Pure elements and intertwining classes of simple strata in local central simple algebras

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by Paul Broussous[1] and Martin Grabitz[2]

Introduction

Let $G = GL_m(D)$ be the general linear group over a division algebra D with centre a p-adic field F. It has been conjectured that the smooth dual of G can be constructed as Bushnell and Kutzko did for $GL_N(F)$ [BK2]. Checking such a conjecture means a lot of constructions and computations. The aim of this paper is to establish some fundamental results on *simple strata*: the "intertwining implies conjugacy" property and the existence of a simple stratum equivalent to a pure one. This virtually enables us to define *simple types* for supercuspidal representations.

Let us be more precise. We want to produce a list of pairs (J, λ) formed of a compact mod center subgroup J and an irreducible smooth representation λ of J, with at least the two following properties:

(a) any supercuspidal representation contains a pair (J, λ) and is then isomorphic to the compactly induced representation $c\text{-}Ind_J^G \lambda$;

(b) If two pairs are contained in the same irreducible representation, they are conjugate under the action of G.

The first step towards proving (a) was done in [Br2]: it was shown that any irreducible positive level supercuspidal representation contains the character attached to a simple stratum $[\mathfrak{A}, n, n-1, \beta]$ (see [BK2] or [Br2] for the missing definitions). Recall [Br2] that a simple stratum is a quadruple

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$[\mathfrak{A}, n, q, \beta]$, where $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in the F-algebra $A = M_m(D)$, n and q are integers and β is an element of A, satisfying a certain list of properties. In particular the stratum must be pure (see the definition below) and in the case $q = n-1$, the character attached to the stratum has a nice intertwining formula.

Recall that a stratum $[\mathfrak{A}, n, q, \beta]$ is pure if $E = F[\beta]$ is a subfield of A whose multiplicative group normalizes $\mathfrak{A}$ (we then say that β is $\mathfrak{A}$-*pure*) and if β has $\mathfrak{A}$-valuation $-n$ (see below). We prove (theorem (5.1) below) that any such pure stratum is equivalent to a simple one. This virtually enables us to construct the list of pairs (J, λ) by an inductive procedure (see [BK2]). Of course, a lot of things have to be checked but the case $m = 1$ ([Br3]) shows that this is reasonable.

The results of [BH] and our theorem (5.1) also enable us to compare a metric on certain sets of pure elements for A with a metric on pure elements for an arbitrary F-division algebra Δ of suitable index. In the work [Z2], it was shown that the latter metric actually agrees with a certain polynomial metric on the the set of minimal polynomials of the pure elements in Δ. So our proposition (7.5) actually shows that our metric (introduced in definition (7.4)) is in some sense independent of our algebra A of fixed reduced degree $N = md$ (where d is the index of D). We conjecture that this can be used to introduce parameters for simple types of different central simple algebras of fixed reduced degree, as it was done in [Z2] for a F-division algebra of index N and $GL_N(F)$.

Property (b) corresponds to a certain rigidity of simple strata which we also prove in this paper as (theorem (4.1.2)). The situation is here more complicated than in the split case of [BK2]. We first have to define a "type of embedding" for the $\mathfrak{A}$-pure elements. The type of embedding of β describes how the field extension $F[\beta]/F$, seen as an abstract field extension, is embedded in A with respect to $\mathfrak{A}$. We show the following fundamental results:

(i) the type of embedding can be described by a finite number of numerical invariants;

(ii) if two $\mathfrak{A}$-pure elements are conjugate under G and have the same type of embedding, they are conjugate under the normalizer of $\mathfrak{A}$;

(iii) given an irreducible polynomial in $F[X]$, there is an effective criterion to decide when it has a root in A which is $\mathfrak{A}$-pure and has a prescribed type of embedding.

All of this relies on the papers [Br1] and [G] where pure elements are studied in detail. We define the type of embedding by using an equivalence relation on the set of pure elements (more precisely, on the corresponding "pure extensions" $F[\beta]/F$). Generalizing an idea of Zink [SZ], we show that in each equivalence class there is an element which is effectively described (a "pearl embedding"). We then use [Br1] to attach numerical invariants to this special element.

The rigidity theorem (4.1.2) takes the following form. Let $[\mathfrak{A}, n, q, \gamma_i]$, $i = 1, 2$, be two simple strata where γ_1 and γ_2 have the same type of embedding and assume they are intertwined by an element of

G. Then they are conjugate by an element of the normalizer of $\mathfrak{A}$ in G. To prove this, we will strongly use a descent lemma (theorem (4.3.9)), comparing intertwinings in A with intertwinings in $A \otimes_F L$, for a finite unramified extension L/F: any pair of strata in A can be lifted to strata in $A \otimes_F L$; then if the latter strata are conjugate (under the normalizer of the order), the former strata are conjugate also. We will apply this to simple strata where the field extensions are totally ramified: they remain pure under unramified base change. To reduce to this case we will prove that we can assume that the unramified parts of the field extensions are the same and then work in the centralizer of this common unramified part. All these procedures introduce a change of ground field so we will need some results from [BH] which give the base change theory for simple strata and characters.

The paper is organized as follows. In §1, we recall some standard notation. In §2, we study the problem of embedding field extensions with respect to hereditary orders and show how to attach to them numerical invariants. We then give a numerical criterion to decide when a field extension can be embedded with prescribed invariants. In §3, we use our results to define a new notion of purity and we prove property (ii) above.

In §4 we establish the "intertwining implies conjugacy" property for simple strata and in §5 we prove the existence of a simple representative for a pure stratum. In §6 we prove the descent lemma relative to unramified base change and intertwining. Finally in §7 we give a short outlook on the possibility of comparing pure elements from different simple algebras of the same reduced degree, in the spirit of [Z2].

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1 Notation

For a ring R with unit element and a subset $S \subset R$, we denote by $C_R(S) = \{x \in R \mid: xs = sx,\ s \in S\}$ the centralizer of S in A and by $N_{R^\times}(S) = \{x \in R^\times \mid xSx^{-1} = S\}$ the normalizer of S in $R^\times$, where $R^\times$ denotes the group of units of R. We denote by $rad(R)$ the Jacobson radical of R, which is the intersection of all maximal left (resp. right) ideals of R and by R^{op} its opposite ring. We denote by $M_m(R)$ the set of all $m \times m$ matrices, by $GL_m(R)$ the subset of all invertible ones, and by I_m the unit matrix in $M_m(R)$.

Let F be a non-archimedean local field and A a finite dimensional central simple F-algebra. Let V be a simple left A-module; then, by Wedderburn's structure theorem $A \simeq \mathrm{End}_{\mathrm{D}}(V) \simeq M_m(D)$, for a natural number m, where $D^{op} = \mathrm{End}_A(V)$ is a central F-division algebra. Moreover $\dim_F(D) = d^2$ is a square and its square root d is called the index of the algebra. The square root N of $dim_F(A) = m^2d^2$ is called the reduced degree of A.

If E/F is a finite field extension embedded in A, then $B := C_A(E)$ is a central simple E-algebra. Moreover for an arbitrary field extension E/F, we denote $f(E/F)$ the inertia degree and $e(E/F)$ the ramification index.

For a F-division algebra D, we denote by $\mathfrak{o}_D$ its ring of integers, by $\mathfrak{p}_D$ the unique maximal ideal of $\mathfrak{o}_D$ and by $\mathfrak{k}_D := \mathfrak{o}_D/\mathfrak{p}_D$ the residual class field. We denote by $\sharp M$ the cardinality of a set M and let $q = p^k := \sharp\mathfrak{k}_F$, where p is the residual characteristic of F.

An $\mathfrak{o}_F$-order $\mathfrak{A}$ in A is a subring of A which contains the same unit element and is a complete $\mathfrak{o}_F$-lattice in A. In this situation we set $\mathcal{K}_{\mathfrak{A}} := N_{A^\times}(\mathfrak{A})$. Moreover $\mathfrak{A}$ is called hereditary, if it is hereditary as a ring. For standard results about hereditary orders, we refer to [R] and [CR]. For non-zero integers $k_1, \ldots, k_u$, we denote by $\langle k_1, \ldots, k_u\rangle$ their greatest common positive divisor.

For an element h of a group H, we denote by $\langle h\rangle$ the cyclic group generated by h.

2 Hereditary orders and embeddings

2.1 Preparations

Let us fix an isomorphism $A \simeq M_m(D)$ and identify A with $M_m(D)$. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A. We denote by $\mathfrak{P}$ its Jacobson radical and by $\mathcal{K}_{\mathfrak{A}}$ its normalizer. We know, from [BF] that there exists an $\mathfrak{o}_D$-lattice chain $\{X_k\}_{k\in\mathbb{Z}}$ in V, unique up to a translation of the index, such that

$$\mathfrak{A} = \mathrm{End}_{\mathfrak{o}_D}\{X_k\}_{k\in\mathbb{Z}} = \{x \in A \mid xX_k \subset X_k,\ k \in \mathbb{Z}\}.$$

Furthermore,

$$\mathfrak{A}/\mathfrak{P} \simeq \bigoplus_{i=1}^{r} \mathrm{End}_{\mathfrak{k}_D}(X_{i-1}/X_i) \simeq M_{n_1}(\mathfrak{k}_D) \times \ldots \times M_{n_r}(\mathfrak{k}_D)$$

for some numbers $n_1, \ldots, n_r$; the conjugacy class of $\mathfrak{A}$ determines them uniquely up to cyclic permutation and they are called the invariants of $\mathfrak{A}$.

These numbers also come from a normal form:

(2.1.1)

$$g\mathfrak{A}g^{-1} = \begin{pmatrix} (\mathfrak{o}_D)\,, (\mathfrak{p}_D) & \cdots, (\mathfrak{p}_D) \\ (\mathfrak{o}_D)\,, (\mathfrak{o}_D) & \cdots, \;\cdots \\ \cdots\;, \;\cdots & \cdots, (\mathfrak{p}_D) \\ (\mathfrak{o}_D)\,, \;\cdots & \cdots, (\mathfrak{o}_D) \end{pmatrix}^{\{n_1,\dots,n_r\}}$$

for a suitable element $g \in A^\times$. Here $(\mathfrak{o}_D)$ (resp. $(\mathfrak{p}_D)$) means $n_i \times n_j$-matrices with entries in $\mathfrak{o}_D$ (resp. in $\mathfrak{p}_D$). The number r is uniquely determined and called the period of $\mathfrak{A}$.

We also have:

$$g\mathfrak{P}g^{-1} = \begin{pmatrix} (\mathfrak{p}_D)\,, (\mathfrak{p}_D)\,, \cdots, (\mathfrak{p}_D)\,, (\mathfrak{p}_D) \\ (\mathfrak{o}_D)\,, (\mathfrak{p}_D)\,, \cdots, \;\cdots\;, \;\cdots \\ \cdots\;, (\mathfrak{o}_D)\,, \cdots, \;\cdots\;, \;\cdots \\ (\mathfrak{o}_D)\,, \;\cdots\;, \cdots, (\mathfrak{p}_D)\,, \;\cdots \\ (\mathfrak{o}_D)\,, \;\cdots\;, \cdots, (\mathfrak{o}_D)\,, (\mathfrak{p}_D) \end{pmatrix}^{\{n_1,\dots,n_r\}}.$$

Let E/F be a field extension of F in A, such that $E^\times \subset \mathcal{K}_{\mathfrak{A}}$. Let $B = C_A(E)$ be its centralizer in A. From the theory of the Brauer group it is well known that we have $B \simeq M_{m'}(D')$, for D' a central E-division algebra of index $d' = \frac{d}{\langle [E:F], d\rangle}$ and $m' = \frac{\langle [E:F], d\rangle m}{[E:F]}$ (see for example [Z](proposition 1)). From [Br1](1.3) we know, that $\mathfrak{B} = \mathfrak{A} \cap B$ is a hereditary $\mathfrak{o}_E$-order in B and, of course, $\mathcal{K}_{\mathfrak{A}} \cap B \subset \mathcal{K}_{\mathfrak{B}}$. If in addition $\mathcal{K}_{\mathfrak{A}} \cap B \supset \mathcal{K}_{\mathfrak{B}}$, $\mathfrak{A}$ is called a continuation of $\mathfrak{B}$. The pair $(E, \mathfrak{A})$ is called *an embedding* (with respect to $\mathfrak{A}$). We have the following notion from [Br1].

(2.1.2) Definition *(i) An embedding $(E, \mathfrak{A})$ is called good if $\mathfrak{A}$ is a continuation of $\mathfrak{B}$ as above.*
(ii) More generally, we call $(\mathcal{K}_{\mathfrak{B}} : \mathcal{K}_{\mathfrak{A}} \cap B)$ the index of the embedding.

From [G](2.2)(ii), we know that all embeddings $(E, \mathfrak{A})$ with $\langle f(E/F), d\rangle = 1$ are good and that the associated continuation is unramified in the sense of pseudovaluations (cf. [Be]). More precisely, let $\nu_{\mathfrak{A}} : A \to \mathbb{Z} \cup \{\infty\}$ be defined by $\nu_{\mathfrak{A}}(x) := max\{j \in \mathbb{Z} \mid x \in \mathfrak{P}^j\}$ for $x \neq 0$ and $\nu_{\mathfrak{A}}(0) := \infty$. From [BF](1.2.8)(i) we know that $\nu_{\mathfrak{A}} \mid_{\mathcal{K}_{\mathfrak{A}}}$ is a group homomorphism from $\mathcal{K}_{\mathfrak{A}}$ to $\mathbb{Z}$ with kernel $\mathfrak{A}^\times$; hence, $\nu_{\mathfrak{A}}(\mathcal{K}_{\mathfrak{A}}) = t\mathbb{Z}$, where t is a positive integer. The same holds true for $\mathfrak{B}$ and we similarly define $\nu_{\mathfrak{B}}$. By [G](2.2)(i) we have $\nu_{\mathfrak{B}}(\mathcal{K}_{\mathfrak{B}}) = t\mathbb{Z}$ as well and for all $j \in \mathbb{Z}$, $\mathfrak{P}^j \cap B = \mathfrak{Q}^j$, where $\mathfrak{Q}$ is the Jacobson radical of $\mathfrak{B}$. Therefore there exists $h \in \mathcal{K}_{\mathfrak{B}}$ such that $\nu_{\mathfrak{A}}(h) = t$; hence, $\mathcal{K}_{\mathfrak{A}} = \langle h\rangle\mathfrak{A}^\times$ and $\mathcal{K}_{\mathfrak{B}} = \langle h\rangle\mathfrak{B}^\times$. This is similar to the theory of valuation of fields and h acts here in the function of the uniformizer. In the case of unramified continuation $\nu_{\mathfrak{A}}$ and $\nu_{\mathfrak{B}}$ have a common uniformizer.

2.2 Equivalence classes of embeddings

In this chapter, we use the case $\langle f(E/F), d\rangle = 1$ to reduce the problem of studying embedding to the 'division algebra part' of the embedding, which will be defined below. We split the set of all embeddings into equivalence classes, such that the condition of good embedding, or more generally the index of the embedding, depends only on the equivalence class.

