + \dim M_{k-6} + \dim M_{k-8} .$$

Since the right-hand side of this inequality is equal to the upper bound (1), we deduce:

Theorem 8.2. The space $J_{2*,2}$ of Jacobi forms of index 2 and even weight is free over M_* on three generators X, Y, Z of weight 4, 6 and 8, respectively. The functions X, Y and Z are related to A and B by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \frac{1}{Q^3-R^2} \begin{pmatrix} Q^2 & R & Q \\ QR & Q^2 & R \\ R^2 & QR & Q^2 \end{pmatrix} \begin{pmatrix} A^2 \\ -2AB \\ B^2 \end{pmatrix} , \qquad \begin{pmatrix} A^2 \\ -2AB \\ B^2 \end{pmatrix} = \begin{pmatrix} Q & 0 & -1 \\ -R & Q & 0 \\ 0 & -R & Q \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} .$$

There are two striking aspects to this result: that the module $J_{2*,2}$ is free over M_*, i.e. that we need no more generators than are required by Corollary 2, and that the only modular form we need to invert in order to express these generators in terms of A and B is the discriminant function $\Delta = \frac{1}{1728}(Q^3 - R^2)$. We now show that these two results hold in general.

Theorem 8.3. The ring $J_{2*,*}$ is contained in $M_*[\frac{1}{\Delta}][A,B]$. In other words, the meromorphic modular forms occurring in Corollary 3 of Theorem 8.1 are holomorphic except at infinity.

Proof: In the proof of Theorem 8.1 we used the fact that $\Delta(\tau)$ is not identically zero. Using the fact that $\Delta(\tau)$ vanishes nowhere in H, we find with the same proof that the functions $E_{4,1}(\tau_0,z)$ and $E_{6,1}(\tau_0,z)$ are algebraically independent for each point $\tau_0 \in H$. Indeed, replacing the functions $g_j(\tau)$ in (2) by complex numbers c_j, we find that the left-hand side equals

$$(\text{const.})\ \Delta(\tau_0)^m c_{j_0} z^{2j_0} + O(z^{2j_0+2})$$

as $z \to 0$, where j_0 is the first j with $c_j \neq 0$, and this cannot vanish because $\Delta(\tau_0) \neq 0$. Now suppose that $\phi \in J_{k,m}$ is a non-zero Jacobi form and let $f(\tau)$ be a modular form of minimal weight such that

$f\phi \in M_*[A,B]$, i.e. such that there is a relation of the form (3) (we have already proved that such an f exists). If f is not a power of Δ, then f vanishes at some point $\tau_0 \in H$, and then $f_j(\tau_0) = 0$ for all j by the algebraic independence of $E_{4,1}(\tau_0,z)$ and $E_{6,1}(\tau_0,z)$. But then f and the f_j have a common factor (namely E_4 or E_6 if τ_0 is equivalent to $e^{\pi i/3}$ or i and $E_4^3 - j(\tau_0)\Delta$ otherwise), contradicting the minimality of f .

<u>Theorem 8.4</u>. $J_{*,*}$ <u>is free as a module over</u> M_* .

<u>Proof</u>: The proof is similar to those of 8.1 and 8.3. Let us assume inductively that for some $k > 0$ we have found Jacobi forms $\phi_1, \dots, \phi_r$ of weight $k_1, \dots, k_r < k$ which are a free basis of $J_{*,m}$ over M_* in weights $< k$, i.e. such that every form in $J_{k',m}$ for $k' < k$ can be written uniquely as $\sum_1^r f_i(\tau)\ \phi_i(\tau,z)$ with $f_i \in M_{k'-k_i}$. (This is certainly true if $k = 1$ or k is the smallest integer with $J_{k,m} \neq 0$.) If we can show that the ϕ_i are linearly independent over M_* in weight k also, i.e. that there is no non-trivial linear combination $\sum f_i\,\phi_i$ in $J_{k,m}$ which vanishes, then we are done, for then the subspace $\phi_1 M_{k-k_1} + \dots + \phi_r M_{k-k_r}$ of $J_{k,m}$ is a direct sum, and choosing a $\mathbb{C}$-basis $\phi_{r+1}, \dots, \phi_s$ for its complement gives us a new collection of Jacobi forms $\phi_1, \dots, \phi_s$ satisfying the induction hypothesis with k replaced by $k+1$. So assume that we have a relation $\sum f_i\ \phi_i = 0$ in $J_{k,m}$. Since $k_i < k$, we have $\text{weight}(f_i) = k-k_i > 0$, so f_i lies in the ideal (Q,R) of $M_* = \mathbb{C}[Q,R]$, i

$$f_i = Q\, g_i + R\, h_i$$

for some modular forms $g_i \in M_{k-4-k_i}$, $h_i \in M_{k-6-k_i}$ (where, of course, form of negative weight are zero). Then our relation becomes $Q \cdot \sum g_i\,\phi_i + R \cdot \sum h_i\,\phi_i = 0$. But a relation $Q\psi_1 + R\psi_2 = 0$ between Jacobi forms ψ_1 and ψ_2 implies that $\psi_1 = R\psi$, $\psi_2 = -Q\psi$ for some holomorphic Jacobi form ψ of smaller weight (to see this, note that $\psi := \frac{\psi_1}{R} = -\frac{\psi_1}{Q}$ transforms like a Jacobi form, is holomorphic at all points τ H not equivalent to i because $R(\tau) \neq 0$ and at all τ not equivalent to $e^{\pi i/3}$ because $Q(\tau) \neq 0$, and satisfies the cusp condition because Q and R are invertible near ∞). Hence we have

$$\sum g_i \phi_i = +R\psi , \qquad \sum h_i \phi_i = -Q\psi$$

for some Jacobi form ψ of index m and weight k-10. Since $k-10 < k$, we can by our induction assumption write ψ as a linear combination $\sum e_i \phi_i$ with $e_i \in M_{k-10-k_i}$; then the identities

$$\sum (g_i - Re_i)\phi_i = 0 , \qquad \sum (h_i + Qe_i)\phi_i = 0$$

in weights k-4 and k-6 (both $<k$) imply by the uniqueness part of the induction assumption that $g_i = Re_i$, $h = -Qe_i$ and hence $f_i = 0$ for all i.

Remark: The method of proof used for Theorem 8.4 would equally show that other spaces of modular forms (e.g. modular forms of half-integral weight, or modular forms of level N, with $M_*(\Gamma_1)$ embedded into $M_*(\Gamma_0(N))$ via either $f(\tau) \mapsto f(\tau)$ or $f(\tau) \mapsto f(N\tau)$) are free $M_*(\Gamma_1)$-modules. This fact, although not at all deep, may be of practical interest in tabulating modular forms, since it means that all modular forms of a given type (e.g. of fixed level but arbitrary even, odd or half-integral weight) can be described by tabulating the Fourier coefficients of a finite system of free generators.

The results we have proved up to now have all been additive, i.e. concerned with points i) and ii) in the introduction of this section. We end with a simple result on the multiplicative nature of $J_{*,*}$.

Theorem 8.5. *The ring of Jacobi forms is an infinitely generated ring of transcendence degree* 4.

Proof: That the transcendence degree of $J_{*,*}$ is 2 over M_*, and hence 4 over $\mathbb{C}$, is clear since $J_{*,*}$ contains the polynomial algebra $M_*[A,B] = \mathbb{C}[Q,R,A,B]$ and is algebraic over it (the square of any Jacobi form of odd weight has even weight, and we have proved that a Jacobi form ϕ of even weight actually satisfies a linear equation $a\phi + b = 0$ over $M_*[A,B]$, where in fact we can take $a \in \mathbb{C}[Q,R]$ or even $a \in \mathbb{C}[Q^3-R^2]$). We show that $J_{*,*}$ is not finitely generated. Consider any finite collection of non-constant Jacobi forms $\phi_i \in J_{k_i,m_i}$ $(i=1,\ldots,r)$. By the results of §1 we know that $m_i \geq 0$ and $k_i > 0$. It follows that any monomial $\phi_1^{n_1}\ldots\phi_r^{n_r}$ $(n_1,\ldots,n_r \geq 0)$ has a ratio m/k bounded by

$$\frac{m}{k} = \frac{n_1m_1+\ldots+n_rm_r}{n_1k_1+\ldots+n_rk_r} \leq \max_{1\leq i\leq r} \frac{m_i}{k_i} .$$

Since m/k is unbounded in $J_{*,*}$ (in §2 we constructed Eisenstein series with $k=4$ and arbitrary m), we deduce that $\phi_1,\ldots,\phi_r$ cannot generate $J_{*,*}$ as a ring.

Theorem 8.5 is a negative result. In the next section we will show how to embed $J_{*,*}$ in a slightly larger ring which is finitely generated.

§9. Explicit description of the space of Jacobi forms

Our main tool for studying the structure of $J_{*,*}$ in §8 was the fact that we had estimates on $\dim J_{k,m}$ from above and below which differed by a bounded amount as $k\to\infty$ with m fixed (at least in the case of even weight; for odd weight we have so far given only an upper bound). In this section we will improve both estimates, obtaining upper and lower bounds which actually coincide for k sufficiently large. This will not only permit a more accurate description of the ring $J_{*,*}$, but will also lead to an algorithm for computing $J_{k,m}$ (actually, two algorithms) which is effective and, for modest m, practical. The results will apply to both even and odd weight.

We begin with the upper bound. The bound used in §8, namely

$$(1)\qquad \dim J_{k,m} \;\le\; \begin{cases} \dim M_k + \sum\limits_{\nu=1}^{m} \dim S_{k+2\nu} & (k \text{ even})\\ \sum\limits_{\nu=1}^{m-1} \dim S_{k+2\nu-1} & (k \text{ odd}), \end{cases}$$

was proved in §3 as a corollary of the injectivity of the map D from $J_{k,m}$ into the direct sum of the spaces whose dimensions appear on the right of (1). We will now give a second proof of (1) which is even more elementary than this proof (in that it does not make use of the Taylor development operators D_ν) and leads to a sharper result; however, it gives less precise information than the first proof in that it gives only a filtration of $J_{k,m}$ with successive quotients mapping injectively into spaces $M_{k+\nu}$, rather than an injective map of the whole of $J_{k,m}$ into a direct sum of spaces $M_{k+\nu}$ as we obtained before. The sharpening of (1) we will obtain is the following:

Theorem 9.1. *The dimension of $J_{k,m}$ is bounded above by*

$$\dim J_{k,m} \;\le\; \sum_{\nu=0}^{m} \max\left(\dim M_{k+2\nu} - \left\lceil \frac{\nu^2}{4m}\right\rceil, 0\right) \qquad (k \text{ even}),$$

$$(3)\qquad \dim J_{k,m} \le \sum_{\nu=1}^{m-1} \max\left(\dim M_{k+2\nu-1} - \left\lceil\frac{\nu^2}{4m}\right\rceil, 0\right) \qquad (k \text{ odd}) .$$