(2.2.1) Definition *Embeddings $(E',\mathfrak{A}')$ and $(E,\mathfrak{A})$ are called equivalent, denoted by $(E',\mathfrak{A}')\sim(E,\mathfrak{A})$, iff $\langle f(E/F), d\rangle = \langle f(E'/F), d\rangle =: f$ and, letting E'_d, E_d be the unique unramified subextensions of F in E', E respectively with $[E'_d : F] = [E_d : F] = f$, then there exists $g \in A^\times$ such that $(gE'_dg^{-1}, g\mathfrak{A}'g^{-1}) = (E_d, \mathfrak{A})$.*

We call f the degree and E_d (resp. E'_d) the division algebra part of the embedding $(E, \mathfrak{A})$ (resp. $(E', \mathfrak{A}')$).

We easily check:

(2.2.2) Lemma *Equivalence of embeddings with respect to hereditary orders is an equivalence relation.*

(2.2.3) Definition *We denote by $[E, \mathfrak{A}]$ the equivalence class of embeddings which contains $(E, \mathfrak{A})$ and by $\mathcal{E}$ the set of all equivalence classes of embeddings with respect to hereditary orders.*

(2.2.4) Corollary *Every equivalence class of embeddings is of the form $[E, \mathfrak{A}]$ with E/F unramified and $[E : F] \mid d$.*

Proof.

Let $[E, \mathfrak{A}] \in \mathcal{E}$ and let E_d be the division algebra part of $(E, \mathfrak{A})$; then clearly $(E, \mathfrak{A}) \sim (E_d, \mathfrak{A})$.
□

(2.2.5) Proposition *The index of an embedding depends only on its equivalence class.*

Proof.

Let $(E', \mathfrak{A}')$ and $(E, \mathfrak{A})$ be two equivalent embeddings and E_d, E'_d be their division algebra parts. There exists $g \in A^\times$ such that $(gE'_dg^{-1}, g\mathfrak{A}'g^{-1}) = (E_d, \mathfrak{A})$. Put $C' := C_A(E'_d)$, $C := C_A(E_d)$, $\mathfrak{C} := \mathfrak{A} \cap C$ and $\mathfrak{C}' := \mathfrak{A}' \cap C'$. Then it is easy to see that $\mathcal{K}_{\mathfrak{A}} = g\mathcal{K}_{\mathfrak{A}'}g^{-1}$, $C = gC'g^{-1}$ and $\mathcal{K}_{\mathfrak{C}} = g\mathcal{K}_{\mathfrak{C}'}g^{-1}$. Therefore we have $(\mathcal{K}_{\mathfrak{C}} : \mathcal{K}_{\mathfrak{A}} \cap C) = (g\mathcal{K}_{\mathfrak{C}}g^{-1} : g\mathcal{K}_{\mathfrak{A}} \cap Cg^{-1}) = (\mathcal{K}_{\mathfrak{C}'} : \mathcal{K}_{\mathfrak{A}'} \cap C')$.

Let $B := C_A(E)$, $B' := C_A(E')$, $\mathfrak{B} := \mathfrak{C} \cap B = \mathfrak{A} \cap B$ and $\mathfrak{B}' := \mathfrak{C}' \cap B' = \mathfrak{A}' \cap B'$. Because $\langle f(E/E_d), d'\rangle = \langle f(E'/E'_d), d'\rangle = 1$ and $d' := \frac{d}{[E_d:F]} = \frac{d}{[E'_d:F]}$ is the index of C resp. C', it follows from

[G](2.2)(i) that $\mathfrak{C}$ is an unramified continuation of $\mathfrak{B}$ and $\mathfrak{C}'$ an unramified continuation of $\mathfrak{B}'$ (see the explanations following (2.1.2)). So there exists $h \in \mathcal{K}_{\mathfrak{B}}$ such that $\mathcal{K}_{\mathfrak{C}} = \langle h \rangle \mathfrak{C}^\times$ and $\mathcal{K}_{\mathfrak{B}} = \langle h \rangle \mathfrak{B}^\times$. If k is the least positive integer such that $h^k \in \mathcal{K}_{\mathfrak{A}}$, then $(\mathcal{K}_{\mathfrak{B}} : \mathcal{K}_{\mathfrak{A}} \cap B) = k = (\mathcal{K}_{\mathfrak{C}} : \mathcal{K}_{\mathfrak{A}} \cap C)$. Similarly one shows $(\mathcal{K}_{\mathfrak{B}'} : \mathcal{K}_{\mathfrak{A}'} \cap B') = (\mathcal{K}_{\mathfrak{C}'} : \mathcal{K}_{\mathfrak{A}'} \cap C')$; hence, $(\mathcal{K}_{\mathfrak{B}} : \mathcal{K}_{\mathfrak{A}} \cap B) = (\mathcal{K}_{\mathfrak{B}'} : \mathcal{K}_{\mathfrak{A}'} \cap B')$ as required.
□

(2.2.6) Definition *The index of an equivalence class of embeddings is by definition the index of a representative. Similarly we will say that a class is good if one representative is good.*

2.3 Pearl-embeddings and numerical invariants of equivalence classes

Now we introduce a system of representatives for our equivalence classes of embeddings.

(2.3.1) Definition

(i) An embedding datum is an element $\lambda = (\lambda^1, \ldots, \lambda^r)$ of $((\mathbb{N} \cup \{0\})^f)^r$, where $f, r \in \mathbb{N}$ and $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_f) \in (\mathbb{N} \cup \{0\})^f$ for $i = 1, \ldots, r$, such that $n_1 + \ldots + n_r = m$, $(n_1, ..., n_r) \in \mathbb{N}^r$ and $\sum_{j=1}^{f} \lambda^i_j = n_i$ for $i = 1, \ldots, r$.

(ii) We call f the degree of λ and $(n_1, \ldots, n_r)$ the partition associated to the datum λ as in (i).

(iii) Let Φ be the set of all embedding data and $\lambda \in \Phi$ of degree f and with associated partition $(n_1, ..., n_r)$ as in (i). We have a permutation L_f on $(\mathbb{N} \cup \{0\})^f$ given by $L_f((m_1, \ldots, m_f)) := (m_f, m_1, \ldots, m_{f-1})$ and set $Z(\lambda) := (L_f(\lambda^r), \lambda^1, \ldots, \lambda^{r-1})$. This definition gives a permutation Z of Φ. Let $\mathcal{I} := \langle Z \rangle \backslash \Phi$ the set of all orbits in Φ under the action of the subgroup of the group of all permutations of the set Φ generated by Z. Clearly $\mathcal{I}$ can be considered as a set of equivalence classes of Φ, where $\lambda, \eta \in \Phi$ are called equivalent, denoted $\lambda \sim \eta$, if they are in the same orbit under $\langle Z \rangle$. Denote by $\bar{\lambda}$ the equivalence class of λ.

(iv) For a datum λ of degree f and with associated partition $(n_1, ..., n_r)$, we construct an embedding as follows: Choose (and fix) a maximal unramified extension L/F in D (thus $[L : F] = d$). For every divisor $f \mid d$ we have a unique intermediate field $L/L_f/F$ such that $[L_f : F] = f$. Let π be a prime element of $\mathfrak{o}_D$ such that $\pi L \pi^{-1} = L$ and let $\sigma \in \mathrm{Gal}(L/F)$ be the map $D \to D$, $x \mapsto \pi x \pi^{-1}$ restricted to L. Denote by $\mathfrak{A}_n$ the hereditary $\mathfrak{o}_F$-order in A given in normal form (2.1.1) with invariant tuple $n = (n_1, \ldots, n_r)$ (appearing in this order!). Let $L_f \hookrightarrow A$ be embedded over F by

(2.3.2) $x \mapsto \mathrm{diag}(\mathrm{diag}(\sigma^{\nu_{1,1}}(x), \ldots, \sigma^{\nu_{1,n_1}}(x)) \times \ldots \times \mathrm{diag}(\sigma^{\nu_{r,1}}(x), \ldots, \sigma^{\nu_{r,n_r}}(x))),$

where $\mathrm{diag}(. \times \ldots \times .)$ *means 'diagonal block matrix',* $\mathrm{diag}(., \ldots, .)$ *means 'diagonal matrix' and the first λ^i_1 $\nu_{i,j}$s are zero, the next λ^i_2 $\nu_{i,j}$s are one, ..., until the last λ^i_f $\nu_{i,j}$s are $f-1$ in each block*

$i = 1, \ldots, r$. We write $(\lambda) := (L_f, \mathfrak{A}_n)$, $[\lambda] := [L_f, \mathfrak{A}_n]$ and call (λ) a standard pearl-embedding. (Note that with these conventions, $L_f^\times \subset \langle \pi_F \rangle (\mathfrak{o}_L^\times \times \ldots \times \mathfrak{o}_L{}^\times) \subset \mathcal{K}_{\mathfrak{A}_n}$, where π_F is a prime element of F and $(\mathfrak{o}_L^\times \times \ldots \times \mathfrak{o}_L{}^\times)$ means the set of all diagonal matrices with entries in $\mathfrak{o}_L^\times$.)

(2.3.3) Lemma *Embedding data λ, η are equivalent if and only if the associated embeddings $(\lambda), (\eta)$ are equivalent.*

Proof.

First, suppose $\lambda \sim \eta$. Then there exists $k \in \mathbb{Z}$ such that $Z^k(\lambda) = \eta$. Let λ be of degree f with associated partition $(n_1, ..., n_r)$ as in definition (2.3.1)(i). Let π be as in definition (2.3.1)(iii),

$$h_i := \begin{pmatrix} 0 & , \pi I_{n_i} \\ I_{m-n_i} \, , & 0 \end{pmatrix}$$

for $i = 1, \ldots, r$, and $\mathfrak{A}_n$ the hereditary order in normal form (2.1.1) such that the invariants $(n_1, \ldots, n_r)$ appear in the normal form in that order as in definition (2.3.1). Then $h_r \mathfrak{A}_n h_r^{-1}$, $h_{r-1} h_r \mathfrak{A}_n h_r^{-1} h_{r-1}^{-1}$, $\ldots$, $h_2 \ldots h_r \mathfrak{A}_n h_r^{-1} \ldots h_2{}^{-1}$ are the hereditary orders in normal form (2.1.1) such that the invariants $n_1, \ldots, n_r$ appear in the cyclic permuted orders $n_r, n_1, \ldots, n_{r-1}$; $n_{r-1}, n_r, n_1, \ldots, n_{r-2}$; $\ldots$; $n_2, \ldots, n_r, n_1$ respectively.

Consider the operation of the h_r on $(\lambda) = (L_f, \mathfrak{A}_n)$. The embedding $L_f \hookrightarrow A$ is given by (2.3.2), so the embedding $(h_r L_f h_r^{-1}, h_r \mathfrak{A}_n h_r^{-1})$ is given by

(2.3.4)

$$\begin{aligned} x \mapsto \operatorname{diag}(&\operatorname{diag}(\sigma^{\nu_{r,1}+1}(x), \ldots, \sigma^{\nu_{r,n_r}+1}(x)) \times \operatorname{diag}(\sigma^{\nu_{1,1}}(x), \ldots, \sigma^{\nu_{1,n_1}}(x)) \times \ldots \\ &\times \operatorname{diag}(\sigma^{\nu_{(r-1),1}}(x), \ldots, \sigma^{\nu_{(r-1),n_{r-1}}}(x))), \end{aligned}$$

where the notations are the same as in definition (2.3.1)(iv).

We can choose an $n_r \times n_r$-permutation matrix P_r such that for $p := \operatorname{diag}(P_r \times I_1 \times \ldots \times I_{r-1})$ the embedding $(p h_r L_f h_r^{-1} p^{-1}, p h_r \mathfrak{A}_n h_r^{-1} p^{-1}) = (p h_r L_f h_r^{-1} p^{-1}, h_r \mathfrak{A}_n h_r^{-1})$ takes a standard form, which means that P_r permutes the $\sigma^{\nu_{r,k}+1}(x)$ (for $k = 1, \ldots, n_r$) in the first block of (2.3.4) to an increasing order of the associated exponents. Then we can associate to this embedding a datum λ'. Hence, because $f \equiv 0(\text{mod f})$, we obtain $(\lambda') = (Z(\lambda)) = (p h_r L_f h_r^{-1} p^{-1}, p h_r \mathfrak{A}_n h_r^{-1} p^{-1}) \sim (L_f, \mathfrak{A}_n) = (\lambda)$.

Let S be the set of all standard pearl-embeddings. Because λ was arbitrarily chosen, we have shown that the map $S \to S$ given by $(\lambda) \mapsto (Z(\lambda))$ is a bijection which preserves equivalence classes of embeddings with respect to hereditary orders. From the construction it is clear that the inverse of this mapping is given by $(\lambda) \mapsto (Z^{-1}(\lambda))$ and also preserves equivalence classes of embeddings with respect to hereditary orders. Hence $(\eta) = (Z^k(\lambda)) \sim (\lambda)$, as required.

Conversely, suppose $(\lambda) \sim (\eta)$. Let λ be of degree f, with associated partition $n = (n_1, ..., n_r)$ as in definition (2.3.1)(i), and $\eta = (\eta^1, \dots, \eta^{r'}) = ((\eta_1^1, \dots, \eta_{f'}^1), \dots, (\eta_1^{r'}, \dots, \eta_{f'}^{r'}))$ be of degree f' with associated partition $n' = (n'_1, \dots, n'_{r'})$. Then clearly $f = f'$. Since $\mathfrak{A}_n$ and $\mathfrak{A}_{n'}$ are conjugate, we have $r = r'$ and $(n'_1, \dots, n'_r)$ is a cyclic permutation of $(n_1, ..., n_r)$. Then because $(Z^k(\lambda)) \sim (\lambda)$, as shown above, we can assume without loss of generality that $n = n'$.

By definition (2.2.1), there exists $g \in A^\times$ such that $(gL_{f,\lambda}g^{-1}, g\mathfrak{A}_n g^{-1}) = (L_{f,\eta}, \mathfrak{A}_n)$, where $L_{f,\lambda}$, $L_{f,\eta}$ denote the images of L_f under the embeddings (λ), (η) respectively. Here in particular we have that $g \in \mathcal{K}_{\mathfrak{A}_n}$.

Let $\theta \in \mathfrak{o}_{L_f}$ be a generator of the extension L_f/F, then the image of θ in $L_{f,\lambda}$ under the standard pearl-embedding attached to λ as in definition (2.3.1)(iii) is $\theta_\lambda := \mathrm{diag}(\mathrm{diag}(\sigma^{\nu_{1,1}}(\theta), \dots, \sigma^{\nu_{1,n_1}}(\theta)) \times \dots \times \mathrm{diag}(\sigma^{\nu_{r,1}}(\theta), \dots, \sigma^{\nu_{r,n_r}}(\theta)))$, where $\sharp\{j \in \{1, \dots, n_i\} \mid \nu_{i,j} = k-1\} = \lambda_k^i$ for $k = 1, \dots, f$ and $i = 1, \dots, r$. The image of θ in $L_{f,\eta}$ under the standard pearl-embedding attached to η as in definition (2.3.1)(iii) is $\theta_\eta := \mathrm{diag}(\mathrm{diag}(\sigma^{\upsilon_{1,1}}(\theta), \dots, \sigma^{\upsilon_{1,n_1}}(\theta)) \times \dots \times \mathrm{diag}(\sigma^{\upsilon_{r,1}}(\theta), \dots, \sigma^{\upsilon_{r,n_r}}(\theta)))$, where $\sharp\{j \in \{1, \dots, n_i\} \mid \upsilon_{i,j} = k-1\} = \eta_k^i$ for $k = 1, \dots, f$ and $i = 1, \dots, r$. Clearly $g\theta_\lambda g^{-1}$ is a generator of the extension $L_{f,\eta}/F$; hence, it is the image of a generating element θ' of L_f. Because θ and θ' are Galois conjugate and conjugation by π induces the generator σ of $Gal(L/F)$, there exists $k \in \mathbb{Z}$ such that $\pi^k\theta'\pi^{-k} = \sigma^k(\theta') = \theta$. Since πI_m normalizes $\mathfrak{A}_n$ and $L_{f,\eta}$, we can assume (replacing g by $(\pi^k I_m)g$) that $g\theta_\lambda g^{-1} = \theta_\eta$.

We know (see for example [G] proposition 1.2 and equation (2) of its proof) that $\mathcal{K}_{\mathfrak{A}_n} = \langle x_s \rangle \mathfrak{A}_n^\times = \mathfrak{A}_n^\times \langle x_s \rangle$, where

$$x_s := \begin{pmatrix} 0 & , \pi I_s \\ I_{m-s} \, , & 0 \end{pmatrix}$$

and where $s = n_1 + \dots + n_t$ and t is the smallest period of the numbers $n_1, \dots, n_r$. Hence there exists $k' \in \mathbb{Z}$ such that $gx_s^{-k'} \in \mathfrak{A}_n^\times$. Since $x_s = h_t \dots h_1$ (with the notation of the proof of the first part of this proposition) and by iteration of the process of the first part, we see that there exists a permutation matrix $p \in \mathfrak{A}_n^\times$ (whence $gx_s^{-k'}p^{-1} \in \mathfrak{A}_n^\times$) such that $(px_s^{k'}E_{f,\lambda}x_s^{-k'}p^{-1}, px_s^{k'}\mathfrak{A}_n x_s^{-k'}p^{-1}) = (Z^{k't}(\lambda))$. Therefore we can reduce to the case where $g \in \mathfrak{A}_n^\times$.

Let $\bar{g} := (g \pmod{1 + \mathfrak{P}_n})$ and identify $\bar{g}$ with its image under the isomorphism

(2.3.5) $\mathfrak{A}_n/\mathfrak{P}_n \simeq M_{n_1}(\mathfrak{k}_D) \times \dots \times M_{n_r}(\mathfrak{k}_D)$.

Moreover, let $\bar{g}_i \in GL_{n_i}(\mathfrak{k}_D)$, for $i = 1, \dots, r$,, be its projections to the components on the right side of the isomorphism (2.3.5). Let $\theta_\lambda^i, \theta_\eta^i$ be the images of $\theta_\lambda, \theta_\eta$ under the reduction map from $\mathfrak{o}_{L_{f,\lambda}}, \mathfrak{o}_{L_{f,\eta}}$ to $\mathfrak{k}_{L_{f,\lambda}}$, $\mathfrak{k}_{L_{f,\eta}}$ respectively, followed by the canonical embeddings into $\mathfrak{A}_n/\mathfrak{P}_n$ of the residual fields $\mathfrak{k}_{L_{f,\lambda}}$, $\mathfrak{k}_{L_{f,\eta}}$ respectively and finally projected to a single component i (in (2.3.5)). Then we have $\theta_\lambda^i = \mathrm{diag}(\bar{\sigma}^{\nu_{i,1}}(\bar{\theta}), \dots, \bar{\sigma}^{\nu_{i,n_i}}(\bar{\theta}))$ (in $M_{n_i}(k_d)$) and $\theta_\eta^i := \mathrm{diag}(\bar{\sigma}^{\upsilon_{i,1}}(\bar{\theta}), \dots, \bar{\sigma}^{\upsilon_{i,n_i}}(\bar{\theta}))$ (in $M_{n_i}(k_D)$) for

$i = 1, \ldots, r$, where $\bar{\sigma}$ is the image of σ under the canonical isomorphisms $\mathrm{Gal}(L/F) \simeq \mathrm{Gal}(\mathfrak{k}_L/\mathfrak{k}_F) \simeq \mathrm{Gal}(\mathfrak{k}_D/\mathfrak{k}_F)$ and $\bar{\theta}$ is the canonical image of θ in $\mathfrak{k}_D$.

Since $\bar{g}_i \bar{\theta}^i_\lambda \bar{g}_i^{-1} = \bar{\theta}^i_\eta$, the characteristic polynomials of $\bar{\theta}^i_\lambda$ and $\bar{\theta}^i_\eta$ agree, so that all roots have the same multiplicities; subsequently, $\lambda^i_k = \eta^i_k$ for $k = 1, \ldots, f$ and $i = 1, \ldots, r$.

So we have reduced to the case where $\lambda = \eta$.

□

To show that our system of representatives is complete, we first need a generalization of a lifting lemma of Fröhlich ([F] 3(II)).

(2.3.6) Lemma *Let K/F be an unramified extension in A and $\mathfrak{A}$ a hereditary order in A with Jacobson radical $\mathfrak{P}$ such that $\mathfrak{o}_K \subset \mathfrak{A}$. Let $j : \mathfrak{o}_K \hookrightarrow \mathfrak{A}$ be another $\mathfrak{o}_F$-embedding. Suppose now that there exist $g_1 \in \mathcal{K}_{\mathfrak{A}}$ such that for all $x \in \mathfrak{o}_K$, $g_1^{-1}(j(x))g_1 \equiv x$ (mod $\mathfrak{P}$). Then there exists $g \in \mathcal{K}_{\mathfrak{A}}$ such that $g \equiv g_1$ (mod $1 + \mathfrak{P}$) and, for all $x \in \mathfrak{o}_K : g^{-1}(j(x))g = x$.*

Proof.

The idea of the proof is the same as in the case of a principal order in [F](3(II)). We construct a sequence $\{g_n\}_{n\in\mathbb{N}}$ such that, for all $n > 1$ and all $x \in \mathfrak{o}_K$, $g_{n-1} \equiv g_n (\mathrm{mod}\ 1 + \mathfrak{P}^{n-1})$ and $g_n^{-1}(j(x))g_n \equiv x (\mathrm{mod}\ \mathfrak{P}^n)$.

Once such a sequence is constructed, since $g_1^{-1} g_n \in \mathfrak{A}^\times$ and $\mathfrak{A}^\times$ is compact), $g := \varinjlim_{n\in\mathbb{N}} g_n$ exists and lies in $\mathcal{K}_{\mathfrak{A}}$.

g_1 exists by the assumption. Let $n > 1$ and assume that $g_1, \ldots, g_{n-1}$ have already been constructed. For all $x \in \mathfrak{o}_K$, we have

(2.3.7) $g_{n-1}^{-1}(j(x))g_{n-1} \equiv x + \delta(x)$ *(mod $\mathfrak{P}^n$),*

where $\delta : \mathfrak{o}_K \to \mathfrak{P}^{n-1}$ is a function whose values are determined modulo $\mathfrak{P}^n$.

For all $x, y \in \mathfrak{o}_K$, we have $g_{n-1}^{-1}(j(xy))g_{n-1} \equiv xy + \delta(xy)$ (mod $\mathfrak{P}^n$) and $g_{n-1}^{-1}(j(xy))g_{n-1} = (g_{n-1}^{-1}(j(x))g_{n-1}g_{n-1}^{-1}(j(y))g_{n-1}) \equiv xy + x\delta(y) + \delta(x)y$ (mod $\mathfrak{P}^n$) and therefore

(2.3.8) $\delta(xy) \equiv x\delta(y) + \delta(x)y$ *(mod $\mathfrak{P}^n$).*

From (2.3.7) (and since $\mathfrak{o}_F$ is in the center of A), we have $\delta(\mathfrak{o}_F) \equiv 0$, and from $\pi_F \mathfrak{P}^{n-1} \subset \mathfrak{P}^n$ (π_F a prime element of $\mathfrak{o}_F$), $\mathfrak{p}_K = \pi_F \mathfrak{o}_K$ and (2.3.8), we also obtain $\delta(\mathfrak{p}_K) \equiv 0$. Thus δ is in fact a $\mathfrak{k}_F$-derivation $\delta : \mathfrak{k}_K \to \mathfrak{P}^{n-1}/\mathfrak{P}^n$, where $\mathfrak{P}^{n-1}/\mathfrak{P}^n$ is endowed with the $(\mathfrak{k}_K, \mathfrak{k}_K)$-bimodule structure induced by the $(\mathfrak{o}_K, \mathfrak{o}_K)$-bimodule structure of $\mathfrak{P}^{n-1}$.

Since $\mathfrak{k}_K \otimes_{\mathfrak{k}_F} \mathfrak{k}_K^{op}$ is a semisimple $\mathfrak{k}_F$-algebra, one knows from cohomology theory (see e.g. [CE](4.1;7.9)), that δ must then split: there exists $u \in \mathfrak{P}^{n-1}$, such that for all $x \in \mathfrak{o}_K$

(2.3.9) $\delta(x) \equiv ux - xu$ *(mod* $\mathfrak{P}^n$*).*

Defining $g_n := g_{n-1}(1-u)$, the congruence $g_n \equiv g_{n-1} \pmod{\mathfrak{P}^{n-1}}$ holds and for all $x \in \mathfrak{o}_K$, $g_n^{-1}(j(x))g_n = (1-u)^{-1}g_{n-1}^{-1}(j(x))g_{n-1}(1-u) \equiv (1+u)(x+\delta(x))(1-u) \pmod{\mathfrak{P}^n} \equiv (1+u)(x+xu-ux)(1-u) \pmod{\mathfrak{P}^n} \equiv x \pmod{\mathfrak{P}^n}$ as required.

□

(2.3.10) Proposition *For* $[E,\mathfrak{A}] \in \mathcal{E}$ *there exists* $\lambda \in \Phi$ *such that* $(E,\mathfrak{A}) \sim (\lambda)$.

Proof.

By corollary (2.2.4), we can assume that E/F is unramified and $[E:F] \mid d$. By conjugation with an element $g \in A^\times$, we can reduce to the case where $\mathfrak{A}$ is given in normal form (2.1.1).

To determine $\lambda \in \Phi$, set $f := [E:F]$ and let $r, n = (n_1, \ldots, n_r)$ be the invariants of $\mathfrak{A}$ as in (2.1.1). The canonical embedding

(2.3.11) $\mathfrak{k}_E \hookrightarrow \mathfrak{A}/\mathfrak{P}$

induced by the inclusion $\mathfrak{o}_E \subset \mathfrak{A}$ splits into r embeddings via the projection onto one component with respect to the isomorphism $\mathfrak{A}/\mathfrak{P} \simeq M_{n_1}(\mathfrak{k}_D) \times \ldots \times M_{n_r}(\mathfrak{k}_D)$. The resulting embeddings

(2.3.12) $\mathfrak{k}_E \hookrightarrow M_{n_i}(\mathfrak{k}_D)$

restricted to $\mathfrak{k}_E^\times$ lead to faithful representations

(2.3.13) $\mathfrak{k}_E^\times \hookrightarrow GL_{n_i}(\mathfrak{k}_D)$

of the finite cyclic group $\mathfrak{k}_E^\times$ over the finite field $\mathfrak{k}_D$ of characteristic p. The representations have the property that the subgroup $\mathfrak{k}_F^\times$ is represented by the scalar matrices $\mathfrak{k}_F^\times I_{n_i}$, via $x \mapsto xI_{n_i}$ for $i = 1, \ldots, r$.

Because $\sharp\mathfrak{k}_E^\times = q^f - 1 = p^{kf} - 1$, we have $char(\mathfrak{k}_D) = p \nmid \sharp\mathfrak{k}_E^\times$. Therefore Maschke's proposition (see [M] 3.1) applies and the representations split into a direct sum of irreducibles. $\mathfrak{k}_E^\times$ is a group of exponent $q^f - 1$ and $\mathfrak{k}_D^\times$ is a cyclic group of order $q^d - 1$. Because $f \mid d$, we have $q^f - 1 \mid q^d - 1$, and therefore $\mathfrak{k}_D$ contains a primitive $(q^f - 1) - th$ root of unity. Consequently, by [M] 3.13, the representations split into direct sums of linear characters, where, as we noted before, each of them represents the elements of $\mathfrak{k}_F^\times$ identically.

Interpreting the embeddings (2.3.11),(2.3.12),(2.3.13) as inclusions, for $i = 1, \ldots, r$, there exists, $\bar{h}_i \in GL_{n_i}(\mathfrak{k}_D)$ such that the conjugated embedding $\bar{h}_i \mathfrak{k}_E \bar{h}_i^{-1} \hookrightarrow M_{n_i}(\mathfrak{k}_D)$ (that is the embedding (2.3.12) followed by the adjoint action of $\bar{h}_i$) restricted to $\bar{h}_i \mathfrak{k}_E^\times \bar{h}_i^{-1}$ takes the form

(2.3.14) $x \mapsto \mathrm{diag}(\chi_1^i(x), \ldots, \chi_{n_i}^i(x)),$

where the χ_j^i's are linear characters which represent $\mathfrak{k}_F^\times$ identically for $j = 1, \ldots, n_i$ and $i = 1, \ldots, r$.

Coming back to $\bar{h}_i \mathfrak{k}_E \bar{h}_i^{-1} \hookrightarrow M_{n_i}(\mathfrak{k}_D)$ for $i = 1, \ldots, r$, we see that this embedding must be diagonal. By projection onto one diagonal component j of (2.3.14), this clearly means that each χ_j^i lifts to an embedding of fields $\theta_j^i : \mathfrak{k}_E \hookrightarrow \mathfrak{k}_D$ over $\mathfrak{k}_F$. Each of the θ_j^i is given by $x \mapsto \bar{\sigma}^{\nu_{i,j}}(x)$, where $\nu_{i,j} \in \{0, .., f-1\}$ for $j = 1, \ldots, n_i$ and $\bar{\sigma} \in \mathrm{Gal}(\mathfrak{k}_D/\mathfrak{k}_F)$ is the image of σ as in definition (2.3.1)(iii) under the canonical isomorphisms $\mathrm{Gal}(L/F) \simeq \mathrm{Gal}(\mathfrak{k}_L/\mathfrak{k}_F) \simeq \mathrm{Gal}(\mathfrak{k}_D/\mathfrak{k}_F)$.