Proof: For $\nu \ge 0$ let

$$J_{k,m}^{(\nu)} = \{ \phi \in J_{k,m} \mid \phi(\tau,z) = O(z^\nu) \text{ for } z \to 0 \} ,$$

i.e. $J_{k,m}^{(\nu)}$ is the intersection of $J_{k,m}$ and the space $M_{k,m}^{(\nu)}$ introduced after the Corollary to Theorem 3.3. Then we have a filtration

$$J_{k,m} = J_{k,m}^{(0)} \supset J_{k,m}^{(1)} \supset J_{k,m}^{(2)} \supset \dots \;;$$

moreover, $J_{k,m}^{(\nu)} = J_{k,m}^{(\nu+1)}$ for $\nu \equiv k \pmod 2$ because the order of vanishing of a Jacobi form at $z=0$ has the same parity as the weight, and $J_{k,m}^{(\nu)} = 0$ for $\nu > 2m$ for all k and for $\nu > 2m-3$ for k odd for the same reason used to prove (1) in §3 (a Jacobi form has $2m$ zeros altogether in $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, and for k odd three of these are at the non-zero 2-division points). On the other hand, from the definition of Jacobi forms we see that if a Jacobi form of weight k has an expansion $\phi(\tau,z) = f(\tau)\, z^\nu + O(z^{\nu+1})$ near $z=0$ the function f is a modular form of weight $k+\nu$, so we have an exact sequence

$$(4)\qquad \begin{array}{ccccccc} 0 & \to & J_{k,m}^{(\nu+1)} & \to & J_{k,m}^{(\nu)} & \to & M_{k+\nu} \\ & & & & \phi & \mapsto & f \end{array}$$

similar to (14) of §3. Together this gives filtrations

$$J_{k,m} \supset J_{k,m}^{(0)} \supset J_{k,m}^{(2)} \supset \dots \supset J_{k,m}^{(2m)} \supset J_{k,m}^{(2m+2)} = 0 , \quad J_{k,m}^{(2\nu)}/J_{k,m}^{(2\nu+2)} \hookrightarrow M_{k+2\nu}$$

for k even and

$$J_{k,m} \supset J_{k,m}^{(1)} \supset J_{k,m}^{(3)} \supset \dots \supset J_{k,m}^{(2m-3)} \supset J_{k,m}^{(2m-1)} = 0 , \quad J_{k,m}^{(2\nu-1)}/J_{k,m}^{(2\nu+1)} \hookrightarrow M_{k+2\nu-1}$$

for k odd, and this immediately gives (1) (with M instead of S). To get the sharper estimates (2) and (3), we must say something about the image of the last map in (4). Consider first the case of even k . The Fourier expansion of a Jacobi form of index m can be written in the form

$$\phi(\tau,z) = \sum_{n=0}^{\infty} P_n(e^{2\pi iz})\, q^n$$

where $P_n(\zeta)$ is a polynomial in ζ and ζ^{-1} of degree $\le \sqrt{4nm}$ by virtue of the condition at infinity. This polynomial is symmetric in ζ and ζ^{-1} for k even and hence can be written as a polynomial in $\zeta+\zeta^{-1}$ or, more conveniently, in $\zeta+\zeta^{-1}-2$ ($= -4\sin^2\pi z$) :

$$P_n(\zeta) = Q_n(\zeta+\zeta^{-1}-2)\,, \qquad Q_n(T)\in\mathbb{C}[T]\,, \qquad \deg Q_n \le \sqrt{4nm}\ .$$

If $\phi\in J_{k,m}^{(2\nu)}$, then each coefficient $P_n(e^{2\pi iz})$ must be $O(z^{2\nu})$ as $z\to 0$, so each polynomial $Q_n(T)$ is divisible by T^ν. Hence Q_n must vanish identically if $\deg Q_n < \nu$, i.e.

$$(5) \qquad \nu > \sqrt{4nm} \Rightarrow Q_n \equiv 0\ .$$

The last map in (4) sends ϕ to $\sum a_n q^n$ with

$$a_n = \lim_{z\to 0} P_n(e^{2\pi iz})/z^{2\nu} = (2\pi)^{2\nu} \lim_{T\to 0} Q_n(T)/T^\nu\ ,$$

and from (5) we see that this is 0 for $n < \frac{\nu^2}{4m}$. Hence the image of the last map in (4) is contained in

$$\{ f\in M_{k+2\nu} \mid f = \sum_{\substack{n\in\mathbb{Z}\\ n\ge\nu^2/4m}} a_n q^n \}\ ,$$

and since the dimension of this space is $\max\left(\dim M_{k+2} - \left\lceil\frac{\nu^2}{4m}\right\rceil, 0\right)$, inequality (2) follows. The case of odd k is similar, the only difference being that now $P_n(\zeta)$ is odd and hence has the form $(\zeta-\zeta^{-1})\,Q_n(\zeta+\zeta^{-1}-2)$ for some polynomial Q_n of degree $\le\sqrt{4nm}-1$; then $\phi\in J_{k,m}^{(2\nu-1)}$ implies $T^{\nu-1}\,|\,Q_n(T)$ and hence $\nu^2\le 4nm$ as before.

<u>Corollary</u>. <u>For</u> $k\ge m$ <u>we have the upper bounds</u>

$$\dim J_{k,m} \le \sum_{\nu=0}^{m} \dim M_{k+2\nu} - N_+(m) \qquad (\ k \text{ even}\ )\,,$$

$$\dim J_{k,m} \le \sum_{\nu=1}^{m-1} \dim M_{k+2\nu-1} - N_-(m) \qquad (\text{ k odd }),$$

where

$$(6)\quad N_+(m) = \sum_{\nu=0}^{m} \left\lceil \frac{\nu^2}{4m} \right\rceil, \qquad N_-(m) = \sum_{\nu=1}^{m-1} \left\lceil \frac{\nu^2}{4m} \right\rceil .$$

Proof: We must show that the "max" in formulas (2) and (3) is always attained by the first argument. For k odd and $1 \le \nu \le m-1$ we have

$$\dim M_{k+2\nu-1} \ge \frac{k+2\nu-3}{12} = \frac{\nu^2}{4m} + \frac{k-m}{12} + \frac{m-3}{12m}(m-\nu) + \frac{\nu}{4m}(m-\nu-1) \ge \frac{\nu^2}{4m}$$

and the assertion follows. The same argument shows that $\dim M_{k+2\nu} \ge \frac{\nu^2}{4m}$ for k even and $0 \le \nu \le m-1$; if $\nu = m$ we use instead

$$k \ge m+2 \Rightarrow \dim M_{k+2m} \ge \frac{k+2m-2}{12} \ge \frac{m}{4},$$

$$k = m, m+1 \Rightarrow k+2m \not\equiv 2 \pmod{12} \Rightarrow \dim M_{k+2m} \ge \frac{k+2m}{12} \ge \frac{m}{4},$$

proving the assertion in this case also.

We now turn to the lower bound; for this we have to construct Jacobi forms. To do this we begin by enlarging the space of forms under consideration.

Definition. A weak Jacobi form of weight k and index m is a function satisfying the transformation laws of Jacobi forms of this weight and index and having a Fourier expansion of the form

$$(7)\quad \tilde{\phi}(\tau,z) = \sum_{n \ge 0} \sum_{r} c(n,r)\, q^n \zeta^r .$$

The space of such forms is denoted $\tilde{J}_{k,m}$.

The space $\tilde{J}_{k,m}$ is, of course, in general larger than $J_{k,m}$, but it is still finite-dimensional. Indeed, a weak Jacobi form also has $2m$ zeros in a fundamental domain for $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ (since it satisfies the same transformation law under $z \to z + \lambda\tau + \mu$ as a true Jacobi form), and its restrictions to $z = \lambda\tau + \mu$ $(\lambda, \mu \in \mathbb{Q})$ give modular forms in the same way as in Theorem 1.3 (the conditions at infinity are satisfied because of the condition $n \geq 0$ in the definition of weak Jacobi forms), so the proof of Theorem 1.1 carries over unchanged. Moreover, all of the content of §3 also still applies, so we get maps

$$(8)\qquad \begin{aligned} D&: \tilde{J}_{k,m} \to M_k \oplus M_{k+2} \oplus \dots \oplus M_{k+2m} \quad (k \text{ even}), \\ D&: \tilde{J}_{k,m} \to M_{k+1} \oplus M_{k+3} \oplus \dots \oplus M_{k+2m-3} \quad (k \text{ odd}) \end{aligned}$$

and they are still injective. The only point that needs to be checked in order for, for example, the map

$$D_0: \sum_{n,r} c(n,r)\, q^n \zeta^r \mapsto \sum_n \Big(\sum_r c(n,r) \Big) q^n$$

to make sense is that for a given n there are onlv finitely many r with $c(n,r) \neq 0$. But this follows from the periodicity condition on the coefficients $c(n,r)$ (Theorem 2.2), which applies unchanged to weak Jacobi forms, for given n and r we can choose an r' with $|r'| \leq m$ and $r' \equiv \pm r \pmod{2m}$, and then

$$c(n,r) = \pm\, c\Big(n + \frac{r'^2 - r^2}{4m}\Big) = 0 \quad \text{if} \quad n - \frac{r^2}{4m} < -\frac{r'^2}{4m}$$

by the condition defining a weak Jacobi form,so $c(n,r)$ vanishes as soon as $r^2 - 4nm > m^2$.

The basic result on weak Jacobi forms is the following.

<u>Theorem 9.2</u>. <u>The map (8) is an isomorphism for all</u> k <u>and</u> m.

Notice that the statement of this theorem makes no reference to k being positive, and hence shows that the weight of a weak Jacobi form can be negative (but not less than $-2m$). As a corollary of Theorem 9.2 we get

<u>Corollary</u>. <u>For all</u> k <u>and</u> m <u>we have the lower bounds</u>

$$\dim J_{k,m} \geq \sum_{\nu=0}^{m} \dim M_{k+2\nu} - N_+(m) \qquad (k \text{ even}),$$

$$\dim J_{k,m} \geq \sum_{\nu=1}^{m-1} \dim M_{k+2\nu-1} - N_-(m) \qquad (\text{ k odd }),$$

with $N_\pm(m)$ as in (6). For $k \geq m$ the inequality signs can be replaced by equalities.

To prove the corollary, we note that there is an exact sequence

(9) $$0 \to J_{k,m} \to \tilde{J}_{k,m} \to \mathbb{C}^{N_\pm(m)}$$

in which the last map sends a weak Jacobi form $\tilde{\phi} = \sum c(n,r)\, q^n \zeta^r$ to the collection of Fourier coefficients $c(n,r)$ with $0 \leq r \leq m$ (resp. $0 < r < m$ if k is odd) and $0 < n < r^2/4m$; if all of these coefficients vanish, then it follows from the periodicity of the coefficients (Theorem 2.2) that all $c(n,r)$ with $4nm < r^2$ vanish and hence that $\tilde{\phi}$ is a true Jacobi form, as asserted in (9). The second statement, of course, follows from the corollary to Theorem 9.1.

Proof of Theorem 9.2: We need only prove the surjectivity of (8). To do this, we will replace $\tilde{J}_{k,m}$ by an a priori larger space $J'_{k,m}$, prove the surjectivity of (8) with J' instead of $\tilde{J}$, and then show that in fact $\tilde{J} = J'$. The space $J'_{k,m}$ is defined as the set of all functions $\phi'(\tau,z)$ satisfying the transformation law of Jacobi forms of index m with respect to $z \to z+\lambda\tau+\mu$ $(\lambda, \mu \in \mathbb{Z})$ and having a Fourier expansion of the form (7), but with the transformation law under the action of $SL_2(\mathbb{Z})$ replaced by the weaker conditions

(10) $$\phi'(\tau,-z) = (-1)^k\, \phi'(\tau,z),$$

(11) $$(\phi'|_{k,m} M)(\tau,z) = \phi(\tau,z) \quad \begin{cases} (\text{mod } z^{2m+2}) & (\text{k even}) \\ (\text{mod } z^{2m-1}) & (\text{k odd}) \end{cases} \quad (\forall M \in \Gamma_1).$$

Equation (10) and the transformation law under $\mathbb{Z}+\mathbb{Z}\tau$ show that the coefficients $c(n,r)$ of ϕ' satisfy the periodicity property of Theorem 2.2, and the condition (11) implies that the development coefficients $D_0\phi', D_2\phi', \ldots, D_{2m}\phi'$ (resp. $D_1\phi', D_3\phi', \ldots, D_{2m-3}\phi'$ if k is odd) are modular forms, since the proof that $D_\nu\phi \in M_{k+\nu}$ in §3 used only the Taylor development of ϕ up to order ν. Hence the map (8) is still defined if we replace $\mathcal{J}$ by J'. To show that it is surjective is now a question of linear algebra. Suppose k is even (the case of odd k is similar). Then given a collection of modular forms $\xi_\nu(\tau) = \sum a_\nu(n)\, q^n$ of weight $k+2\nu$ $(0 \le \nu \le m)$ we want to find a $\phi' \in J'_{k,m}$ with $D_{2\nu}\phi' = \xi_\nu$ for all ν, i.e. to find coefficients $c(n,r)$ satisfying

a) $c(n,r) = 0$ for $n < 0$,

b) $c(n,r) = c(n',r')$ for $r' \equiv \pm r \pmod{2m}$, $r'^2 - 4n'm = r^2 - 4nm$,

c) $\sum\limits_r p_{2\nu}^{(k-1)}(r,nm)\, c(n,r) = a_\nu(n)$ for $0 \le \nu \le m$, $n \ge 0$.