Changing the $\bar{h}_i$ by certain permutation matrices from $GL_{n_i}(\mathfrak{k}_D)$, we can assume that $\nu_{i,1} \leq \ldots \leq \nu_{i,n_i}$ for $i = 1, \ldots, r$. Fitting the elements $\bar{h}_i$ together, we get an element $\bar{h}$ of $\mathfrak{A}^\times/(1+\mathfrak{P})$. Choose $h \in \mathfrak{A}^\times$ such that $\bar{h} = (h \pmod{1+\mathfrak{P}})$. Then, we can replace E by hEh^{-1} without leaving the equivalence class of $(E, \mathfrak{A})$ and so that the assumptions of our further reductions hold.

We are thus reduced to the case where $(E, \mathfrak{A})$ has a canonical embedding of the residual field $\mathfrak{k}_E \hookrightarrow \mathfrak{A}/\mathfrak{P} \simeq M_{n_1}(\mathfrak{k}_D) \times \ldots \times M_{n_r}(\mathfrak{k}_D)$ given by $x \mapsto \mathrm{diag}(\bar{\sigma}^{\nu_{1,1}}(x), \ldots, \bar{\sigma}^{\nu_{1,n_1}}(x)) \times \ldots \times \mathrm{diag}(\bar{\sigma}^{\nu_{r,1}}(x), \ldots, \bar{\sigma}^{\nu_{1,n_r}}(x))$, where $\nu_{i,j} \in \{0, \ldots, f-1\}$ for $j = 1, \ldots, n_i$ and $\nu_{i,1} \leq \ldots \leq \nu_{i,n_i}$ for $i = 1, \ldots, r$.

Let $\lambda_k^i := \sharp\{j \in \{1, \ldots, n_i\} \mid \nu_{i,j} = k-1\}$ for $i = 1, \ldots, r$ and $k = 1, \ldots, f$. Then $\lambda = ((\lambda_1^1, \ldots, \lambda_f^1), \ldots, (\lambda_1^r, \ldots, \lambda_f^r))$ is an embedding datum, and, by definition, (λ) has the same canonical embedding of its residual field as $(E, \mathfrak{A})$, whence $(E, \mathfrak{A}) \sim (\lambda)$ follows easily from lemma (2.3.6).

□

(2.3.15) Theorem *The map $\mathcal{I} \to \mathcal{E}$ given by $\bar{\lambda} \mapsto [\lambda]$ is a canonical bijection; i.e. it is independent of our fixed identification $A \simeq M_m(D)$ and of the choices in definition (2.3.1)(iv).*

Proof.

By lemma (2.3.3) the map is well-defined and injective. By proposition (2.3.10) it is surjective.

Two isomorphisms $A \simeq M_m(D)$ differ by the adjoint action of some $g \in A^\times$, whence leave all classes of $\mathcal{E}$ stable (by definition (2.2.1)). Note that if $A \simeq M_m(D')$ then $D' \simeq D$; hence we can assume that $D = D'$.

Without loss of generality we can fix our identification $A \simeq M_m(D)$. Another extension L'/F differs from L/F only by conjugation of an element $x \in D^\times$; hence, for $\lambda \in \Phi$ the pearl-embedding $(\lambda)'$ with respect to L'/F differs from the pearl-embedding (λ) with respect to L/F only by conjugation with xI_m in $A^\times = GL_m(D)$, whence lies in the same equivalence class (again by definition (2.2.1)).

Thus we can fix the extension L/F in D and just look at a different prime element π' of $\mathfrak{o}_D$ which normalizes L. But it is well known that π' and π induce the same automorphism $\sigma \in \mathrm{Gal}(L/F)$, which implies that they lead to the same standard pearl-embeddings.

$\square$

2.4 An explicit version of the criterion of [Br1]

In this section we apply the results of [Br1] to decide which classes of embeddings with respect to hereditary orders are good and which are "bad". To do this, we first have to explain the concept of [Br1] and then apply it to the special case of pearl-embeddings.

(2.4.1) Definition *Let $\gamma = (\gamma_j)_{j\in\mathbb{Z}}$ be a sequence with $\gamma_j \in \mathbb{N}\cup\{0\}$ for all $j \in \mathbb{Z}$ and let $\gamma' = (\gamma_{i_j})_{j\in\mathbb{Z}}$ with $(i_j)_{j\in\mathbb{Z}}$ be a $\mathbb{Z}$-valued sequence such that $i_k < i_l$ if $k < l$ and such that γ' is the sequence of all non-zero numbers from the sequence γ.*

Let t a period of the sequence γ'. As in [Br1]*, we call the sequence γ t-zero-periodic, if the number of zeros occurs t-periodically in γ, which means that for all $j \in \mathbb{Z}$, $i_{j+1} - i_j = i_{j+t+1} - i_{j+t}$ (note that $i_{j+1} - i_j - 1$ is the number of zeros in the sequence γ between the non-zero numbers γ_{i_j} and $\gamma_{i_{j+1}}$).*

Let V be a simple A module, L/F as in definition (2.3.1), and consider the central simple L-algebra $A \otimes_F L \simeq M_{md}(L)$. V is also a simple $A \otimes_F L$ module with the action $(a \otimes l)v := avl$ (note that V is a right D-vector space and $L \subset D$), and, with this action, we have $A \otimes_F L \simeq \mathrm{End}_L(V)$.

Let $(E, \mathfrak{A})$ be an arbitrary embedding in A and consider the algebra $E \otimes_F L \subset A \otimes_F L$. This is a semisimple commutative E-algebra; hence,

(2.4.2) $E \otimes_F L \simeq \bigoplus_{i=1}^{l} K_i$

for some field extensions K_i/E in some closure of E and where $E\otimes_F L$ acts on each K_i irreducibly for $i = 1, \ldots, r$ (this just means in this context that the algebra isomorphism (2.4.2) identifies the K_i's with the simple components of $E \otimes_F L$).

From [Br1] chap. 2 we know that, if $1_A = 1_{E\otimes_F L} = e_1 + \ldots + e_l$ is the representation of the 1 as a sum of central simple idempotents with respect to (2.4.2), then $e_i \in \mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ for $i = 1, \ldots, l$. Moreover $V^i := e_i V$ is a simple $e_i(B \otimes_F L)e_i$-module (where $B = C_A(E)$) and $j \mapsto X^i_j := e_i X_j$ ($j \in \mathbb{Z}$) is a lattice sequence in the sense of [BK3] such that the inclusion ordered set $\{X^i_j \mid j \in \mathbb{Z}\}$ becomes a lattice chain in V^i for the hereditary $e_i(\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)e_i$-orders $e_i(\mathfrak{B} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)e_i$ for $i = 1, \ldots, l$, where $\mathfrak{B} = \mathfrak{A} \cap B$.

From [Br1] we derive:

(2.4.3) Proposition *Let $(E', \mathfrak{A})$ be an arbitrary embedding in A with the same canonical embedding of residual field as $(E, \mathfrak{A})$ (or, more precisely, their images agree). For $i = 1, \ldots, l$ let*

$\gamma^{(i)} := (\dim_{\mathfrak{k}_L}(X^i_j/X^i_{j+1}))_{j\in\mathbb{Z}}$ *and let* $\gamma'^{(i)} = (\gamma'^{(i)}_j)_{j\in\mathbb{Z}}$ *be the subsequence of* $\gamma^{(i)}$ *consisting of all non-zero members of* $\gamma^{(i)}$ *as in definition (2.4.1); let* t_i *be the smallest period of* $\gamma'^{(i)}$*. Then:*

(i) The embedding $(E', \mathfrak{A})$ *is good if and only if the sequence* $\gamma^{(i)}$ *is* t_i*-zero-periodic for one (equivalently for all)* $i = 1, \ldots, l$.

(ii) Let $i \in \{1, \ldots, l\}$. $\mathfrak{C} = \mathfrak{A} \cap C$ *is a hereditary* $\mathfrak{o}_{E'}$*-order in* $C = C_A(E')$*, and for all* $j \in \mathbb{Z}$ *the sequence* $\gamma'^{(i)}_{j+1}, \gamma'^{(i)}_{j+2}, \ldots, \gamma'^{(i)}_{j+r'}$ *are the invariants of* $\mathfrak{C}$*, where* $r' \in \mathbb{N}$ *is the unique number such that for all* $j \in \mathbb{Z}$: $\gamma'^{(i)}_{j+1} + \gamma'^{(i)}_{j+2} + \ldots + \gamma'^{(i)}_{j+r'} = m'$*, where* $m' = \frac{\langle [E':F], d\rangle m}{[E':F]}$ *(in particular,* $t_i = t_j$*, for* $1 \le i, j \le l$*).*

(iii) Let $i \in \{1, \ldots, l\}$ *and use the assumptions and notation of (ii). Let* $\upsilon \in \mathbb{N}$ *be the smallest number such that* $\gamma^{(i)}$ *is* $t_i\upsilon$*-zero-periodic and* $z \in \mathcal{K}_{\mathfrak{C}}$ *such that* $\mathcal{K}_{\mathfrak{C}} = \langle z \rangle \mathfrak{C}^\times$*. Then* $\mathcal{K}_{\mathfrak{A}} \cap C = \langle z^\upsilon \rangle \mathfrak{C}^\times$ *(and in fact* υ *is independent of* i*).*

Proof.

Let $E' \otimes_F L \simeq \bigoplus_{i=1}^l K'_i$ be the decomposition into simple components as in (2.4.2) and $1_A = e'_1 + \ldots + e'_l$ the decomposition of 1_A as a sum of central simple idempotents with respect to the latter decomposition. Let $'X^i_j := e'_i X_j$ for $i = 1, \ldots, l$ and $j \in \mathbb{Z}$. By the proof of [Br1](proposition 4.2.), we have $\gamma^{(i)} = (\dim_{\mathfrak{k}_L}('X^i_j/'X^i_{j+1}))_{j\in\mathbb{Z}}$ for suitable choices of the indices. The assertions (i) and (ii) are reformulations of [Br1](theorem 1.3., proposition 3.2., proposition 4.2., lemma 4.3. and corollary 4.4.). The assertion (iii) is a generalization of [Br1](lemma 4.3.) and can be seen if we take into account that $\mathcal{K}_{\mathfrak{A}} \cap C = \mathcal{K}(\{'X^i_j\}_{j\in\mathbb{Z}}) := \{x \in C^\times \mid$ there exists $\nu \in \mathbb{Z}$ such that for all $j \in \mathbb{Z}$: $x'X^i_j = {}'X^i_{j+\nu}\}$.

Clearly we have $\mathfrak{C}^\times \subset \mathcal{K}(\{'X^i_j\}_{j\in\mathbb{Z}})$ and this containment is proper because $F^\times \subset \mathcal{K}(\{'X^i_j\}_{j\in\mathbb{Z}})$, whence $\mathcal{K}(\{'X^i_j\}_{j\in\mathbb{Z}}) = \langle z^\upsilon \rangle \mathfrak{C}^\times$ for some $\upsilon \in \mathbb{Z}\backslash\{0\}$. Because all powers of elements of $\mathcal{K}(\{'X^i_j\}_{j\in\mathbb{Z}})$ are elements of the latter group as well, υ is just the smallest power of z such that there exists $\nu \in \mathbb{Z}$ such that for all $j \in \mathbb{Z}$ we have $z^{\upsilon\prime}X^i_j = {}'X^i_{j+\nu}$. But ν exists if and only if the repetitions in the lattice sequence are such that, for all $j \in \mathbb{Z}$ the modules $'X^i_j$ and $z^{\upsilon\prime}X^i_j$ appear with the same muliplicity in the sequence $\{'X^i_j\}_{j\in\mathbb{Z}}$. Moreover, by [G](proposition 1.2(i)), t_i is the partition period of $e'_i(\mathfrak{C} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)e'_i \simeq \mathfrak{C} \otimes_{\mathfrak{o}_F} \mathfrak{o}_{K'_i}$; this order has the same partition period as $\mathfrak{C}$ and $\mathcal{K}(\{'X^i_j\}_{j\in\mathbb{Z}}) \subset \mathcal{K}_{e'_i(\mathfrak{C}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L)e'_i}$. Hence z^υ has period υt_i on the lattice chain of $e'_i(\mathfrak{C} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)e'_i$ in the simple $e'_i(C \otimes_F L)e'_i$-module e'_iV so the repetitions have to appear υt_i-periodically. This leads to the asserted condition for the sequence $\gamma^{(i)}$.

□

(2.4.4) Corollary *Let* $\lambda = ((\lambda^1_1, \ldots, \lambda^1_f), \ldots, (\lambda^r_1, \ldots, \lambda^r_f)) \in \Phi$ *be an arbitrary embedding datum of degree* f*, with associated partition* $(n_1, \ldots, n_r)$*, as in definition (2.3.1)(i). Let* r' *be the number of couples* (i, j) *such that* $\lambda^i_j \neq 0$*,* $n'_1, \ldots, n'_{r'}$ *be the subsequence of non-zero numbers of* $\lambda^1_1, \ldots, \lambda^r_1, \ldots, \lambda^1_f, \ldots, \lambda^r_f$ *(in that order!),* $t(\lambda)$ *the smallest period of the sequence* $n'_1, \ldots, n'_{r'}$*,* $\upsilon(\lambda) \in \mathbb{N}$

the smallest number such that the sequence $\dots, \lambda_f^r, \lambda_1^1, \dots, \lambda_1^r, \lambda_2^1, \dots, \lambda_2^r, \dots, \lambda_f^1, \dots, \lambda_f^r, \lambda_1^1, \dots$ *is* $\upsilon(\lambda)t(\lambda)$*-zero-periodical and* $d' := \frac{d}{f}$.

Then:

(i) $\upsilon(\lambda)$ *is the index of* $[\lambda]$.

(ii) If $(E, \mathfrak{A}) \in [\lambda]$ *is an arbitrary member of this equivalence class, then the* $\mathfrak{o}_E$*-order* $\mathfrak{B} := \mathfrak{A} \cap B$ *(with* $B = C_A(E)$*) is hereditary and has invariants* $\bar{n}_1, \dots, \bar{n}_{\bar{r}}, \bar{r}$, *with* $\bar{r} = \frac{\langle e(E/F), d' \rangle r'}{e(E/F)}$ *and* $\bar{n}_i = \frac{n_i'}{f(E/E_d)}$ *for* $i = 1, \dots, \bar{r}$, *where* E_d *is the division algebra part of* E.

(iii) Let E/F *be a field extension and* $\mathfrak{A}$ *a hereditary* $\mathfrak{o}_F$*-order in* A *with invariants* $n_1, \dots, n_r$. *Then there exists an embedding* $i : E \hookrightarrow A$ *such that* $i(E^\times) \subset \mathcal{K}_{\mathfrak{A}}$ *and* $(i(E), \mathfrak{A}) \in [\lambda]$ *if and only if the following conditions hold:*

(a) $[E_d : F] = f$,

(b) $f(E/E_d) \mid \langle \lambda_1^1, \dots, \lambda_1^r, \dots, \lambda_f^1, \dots, \lambda_f^r \rangle$ *and*

(c) $\upsilon(\lambda)e(E/F) \mid \frac{r'}{t(\lambda)} d'$,

where E_d *is the division algebra part of* E.