Because of b), we need only find $c(n,r)$ for $0 \le r \le m$, since the rest are then determined by the periodicity condition; in terms of this we can rewrite c) as

$$(12) \qquad \sum_{r=0}^{m} \varepsilon_r\, p_{2\nu}^{(k-1)}(r,nm)\, c(n,r) + \dots = a_\nu(n) \quad ,$$

where ε_r is 1 or 2 depending whether r is 0 or not and "..." denotes a linear combination of coefficients $c(n',r)$ with $0 \le r \le m$ and $n' < n$. We can assume inductively that the equations have been solved for $n' < n$ and hence that the "..." denotes a known quantity; then (12) becomes an $(m+1) \times (m+1)$ system of linear equations in the $m+1$ unknowns $c(n,r)$ $(0 \le r \le m)$ with coefficients $\varepsilon_r\, p_{2\nu}^{(k-1)}(r,nm)$ $(0 \le r, \nu \le m)$. But this matrix is invertible, since it is the product of the non-singular diagonal matrix $(\varepsilon_r\, \delta_{r\nu})_{r,\nu}$ and the matrix $(p_{2\nu}^{(k-1)}(r,nm))_{r,\nu}$, which can be reduced to a Vandermonde determinant $(r^{2\nu})_{r,\nu}$ by elementary row operations because $p_{2\nu}^{(k-1)}(r,nm)$ is a polynomial of degree exactly 2ν in r. Hence the equations can be solved inductively for all coefficients $c(n,r)$.

It remains to see that $\mathcal{J} = J'$. But this is easy, because if $\phi' \in J'$ and $M \in \Gamma_1$, then the difference $\phi' - \phi'|_{k,m} M$ transforms like a Jacobi form with respect to translations $z \to z + \lambda\tau + \mu$ (because of the compatibility of the actions of $\mathbb{Z}^2$ and $SL_2(\mathbb{Z})$ proved in §1) and vanishes to order $>2\nu$ at $z=0$ by (11), so Theorem 1.2 shows that it vanishes identically and hence that ϕ' transforms correctly under Γ_1.

We now prove a theorem which determines the structure of the bigraded ring $\tilde{J}_{ev,*} = \bigoplus_{\substack{k,m \\ k \text{ even}}} \tilde{J}_{k,m}$ completely.

<u>Theorem 9.3. The ring</u> $\tilde{J}_{ev,*}$ <u>is a polynomial algebra over</u> M_* <u>on two generators</u>

$$\tilde{\phi}_{-2,1} = \frac{\phi_{10,1}}{\Delta} \in \tilde{J}_{-2,1}, \qquad \tilde{\phi}_{0,1} = \frac{\phi_{12,1}}{\Delta} \in \tilde{J}_{0,1},$$

<u>where</u> $\phi_{k,1} \in J^{cusp}_{k,1}$ $(k=10,12)$ <u>are the Jacobi forms constructed in §3</u>.

<u>Proof</u>: That $\tilde{\phi}_{-2,1}$ and $\tilde{\phi}_{0,1}$ are weak Jacobi forms is clear, since $\phi_{10,1}$ and $\phi_{12,1}$, being cusp forms, have Fourier developments containing only positive powers of q and $\Delta(\tau) = q + O(q^2)$. Since $\tilde{\phi}_{-2,1}$ and $\tilde{\phi}_{0,1}$ are algebraically independent over M_* by Theorem 8.1, the map

$$\begin{aligned} P:\ M_k \oplus M_{k+2} \oplus \dots \oplus M_{k+2m} &\longrightarrow \tilde{J}_{k,m} \\ (f_0,\ f_1,\ \dots,\ f_m) &\longmapsto \sum f_i\, \tilde{\phi}^{i}_{-2,1}\, \tilde{\phi}^{m-i}_{0,1} \end{aligned}$$

is injective, and combining this with the injectivity of the map (8) in the other direction shows that both are in fact isomorphisms, thus proving Theorems 9.2 (for even weight) and 9.3 at one blow.

This proof of Theorem 9.2 is of course, much shorter than the one we gave before (and a similar proof works for odd weight, as we shall see in a moment). Nevertheless, we preferred to give the direct proof of the surjectivity of D because

a) in other situations (e.g. for congruence subgroups) one might not happen to have enough explicit generators to deduce the surjectivity of D purely by dimension considerations, and

b) the first proof given shows explicitly how to obtain a weak Jacobi form mapping to a given (m+1)-tuple of modular forms by solving a system of recurrence relations. This and the proof just given for Theorem 9.3 then give two algorithms for computing the space of Jacobi forms of given weight and index: we construct a basis of $\tilde{J}_{k,m}$ either by picking a basis of $M_k \oplus \ldots \oplus M_{k+2m}$ and applying P or by picking a basis of $M_k \oplus \ldots \oplus M_{k+2m}$ and applying D^{-1} as in the proof of Theorem 9.2; then in both cases we compute the Fourier coefficients $c(n,r)$ $(0 \le r \le m,$ $0 \le n \le r^2/4m)$ of our basis elements and obtain $J_{k,m}$ as the kernel of the map $\tilde{J}_{k,m} \to \mathbb{C}^{N_+(m)}$. Both methods will be illustrated later.

Observe that Theorem 9.3 gives a considerable sharpening of Theorem 8.3. There we showed that any Jacobi form, multiplied by a suitable power of Δ, could be expressed as a polynomial in A and B, and that in fact one can take $\frac{m(m-1)}{2}$ as the exponent of Δ, where m is the index of the form. Here we show that $\Delta^m\phi$ can be written as a polynomial in A and B, and in fact as a polynomial in $RA-QB$ and Q^2A-RB, for any Jacobi form or weak Jacobi form ϕ of index m.

Theorem 9.3 gives the structure of $\tilde{J}$ for even weights. We now find the corresponding result for odd weight. The upper and lower bounds given in Theorems 9.1 and 9.2 and their corollaries agree for $k \ge m$ and also for $m \le 5$; for $m \le 6$ we obtain the table

$\dim J_{k,m}$	k: 3	5	7	9	11	13	15	17
m = 1	(0 for all odd k)							
2	0	0	0	0	1	0	1	1
3	0	0	0	1	1	1	2	2
4	0	0	1	1	2	2	3	3
5	0	1	1	2	3	3	4	5
6	0	0 or 1	1	2	3	3	5	5

In particular, there is a certain Jacobi form $\phi_{11,2}$ of weight 11 and index 2 (we shall see how to construct it explicitly a little later).

Since $\phi_{11,2}$ is a cusp form (it follows from §2 that the smallest index of a non-cusp form of odd weight is 9), the quotient $\tilde{\phi}_{-1,2} = \phi_{11,2}/\Delta$ is a weak Jacobi form. Hence $\tilde{J}_{-1,2} \neq \{0\}$; of course, this also follows directly from Theorem 9.2. Then $\tilde{\phi}_{-1,2}\cdot\tilde{J}_{k+1,m-2} \subset \tilde{J}_{k,m}$ for all k and m, and since Theorem 9.2 gives

$$\dim \tilde{J}_{k+1,m-2} = \dim M_{k+1} + \dim M_{k+3} + \ldots + \dim M_{k+2m-3} = \dim \tilde{J}_{k,m}$$

for k odd, this inclusion is in fact an equality. Hence we have $J_{od,*} = \tilde{\phi}_{-1,2}\cdot\tilde{J}_{ev,*}$, i.e. the weak Jacobi forms of odd weight form a free module of rank 1 over the ring of weak Jacobi forms of even weight, with generator $\tilde{\phi}_{-1,2}$. This gives the ring structure, too, for $\tilde{\phi}^2_{-1,2}$ lies in $\tilde{J}_{ev,*}$ and can therefore be expressed as a polynomial in $\tilde{\phi}_{-2,1}$ and $\tilde{\phi}_{0,1}$ with coefficients in M_*. Comparing indices and weights, we see that this polynomial has the form $\sum_{i=1}^{4} f_i\, \tilde{\phi}^{i}_{-2,1}\, \tilde{\phi}^{4-i}_{0,1}$ with f_i in M_{2i-2} and hence

$$f_1 = \alpha\,, \qquad f_2 = 0\,, \quad f_3 = \beta\, E_4\,, \quad f_4 = \gamma\, E_6$$

for some constants α, β, γ. To find these constants, we observe that $D_1 : \tilde{J}_{-1,2} \xrightarrow{\sim} M_0 = \mathbb{C}$ by Theorem 9.2, so $D_1\tilde{\phi}_{-1,2}$ is a constant and this constant determines $\tilde{\phi}_{-1,2}$. We normalize by choosing $D_1\tilde{\phi}_{-1,2} = 2$; then

$$\tilde{\phi}_{-1,2}(\tau,z) = \sum_{n\geq 0}\sum_{r} c(n,r)\, q^n \zeta^r, \qquad \sum_{r>0} r\, c(n,r) = \begin{cases} 2 & (n=0)\,, \\ 0 & (n\neq 0)\,. \end{cases}$$

By the argument preceding the statement of Theorem 9.2, $c(n,r)=0$ for $r^2 > 8n+1$; in particular $c(0,r)=0$ for $|r|>1$ and therefore

$$\tilde{\phi}_{-1,2} = (\zeta - \zeta^{-1}) + \sum_{n\geq 1} (\text{polynomial in } \zeta,\ \zeta^{-1})\, q^n\,.$$

This already suffices to determine α, β and γ, for comparing the coefficients of q^0 in the two expressions for $\tilde{\phi}^2_{-1,2}$ gives

$$(\zeta - \zeta^{-1})^2 = \alpha(\zeta-2+\zeta^{-1})(\zeta+10+\zeta^{-1})^3 + \beta(\zeta-2+\zeta^{-1})^3(\zeta+10+\zeta^{-1}) + \gamma(\zeta-2+\zeta^{-1})$$

or

$$T^2+4T = \alpha T(T+12)^3 + \beta T^3(T+12) + \gamma T^4$$

$(T = \zeta - 2 + \zeta^{-1})$, from which $\alpha = \frac{1}{432}$, $\beta = -3\alpha$, $\gamma = 2\alpha$. We have proved:

<u>Theorem 9.4.</u> <u>The ring</u> $\tilde{J}_{*,*}$ <u>has the structure</u>

$$\tilde{J}_{*,*} = M_*[a,b,c]/\ c^2 = \frac{1}{432}\, a\,(b^3 - 3Q\,a^2 b + 2R\,a^3)\ ,$$

<u>where</u> $a \in \tilde{J}_{-2,1}$, $b \in \tilde{J}_{0,1}$, $c \in \tilde{J}_{-1,2}$.

We make one remark about the relation just obtained. We can write it in the form

$$c^2 = \frac{1}{432}\, a^4\, [(\tfrac{b}{a})^3 - 3Q\tfrac{b}{a} + 2R]\ .$$

On the other hand, by Theorem 3.6 we have

$$\frac{b}{a} = \frac{\tilde{\phi}_{0,1}}{\tilde{\phi}_{-2,1}} = \frac{\phi_{12,1}}{\phi_{10,1}} = -\frac{3}{\pi^2}\, p\ ,$$

where p is the Weierstrass p-function. Hence

$$c^2 = \frac{a^4}{(2\pi i)^6}\,(4p^3 - \frac{4\pi^4}{3}E_4\, p - \frac{8\pi^6}{27}E_6)\ ,$$

where the expression in parentheses is equal to p'^2 by Weierstrass' equation. Thus the relation in Theorem 9.4 is just Weierstrass' equation for p'^2 as a cubic in p and the theorem can be interpreted as saying that $\tilde{J}_{*,*}$ is the coordinate ring of the universal elliptic curve over the modular curve (we leave this purposely vague); $J_{*,*}$ itself, of course, is not finitely generated and hence not the coordinate ring of anything. The square root of the relation just obtained is $c = \frac{-a^2}{(2\pi i)^3}\, p'(\tau,z)$ or

$$\begin{aligned}\tilde{\phi}_{-1,2} &= \frac{-1}{(2\pi i)^3}\,(\frac{\phi_{10,1}}{\Delta})^2\, p'(\tau,z)\\ &= \frac{-1}{12\,(2\pi i)\,\Delta^2}\,\phi_{10,1}^2\,\frac{\partial}{\partial z}\,\frac{\phi_{12,1}(\tau,z)}{\phi_{10,1}(\tau,z)}\\ &= \frac{-1}{12\,\Delta^2}\,\frac{1}{2\pi i}\,(\phi'_{12,1}\,\phi_{10,1} - \phi'_{10,1}\,\phi_{12,1})\ .\end{aligned}$$

Substituting into this the expressions for $\phi_{10,1}$ and $\phi_{12,1}$ in terms of $A = E_{4,1}$ and $B = E_{6,1}$, we find that the right-hand side is equal to $\frac{12^{-2}}{2\pi i}\frac{1}{\Delta}(A'B - AB')$. This proves the relation

(13) $288\pi i\,\phi$[illegible] $= E'$[illegible] E[illegible] $- E$[illegible] E'[illegible] .