Proof.

Note that A has been identified with $M_m(D)$. Set $V = D^m$ (column vectors) and $D' := C_D(L_f)$ (where L_f is as in definition 2.3.1(iv) and f is the degree of λ). Using the notation from the beginning of this chapter for the special case $(E, \mathfrak{A}) = (\lambda)$, it is easy to see that in (2.4.2), we have $l = f$ and $K_i \simeq L$ for $i = 1, \dots, f$.

Because $K_i \simeq L$, for $i = 1, \dots, f$ the K_i form a system of representatives of the simple $E \otimes_F L$-modules and there are f isomorphism classes of them. But $L, L\pi, \dots, L\pi^{f-1} \subset D$ is also a set of pairwise non-isomorphic simple $E \otimes_F L$-modules, where π is as in definition (2.3.1)(iii) and $E \otimes_F L$ acts on D by $(e \otimes l)d := edl$. Thus the e_i can be indexed in such a way that

(2.4.5) e_i *acts on* D *(under this action of* $E \otimes_F L$*) as a projector on the isotypic component of* $L\pi^{i-1}$ *for* $i = 1, \dots, f$.

Since $D = L \oplus L\pi \oplus \dots \oplus L\pi^{d-1}$ and $D' = L \oplus L\pi^f \oplus \dots \oplus L\pi^{f(\frac{d}{f}-1)}$, D may be written as $D = D' \oplus D'\pi \oplus \dots \oplus D'\pi^{f-1}$. Moreover with the above action of $E \otimes_F L$ on D, we have $L\pi^{i-1} \simeq L\pi^{j-1}$ iff $i \equiv j \ (mod\ f)$ and with the indexing (2.4.5) e_i acts as a projector to $D'\pi^{i-1}$ for $i = 1, \dots, f$.

Consider more generally the action of $E \otimes_F L$ on D given by $(e \otimes l)d := \sigma^{j-1}(e)dl$ for $j = 1, \dots, f$ and σ as in definition (2.3.1)(iii). Denote the module with the underlying set D and the latter action by $(D)^{\sigma^{j-1}}$. Denote the action of $E \otimes_F L$ in $(D)^{\sigma^{j-1}}$ by $\star^{\sigma^{j-1}}$.

Now let $d = \sum_{\nu=1}^d l_\nu \pi^{\nu-1}$ be an element of the set D and define a map $\rho^{\sigma^{j-1}} : D \to D$ by $\rho^{\sigma^{j-1}}(d) := \sum_{\nu=1}^d \sigma^{1-j}(l_\nu)\pi^{\nu-j}$. This map is clearly an F-vectorspace isomorphism. Moreover for

$\sum_{i=1}^d e_i \otimes l'_i \in E \otimes_F L$ we have

$$\rho^{\sigma^{j-1}}((\sum_{i=1}^d e_i \otimes l'_i) \star^{\sigma^{j-1}} (\sum_{\nu=1}^d l_\nu \pi^{\nu-1})) = \rho^{\sigma^{j-1}}(\sum_{i=1}^d \sum_{\nu=1}^d \sigma^{j-1}(e_i) l_\nu \pi^{\nu-1} l'_i) =$$
$$\rho^{\sigma^{j-1}}(\sum_{i=1}^d \sum_{\nu=1}^d \sigma^{j-1}(e_i) l_\nu \sigma^{\nu-1}(l'_i)\pi^{\nu-1}) = \sum_{i=1}^d \sum_{\nu=1}^d e_i \sigma^{1-j}(l_\nu)\sigma^{\nu-j}(l'_i)\pi^{\nu-j} =$$
$$\sum_{i=1}^d \sum_{\nu=1}^d e_i \sigma^{1-j}(l_\nu)\pi^{\nu-j} l'_i = (\sum_{i=1}^d e_i \otimes l'_i) \star^{id} (\sum_{\nu=1}^d \sigma^{1-j}(l_\nu)\pi^{\nu-j}) =$$
$$(\sum_{i=1}^d e_i \otimes l'_i) \star^{id} \rho^{\sigma^{j-1}}(\sum_{\nu=1}^d l_\nu \pi^{\nu-1}).$$

Hence, in fact $\rho^{\sigma^{j-1}}$ is an isomorphism of $E \otimes_F L$-modules $(D)^{\sigma^{j-1}} \simeq D$. If we denote the projection induced by the multiplication of e_1 in $(D)^{\sigma^{j-1}}$ by $e_1^{\sigma^{j-1}}$ and in D $(= (D)^{id})$ by e_1^{id}, then we obtain an commutative diagramm:

$$\begin{array}{ccc} (D)^{\sigma^{j-1}} & \overset{\rho^{\sigma^{j-1}}}{\rightarrow} & D \\ \downarrow e_1^{\sigma^{j-1}} & & \downarrow e_1^{id} \\ (D)^{\sigma^{j-1}} & \overset{\rho^{\sigma^{j-1}}}{\rightarrow} & D. \end{array}$$

The map $\rho^{\sigma^{j-1}}$ maps the subset $D'\pi^{j-1}$ of D onto D', the map e_1^{id} is a projector onto D' and the map $(\rho^{\sigma^{j-1}})^{-1}$ maps the subset D' of D onto $D'\pi^{j-1}$. Therefore the map $e_1^{\sigma^{j-1}} = (\rho^{\sigma^{j-1}})^{-1} \circ e_1^{id} \circ \rho^{\sigma^{j-1}}$ is a projetor to $D'\pi^{j-1}$ for $j = 1, \ldots, f$.

We conclude that

(2.4.6) *e_1 acts under the module structure $(D)^{\sigma^{j-1}}$ as an projector to $D'\pi^{j-1}$ for $j = 1, \ldots, f$.*

Because of the definition of the action of $A \otimes_F L$ on V, and since the embedding of E in A is given by equation (2.3.2) in definition (2.3.1), we have that $E \otimes_F L$ acts componentwise on $V = D^m$. Furthermore, if we index the components of V in the same way as the diagonal elements of A in the embedding (2.3.2), then $E \otimes_F L$ acts on the (i, j)-component on the underlying set D with the module structure of $(D)^{\sigma^{\nu_{i,j}}}$ for $j = 1, \ldots, n_i$ and $i = 1, \ldots, r$.

By (2.4.6)

(2.4.7) *e_1 acts on the (i, j)-component as a projection onto $D'\pi^{\nu_{i,j}}$*

for $j = 1, \ldots, n_i$ and $i = 1, \ldots, r$.

Let $\epsilon_{i,j} : D \to D'\pi^{\nu_{i,j}} \to D'$ be the associated projection from D to the $(L)^{\sigma^{\nu_{i,j}}}$-isotypic component followed by the map given by $d'\pi^{\nu_{i,j}} \mapsto d'$ for $j = 1, \ldots, n_i$ and $i = 1, \ldots, r$.

Since $\mathfrak{A}_n$ is given in the normal form 2.1.1, for a suitable choice of the index, the associated chain $\{X_j\}_{j \in \mathbb{Z}}$ takes the form: ${}^tX_0 = \mathfrak{o}_D \oplus \mathfrak{o}_D \oplus \ldots \oplus \mathfrak{o}_D$ (sum of m copies of $\mathfrak{o}_D$); ${}^tX_1 = \mathfrak{p}_D \oplus \ldots \oplus \mathfrak{p}_D \oplus \mathfrak{o}_D \oplus \ldots \oplus \mathfrak{o}_D$ (sum of n_1 copies of $\mathfrak{p}_D$ and of $m - n_1$ copies of $\mathfrak{o}_D$); $\ldots$; ${}^tX_{r-1} = \mathfrak{p}_D \oplus \ldots \oplus \mathfrak{p}_D \oplus \mathfrak{o}_D \oplus \ldots \oplus \mathfrak{o}_D$ (sum of $n_1 + n_2 + \ldots + n_{r-1}$ copies of $\mathfrak{p}_D$ and of $n_r = m - n_1 - \ldots - n_{r-1}$ copies of $\mathfrak{o}_D$); ${}^tX_r = \mathfrak{p}_D \oplus \ldots \oplus \mathfrak{p}_D = {}^t(X_0\mathfrak{p}_D)$, where ${}^t(.)$ means a transposed vector. Therefore

(2.4.8) $\dim_{\mathfrak{k}_{D'}}(X^1_\nu/X^1_{\nu+1}) = \sum_{i=1}^{r}\sum_{j=1}^{n_i}\dim_{\mathfrak{k}_{D'}}(\epsilon_{i,j}(\mathfrak{p}_D^{k+\delta_{i,l}})/\epsilon_{i,j}(\mathfrak{p}_D^{k+\delta_{i,l+1}})) = \sum_{j=1}^{n_{l+1}}\dim_{\mathfrak{k}_{D'}}(\epsilon_{l+1,j}(\mathfrak{p}_D^{k})/\epsilon_{l+1,j}(\mathfrak{p}_D^{k+1}))$

for all $\nu \in \mathbb{Z}$, where $\delta_{s,t} := \begin{cases} 1 & \text{for } s \le t \\ 0 & \textit{otherwise} \end{cases}$ and $k, l \in \mathbb{Z}$ are the unique integers such that $\nu = rk + l$ and $0 \le l \le r-1$.

We have $\ldots$; $\mathfrak{p}_D^{-1} = \mathfrak{o}_{D'} \oplus \ldots \oplus \mathfrak{o}_{D'}\pi^{f-2} \oplus \mathfrak{p}_{D'}^{-1}\pi^{f-1}$; $\mathfrak{o}_D = \mathfrak{o}_{D'} \oplus \ldots \oplus \mathfrak{o}_{D'}\pi^{f-1}$; $\mathfrak{p}_D = \mathfrak{p}_{D'} \oplus \mathfrak{o}_{D'}\pi \oplus \ldots \oplus \mathfrak{o}_{D'}\pi^{f-1}$; $\ldots$; $\mathfrak{p}_D^f = \mathfrak{p}_{D'} \oplus \ldots \oplus \mathfrak{p}_{D'}\pi^{f-1}$; $\mathfrak{p}_D^{f+1} = \mathfrak{p}_{D'}^2 \oplus \mathfrak{p}_{D'}\pi \oplus \ldots \oplus \mathfrak{p}_{D'}\pi^{f-1}$; $\ldots$. Hence by (2.4.7), for $k \in \mathbb{Z}$, $0 \le l \le r-1$ and u the unique integer $0 \le u \le f-1$ such that $u \equiv k \pmod f$, we have

(2.4.9) $\dim_{\mathfrak{k}_{D'}}(\epsilon_{l+1,j}(\mathfrak{p}_D^{k})/\epsilon_{l+1,j}(\mathfrak{p}_D^{k+1})) = \begin{cases} 1 & \textit{for } \nu_{l+1,j} = u \\ 0 & \textit{otherwise} \end{cases}$

for $j = 1, \ldots, n_{l+1}$.

Therefore, for $k \in \mathbb{Z}$ and $0 \le l \le r-1$, we have

(2.4.10) $\sum_{j=1}^{n_{l+1}}\dim_{\mathfrak{k}_{D'}}(\epsilon_{l+1,j}(\mathfrak{p}_D^{k})/\epsilon_{l+1,j}(\mathfrak{p}_D^{k+1})) = \sharp\{j \mid j \in \{1, \ldots, n_{l+1}\} \wedge \nu_{l+1,j} = u\} = \lambda^{l+1}_{u+1}$,

.

Putting (2.4.8),(2.4.9) and (2.4.10) together, for all $\nu \in \mathbb{Z}$, we have

(2.4.11) $\dim_{\mathfrak{k}_{D'}}(X^1_\nu/X^1_{\nu+1}) = \lambda^{l+1}_{u+1}$,

where u, k, l are the unique integers such that $0 \le l \le r-1$, $k \in \mathbb{Z}$, $\nu = rk + l$, $0 \le u \le f-1$ and $u \equiv k \pmod f$.

Now note that $C' := C_A(E) \simeq M_m(\tilde{D})$, where $\tilde{D}$ is a central E-division algebra of index $d' := \frac{d}{f}$ and $\tilde{D} \simeq D'$ if we choose a suitable isomorphism $E \simeq L_f$ and view D' as an E-algebra. Moreover, set $\mathfrak{C}' := \mathfrak{A}_n \cap C$.

(i) Follows now from (2.4.11) and proposition (2.4.3)(iii).

(ii) If $(E, \mathfrak{A}) \sim (\lambda) =: (E_f, \mathfrak{A}_n)$, there exists $g \in A^\times$ such that $(gE_dg^{-1}, g\mathfrak{A}g^{-1}) = (gE_dg^{-1}, \mathfrak{A}_n) = (E_f, \mathfrak{A}_n)$. Subsequently, by proposition (2.4.3)(ii), [Br1] (theorem (1.1)) and (2.4.11), $\mathfrak{C}' = \mathfrak{A}_n \cap C_A(E_f)$ is a hereditary order with invariants $n'_1, \ldots, n'_{r'}$.

We have $C := g^{-1}C_A(E_f)g = C_A(E_d)$, $\mathfrak{C} := \mathfrak{A} \cap C = g^{-1}\mathfrak{C}'g$, $\mathcal{K}_{\mathfrak{C}} = g^{-1}\mathcal{K}_{\mathfrak{C}'}g$ and $\mathcal{K}_{\mathfrak{A}} = g^{-1}\mathcal{K}_{\mathfrak{A}_n}g$; hence $E^\times \subset \mathcal{K}_{\mathfrak{A}} \cap C \subset \mathcal{K}_{\mathfrak{C}}$ and $\mathfrak{C}$ is a hereditary $\mathfrak{o}_{E_d}$-order in C with invariants $n'_1, \ldots, n'_{r'}$.

Since $C \simeq M_m(\hat{D})$ for some central E_d-division algebra $\hat{D}$ of index d' and clearly $\langle f(E/E_d), d' \rangle = 1$, [G] (theorem (2.2)(i)(a),(ii)) applies. We conclude that $(E, \mathfrak{C})$ is a good embedding and $\mathfrak{B} = \mathfrak{C} \cap B$ is a hereditary $\mathfrak{o}_E$-order in B with invariants $\bar{n}_1, \ldots, \bar{n}_{\bar{r}}$, as required.

(iii) ($\Rightarrow$) (a) follows from definition (2.2.1) and (b) from part (ii). Because $\pi_E \in \mathcal{K}_{\mathfrak{A}} \cap C$, where $C = C_A(E_d)$, by proposition (2.4.3)(iii) there exists $k \in \mathbb{Z}$ and $\epsilon \in \mathfrak{C}^\times$ ($\mathfrak{C} = \mathfrak{A} \cap C$) such that $z^{kv(\lambda)}\epsilon = \pi_E$, where π_E is a prime element of $\mathfrak{o}_E$ and z a uniformizer for $\mathfrak{C}$. Therefore by [G](proposition 1.2(i)) we get $v(\lambda)t(\lambda) \mid \frac{r'd'}{e(E/E_d)}$ and (c) follows (because E_d/F is unramified).

($\Leftarrow$) Because of (a) we have $E_d \simeq L_f$, where L_f is as in definition (2.3.1)(iii) and we can embed E_d/F via (λ). From part (ii) and its proof we obtain (by identifying E_d with its image in A) that $\mathfrak{C}' := \mathfrak{A}_n \cap C'$ is a hereditary $\mathfrak{o}_{E_d}$-order in $C' := C_A(E_d)$ with invariants $n'_1, \ldots, n'_{r'}$ continued by $\mathfrak{A}_n$ to A.

As in the proof of (i), we have $C' \simeq M_m(\tilde{D})$ for some central E_d-division algebra $\tilde{D}$ of index d' and clearly $\langle f(E/E_d), d' \rangle = 1$. Thus we may conclude by (b),(c) [G](theorem 2.2 (ii)), that there exists an embedding $E \hookrightarrow C'$ such that $E^\times \subset \mathcal{K}_{\mathfrak{C}'}$.

Clearly $\mathfrak{o}_E^\times \subset \mathfrak{C}'^\times \subset \mathfrak{A}_n^\times \subset \mathcal{K}_{\mathfrak{A}_n}$ and by (c), [G](proposition 1.2(i)) and proposition (2.4.3)(iii), we get $\pi_E \in \mathcal{K}_{\mathfrak{A}_n} \cap C'$. Consequently, there exists an embedding $(E, \mathfrak{A}_n) \sim (\lambda)$ and because $\mathfrak{A}$ and $\mathfrak{A}_n$ have the same invariants, whence are conjugate, we can find $g \in A^\times$ such that $(gEg^{-1}, g\mathfrak{A}_n g^{-1}) = (gEg^{-1}, \mathfrak{A})$.

Thus $(gEg^{-1}, \mathfrak{A}) \sim (\lambda)$, as required.

□

Remark Since the bijection in theorem (2.3.15) is canonical, the results of corollary (2.4.4) are also independent of all choices made.

3 Pure elements

We now apply our results to get a new notion of purity.

(3.1) Definition *(i) Let $\lambda \in \Phi$ be an embedding datum for a hereditary order $\mathfrak{A}_n$ (whence $\bar{\lambda} \in \mathcal{I}$ is its equivalence class) of degree f and with associated partition $n = (n_1, \ldots, n_r)$. Let f' be a divisor of f and denote by $(\lambda)_{f'}$ the embedding of $L_{f'}$ obtained by restricting the standard pearl-embedding (λ) of L_f to $L_{f'}$ (with notations of definition (2.3.1)(iii)). We say that $(E, \mathfrak{A})$ is of type $\bar{\lambda}$ if there exists a divisor f' of f such that $(E, \mathfrak{A}) \sim (\lambda)_{f'}$.*

(ii) An equivalence class of embedding data $\bar{\tau}$, where τ is an embedding datum of maximal degree d, is called a type of embedding.

(iii) Let $\bar{\tau}$ be a type of embedding, then $\alpha \in A$ is called $(\mathfrak{A}, \bar{\tau})$-pure if

(a) $E := F[\alpha]$ *is a field with* $E^\times \subset \mathcal{K}_{\mathfrak{A}}$ *and*

(b) $(E, \mathfrak{A})$ *is of type* $\bar{\tau}$.

(3.2) Lemma *Let* $\bar{\tau}$ *a type of embedding,* $\alpha, \beta \in A$ *be* $(\mathfrak{A}, \bar{\tau})$*-pure and suppose that there exists* $g \in A^\times$ *such that* $\alpha = g\beta g^{-1}$. *Then there exists* $h \in \mathcal{K}_{\mathfrak{A}}$ *such that* $\alpha = h\beta h^{-1}$.

Proof.

Let $E^\alpha := F[\alpha], E^\beta := F[\beta]$ and let E_d^α, E_d^β be the division algebra parts of E^α, E^β respectively; then $\alpha = g\beta g^{-1}$ implies $[E_d^\alpha : F] = [E_d^\beta : F] =: f$, hence there exists an embedding datum λ of degree f and with associated partition $n = (n_1, \ldots, n_r)$, where $n_1, \ldots, n_r$ are the invariants of $\mathfrak{A}$, such that $(E_d^\alpha, \mathfrak{A}) \sim (E^\alpha, \mathfrak{A}) \sim (\lambda) \sim (E^\beta, \mathfrak{A}) \sim (E_d^\beta, \mathfrak{A})$.

Therefore, there exists $h_1 \in \mathcal{K}_{\mathfrak{A}}$ such that $h_1 E_d^\beta h_1^{-1} = E_d^\alpha$ and we can (replacing β by $h_1\beta {h_1}^{-1}$ and g by gh_1^{-1}) reduce to the case where $E_d := E_d^\alpha = E_d^\beta$.

Now choose $z \in A^\times$, such that $z\mathfrak{A}z^{-1} = \mathfrak{A}_n$ where $\mathfrak{A}_n$ is the hereditary order given in normal form (2.1.1) with invariants $n = (n_1, \ldots, n_r)$. The embedding $(L_f, \mathfrak{A}_n) := (\lambda)$ is then given by $x \mapsto \mathrm{diag}(\mathrm{diag}(\sigma^{\nu_{1,1}}(x), \ldots, \sigma^{\nu_{1,n_1}}(x)) \times \ldots \times \mathrm{diag}(\sigma^{\nu_{r,1}}(x), \ldots, \sigma^{\nu_{r,n_r}}(x)))$, using the notation of definition (2.3.1)(iii).

Conjugating $\mathfrak{A}_n$ by $x_\lambda := \mathrm{diag}(\mathrm{diag}(\pi^{\nu_{1,1}}, \ldots, \pi^{\nu_{1,n_1}}) \times \ldots \times \mathrm{diag}(\pi^{\nu_{r,1}}, \ldots, \pi^{\nu_{r,n_r}}))$ (with π as in definition (2.3.1)(iii)) and letting $\mathfrak{A}_{n,\lambda} := x_\lambda \mathfrak{A}_n x_\lambda^{-1}$, we obtain

(3.3) $\mathfrak{A}_{n,\lambda} = (\mathfrak{p}_D^{a_{i,j}})_{i,j=1,\ldots,m}$,

where the last notation means the set of $m \times m$-matrices with entries in $\mathfrak{p}_D^{a_{i,j}}$ in the (i,j) place for suitable $a_{i,j} \in \mathbb{Z}$, $i = 1, \ldots, m$, $j = 1, \ldots, m$.

(3.4) *The embedding* $(x_\lambda L_f x_\lambda^{-1}, \mathfrak{A}_{n,\lambda})$ *is then given by* $x \to diag(x, \ldots, x) = xI_m$.

Because $\mathcal{K}_{x_\lambda z \mathfrak{A} z^{-1} x_\lambda{}^{-1}} = x_\lambda z \mathcal{K}_{\mathfrak{A}} z^{-1} x_\lambda^{-1}$, we can assume (replacing $\mathfrak{A}$ by $\mathfrak{A}_{n,\lambda}$, α by $x_\lambda z\alpha z^{-1}x_\lambda^{-1}$, β by $x_\lambda z\beta z^{-1}x_\lambda^{-1}$ and g by $x_\lambda zgz^{-1}x_\lambda^{-1}$) that

(3.5) $\mathfrak{A} = \mathfrak{A}_{n,\lambda}$.

The equivalences $(E_d, \mathfrak{A}) \sim (\lambda) \sim (x_\lambda L_f x_\lambda^{-1}, \mathfrak{A})$ imply that there exists $h_2 \in \mathcal{K}_{\mathfrak{A}}$ such that $h_2 E_d h_2^{-1} = x_\lambda L_f x_\lambda^{-1}$. We may thus assume (replacing α by $h_2\alpha h_2^{-1}$, β by $h_2\beta h_2^{-1}$ and g by $h_2 g h_2^{-1}$) that

(3.6) $E_d = x_\lambda L_f x_\lambda^{-1}$.

Now let $\mathfrak{B}_\alpha := \mathfrak{A} \cap C_A(\alpha)$, $\mathfrak{B}_\beta := \mathfrak{A} \cap C_A(\beta)$ and $\mathfrak{C} := \mathfrak{A} \cap C_A(E_d)$. By [Br1](theorem (1.1)), $e(E^\alpha/F) = e(E^\beta/F)$, $f(E^\alpha/E_d^\alpha) = f(E^\beta/E_d^\beta)$, and, by corollary (2.4.4)(ii), we have:

(3.7) *$\mathfrak{B}_\alpha$ and $\mathfrak{B}_\beta$ are hereditary orders with the same invariants.*

Since $C_A(E_d) \simeq M_m(\tilde{D})$, for some central E_d-division algebra $\tilde{D}$ of index $d' = \frac{d}{f}$ and since $\langle f(E/E_d), d' \rangle = 1$ and $E^{\alpha\times} \subset \mathcal{K}_{\mathfrak{A}} \cap C_A(E_d) \subset \mathcal{K}_{\mathfrak{C}}$, $E^{\beta\times} \subset \mathcal{K}_{\mathfrak{A}} \cap C_A(E_d) \subset \mathcal{K}_{\mathfrak{C}}$, we get by [G](theorem 2.2(ii)) that the embeddings $(E^\alpha, \mathfrak{C})$, $(E^\beta, \mathfrak{C})$ are good. Moreover, by [G](theorem 2.2(i)), we conclude in particular that

(3.8) *$\mathfrak{C}$ is the unique continuation of $\mathfrak{B}_\alpha$ from $C_A(\alpha)$ to $C_A(E_d)$.*

By [G](theorem 2.2 (i)(a)(b)),

(3.9) *there exists $z \in \mathcal{K}_{\mathfrak{B}_\beta}$ such that $\mathcal{K}_{\mathfrak{C}} = \mathfrak{C}^\times \langle z \rangle$,*

which just means that the continuation of $\mathfrak{B}_\beta$ by $\mathfrak{C}$ is unramified in the sense of the associated pseudo-valuations.

Since $g\mathfrak{A}g^{-1} \cap C_A(\alpha) = g\mathfrak{A}g^{-1} \cap C_A(g\beta g^{-1}) = g\mathfrak{B}_\beta g^{-1}$ and because of (3.7), there exists $y \in C_A(\alpha)$ such that

(3.10) $yg\mathfrak{B}_\beta g^{-1}y^{-1} = \mathfrak{B}_\alpha$.

The field $ygE_dg^{-1}y^{-1}/F$ is an unramified subextension of degree f of E^α/F, whence $ygE_dg^{-1}y^{-1} = E_d$ and therefore

(3.11) $yg \in N_{A^\times}(E_d)$,

where $N_{A^\times}$ means the $A^\times$-normalizer. .

Letting $\mathfrak{C}' := yg\mathfrak{A}g^{-1}y^{-1} \cap C_A(E_d)$ and applying (3.11), we have $ygC_A(E_d)g^{-1}y^{-1} = C_A(ygE_dg^{-1}y^{-1}) = C_A(E_d)$. Therefore $\mathfrak{C}' = yg\mathfrak{C}g^{-1}y^{-1}$; hence, from (3.10) and since $\mathcal{K}_{yg\mathfrak{C}g^{-1}y^{-1}} = yg\mathcal{K}_{\mathfrak{C}}g^{-1}y^{-1}$, we get that $\mathfrak{C}'$ and $\mathfrak{C}$ are both continuations of $\mathfrak{B}_\alpha$. As a consequence of (3.8), $\mathfrak{C}' = \mathfrak{C}$ and

(3.12) $yg \in N_{A^\times}(\mathfrak{C})$.

From (3.11), there exists $k \in \mathbb{Z}$ such that, for all $e \in E_d$, $ygeg^{-1}y^{-1} = \sigma^k(e)$ (where σ is as in definition (2.3.1)(iii)); hence by reductions (3.6) and (3.4)

(3.13) $\pi^k yg \in C_A(E_d)$.

By reductions (3.5) and (3.3), we also have $\mathfrak{C} = \mathfrak{A} \cap C_A(E_d) = (\mathfrak{p}_D^{a_{i,j}} \cap C_D(E_d))_{i,j=1,\dots,m}$ and $\pi \in N_{A^\times}(E_d)$; hence $\pi \in N_{A^\times}(\mathfrak{C})$. But this, together with (3.12) and (3.13), leads to $\pi^k yg \in N_{A^\times}(\mathfrak{C}) \cap C_A(E_d) = \mathcal{K}_{\mathfrak{C}}$. By (3.9) there exists $x \in \mathcal{K}_{\mathfrak{B}_\beta}$ such that $\pi^k ygx \in \mathfrak{C}^\times \subset \mathfrak{A}^\times \subset \mathcal{K}_{\mathfrak{A}}$. Because of the reductions (3.5) and (3.3), we have $\pi \in \mathcal{K}_{\mathfrak{A}}$; hence $ygx \in \mathcal{K}_{\mathfrak{A}}$.

Setting $h := ygx$, we get $h\beta h^{-1} = ygx\beta x^{-1}g^{-1}y^{-1} = yg\beta g^{-1}y^{-1} = y\alpha y^{-1} = \alpha$ (because $x \in C_A(\beta)$ and $y \in C_A(\alpha)$), as required.

□

(3.14) Definition *(i) For a type of embedding $\bar{\tau}$, we denote by $A(\mathfrak{A}, \bar{\tau})$ the set of all $(\mathfrak{A}, \bar{\tau})$-pure elements and by $Ad_{\mathcal{K}_{\mathfrak{A}}} \backslash A(\mathfrak{A}, \bar{\tau})$ the set of all $\mathcal{K}_{\mathfrak{A}}$-orbits of $A(\mathfrak{A}, \bar{\tau})$.*

(ii) Let $\bar{\tau}$ be a type of embedding for a hereditary order $\mathfrak{A}$ and, for each divisor f of d, fix an embedding datum λ_f such that $(\tau)_f \simeq (\lambda_f)$. Let $t(\lambda_f), \upsilon(\lambda_f)$ be as in the assumptions of corollary (2.4.4) for λ_f and $\mathfrak{A}$. Denote by $F_{\mathfrak{A},\bar{\tau},f}[T]$ the set of all irreducible polynomials $f(T)$ over F such that for $E := F[T]/(f(T))$ the numerical conditions (a)-(c) in corollary (2.4.4)(iii) are fulfilled. Set $F_{\mathfrak{A},\bar{\tau}}[T] := \bigcup_{f|d} F_{\mathfrak{A},\bar{\tau},f}[T]$.

(3.15) Theorem *Let $\bar{\tau}$ be a type of embedding for a hereditary order $\mathfrak{A}$. The map which takes each pure element to its minimal polynomial over F induces a bijection*

$$Ad_{\mathcal{K}_{\mathfrak{A}}} \backslash A(\mathfrak{A}, \bar{\tau}) \leftrightarrow F_{\mathfrak{A},\bar{\tau}}[T].$$

Proof.

Clearly this is a well-defined map from $Ad_{\mathcal{K}_{\mathfrak{A}}} \backslash A(\mathfrak{A}, \bar{\tau})$ to a subset of all irreducible polynomials defined over F. By lemma (3.2) it is injective and by corollary (2.4.4)(iii) it is surjective.