This is a special case of the following easy fact, which gives a general construction for Jacobi forms of odd weight from forms of even weight.

Theorem 9.5. Let ϕ_1 and ϕ_2 be Jacobi forms of weight k_1 and k_2 and index m_1 and m_2, respectively. Then $m_2\phi_1'\phi_2 - m_1\phi_1\phi_2'$, where ' denotes differentiation with respect to z, is a Jacobi form of index m_1+m_2 and weight k_1+k_2+1.

Proof: The derivative of a meromorphic Jacobi form of index 0 and weight k is a meromorphic Jacobi form of index 0 and weight $k+1$. Applying this to $\phi_1^{m_2}/\phi_2^{m_1}$ proves the theorem (the conditions at infinity are trivial).

Examples of Jacobi forms of index > 1

We end with some numerical examples to illustrate the techniques for computing Jacobi forms implicit in the theorems of this section. We give several types of examples with $m>1$ (the case $m=1$ having been treated in §3) to illustrate the results described at the end of §5. We look only at cusp forms.

1. k even, m prime.

This situation, the one treated in Theorem 5.6, is the simplest case after $m=1$ (because in the decomposition (17) of §5 there is only the term $\varepsilon=1$). From the dimension formulas we see that the first three cases are $(m,k) = (2,8)$, $(2,10)$ and $(3,10)$ and that $\dim J_{k,m}^{cusp} = 1$ in all three cases.

For $\phi = \sum c(n,r)\,q^n\zeta^r \in J_{8,2}^{cusp}$, the first two Taylor coefficients $D_0\phi$ and $D_2\phi$ are cusp forms of weight 8 and 10 and hence vanish, while $D_4\phi$ must be a multiple of $\Delta(\tau)$. Also, $c(n,r)$ depends only on $8n-r^2$ by Theorem 2.2, so $c(n,r) = c(8n-r^2)$ for some sequence of coefficients $c(N) = c_{8,2}(N)$ satisfying

$$\sum_r c(8n-r^2) = 0\,, \quad \sum_r r^2\, c(8n-r^2) = 0\,, \quad \sum_r r^4\, c(8n-r^2) = 24\,\tau(n)$$

for all n (the constant 24 was chosen for convenience). These recursions can be solved uniquely (at each stage the numbers $c(8n)$, $c(8n-1)$ and $c(8n-4)$ are expressed in terms of $c(N)$ with $N \le 8(n-1)$). We find the values

N	4	7	8	12	15	16	20	23	24	28	31	32
$c_{8,2}(N)$	1	-4	6	0	36	-64	-84	-84	252	512	-168	-384

N	36	39	40	44	47	48
$c_{8,2}(N)$	-1107	972	28	0	-504	0

These coefficients could also be calculated from the fact that $\phi = \frac{1}{12} Z = \phi_{10,1}^2/\Delta$, where Z is the form of Theorem 8.2 and $\phi_{10,1} \in J_{10,1}$ the cusp form given in §3. The most striking thing in the table is the occurrence of zero entries. To explain them, we note that the form $\sum c(N)\, q^N$ lies in the space

$$M^+_{15/2}(2) = \{ h \in M_{15/2}(\Gamma_0(8)) \mid h = \sum_{\substack{N \ge 0 \\ N \equiv 0,4,7\ (8)}} c(N)\, q^N \}$$

by Theorem 5.6 and is a new form (since $M_{15/2}(1) = \{0\}$); it then follows from the results of Kohnen [12] that $c(N) = 0$ for all N of the form $4^s(4r+3)$.

For $\phi \in J^{cusp}_{10,2}$ the same method works; now $D_0\phi = 0$, $D_2\phi = \text{const.}\cdot\Delta$, $D_4\phi = 0$ and we find $\phi = \sum c_{10,2}(8n-r^2)\, q^n \zeta^r$ with the first coefficients given by

N	4	7	8	12	15	16	20	23	24	28	31	32	36	39	40
$c_{10,2}(N)$	1	8	-18	-120	120	-16	900	-1368	732	-2176	3408	288	2277	3432	-19940

This time there are no zero coefficients because ϕ is not a new form: it is $-\frac{1}{2}\phi_{10,1}|V_2$. By Theorem 4.2, this is equivalent to the formula

$$c_{10,2}(N) = -\frac{1}{2}\left(c_{10,1}(N) - 2^9 c_{10,1}(N/4)\right)$$

(with the convention $c_{10,1}(n) = 0$ for $n \notin \mathbb{Z}$), and one can check this by comparing the above values with those given in the Table in §3.

Finally, for $m=3$ the first cusp form has weight 6; for this form ϕ we must have $D_0\phi = D_2\phi = D_4\phi = 0$ and $D_6\phi$ a multiple of Δ, so

$$\phi = \sum c(12n-r^2)\, q^n \zeta^r , \qquad \sum_r r^{2\nu}\, c(12n-r^2) = \begin{cases} 0 & (\nu \le 2) \\ 360\ \tau(n) & (\nu = 3) \end{cases}$$

(again with 360 chosen for convenience), from which we get the values

N	3	8	11	12	15	20	23	24	27	32	35	36
$c_{6,3}(N)$	1	-6	15	-20	24	-24	-24	60	-81	216	-126	0

We could also compute ϕ as $\phi_{10,1}^3/\Delta^2$. Again the zero coefficient for $N=36$ can be explained by the fact that $\sum c(N)\, q^N \in M^+_{11/2}(3)$ is a new form and the theory given in [12]; more generally, we would have $c(N)=0$ for all N of the form $9^s(3r+1)$.

2. <u>k odd</u>, <u>m prime</u>.

This is the next simplest example, since again the decomposition of $J_{k,m}$ given in (17) of §5 reduces to a single term, but no longer with ε trivial. The first case occurring, as we saw in the discussion preceding Theorem 9.4, is $m=2$, $k=11$. The corresponding form $\phi = \phi_{11,2}$ is given explicitly by (13), but although this formula can be used to compute the Fourier coefficients of ϕ it is easier to proceed as in the examples above. Indeed, if we write $\phi = \sum c(n,r)\, q^n \zeta^r$ then $c(n,r) = -c(n,-r)$ (since the weight is odd) and $c(n,r)$ depends only on $8n-r^2$ and on $r \pmod 4$; together these facts imply that $c(n,r) = 0$ for r even and that $c(n,r)$ can be written as $\left(\frac{-4}{r}\right) c(8n-r^2)$ for some coefficients $c(N)$ which are non-zero at most for $N \equiv 7 \pmod 8$. Then $D_1\phi \in S_{12} = \mathbb{C}\cdot\Delta$ implies that $\sum r\, c(n,r)$ is a multiple of $\tau(n)$, so (with a suitable normalization)

$$\sum_{0<r<\sqrt{8n}} \left(\frac{-4}{r}\right) r\, c(8n-r^2) = \tau(n) .$$

This recursion determines the $c(N)$ uniquely and we get the table

N	7	15	23	31	39	47
$c_{11,2}(N)$	1	-21	189	-910	2205	-378

.

But here we can do more. Indeed, equation (14) is equivalent to

$$\Big(\sum_{\substack{N>0 \\ N\equiv 7 \ (\mathrm{mod}\ 8)}} c(n)\, q^{\frac{N}{8}}\Big)\Big(\sum_{r>0} \big(\tfrac{-4}{r}\big)\, r\, q^{\frac{r^2}{8}}\Big) \;=\; \sum_{n>0} \tau(n)\, q^n \;.$$

The first factor on the left is $\eta(\tau)^3$ by Jacobi's identity and the expression on the right is $\Delta(\tau) = \eta(\tau)^{24}$. Hence we get, instead of just a table, the closed formula

$$h_{\tilde{\varepsilon}}(\tau) \;:=\; \sum_{N>0} c_{11,2}(N)\, q^N \;=\; \eta(8\tau)^{21} \quad ,$$

where $\tilde{\varepsilon}$ as in §5 is an extension of ε to a Dirichlet character modulo 4 (here necessarily $\tilde{\varepsilon} = (\frac{-4}{\cdot})$) and $h_{\tilde{\varepsilon}}$ the form of weight $10\frac{1}{2}$ defined in §5. The fact that $h_{\tilde{\varepsilon}}(\tau/8)$ is a modular form on all $SL_2(\mathbb{Z})$ (with multiplier) rather than on $\Gamma_0(16)$ is due to the fact that the representation C_ε of $SL_2(\mathbb{Z})$ mentioned at the end of §5 is one-dimensional in this special case (in general for m prime and k odd it would have dimension $m-1$).

3. <u>k even</u>, <u>m a product of two primes</u>

This is the first case where the decomposition of $J_{k,m}$ has more than one piece; we are interested in the summand $J_{k,m}^{--}$ corresponding to non-trivial ε , since the space $J_{k,m}^{++}$ can be understood by Theorem 5.6 and the following remarks. The first composite m is 6 and the first weight k with $J_{k,6}^{--} \neq \{0\}$ is 12 . To see this, note that a function $\phi \in J_{k,6}^{--}$ has a Fourier development of the form $\sum c_r(24n-r^2)\, q^n \zeta^r$ where $c_\mu(N)$ depends only on $\mu \pmod{12}$ and changes sign if μ is replaced by 5μ or 7μ . It follows that $c_\mu(N) = 0$ for μ not prime to 12 (since then μ is congruent to 5μ or 7μ modulo 12) and that in general $c_\mu(N)$ can be written as $(\frac{12}{\mu})\, c(N)$ for some coefficients $c(N)$ depending only on N and vanishing unless $N \equiv 23 \pmod{24}$. In particular, $c_\mu(0) = 0$ for all μ , so ϕ is automatically a cusp form, and since ϕ is determined by $D_0\phi = \phi(\tau,0) \in S_k$ the first possible weight is $k = 12$. For the form with $k = 12$, and normalized by $\phi(\tau,0) = 2\,\Delta(\tau)$, we have

$$\phi = \sum_{n,r} \left(\frac{12}{r}\right) c(24n-r^2)\, q^n \zeta^r \;, \qquad \sum_{r>0} \left(\frac{12}{r}\right) c(24n-r^2) = \tau(n) \;,$$

which can be solved recursively to give the numerical values

N	23	47	71	95	119
$c_{12,6}(N)$	1	−23	230	−1265	3795

or explicitly in the same way as was done for the last example: we write the recursion for c as

$$\left(\sum_{r>0} \left(\frac{12}{r}\right) q^{\frac{r^2}{24}}\right)\left(\sum_{N>0} c(N)\, q^{\frac{N}{24}}\right) = \sum_{n>0} \tau(n)\, q^n$$

and observe that the first factor is $\eta(\tau)$ (Euler's identity), so

$$h_{\tilde\varepsilon}(\tau) := \sum_{\substack{N>0 \\ N \equiv 23\ (24)}} c(N)\, q^N = \eta(24\tau)^{23} \qquad \left(\tilde\varepsilon = \left(\frac{12}{\cdot}\right)\right)$$

(again $h_{\tilde\varepsilon}(\tau/24)$ is a modular form on all $SL_2(\mathbb{Z})$ because C_ε is one-dimensional). Similarly, if $\phi \in J_{k,6}^{\sim}$ with $\frac{1}{2}\,\phi(\tau,0) = f(\tau) = \sum a(n)\, q^n \in M_k$ then the recursion relation for the $c(N)$ gives

$$\left(\sum_r \left(\frac{12}{r}\right) q^{r^2/24}\right)\left(\sum_N c(N)\, q^{N/24}\right) = \sum_n a(n)\, q^n$$

or $h_{\tilde\varepsilon}(\frac{\tau}{24}) = \eta(\tau)^{-1} f(\tau)$. Since f is a cusp form we have $f = \Delta g$ for some $g \in M_{k-12}$, and then $h_{\tilde\varepsilon}(\frac{\tau}{24}) = \eta(\tau)^{23} g(\tau)$. It follows that $\phi(\tau,z)$ is $g(\tau)$ times $\phi_{12,6}(\tau,z)$, i.e. $J_{*,6}^{\sim}$ is a free module of rank 1 over M_* with generator $\phi_{12,6}$. We could also have seen this by a direct argument; more generally, it is not hard to modify the proof of Theorem 8.4 to show that $J_{*,m}^{\varepsilon}$ is a free module over M_* for every character ε and to give a formula for its rank.