□

(3.16) Corollary *Let $\bar{\tau}$ be a type of embedding for a hereditary order $\mathfrak{A}$ and $\alpha \in A$ be $(\mathfrak{A}, \bar{\tau})$-pure.*

(i) Let $F[\alpha]/E/F$ be an arbitrary intermediate extension; then each $\beta \in A$ such that $F[\beta] = E$ is $(\mathfrak{A}, \bar{\tau})$-pure.

(ii) Let f be a divisor of $f(F[\alpha]/F)$ and e a divisor $e(F[\beta]/F)$. Then for each irreducible polynomial $p(T)$ over F such that $f(K(p)/F) = f$ and $e(K(p)/F) = e$, where $K(p) := F[T]/(p(T))$, there exists an $(\mathfrak{A}, \bar{\tau})$-pure element β with minimal polynomial $p(T)$.

Proof.

Let λ be an embedding datum such that $(\lambda) \sim (F[\alpha], \mathfrak{A})$ and let $f' := \langle f(F[\alpha]/F), d \rangle$. Clearly (λ) is of type $\bar{\tau}$.

(i) Let $\bar{f} := \langle f(E/F), d \rangle$, then $\bar{f} \mid f'$ and it is easy to see that $(\lambda)_{\bar{f}}$ is of type $\bar{\tau}$ as well.

Let $F[\alpha]_d, E_d$ be the division algebra parts of $F[\alpha], E$ respectively; then $[F[\alpha]_d : F] = f'$, $[E_d : F] = \bar{f}$ and $F[\alpha]_d/E_d$ is an unramified extension of degree $\frac{f'}{\bar{f}}$. Let M be the compositum of E and $F[\alpha]_d$ in $F[\alpha]$; then, because $\langle f(E/E_d), [F[\alpha]_d : E_d]\rangle = 1$, we have $f(M/E) = [M : E] = \frac{f'}{\bar{f}}$. Let M_d be the division algebra part of M; then $M_d = F[\alpha]_d$. Hence $(M, \mathfrak{A}) \sim (F[\alpha]_d, \mathfrak{A}) \sim (\lambda)$.

Let $g \in A^\times$ such that $(gM_dg^{-1}, g\mathfrak{A}g^{-1}) = (L_{f'}, \mathfrak{A}_n) := (\lambda)$; then $L_{f'}/gE_dg^{-1}/F$ is the unique unramified intermediate extension of degree $\bar{f}$ over F. Hence, $(gE_dg^{-1}, \mathfrak{A}_n) = (\lambda)_{\bar{f}}$. This implies $(E, \mathfrak{A}) \sim (\lambda)_{\bar{f}}$ and hence $(E, \mathfrak{A})$ is of type $\bar{\tau}$ and each $\beta \in A$ such that $F[\beta] = E$ is $(\mathfrak{A}, \bar{\lambda})$-pure, as required.

(ii) For this part let $M/K(p)$ be an unramified extension of degree $\frac{f(F[\alpha]/F)}{f(K(p)/F)}$, then we have $f(M/F) = f(F[\alpha]/F)$ and $e(M/F) = e(K(p)/F)$. Because $F[\alpha]/F$ fulfills all the relations of (2.4.4)(iii)(a)-(c) with respect to λ, we have that M/F fulfills those relations as well. Therefore there exists an embedding $M \hookrightarrow A$ over F such that, if M' is the image of M under this embedding, $(M', \mathfrak{A})$ is an embedding equivalent to (λ). Let $\alpha' \in A$ be such that $F[\alpha'] = M'$; then α' is $(\mathfrak{A}, \bar{\tau})$-pure. Let K' be the image of $K(p)$ of our embedding in M'. Then clearly, there exists $\beta \in A$ such that $F[\beta] = K'$ and β has minimal polynomial $p(T)$. Now the first part applies and shows that β is $(\mathfrak{A}, \bar{\tau})$-pure as required.

□

4 Intertwining implies conjugacy

4.1 The result

We first recall some notions from [Br2] (see §1.1 and §4.1). In this section and in sections 5, 6, the symbol q does not denote the cardinal of the residue field of F.

A *stratum* in A is a quadruple $\sigma = [\mathfrak{A}, n, q, \beta]$, where $\mathfrak{A}$ is an hereditary $\mathfrak{o}_F$-order in A, n, q are integers satisfying $q < n$, and β is an element of $\mathfrak{P}^{-n}$, $\mathfrak{P} = rad(\mathfrak{A})$. We say that σ is *pure* if β is $\mathfrak{A}$-pure and $\beta\mathfrak{A} = \mathfrak{P}^{-n}$; in particular $E = F[\beta]$ is a field and $E^\times$ is contained in $\mathcal{K}$, the normalizer of $\mathfrak{A}$ in G.

Let $\tilde{A} = \mathrm{End}_F(V)$. For each hereditary order $\mathfrak{A}$ in A (attached to an $\mathfrak{o}_D$-lattice chain $\{X_k\}$ in V), define a hereditary order in $\tilde{A}$ by:

$$\tilde{\mathfrak{A}} = \mathrm{End}_{\mathfrak{o}_F}\{X_k\} = \{a \in \tilde{A};\ aX_k \subset X_k\}.$$

We easily check that $\tilde{\mathfrak{P}}^k \cap A = \mathfrak{P}^k$, $k \in \mathbb{Z}$, and $\tilde{\mathcal{K}} \cap A = \mathcal{K}$, where $\tilde{\mathfrak{P}} = rad(\tilde{\mathfrak{A}})$ and $\tilde{\mathcal{K}}$ is the normalizer of $\tilde{\mathfrak{A}}$ in $\tilde{A}^\times$. If r is the period of (X_k), then the period of $\tilde{\mathfrak{A}}$ is $e(\tilde{\mathfrak{A}} \mid \mathfrak{o}_F) = rd$, i.e. $\pi_F\tilde{\mathfrak{A}} = \tilde{\mathfrak{P}}^{rd}$ for any uniformizer π_F of F. Moreover, for all $a \in A$, we have $\nu_{\tilde{\mathfrak{A}}}(a) = \nu_{\mathfrak{A}}(a)$.

To any stratum $\sigma = [\mathfrak{A}, n, q, \beta]$, we attach the stratum $\tilde{\sigma} = [\tilde{\mathfrak{A}}, n, q, \beta]$ of $\tilde{A}$. Then σ is pure if and only if $\tilde{\sigma}$ is. We say that σ is *simple* if $\tilde{\sigma}$ is simple in the sense of [BK2](1.5.5).

Two strata $\sigma_i = [\mathfrak{A}, n, q, \gamma_i]$, $i = 1, 2$, are said to be *equivalent* if $\gamma_1 \equiv \gamma_2 \bmod \mathfrak{P}^{-q}$. We say that they *intertwine* in G if there exists g in G such that

$$g(\gamma_1 + \mathfrak{P}^{-q})g^{-1} \cap (\gamma_2 + \mathfrak{P}^{-q}) \neq \emptyset.$$

Finally we say that a pure stratum $[\mathfrak{A}, n, q, \beta]$ is $\bar{\tau}$-pure, for a type of embedding $\bar{\tau}$, if β is $(\mathfrak{A}, \bar{\tau})$-pure.

Suppose for a moment that A is split. Then we have the following fundamental result ([BK2] theorem (2.6.1)):

(4.1.1) Theorem (Bushnell-Kutzko) *Let $[\mathfrak{A}, n, q, \gamma_i]$, $i = 1, 2$, be two simple strata and assume they intertwine in G. Then there exists $k \in \mathfrak{A}^\times$ such that $k\gamma_1 k^{-1} - \gamma_2 \in \mathfrak{P}^{-q}$.*

When A is not split, this result is no longer true. However in this section we are going to prove the following generalization.

(4.1.2) Theorem *Let τ be a type of embedding and $[\mathfrak{A}, n, q, \gamma_i]$, $i = 1, 2$, be two $\bar{\tau}$-pure simple strata. Assume that they intertwine in G. Then there exists $k \in \mathcal{K}$ such that $k\gamma_1 k^{-1} - \gamma_2 \in \mathfrak{P}^{-q}$.*

Remark. If $q = n-1$, then (fixing a smooth character of F with conductor $\mathfrak{p}_F$) the strata $[\mathfrak{A}, n, q, \gamma_i]$ give rise to characters ψ_{γ_1}, ψ_{γ_2} of the group $U^n(\mathfrak{A}) := 1 + \mathfrak{P}^n \subset G$ (see [Br2](1.1.2)). As a corollary of the theorem, we have that if the ψ_{γ_i} intertwine in G, they are actually conjugate by an element of $\mathcal{K}$.

Note that if $g \in G$ intertwines the strata $[\mathfrak{A}, n, q, \gamma_i]$, $i = 1, 2$, it intertwines the strata $[\tilde{\mathfrak{A}}, n, q, \gamma_i]$, $i = 1, 2$ as well. Indeed we have the containment:

$$g(\gamma_1 + \mathfrak{P}^{-q})g^{-1} \cap (\gamma_2 + \mathfrak{P}^{-q}) \subset g(\gamma_1 + \tilde{\mathfrak{P}}^{-q})g^{-1} \cap (\gamma_2 + \tilde{\mathfrak{P}}^{-q}).$$

Therefore our result follows from the following one:

(4.1.3) Proposition *With the assumptions of (4.1.2), assume that the simple strata $[\tilde{\mathfrak{A}}, n, q, \gamma_i]$, $i = 1, 2$, intertwine in $\tilde{A}^\times$. Then there exists $k \in \mathcal{K}$ such that $k\gamma_1 k^{-1} - \gamma_2 \in \mathfrak{P}^{-q}$.*

4.2 Simple pairs and change of base field

Before proving (4.1.3), we need to recall some concepts from [BH]. Let $\mathfrak{U}$ be a hereditary order in some split F-algebra U. Let β be a $\mathfrak{U}$-pure element of U and write $E = F[\beta] \subset U$. Define $e_\beta(\mathfrak{U})$ to be the integer $e(\mathfrak{U} \mid \mathfrak{o}_F)/e(E/F)$. Then we have

$$\nu_{\mathfrak{U}}(\beta) = e_\beta(\mathfrak{U})\nu_E(\beta),$$

where ν_E is the normalized valuation of E.

Assume that $[\mathfrak{U}, n, q, \beta]$ is a simple stratum in U. Then *the simple pair* attached to this stratum is the pair $[m, P]$, where $m = [q/e_\beta(\mathfrak{U})] \in \mathbb{Z}$ and P is the minimal polynomial of β over F. *A simple pair* is any pair $[m, P]$ arising from a simple stratum in some split simple algebra; such a stratum is called a *realization* of $[m, P]$. Actually in [BH] simple pairs are defined in an intrinsic way that we do not need for our purposes. Moreover, our language differs a little bit but the reader will check that it is rigorously equivalent.

If $[\mathfrak{U}, n, q, \beta]$ is a pure stratum such that $[[q/e_\beta(\mathfrak{U})], P]$ is a simple pair, P being the minimal polynomial of β over F, then it is automatically simple ([BH]p.115), i.e. it is a realization of $[[q/e_\beta(\mathfrak{U})], P]$.

The following result will be crucial for our proof.

(4.2.1) Theorem ([BH](1.10)) *Let $[m_i, P_i]$, $i = 1, 2$ be two simple pairs. Assume that they have some respective realizations $[\mathfrak{U}, n_i, q_i, \beta_i]$, $i = 1, 2$, which intertwine in $U^\times$. Then any other pair of realizations with a common order intertwine as well.*

Let us fix a split algebra U, a hereditary order $\mathfrak{U} \subset U$ as well as a finite, Galois, tamely ramified extension $K/F \subset U$, such that $K^\times$ normalizes $\mathfrak{U}$. Let W be the centralizer of K in U and $\mathfrak{W} = \mathfrak{U} \cap W$. Then $\mathfrak{W}$ is a hereditary $\mathfrak{o}_K$-order in the K-algebra W. Let $[\mathfrak{W}, n, q, \beta_i]$, $i = 1, 2$, be simple strata in W. This automatically implies that the strata $[\mathfrak{U}, n, q, \beta_i]$ are at least pure (cf. [BH] §2). Let $\mathcal{N}_{\mathfrak{U}}(K)$ be the set of elements in $U^\times$ which normalize $\mathfrak{U}$ and K.

(4.2.2) Proposition ([BH](6.7)) *Assume that the strata $[\mathfrak{U}, n, q, \beta_i]$, $i = 1, 2$, are simple and intertwine in $U^\times$. Then there exists $x \in \mathcal{N}_{\mathfrak{U}}(K)$ such that $[\mathfrak{W}, n, q, x^{-1}\beta_1 x]$ is equivalent to $[\mathfrak{W}, n, q, \beta_2]$.*

Finally we will need:

(4.2.3) Proposition ([BH](4.3)) *With the notation as before, assume we are given a pure stratum* $[\mathfrak{W}, n, q, \beta]$ *(so that* $[\mathfrak{U}, n, q, \beta]$ *is automatically pure). Then it is equivalent to a simple stratum* $[\mathfrak{W}, n, q, \gamma]$ *such that* $[\mathfrak{U}, n, q, \gamma]$ *is simple as well.*

4.3 Proof of the theorem

We start by making some technical reductions. Since the strata $[\tilde{\mathfrak{A}}, n, q, \gamma_i]$ are simple and intertwine, they intertwine in $\tilde{\mathcal{K}}$: there exists $k \in \tilde{\mathcal{K}}$ such that $[\tilde{\mathfrak{A}}, n, q, k\gamma_1 k^{-1}]$ is equivalent to $[\tilde{\mathfrak{A}}, n, q, \gamma_2]$. We can then apply [BK2] theorem (2.4.1) to get:

$$e(F[\gamma_1]/F) = e(F[\gamma_2]/F), \; f(F[\gamma_1]/F) = f(F[\gamma_2]/F).$$

Let $F[\gamma_i]^{unr}$ be the unramified part of $F[\gamma_i]/F$, $i = 1, 2$. By assumption the extensions $F[\gamma_i]^{unr}/F$, $i = 1, 2$, are $(\mathfrak{A}, \bar{\tau})$-pure. Moreover they are conjugate under $A^\times$ since they have the same degree over F. We can therefore write $F[\gamma_i]^{unr} = F[\zeta_i]$, $i = 1, 2$, with $\zeta_1 = g\zeta_2 g^{-1}$ for some $g \in G$. By applying lemma (3.2), we have that the ζ_i, whence the $F[\gamma_i]^{unr}$, are conjugate under $\mathcal{K}$. Therefore:

(4.3.1) *By conjugating* γ_1 *by an element of* $\mathcal{K}$*, we may and will assume that* $F[\gamma_1]^{unr} = F[\gamma_2]^{unr} =: K$*.*

Let $\tilde{B}$ be the centralizer of K in $\tilde{A}$, $\tilde{\mathfrak{B}}$ be the hereditary $\mathfrak{o}_K$-order $\tilde{B} \cap \tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{Q}} = rad(\tilde{\mathfrak{B}})$. The algebras $K[\gamma_i] = F[\gamma_i]$, $i = 1, 2$, are fields and $K[\gamma_i]^\times \subset \tilde{\mathcal{K}} \cap \tilde{B} = \mathcal{K}_{\tilde{\mathfrak{B}}}$ (see [BK2]§ 1.2). Therefore the strata $[\tilde{\mathfrak{B}}, n, q, \gamma_i]$ are pure. Moreover by [BH] theorem (2.4), these strata are simple because the $[\tilde{\mathfrak{A}}, n, q, \gamma_i]$ are so.