4. <u>k = 2</u>

The case of Jacobi forms of weight 2 is particular interesting, both because this is the smallest even weight occurring and because of the connecti with Heegner points mentioned briefly in the Introduction. However, it is als the most difficult case, since we do not even have the Eisenstein series $E_{k,1}$

to get things off the ground. The lower bound for $J_{2,m}$ obtained from the Corollary to Theorem 9.2 is

(14)

m	25	37	43	49	53	61	64	67	73	79	81	83	85	89	91	93	97	98	100
$\dim J_{2,m} \geq$	1	1	1	2	1	1	1	2	2	1	2	1	1	1	1	1	3	1	1

the bounds for the omitted m being zero or negative. The cases $m = 25, 49, 64, 81, 100$ are accounted for by the Eisenstein series of weight 2 (cf. the remarks following (11) of §2). The other dimensions are equal to those of $S_2^{new}(\Gamma^*(m))$ (given in [5], Table 5) when m is prime and one less when m is a product of 2 primes, the first in accordance with the relation between $J_{k,m}$ and $M_{2k-2}(\Gamma^*(m))$ discussed in the Introduction (eq. (9)) and in §10. This relation will be proved in Chapter IV by trace formula methods for $k>2$, but the case $k=2$ presents extra difficulties, so the numbers (14) are not necessarily the true values of $J_{2,m}$. In any case, they are lower bounds, so we should be able to find, for instance, a Jacobi cusp form of weight 2 and index 37.

Let us look for a cusp form $\phi \in J_{2,37}$. By Theorem 2.2, the coefficient of $q^n\zeta^r$ in ϕ depends only on $148n - r^2$, i.e. we have

$$\phi(\tau,z) = \sum_{\substack{n,r \\ 148n>r^2}} c(148n - r^2)\, q^n \zeta^r$$

with some coefficients $c(N)$ ($N>0$, $N \equiv 0$ or $3 \pmod 4$, $(\frac{N}{37}) = 0$ or 1). However, to find these coefficients by either of the algorithms mentioned after Theorem 9.3 would involve an impossible amount of computing. For instance, using the Taylor-development operator in (8) would involve constructing an explicit map into the space $S_2 \oplus S_4 \oplus \ldots \oplus S_{76}$ of dimension 102 and then solving a 102×102 system of equations with coefficients which are very large and hard to compute. We therefore use another method. The function

$$h(\tau) = \sum_{N>0} c(N)\, q^N$$

belongs to $M_{3/2}(37)$ by Theorem 5.6, and an easy calculation shows that the form

$$(15) \qquad h(\tau)\theta(\tau)|U_4 = \sum_{n=1}^{\infty} a(n)\, q^n \ , \quad a(n) = \sum_{|r|<\sqrt{4n}} c(4n-r^2) \ ,$$

is a cusp form of level 37 (and weight 2). Since $S_2(\Gamma_0(37))$ is only two-dimensional, and the Fourier coefficients of a basis have been tabulated (e.g. in [5]), we have a good chance to get the coefficients $a(n)$, but at first sight this seems insufficient to determine the $c(N)$, since the identi (15) gives a recursion which at each new step involves two new coefficients $c(4n)$ and $c(4n-1)$. What saves us is that we know a priori that about 1/2 of the $c(N)$ are 0 (namely those with N a quadratic non-residue of 37) so that <u>on the average</u> this one recursion for two coefficients $c(4n)$, c(4n- will suffice. Let us see how this works in practice. The first cases of our recursion, combined with the vanishing of $c(N)$ for $\left(\frac{N}{37}\right) = -1$, give

$$\begin{aligned}
c(4) + 2c(3) &= a(1) \ , \\
2c(7) + 2c(4) &= a(2) \ , \\
c(12) + 2c(11) + 2c(3) &= a(3) \ , \\
c(16) + 2c(12) + 2c(7) &= a(4) \ , \\
2c(16) + 2c(11) + 2c(4) &= a(5) \ , \\
0 &= a(6) \ .
\end{aligned}$$

Looking at the tables in [5], we see that the last equation determines the form $\sum a(n)q^n \in S_2(\Gamma_0(37))$ up to a constant: it must be a multiple of the eigenform

$$f_1(z) = q + q^3 - 2q^4 - q^7 - 2q^9 + 3q^{11} - 2q^{12} - 4q^{13} + 4q^{16} + \dots$$

At this stage we have 5 non-trivial equations involving 7 unknowns a(1),

$c(3)$, $c(4)$, $c(7)$, $c(11)$, $c(12)$, $c(16)$. Proceeding further, we find that at $n=15$ the number of equations overtakes the number of unknowns, so that we suddenly get all $c(N)$ up to $N=60$. For various higher n, we again have enough equations. However, eventually we will have too few, since (15) provides an average of one equation for every two $c(N)$ with $N \equiv 0$ or $3 \pmod 4$, while the proportion of $c(N)$ known a priori to vanish is only $\frac{18}{37} < \frac{1}{2}$. However, we have some extra information. Namely, for m prime the space $M^{+}_{3/2}(m)$ as defined in Theorem 5.6 is by Kohnen's work isomorphic as a Hecke module to $M_2(\Gamma^*(m)) = \{f \in M_2(\Gamma_0(m)) \mid f|W_m = f\}$ (cf. proof of Theorem 5.6). For $m=37$ both spaces are 1-dimensional, so our form $h(\tau) \in M^{+}_{3/2}(37)$ is a Hecke eigenform with the same eigenvalues as the second eigenform

$$f_2(z) = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + \dots$$

in $S_2(\Gamma_0(37))$. This means that we can write all coefficients $c(N)$ as multiples of those with $-N$ a fundamental discriminant, thus reducing the number of unknown coefficients by a further factor of $\zeta(2)^{-1} \approx 0.61$. Using this, we find that we now have a surfeit of equations to determine the coefficients $c(N)$ (so many, in fact, that we can simultaneously solve for the coefficients of q^p, p prime, in f_1 and f_2 and hence dispense with the Antwerp tables entirely!). We tabulate the resulting values up to $N=250$, since the example is of interest in connection with recent work of B. Gross and the second author on "Heegner points" and will be cited in a forthcoming paper of theirs.

n	$c_{2,37}(n)$	n	$c_{2,37}(n)$
3	1	127	1
4	1	132	3
7	-1	136	4
11	1	139	0
12	-1	144	4
16	-2	147	-2
27	-3	148	-3
28	3	151	-2
36	-2	152	-2
40	2	155	2
44	-1	159	1
47	-1	160	-4
48	0	164	-1
63	2	175	1
64	2	176	0
67	6	184	0
71	1	188	3
75	-1	192	2
83	-1	195	2
84	-1	196	0
95	0	211	3
99	-4	212	3
100	-3	215	0
104	0	219	1
107	0	223	-3
108	3	231	-1
111	1	232	6
112	-4	243	6
115	-6	247	-4
120	-2	248	0
123	3		

Table 3. Coefficients of the form in $J_{2,37}$

§10. Discussion of the formula for $\dim J_{k,m}$

For $k, m \in \mathbb{N}$ set

$$j(k,m) = \begin{cases} \sum_{\nu=0}^{m} \left(\dim M_{k+2\nu} - \left\lceil \frac{\nu^2}{4m} \right\rceil \right) & (k \text{ even}) , \\ \sum_{\nu=1}^{m-1} \left(\dim M_{k+2\nu-1} - \left\lceil \frac{\nu^2}{4m} \right\rceil \right) & (k \text{ odd}) . \end{cases}$$

In §9 we showed that $\dim J_{k,m} \geq j(k,m)$ for all k and $\dim J_{k,m} = j(k,m)$ for k sufficiently large (namely for $k \geq m$; this will be improved to $k \geq 3$ in Chapter IV by means of a trace formula). The purpose of this section is to write $j(k,m)$ in terms of standard arithmetical functions and to relate it to the dimensions of known spaces of modular forms. More precisely, we will prove the following result.

Theorem 10.1. For $k > 2$ and $m \in \mathbb{N}$ define $j_0(k,m)$ inductively by

$$(1) \qquad j(k,m) = \sum_{d|m} \Big(\sum_{\ell^2 | d} 1 \Big) \, j_0(k, \tfrac{m}{d}) \; .$$

Then $j_0(k,m)$ equals the dimension of the space $M_{2k-2}^{new}(\Gamma_0(m))^{(-1)^k}$ of new forms of weight $2k-2$ on $\Gamma_0(m)$ with eigenvalue $(-1)^k$ under the Fricke-Atkin-Lehner involution

$$W_m : \; f(\tau) \mapsto m^{-k+1} \, \tau^{-2k+2} \, f(-\tfrac{1}{m\tau}) \; .$$

Similarly, if $j^{cusp}(k,m)$ is defined like $j(k,m)$ but with $\left\lceil \frac{\nu^2}{4m} \right\rceil$ replaced by $\left\lfloor \frac{\nu^2}{4m} \right\rfloor + 1$, and j_0^{cusp} by the obvious analogue of (1), then $j_0^{cusp}(k,m)$ equals $\dim S_{2k-2}^{new}(\Gamma_0(m))^{(-1)^k}$, the dimension of the corresponding space of cusp forms.

Note that the difference $j(k,m) - j^{cusp}(k,m)$ is the number of ν with $0 \leq \nu \leq m$ (resp. $0 < \nu < m$ if k is odd) and $\nu^2 \equiv 0 \pmod{4m}$, which equals $\left\lfloor \frac{b+2}{2} \right\rfloor$ (resp. $\left\lfloor \frac{b-1}{2} \right\rfloor$), where b is the largest integer whose square divides m, and this is just the dimension of $J_{k,m}^{Eis}$ (Theorems 2.3 and 2.4). In view of Theorem 4.4, Theorem 10.1 has a plausible interpretation as saying that the decomposition

$$J_{k,m} = \sum_{d|m} \sum_{\ell^2|d} J_{k,m}^{new}|U_\ell V_{d\ell^{-2}}$$

given there is direct and that there is an isomorphism between $J_{k,m}^{new}$ and $M_{2k-2}^{new}(\Gamma_0(m))$ preserving cusp forms. In Chapter IV we will prove that this is indeed the case by proving the analogue of Theorem 10.1 with dimensions replaced by traces of Hecke operators. Here we restrict ourselves to the equality of dimensions.

Proof: We carry out the proof only for k even, leaving the entirely analogous case of odd k to the reader.

Our first objective is to write the formula defining $j(k,m)$ in terms of familiar arithmetic functions. It is convenient to replace the different rounding functions in the definitions of j and of j^{cusp} by their average. We therefore use the notation

$$((x)) = x - \frac{\lceil x\rceil + \lfloor x\rfloor}{2} = \begin{cases} \alpha - \frac{1}{2} & \text{if } x \in \mathbb{Z}+\alpha,\ 0<\alpha<1, \\ 0 & \text{if } x \in \mathbb{Z}; \end{cases}$$

then

$$(2) \quad j(k,m) = \sum_{\nu=0}^{m} \{ \dim M_{k+2\nu} - \frac{\nu^2}{4m} - \frac{1}{2} + ((\frac{\nu^2}{4m})) \} + \frac{1}{2}\left\lfloor\frac{b+2}{2}\right\rfloor$$

(b as above) and $j^{cusp}(k,m)$ is given by a similar formula with the sign of the last term reversed. We substitute into this the formula

$$\dim M_k = \frac{k+5}{12} - \frac{1}{3}\chi_3(k-1) - \frac{1}{4}\chi_4(k-1),$$

where χ_3 and χ_4 denote the non-trivial Dirichlet characters modulo 3 and modulo 4, and calculate:

$$\sum_{\nu=0}^{m} (\frac{k+2\nu+5}{12} - \frac{\nu^2}{4m} - \frac{1}{2}) = \frac{(2k-3)(m+1)}{24},$$

$$(3) \quad -\frac{1}{3}\sum_{\nu=0}^{m} \chi_3(k+2\nu-1) = \begin{cases} 1/3 & \text{if } m \equiv k \not\equiv 2 \pmod 3, \\ -1/3 & \text{if } m \not\equiv k \equiv 2 \pmod 3, \\ 0 & \text{otherwise}, \end{cases}$$

$$(4) \quad -\frac{1}{4}\sum_{\nu=0}^{m} \chi_4(k+2\nu-1) = \begin{cases} 1/4 & \text{if } m \text{ is even, } k \equiv 0 \pmod 4, \\ -1/4 & \text{if } m \text{ is even, } k \equiv 2 \pmod 4, \\ 0 & \text{if } m \text{ is odd}. \end{cases}$$

Also, since $((x))$ is periodic with period 1, the function $((\frac{\nu^2}{4m}))$ is periodic with period $2m$, so we can write

$$\sum_{\nu=0}^{m} ((\frac{\nu^2}{4m})) = \frac{1}{2} \sum_{\nu \,(\mathrm{mod}\ 2m)} ((\frac{\nu^2}{4m})) + \frac{1}{2}((\frac{m}{4})) .$$

Finally, the last term in (2) equals $\frac{b}{4}+\frac{1}{2}$ or $\frac{b}{4}+\frac{1}{4}$ depending whether or not 4 divides m. Hence (2) can be written

$$(5) \qquad j(k,m) = \sum_{i=1}^{8} j_i(k,m)$$

with

$$j_1(k,m) = \frac{2k-3}{24}\, m , \qquad j_2(k,m) = \frac{2k-3}{24} ,$$

$$j_3(k,m) = \text{RHS of (3)} , \qquad j_4(k,m) = \text{RHS of (4)} ,$$

$$j_5(k,m) = \frac{1}{2} \sum_{\nu \,(\mathrm{mod}\ 2m)} ((\frac{\nu^2}{4m})) , \qquad j_6(k,m) = \frac{1}{2}((\frac{m}{4})) = -\frac{1}{8}\chi_4(m) ,$$

$$j_7(k,m) = \frac{1}{4}\, b , \qquad j_8(k,m) = \begin{cases} 1/4 & \text{if } 4\nmid m , \\ 1/2 & \text{if } 4\mid m , \end{cases}$$

and $j^{\mathrm{cusp}}(k,m)$ is given by the same formula but with the signs of j_7 and j_8 reversed. We have now practically achieved our goal of writing $j(k,m)$ in terms of familiar arithmetical functions, for all the j_i except j_5 are extremely simple functions of m and k (periodic functions of m and k or products of a polynomial in k and a multiplicative function of m). We shall now see that the function j_5, which depends only on m, can also be expressed in terms of a well-known — though less elementary — arithmetic function.

<u>Lemma</u>. <u>Define</u> $h'(d)$ <u>for</u> $d \neq -3,-4$ <u>as the class number of positive definite binary quadratic forms of discriminant</u> d (<u>so</u> $h'(d)=0$ <u>if</u> $d>0$ <u>or</u> $d \not\equiv 0,1 \pmod 4$), $h'(-3)=1/3$, $h'(-4)=1/2$. <u>Then for any natural number</u> N,

$$(6) \qquad \sum_{\nu \,(\mathrm{mod}\ N)} ((\frac{\nu^2}{N})) = -\sum_{d \mid N} h'(-d) .$$

<u>In particular</u>,

$$j_5(k,m) = -\frac{1}{4} \sum_{d|4m} h'(-d) .$$

Proof: Suppose $N = p$ is a prime. If $p = 2$ or $p \equiv 1 \pmod 4$, then both sides of (6) are zero, the left side because $((\ ))$ is an odd function and -1 is a quadratic residue of p, the right side because p has no divisors congruent to 0 or 3 (mod 4). If $p \equiv 3 \pmod 4$, then the left-hand side of (6) equals

$$\sum_{n \pmod p} \left(\left(\frac{n}{p}\right)\right) \sum_{\substack{\nu \pmod p \\ \nu^2 \equiv n \pmod p}} 1 = \sum_{n \pmod p} \left(\left(\frac{n}{p}\right)\right) \left(1 + \left(\frac{n}{p}\right)\right)$$

$$= \sum_{n \pmod p} \left(\left(\frac{n}{p}\right)\right) \left(\frac{n}{p}\right)$$

$$= \sum_{n=1}^{p-1} \left(\frac{n}{p} - \frac{1}{2}\right) \left(\frac{n}{p}\right)$$

$$= \frac{1}{p} \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) n ,$$

and this equals $-h'(-p)$ by Dirichlet's class number formula. The case of composite N can be handled similarly by expressing the square-roots of $n \pmod N$ as $\sum_\chi \chi(n)$, where the sum is over all quadratic Dirichlet characters $\chi \pmod N$ (this formula must be modified if $(n,N) > 1$) and applying Dirichlet's formula to each χ separately.

Equation (5) and the Lemma give a very explicit formula for $j(k,m)$; to prove Theorem 10.1 we must relate it to the formula for the dimension of $M_{2k-2}(\Gamma_0(m))^{+1}$. Set

$$d(k,m) = \dim M_k(\Gamma_0(m)) , \qquad w(k,m) = \mathrm{tr}(W_m, M_k(\Gamma_0(m)))$$

and write d_0, w_0 for the corresponding numbers with M_k replaced by M_k^{new}. We have to show that

$$j_0(k,m) = \frac{1}{2}\left(d_0(2k-2,m) + w_0(2k-2,m) \right) ,$$

since this is the dimension of the (+1)-eigenspace of W_m. Equivalently, if we define

$$j'(k,m) = \frac{1}{2} \sum_{d|m} \Big(\sum_{\ell^2|d} 1 \Big) \, d_0(2k-2, m/d) ,$$

(7)

$$j''(k,m) = \frac{1}{2} \sum_{d|m} \Big(\sum_{\ell^2|d} 1 \Big) \, w_0(2k-2, m/d)$$

(which equal one-half of the dimension and trace of W_m on a space lying between $M_{2k-2}^{new}(\Gamma_0(m))$ and $M_{2k-2}(\Gamma_0(m))$), then we must show that j is the sum of j' and j''. Rather than work out everything in terms of new forms, it is convenient to write (7) directly in terms of d and w instead of d_0 and w_0. As is well-known, d and d_0 are related by

$$d(k,m) = \sum_{m'|m} \Big(\sum_{\ell|\frac{m}{m'}} 1 \Big) \, d_0(k,m') \quad ;$$

indeed, this is clear because $M_k(\Gamma_0(m))$ has a basis consisting of the functions $f(\tau) = h(\ell\tau)$ with h a new (eigen)form of some level m' dividing m and ℓ a divisor of $\frac{m}{m'}$. Similarly, w and w_0 are related by

$$w(k,m) = \sum_{\substack{m'|m \\ m/m' = \square}} w_0(k,m') ,$$

because for h and f as above we have

$$h|W_{m'} = \varepsilon h \quad \Rightarrow \quad f|W_m = \varepsilon \left(\frac{m}{m'\ell^2}\right)^{k/2} h\left(\frac{m}{m'\ell}\tau\right)$$

and therefore h gives a contribution ε to $\mathrm{tr}(W_m)$ if $\ell = \frac{m}{m'\ell}$ and 0 otherwise (since then the sum and difference of $\ell^{k/2} h(\ell\tau)$ and $\left(\frac{m}{m'\ell}\right)^{k/2} h\left(\frac{m}{m'\ell}\tau\right)$ are non-zero eigenfunctions of W_m with opposite eigenvalues). Substituting these two formulas into (7) and carrying out an easy exercise in multiplicative functions, we deduce

(8) $$j'(k,m) = \frac{1}{2} \sum_{m'|m} \lambda\left(\frac{m}{m'}\right) d(2k-2, m') \quad ,$$

(9) $$j''(k,m) = \frac{1}{2} \sum_{m'|m} w(2k-2, m') \, ,$$

where λ is the Liouville function ($\lambda(\Pi p_i^{\nu_i}) = (-1)^{\Sigma \nu_i}$). Into these formulas we have to substitute the known formulas for $d(k,m)$ and $w(k,m)$ and then compare $j'+j''$ with our formula for j .

The formula for $d(k,m)$ is well-known (see e.g. [8]); for $k>2$ we can write it as

$$(10) \qquad d(k,m) = \sum_{i=1}^{4} c_i(k)\, f_i(m)$$

where

$$c_1(k) = \frac{k-1}{12}\,, \quad c_2(k) = \frac{1}{2}\,, \quad c_3(k) = -\frac{1}{3}\chi_3(k-1)\,, \quad c_4(k) = -\frac{1}{4}\chi_4(k$$

and the f_i are multiplicative functions:

$$f_1(m) = m \prod_{p|m} \left(1+\frac{1}{p}\right),$$

$$f_2(m) = \prod_{\substack{p^\nu \| m \\ \nu>0}} \left(p^{\lfloor \nu/2 \rfloor} + p^{\lfloor (\nu-1)/2 \rfloor}\right),$$

$$f_3(m) = \prod_{p|m} (1+\chi_3(p)) \times \begin{cases} 1 & \text{if } 9 \nmid m \\ 0 & \text{if } 9 | m \end{cases},$$

$$f_4(m) = \prod_{p|m} (1+\chi_4(p)) \times \begin{cases} 1 & \text{if } 4 \nmid m \\ 0 & \text{if } 4 | m \end{cases}.$$

These formulas can be written more uniformly as

$$f_1 = \prod_p (p^\nu + p^{\nu-1}), \quad f_2 = \prod_p \left(p^{\lfloor \frac{\nu}{2} \rfloor} + p^{\lfloor \frac{\nu-1}{2} \rfloor}\right), \quad f_3 = \prod_p (\chi_3(p^\nu) + \chi_3(p^{\nu-1})),$$

$$f_4 = \prod_p (\chi_4(p^\nu) + \chi_4(p^{\nu-1})),$$

where in each case the product is over primes p dividing m and ν is the exponent of p in m . Thus each f_i has the form

$$f_i(m) = \prod_p (g_i(p^\nu) + g_i(p^{\nu-1})) = \sum_{\substack{m'|m \\ m/m' \text{ squarefree}}} g_i(m)$$

for some much simpler multiplicative function g_i :

$$g_1(m) = m, \quad g_2(m) = \prod_p p^{\lfloor \frac{\nu}{2} \rfloor} = b, \quad g_3(m) = \chi_3(m), \quad g_4(m) = \chi_4(m)$$

(b as before the largest integer such that $b^2|m$). On the other hand, the inversion of (8) is

$$(11)\qquad d(2k-2,m) = 2 \sum_{\substack{m'|m \\ m/m' \text{ squarefree}}} j'(k,m) ,$$

so this means that $j'(k,m)$ is given by

$$(12)\qquad \begin{aligned} j'(k,m) &= \frac{1}{2}\sum_{i=1}^{4} c_i(2k-2)\ g_i(m) \\ &= \frac{2k-3}{24}m + \frac{1}{4}b + \frac{1}{6}\chi_3(k)\chi_3(m) - \frac{1}{8}\chi_4(m) . \end{aligned}$$

The first two terms and the last are the functions j_1, j_7 and j_6 of (5), respectively. (Note how much simpler (12) is than the more familiar formula (10); the fact that each of the four multiplicative functions $f_i(m)$ occurring in the dimension formula (10) has a natural decomposition parallel to (11) suggests that, in some respects at least, the theory of Jacobi forms is <u>simpler</u> than the usual theory of modular forms of higher level. We shall see the same phenomenon again in Chapter IV, where almost every term of the trace formula is simpler for $J_{k,m}$ than for either $M_{2k-2}(\Gamma_0(m))$ or $M_{2k-2}^{new}(\Gamma_0(m))$.)

We still have to look at the formula for $tr(W_m)$. Since this is given in the literature only for cusp forms, we first consider the Eisenstein series contribution separately. The term $i=2$ in (10) is the contribution from the cusps ($f_2(m)$ is the number of cusps of $\Gamma_0(m)$) and would change sign if we replaced $d(k,m)$ by $\dim S_{2k-2}(\Gamma_0(m))$, just as the term $j_7(k,m)$ in (5) to which it corresponded changes sign when we replace j by j^{cusp} . The number $f_2(m)$ can be written $\sum_{d|m} \phi((d,\frac{m}{d}))$, where ϕ is the Euler function; this formula arises by attaching to a cusp $\frac{x}{y}\in(\mathbb{Q}\cup\{\infty\})/\Gamma_0(m)$ (x,y coprime integers) the invariants $d=(y,m)$ and $x\cdot\frac{y}{d} \pmod{(d,\frac{m}{d})}$. To each cusp corresponds an Eisenstein series, so the trace of W_m on M_{2k-2}^{Eis} is just the number of fixed points of W_m on the set of cusps. Since W_m sends $\frac{x}{y}$ to $\frac{-y}{mx}=\frac{x'}{y'}$ with $x'=-\frac{y}{d}$, $y'=\frac{m}{d}x$, $d'=(y',m)=\frac{m}{d}$, we see that a fixed point occurs only if $d=\frac{m}{d}$ (so $m=d^2$) and $x'\frac{y'}{d}\equiv x\frac{y}{d} \pmod d$; since on the other hand $xy=-x'y'$ and $x\frac{y}{d}$ is prime to d, this can only happen for $d=1$ or $d=2$. Hence

$$\mathrm{tr}(W_m, M_k^{Eis}(\Gamma_0(m))) = \begin{cases} 1 & \text{if } m=1 \text{ or } 4 , \\ 0 & \text{otherwise} , \end{cases}$$

and putting this into (9) gives a contribution

$$\begin{cases} 1/2 & \text{if } 4\nmid m \\ 1 & \text{if } 4|m \end{cases} = 2\, j_8(k,m)$$

to $j''(k,m)$, as required (notice that $j^{Eis}=2(j_7+j_8)$ in the notation of equation (5)).

Finally, the formula for the trace of W_m on $S_k(\Gamma_0(m))$ (given in [38] and for $k=2$ essentially going back to Fricke, who computed the number of fixed points of $\begin{pmatrix}0&-1\\m&0\end{pmatrix}$ on $H/\Gamma_0(m)$) can be written

$$\mathrm{tr}(W_m,M_{2k-2}(\Gamma_0(m))) = \frac{1}{2}\,\mathrm{tr}(W_m, M_{2k-2}^{Eis}(\Gamma_0(m)))$$

$$+ \begin{cases} -\frac{1}{2}h'(-4m)-\frac{1}{2}h'(-m) & \text{if } m\equiv 3 \pmod 4 \\ -\frac{1}{2}h'(-4m) & \text{otherwise} \end{cases}$$

$$+ \begin{cases} 2/3 & \text{if } m=3,\ 3|k \\ -1/3 & \text{if } m=3,\ 3\nmid k \\ 1/2 & \text{if } m=2,\ k\equiv 0 \pmod 4 \\ -1/2 & \text{if } m=2,\ k\equiv 2 \pmod 4 \\ \frac{2k-3}{12}+\frac{1}{3}\chi_3(k) & \text{if } m=1 \end{cases}$$

(the correction terms for $m=2$ and 3 come from extra fixed points of W_m ; the large correction for $m=1$, of course, comes from the fact that $W_1=1$ and hence $w(k,1)=d(k,1)$). Inserting this into (9) we find

$$j''(k,m) = \frac{1}{2}j''^{Eis}(k,m) - \frac{1}{4}\sum_{d|4m} h'(-4d) + \begin{cases} (-1)^{k/2}/4 & \text{if } 2|m \\ 0 & \text{if } 2\nmid m \end{cases}$$

$$+ \frac{2k-3}{24} + \frac{1}{6}\chi_3(k) + \begin{cases} 1/3 & \text{if } 3|m,\ 3|k \\ -1/6 & \text{if } 3|m,\ 3\nmid k \\ 0 & \text{if } 3\nmid m \end{cases}$$

The first term on the right is j_8, by the discussion above, the second equals j_5 by the Lemma, and the third and fourth terms are j_4 and j_2, respectively. Finally, the last two terms together with the so far unaccounted-for term $\frac{1}{6}\chi_3(k)\,\chi_3(m)$ in (12) together add up to j_3, as one checks easily. This completes the proof of Theorem 10.1.

Notice that for m prime the Theorem gives

$$\begin{aligned} j(k,m) &= j_0(k,m) + j_0(k,1) \\ &= \frac{1}{2}\left(d_0(2k-2,m) + w(2k-2,m) \right) + d_0(2k-2,1) \\ &= \frac{1}{2}\left(d(2k-2,m) + w(2k-2,m) \right) \\ &= \dim M_{2k-2}(\Gamma_0(m))^{+1} \\ &= \dim M_{2k-2}(\Gamma^*(m)) , \end{aligned}$$

where $\Gamma^*(m) = \Gamma_0(m) \cup \Gamma_0(m)W_m$ is the normalizer of $\Gamma_0(m)$ in $SL_2(\mathbb{R})$; this is the equation which we used in §4 (eq. (19)).

§11. Zeros of Jacobi forms

To complete the theory developed in this section, we discuss the sets of zeros of Jacobi forms. These zeros are divisors on the algebraic surface $(\mathbb{H}\times\mathbb{C})/\Gamma_1^J$, and we could give a discussion in algebraic-geometric terms, but we prefer to take a fuction-theoretic point of view more in accordance with the rest of the paper.

Let, then, $\phi\in J_{k,m}$ be a non-zero Jacobi form. If $\phi(\tau_0,z)$ vanishes identically in z for some $\tau_0\in\mathbb{H}$, then we can write $\phi = f\phi_1$ where $f(\tau)$ is a modular form vanishing at τ_0 and ϕ_1 is still a holomorphic Jacobi form. Hence we can assume that the zero-divisor of ϕ contains no fibres of $\pi\colon (\mathbb{H}\times\mathbb{C})/\Gamma_1^J \longrightarrow \mathbb{H}/\Gamma_1$. We denote by $\nu(\tau)$ the values of z at which $\phi(\tau,z)$ vanishes; then $\nu(\tau)$ is an infinitely many-valued function. By Theorem 1.2, we know that $\nu(\tau)$ has $2m$ values (counting multiplicity) modulo $\mathbb{Z}\tau+\mathbb{Z}$; near a given point τ_0 of $\mathbb{H}$ we can number these $\pm\nu_1(\tau)$, ..., $\pm\nu_m(\tau)$ so that

$$\phi(\tau,z) = 0 \iff z = \lambda\tau+\mu\pm\nu_j(\tau) \quad \text{for some } \lambda,\ \mu\in\mathbb{Z},\ j\in\{1,\dots,m\}$$

(this numbering cannot be done globally because the restriction of π to the divisor of ϕ is ramified). We want to see what kind of functions ν and the ν_j are. In particular, we will show that they are well-behaved at infinity and that their second derivations are algebraic modular forms (i.e. satisfy algebraic equations over the ring of modular forms) of weight 3.

<u>Theorem 11.1.</u> <u>As</u> $\tau\longrightarrow\infty$, <u>each branch of</u> $\nu(\tau)$ <u>has an expansion</u>

$$(1) \qquad \nu_j(\tau) = \alpha\tau+\beta+\sum_{n=1}^{\infty} c_n\, e^{2\pi i n\delta\tau}$$

<u>for some</u> $\alpha\in\mathbb{Q}$, $\beta\in\mathbb{C}$, <u>and</u> $\delta\in\mathbb{Q}_{>0}$. <u>If</u> M <u>is the denominator of</u> α, <u>then there are</u> $M-1$ <u>further branches of</u> $\nu(\tau)$ <u>with the same</u> α <u>but</u> β <u>replaced by</u> $\beta+\frac{L}{M}$ $(L=1,\dots,M-1)$.

In other words, the branches of $v(\tau)$ near infinity tend with exponential rapidity to a subset of $\mathbb{C}$ of the form

$$\bigcup_s \left((\alpha_s + \mathbb{Z})\tau + \beta_s + M_s^{-1}\mathbb{Z}\right)$$

for some numbers $M_s \in \mathbb{N}$, $\alpha_s \in M_s^{-1}\mathbb{Z}$ and $\beta_s \in \mathbb{C}$ with $\sum_s M_s = 2m$; in a fundamental parallelogram for the action of $\mathbb{Z}\tau+\mathbb{Z}$ on $\mathbb{C}$ such a set could look as in Figure 1 (where $m=5$, $\alpha_1 = \frac{1}{5}$, $\alpha_2 = -\frac{1}{5}$).

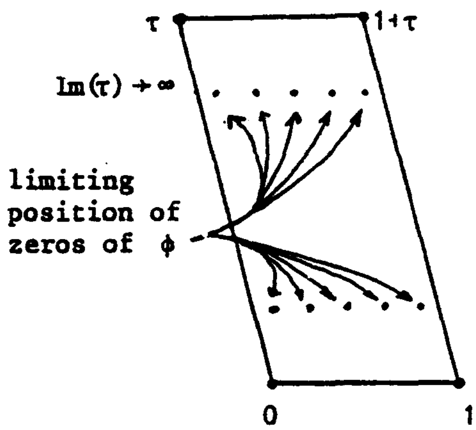

Figure 1

Proof. The method of proof will be to take the expansion (1) as an "Ansatz," substitute it into the expansion of $\phi(\tau,z) = 0$, and show that we obtain the full number (i.e. $2m$ modulo translations by $\mathbb{Z}\tau+\mathbb{Z}$) in this way. Let

$$\phi(\tau,z) = \sum_{(r,n)\in C} c(n,r)\, q^n \zeta^r$$

be the Fourier expansion of ϕ, where $C = \{(r,n) \in \mathbb{Z}^2 \mid c(n,r) \neq 0\}$. Then we want

$$(2) \qquad 0 = \phi(\tau, \alpha\tau + \beta + o(1)) = \sum_{(r,n)\in C} e^{2\pi i r\beta}\, c(n,r)\, q^{\alpha r+n}(1 + o(1)) ,$$

where we have used $o(1)$ to denote any function which is exponentially small as $\tau \longrightarrow \infty$. This equation can hold only if the mimimum value of $\alpha r + n$ ($(n,r) \in C$) is attained for at least two distinct pairs $(n,r) \in C$, i.e. only if $n = \gamma - \alpha r$ is a supporting line of C. Hence $-\alpha$ must be the slope of one of the segments bounding the convex hull $\overline{C}$ of C. By the properties of the coefficients $c(n,r)$, we know that C satisfies the two properties

i) $C \subseteq \{(r,n) \mid n \geq \frac{r^2}{4m}\}$,

ii) C is invariant under the map $\theta: (r,n) \to (r+2m, n+r+m)$

(cf. Figure 2). It follows that the boundary of C consists of an infinite sequence of line segments $(P_s, P_{s+1})_{s \in \mathbb{Z}}$ with $\theta(P_s) = P_{s+S}$ for some $S \in \mathbb{N}$ and the numbering chosen so that $P_s = (r_s, n_s)$ with $\ldots < r_s < r_{s+1} < \ldots$.

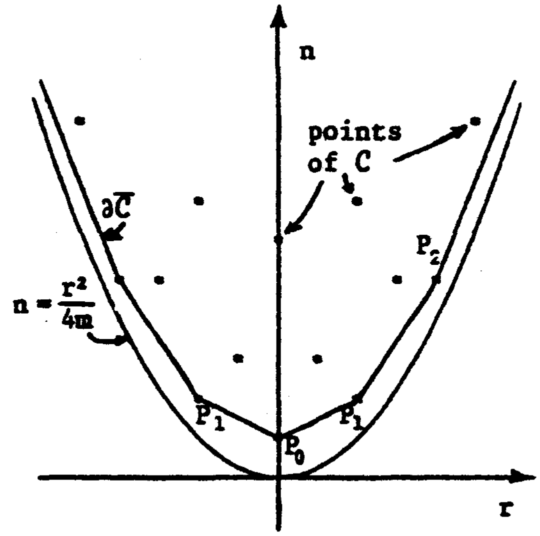

Figure 2

Let α_s denote the negative slope of this segment, i.e.

$$\alpha_s = -\frac{n_{s+1} - n_s}{r_{s+1} - r_s} .$$

Then

$$\alpha_{s+S} = -\frac{(n_{s+1}+r_{s+1}+m) - (n_s+r_s+m)}{(r_{s+1}+2m) - (r_s+2m)} = \alpha_s - 1 ,$$

so the numbers α_s repeat periodically (modulo 1) with period S . If we set

$$\gamma_s = \alpha_s r_s + n_s = \alpha_s r_{s+1} + n_{s+1} ,$$

then the right-hand side of (2) with $\alpha = \alpha_s$ equals

(3) $$q^{\gamma_s} \left(\sum_{(r,n) \in [P_s, P_{s+1}]} c(n,r)\, e^{2\pi i r \beta} + o(1) \right) .$$

This can vanish only if $e^{2\pi i \beta}$ is a root of the polynomial

(4) $$\sum_{j=0}^{r_{s+1}-r_s} c(n_s - \alpha_s j\, , r_s + j)\, x^j$$

of degree $r_{s+1} - r_s$. Conversely, given any such β , we can choose the

δ and c_n in (1) to make the $o(1)$ terms in (3) vanish (Puiseux expansion principle). Thus we get exactly $r_{s+1}-r_s$ solutions (modulo translation of β by $\mathbf{Z}$) of the form (1) with $\alpha=\alpha_s$. Since $\sum_{s \,(\mathrm{mod}\, S)}(r_{s+1}-r_s)=r_{s+S}-r_s=2m$, this accounts for all branches of $\nu(\tau)$ (mod $\mathbf{Z}\tau+\mathbf{Z}$) . The translation invariance $\beta \longrightarrow \beta+\frac{1}{M_s}$ (M_s = denom(α_s)) follows from the fact that the only non-zero coefficients in (4) are those with $n_s-\alpha_s j \in \mathbf{Z}$ or $j \equiv 0 \pmod{M_s}$, so that (4) is actually a polynomial of degree $\frac{r_{s+1}-r_s}{M_s}$ in x^{M_s} and its roots therefore invariant under multiplication by $e^{2\pi i/M_s}$. This completes the proof.

It is not hard to see from the proof of Theorem 11.1 that we have the following converse: any collection of 2m points $(\alpha,\beta) \in \mathbf{Q}\times\mathbf{C}/\mathbf{Z}^2$ invariant under $(\alpha,\beta) \longrightarrow (-\alpha,-\beta)$ and $(\alpha,\beta) \longrightarrow (\alpha,\beta+\alpha)$ occurs as the limiting position of the zeros of some Jacobi form of index m . Another converse is that any m-valued section of $(\mathcal{H}\times\mathbf{C})/\Gamma_1^J$ (i.e. any m-valued function $\nu(\tau) \in (\mathbf{C}/\mathbf{Z}\tau+\mathbf{Z})/(\pm 1)$ satisfying the transformation equation (5) below) whose behaviour at infinity is as described in Theorem 11.1, is the zero-divisor of some holomorphic Jacobi form ϕ of index m . Indeed, if the branches of $\pm\nu(\tau)$ (mod $\mathbf{Z}\tau+\mathbf{Z}$) are numbered $\nu_1(\tau), \ldots, \nu_m(\tau)$ with $\nu_j(\tau)$ identically zero for $j = m'+1, \ldots, m$, then Theorem 3.6 and a little thought imply that we can take

$$\phi(\tau,z) = f(\tau)\phi_{10,1}(\tau,z)^{m-m'}\prod_{j=1}^{m'}\left(\phi_{12,1}(\tau,z)+\frac{3}{\pi^2}p(\tau,\nu_j(\tau))\phi_{10,1}(\tau,z)\right)$$

for some modular form f .

We now consider the behaviour of $\nu(\tau)$ in the interior of $\mathcal{H}$. From the transformation law of ϕ under Γ_1 it is clear that $\phi(\tau,z)=0 \Longleftrightarrow \phi(M\tau,\frac{z}{c\tau+d})=0$ for any $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1$, so we have the equality of many-valued functions

$$(5)\qquad \nu\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-1}\,\nu(\tau)$$

or, written out explicitly,

(6) $\quad \nu_j(M\tau) = \pm(c\tau+d)^{-1}\,\nu_{j'}(\tau) + \lambda M\tau + \mu \qquad (j=1,\ldots,m)$

for some $j' \in \{1,\ldots,m\}$ and $\lambda, \mu \in \mathbb{Z}$. Since the infinitely-many-valuedness of ν arises from the addition of linear functions $\lambda\tau+\mu$ $(\lambda, \mu \in \mathbb{Z})$, the second derivative $\nu''(\tau) = \frac{d^2}{d\tau^2}\nu(\tau)$ is only finitely many-valued. An easy calculation from (5) or (6) gives

$$\nu''\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^3\,\nu(\tau)$$

or

$$\nu_j''\left(\frac{a\tau+b}{c\tau+d}\right) = \pm(c\tau+d)^3\,\nu_{j'}''(\tau) \quad .$$

In other words, $\nu''(\tau)$ transforms under Γ_1 like a (2m-valued) modular form of weight 3 and $\nu''(\tau)^2$ like an m-valued modular form of weight 6. We can eliminate the remaining many-valuedness by setting

(7) $\quad \psi_j(\tau) = \sigma_j(\nu_1''(\tau)^2,\ldots,\nu_m''(\tau)^2) \qquad (j=1,\ldots,m)$,

where σ_j denotes the j^{th} elementary-symmetric polynomial; then ψ_j is single-valued and transforms under Γ_1 like a modular form of weight $6j$. However, ψ_j is not necessarily holomorphic in general: at a ramification point of $\nu(\tau)_0$ the function $\nu'(\tau)$ can have a pole of rational order >-1 and $\nu''(\tau)$ a pole of rational order >-2 (locally, $\nu = c_0 + c_1(\tau-\tau_0)^\lambda + \ldots$ with λ a positive rational number, $\nu'' = c_1\lambda(\lambda-1)(\tau-\tau_0)^{-2+\lambda}$). Hence the function (7) can have a pole of order $>-4j$ at such a point τ_0. Near ∞, it follows from Theorem 11.1 that all $\nu_j''(\tau)^2$ are exponentially small, so the same holds for $\psi_j(\tau)$. We have proved:

<u>Theorem 11.2</u>. <u>Let</u> $\phi(\tau,z)$ <u>be a Jacobi form of index</u> m <u>with zeros given by</u> $z = \pm\nu_j(\tau) \pmod{\mathbb{Z}\tau+\mathbb{Z}}$ $(j=1,\ldots,m)$, <u>and define functions</u> $\psi_j(\tau)$ $(j=1,\ldots,m)$ <u>by (7)</u>. <u>Then each</u> $\psi_j(\tau)$ <u>is a meromorphic modular form of weight</u> $6j$ <u>which vanishes at</u> ∞ <u>and has no poles of order</u> $\geq 4j$.

We can also state this by saying that the zeros of any Jacobi form of index m have the form

$$z = (\lambda+\alpha)\tau + \mu + \beta \pm \int_\tau^{i\infty} (t-\tau)\ \sqrt{\rho(t)}\ dt \qquad (\lambda,\ \mu \in \mathbf{Z})$$

where $\rho(t)$ $(=\nu''(\tau)^2)$ is a root of

$$\rho(\tau)^m - \psi_1(\tau)\rho(\tau)^{m-1} + \ldots \pm \psi_m(\tau) = 0 \quad ,$$

i.e. ρ has weight 6 and is algebraic of degree m over the graded field of modular functions. In the particular case $m=1$, the function $\rho(\tau)=\psi_1(\tau)$ is itself a meromorphic modular form of weight 6. In this case we can make Theorem 11.2 more precise:

<u>Theorem 11.3</u>. <u>Let</u> $\phi(\tau,z) \in J_{k,1}$ <u>be a Jacobi form of index</u> 1 <u>which is not divisible in the ring of Jacobi forms by any modular form of positive weight. Then the zero-set of</u> ϕ <u>has the form</u>

$$\phi(\tau,z)=0 \iff z = \pm\left(\alpha\tau+\beta+\int_\tau^{i\infty}(t-\tau)\ \frac{G(t)}{H(t)^{3/2}}\ dt\right) \qquad (\mathrm{mod}\ \mathbf{Z}\tau+\mathbf{Z})$$

<u>for some</u> $\alpha \in \{0,\frac{1}{2}\}$ <u>and</u> $\beta \in \mathbf{C}$, <u>where</u> H <u>is a modular form of weight</u>

$$k' = \begin{cases} 3k-30 & \text{if } \phi \text{ is a cusp form} \\ 3k-6 & \text{otherwise} \end{cases}$$

<u>and</u> G <u>a modular form of weight</u> $\frac{3}{2}k'+3$ <u>with</u> $\mathrm{ord}_\infty(G) > \frac{3}{2}\,\mathrm{ord}_\infty(H)$.

<u>Proof</u>. The particular case $\phi=\phi_{12,1}$ was treated in [9] (note that the zeros of $\phi_{12,1}$ and of p are the same, by Theorem 3.6): There $k=12$, $k'=6$ (since ϕ is a cusp form), so H and G must be multiples of E_6 and Δ, respectively. The formula actually obtained was

$$p(\tau,z)=0 \iff z=\lambda\tau+\mu\pm\left(\frac{1}{2}+\frac{\log(5+2\sqrt{6})}{2\pi i}+144\pi i\sqrt{6}\int_\tau^{i\infty}(t-\tau)\ \frac{\Delta(t)}{E_6(t)^{3/2}}\ dt\right)\ (\lambda,\mu\in\mathbf{Z})$$

The general case is similar. We write

$$\phi(\tau,z) = f(\tau)E_{4,1}(\tau,z) + g(\tau)E_{6,1}(\tau,z) \qquad (f\in M_{k-4},\ g\in M_{k-6})$$

(Theorem 3.5) to get

$$\phi(\tau,z) = 0 \iff -\frac{3}{\pi^2}\,p(\tau,z) = \frac{\phi_{12}(\tau,z)}{\phi_{10}(\tau,z)} = \frac{E_4(\tau)^2 g(\tau)+E_6(\tau)f(\tau)}{E_6(\tau)g(\tau)+E_4(\tau)f(\tau)}$$

(Theorem 3.6). The only poles of $\nu''(\tau)^2$ occur where the two branches $\pm\nu(\tau)$ (mod $\mathbb{Z}\tau+\mathbb{Z}$) meet, i.e. at points τ_0 where the zero of $\phi(\tau,z)$ is at a 2-division point $z\in\frac{1}{2}(\mathbb{Z}\tau+\mathbb{Z})$; at such a point, the cubic polynomial

$$4p^3(\tau) - \frac{4\pi^4}{3}E_4(\tau)p(\tau,z) - \frac{8\pi^6}{27}E_6(\tau,z) = p'(\tau)^2$$

vanishes. Substituting this into the above formula and clearing denominators, we see that this is equivalent to the vanishing of

$$(E_4^2g+E_6f)^3 - 3E_4(E_4^2g+E_6f)(E_6g+E_4f)^2 + 2E_6(E_6g+E_4f)^2 = (E_6^2-E_4^3)H(\tau) \quad ,$$

where

$$H(\tau) := E_6f^3 + 3E_4^2f^2g + 3E_4E_6fg^2 + (2E_6^2-E_4^3)g^3 \quad ,$$

a modular form of weight $k'=3k-6$. Any pole of $\nu''(\tau)^2$ is simple or triple (since ν looks like $(\tau-\tau_0)^{1/2}$ or $(\tau-\tau_0)^{3/2}$ locally), so H^3 annihilates the poles of $\nu''(\tau)^2$ or turns them into double zeros; away from ramification points, $\nu(\tau)$ is locally holomorphic, so $\nu''(\tau)^2$ has zeros of even order. Hence $\nu''(\tau)^2$ has the form $G(\tau)^2/H(\tau)^3$ for some G of weight $3+\frac{3}{2}k'$. If ϕ is not a cusp form, then $f(i\infty)\neq -g(i\infty)$ so $H(i\infty)\neq 0$ and we are done (G must be a cusp form because $\nu''(\tau)$ vanishes at ∞). If $\phi(\tau,0)$ is a cusp form, then we can write ϕ as $f\phi_{10,1}+g\phi_{12,1}$ with $f\in M_{k-10}$, $g\in M_{k-12}$ (cf. comments at the end of §3) and repeat the argument to get

$$\phi(\tau,z)=0,\ \text{ramified} \iff H := f^3 - 3E_4fg^2 - 2E_6g^3 = 0 \quad ,$$

where now H has degree $3k-30$. This completes the proof, and we have even given an explicit formula for H in terms of f and g ; a formula for G in terms of f and g and their derivatives of order ≤ 2 is given implicitly in the last paragraph of [9].

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