Now since the $[\tilde{\mathfrak{A}}, n, q, \gamma_i]$ intertwine, we can apply (4.2.2) to get an element $x \in \mathcal{N}_{\tilde{\mathfrak{A}}}(K)$ such that

$$x^{-1}\gamma_1 x - \gamma_2 \in \tilde{\mathfrak{Q}}^{-q},$$

where $\mathcal{N}_{\tilde{\mathfrak{A}}}(K)$ is the normalizer of K in $\tilde{\mathcal{K}}$.

By conjugation, x induces on K an element σ of $Gal(K/F)$: $\sigma(u) = xux^{-1}$, $u \in K$. Pick elements $u_i \in K$, $i = 1, 2$, such that $K = F[u_i]$ and such that σ is given by $\sigma(u_1) = u_2$. Since the u_i are conjugate under $A^\times$ and are $(\mathfrak{A}, \bar{\tau})$-pure, there exists $x_o \in \mathcal{K}$ such that $u_2 = x_o u_1 x_o^{-1}$. It follows that $x^{-1}x_o$ is an element of $\tilde{\mathcal{K}}$ which centralizes K, i.e this is an element of $\mathcal{K}_{\tilde{\mathfrak{B}}}$. Moreover $(x^{-1}x_o)(x_o^{-1}\gamma_1 x_o)(x^{-1}x_o)^{-1} - \gamma_2 \in \tilde{\mathfrak{Q}}^{-q}$; we still have $F[x_o^{-1}\gamma_1 x_o]^{unr} = K$ and the stratum $[\mathfrak{A}, n, q, x_o^{-1}\gamma_1 x_o]$ remains simple because it is conjugate under $\mathcal{K}$ to a simple stratum. Therefore:

(4.3.2) *By conjugating γ_1 by an element of $\mathcal{K}$, we may and will assume that the strata $[\tilde{\mathfrak{B}}, n, q, \gamma_i]$, $i = 1, 2$, intertwine in $\tilde{B}^\times$.*

Recall that $e_{\gamma_i}(\tilde{\mathfrak{A}}) = e(\tilde{\mathfrak{A}} \mid \mathfrak{o}_F)/e(F[\gamma_i]/F) = rd/e(F[\gamma_i]/F)$; this is an integer independent of i. If $q_o = e_{\gamma_i}(\tilde{\mathfrak{A}})[q/e_{\gamma_i}(\tilde{\mathfrak{A}})]$, then $[\tilde{\mathfrak{A}}, n, q_o, \gamma_i]$ is simple because this is a realization of the simple pair of $[\tilde{\mathfrak{A}}, n, q, \gamma_i]$. Moreover by (4.2.1), the simple strata $[\tilde{\mathfrak{A}}, n, q_o, \gamma_i]$, $i = 1, 2$, intertwine. If we prove that the $[\mathfrak{A}, n, q_o, \gamma_i]$, $i = 1, 2$, intertwine in $\mathcal{K}$, we will be done. Indeed if $k\gamma_1 k^{-1} - \gamma_2 \in \mathfrak{P}^{-q_o}$ for some k in $\mathcal{K}$, then k intertwines the $[\mathfrak{A}, n, q, \gamma_i]$, $i = 1, 2$, for $\mathfrak{P}^{-q_o} \subset \mathfrak{P}^{-q}$ $(q_o \leq q)$. Therefore:

(4.3.3) *We may and will assume that q is a multiple of $e_{\gamma_i}(\tilde{\mathfrak{A}})$.*

This restriction will be useful for reasons which will become clear.

Let B be the centralizer of K in A and $\mathfrak{B} = B \cap \mathfrak{A}$. According to [Br1] theorem (1.3), $\mathfrak{B}$ is a hereditary $\mathfrak{o}_K$-order in B. We certainly have $\mathcal{K} \cap B \subset \mathcal{K}_{\mathfrak{B}}$ (the normalizer of $\mathfrak{B}$ in $B^\times$) and the elements γ_i, $i = 1, 2$, are pure with respect to $\mathfrak{B}$. Put $\mathfrak{Q} = rad(\mathfrak{B})$.

(4.3.4) Lemma *Write $\mathcal{K} = \langle h \rangle \mathfrak{A}^\times$ for some $h \in \mathcal{K}$, so that $h\mathfrak{A} = \mathfrak{P}^t$, where $\nu_{\mathfrak{A}}(\mathcal{K}) = t\mathbb{Z}$. Let $\rho \in \mathbb{N}$ be the smallest positive integer such that there exists $\epsilon \in \mathfrak{A}^\times$ with $h^\rho \epsilon \in \mathcal{K}_{\mathfrak{B}}$. Then:*

(i) $\mathfrak{P}^{t\rho k} \cap B = \mathfrak{Q}^{t\rho k r' d'/rd}$, $k \in \mathbb{Z}$,

where r' is the period of $\mathfrak{B}$ and d' is the index over E of a K-division algebra Δ such that $B \simeq M_{m'}(\Delta)$.

(ii) We have $t\rho \mid e_{\gamma_i}(\tilde{\mathfrak{A}}) = rd/e(F[\gamma_i]/F)$. In particular, we have $t\rho \mid q$.

Proof. (i) Write $z = h^\rho \epsilon$, $\epsilon \in \mathfrak{A}^\times$, $z \in \mathcal{K}_{\mathfrak{B}}$. we have $\mathfrak{P}^{t\rho k} \cap B = (z^k \mathfrak{A}) \cap B = z^k(\mathfrak{A} \cap B) = \mathfrak{Q}^{\nu_{\mathfrak{B}}(z^k)} = \mathfrak{Q}^{k\nu_{\mathfrak{B}}(z)}$. For all $u \in \mathcal{K} \cap B$, we have $\nu_{\mathfrak{B}}(u) = C\nu_{\mathfrak{A}}(u)$, where C does not depend on u. We have $\pi_F \mathfrak{A} = \mathfrak{P}^{rd}$, for any uniformizer π_F of F, and, since K/F is unramified, we have $\pi_F \mathfrak{B} = \mathfrak{Q}^{r'd'}$. Therefore $C = r'd'/rd$ and (i) follows from the equality $\nu_{\mathfrak{A}}(z) = \rho t$.

(ii) Let π_i be a uniformizer of $F[\gamma_i]$; then there exists $\epsilon \in \mathfrak{o}_{F[\gamma_i]}^\times \subset \mathfrak{B}^\times$ such that $\pi_i^{e(F[\gamma_i]/F)} \epsilon = \pi_F$ is a uniformizer of F. Then $\pi_i \in \mathcal{K} \cap B$ and $\nu_{\mathfrak{A}}(\pi_i) = \nu_{\mathfrak{A}}(\pi_F)/e(F[\gamma_i]/F) = rd/e(F[\gamma_i]/F) = e_{\gamma_i}(\tilde{\mathfrak{A}})$. Since all elements of $\mathcal{K} \cap B$ have as $\mathfrak{B}$-valuation a multiple of $\nu_{\mathfrak{B}}(z)$, we have $\rho t(r'd'/rd) \mid rd/e(F[\gamma_i]/F)(r'd'/rd)$, i.e. $\rho t \mid rd/e_{\gamma_i}(\tilde{\mathfrak{A}})$, as required.

□

Put $n' = n(r'd'/rd)$ and $q' = qr'd'/rd$. Then the strata $[\mathfrak{B}, n', q', \gamma_i]$ are pure and $\mathfrak{Q}^{-q'} \subset \mathfrak{P}^{-q}$ by the previous lemma. We are actually going to prove that these strata intertwine in $\mathfrak{B}^\times$. This will give the existence of $k \in \mathfrak{B}^\times \subset \mathfrak{A}^\times$ such that $k\gamma_1 k^{-1} - \gamma_2 \in \mathfrak{Q}^{-q'}$, whence $k\gamma_1 k^{-1} - \gamma_2 \in \mathfrak{P}^{-q}$ and we will be done.

Let L/K be an unramified extension which splits the K-algebra B. Put $\bar{B} = B \otimes_K L$. For each $\mathfrak{o}_K$-submodule (resp. $\mathfrak{o}_K$-order) X of B, we see $X \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$ as an $\mathfrak{o}_L$-submodule (resp. $\mathfrak{o}_L$-order) of $\bar{B}$ in a canonical way. In particular $\bar{\mathfrak{B}} = \mathfrak{B} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$ is a hereditary $\mathfrak{o}_L$-order in $\bar{B}$. Let $\bar{\mathfrak{Q}} = rad(\bar{\mathfrak{B}})$. Then for all $k \in \mathbb{Z}$, we have $\bar{\mathfrak{Q}}^k = \mathfrak{Q}^k \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$.

(4.3.5) Lemma *The elements $\gamma_i \otimes 1 \in \bar{B}$ are pure with respect to $\bar{\mathfrak{B}}$. Moreover $\nu_{\bar{\mathfrak{B}}}(\gamma_i \otimes 1) = n'$, for $i = 1, 2$.*

Proof. Fix i. Put $E := L[\gamma_i \otimes 1] = K[\gamma_i] \otimes_K L$. By construction, the extension $K[\gamma_i]/K$ is totally ramified and is therefore linearly disjoint from L. It follows that E is a field (and that the minimal polynomial of $\gamma_i \otimes 1$ over L is that of γ_i over K). Moreover, since L/F is unramified, we have $\mathfrak{o}_E = \mathfrak{o}_{K[\gamma_i]} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$. Since γ_i is $\mathfrak{B}$-pure, we have $\mathfrak{o}_{K[\gamma_i]} \subset \mathfrak{B}$ and therefore $\mathfrak{o}_E \subset \bar{\mathfrak{B}}$, as well as $\mathfrak{o}_E^\times \subset \mathcal{K}_{\bar{\mathfrak{B}}}$. Let π be a uniformizer of $K[\gamma_i]$; then $\pi \otimes 1$ is a uniformizer of E. Since π normalizes $\mathfrak{B}$, we have that $\pi \otimes 1$ normalizes $\bar{\mathfrak{B}}$. We have obtained that $E^\times = \langle \pi \otimes 1 \rangle \mathfrak{o}_E^\times$ is contained in $\mathcal{K}_{\bar{\mathfrak{B}}}$; as required.

We have $(\gamma_i \otimes 1)\bar{\mathfrak{B}} = (\gamma_i \mathfrak{B}) \otimes_{\mathfrak{o}_K} \mathfrak{o}_L = \mathfrak{Q}^{n'} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L = \bar{\mathfrak{Q}}^{n'}$, whence $\nu_{\bar{\mathfrak{B}}}(\gamma_i \otimes 1) = n'$.
□

Similarly we can consider the L-algebra $\tilde{C} = \tilde{B} \otimes_K L$ and the hereditary $\mathfrak{o}_L$-order $\tilde{\mathfrak{C}} = \tilde{\mathfrak{B}} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$. By arguments similar to those of the previous lemma, we have that the strata $[\tilde{\mathfrak{C}}, n, q, \gamma_i \otimes 1]$, $i = 1, 2$, are pure.

(4.3.6) Lemma *The strata $[\tilde{\mathfrak{C}}, n, q, \gamma_i \otimes 1]$ are simple.*

Proof. Let $\tilde{\mathfrak{C}}_1$ be the centralizer of $\gamma_1 \otimes 1$ in $\tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{B}}_1$ the centralizer of γ_1 in $\tilde{\mathfrak{B}}$. We observe that $\tilde{\mathfrak{C}}_1$ is stable under the action of $Gal(L/K)$ and that the set of fixed points is $\tilde{\mathfrak{B}}_1 \otimes 1$. It follows (see e.g. [BH] (11.15)) that $\tilde{\mathfrak{C}}_1 = \tilde{\mathfrak{B}}_1 \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$. Put $\mathfrak{N}_k(\gamma_1 \otimes 1, \tilde{\mathfrak{C}}) = \{x \in \tilde{\mathfrak{C}};\ x(\gamma_1 \otimes 1) - (\gamma_1 \otimes 1)x \in rad(\tilde{\mathfrak{C}})^k\}$ and $\mathfrak{N}_k(\gamma_1, \tilde{\mathfrak{B}}) = \{x \in \tilde{\mathfrak{B}};\ x\gamma_1 - \gamma_1 x \in rad(\tilde{\mathfrak{B}})^k\}$, for $k \in \mathbb{Z}$. Similarly we check that $\mathfrak{N}_k(\gamma_1 \otimes 1, \tilde{\mathfrak{C}}) = \mathfrak{N}_k(\gamma_1, \tilde{\mathfrak{B}}) \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$.

Put $k_0(\gamma_1 \otimes 1, \tilde{\mathfrak{C}}) = max\{k \in \mathbb{Z};\ \mathfrak{N}_k(\gamma_1 \otimes 1, \tilde{\mathfrak{C}}) \not\subset \tilde{\mathfrak{C}}_1 + rad(\tilde{\mathfrak{C}})\}$, and $k_0(\gamma_1, \tilde{\mathfrak{B}}) = max\{k \in \mathbb{Z};\ \mathfrak{N}_k(\gamma_1, \tilde{\mathfrak{B}}) \not\subset \tilde{\mathfrak{B}}_1 + rad(\tilde{\mathfrak{B}})\}$. Then, by definition ([BK2](1.5.5)), the fact that $[\tilde{\mathfrak{B}}, n, q, \gamma_i]$ is simple means that $-q > k_0(\gamma_1, \tilde{\mathfrak{B}})$. Let us prove that $k_0(\gamma_1 \otimes 1, \tilde{\mathfrak{C}}) = k_0(\gamma_1, \tilde{\mathfrak{B}})$; this will imply that $[\tilde{\mathfrak{C}}, n, q, \gamma_1 \otimes 1]$ is simple as well. For all $k \in \mathbb{Z}$, we have $k \geq k_0(\gamma_1 \otimes 1, \tilde{\mathfrak{C}})$ if and only if $\mathfrak{N}_k(\gamma_1, \tilde{\mathfrak{B}}) \otimes_{\mathfrak{o}_K} \mathfrak{o}_L \subset (\tilde{\mathfrak{B}}_1 + rad(\tilde{\mathfrak{B}})) \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$, which is equivalent to $\mathfrak{N}_k(\gamma_1, \tilde{\mathfrak{B}}) \subset \tilde{\mathfrak{B}}_1 + rad(\tilde{\mathfrak{B}})$, i.e. $k \geq k_0(\gamma_1, \tilde{\mathfrak{B}})$. Hence $k_0(\gamma_1 \otimes 1, \tilde{\mathfrak{C}}) = k_0(\gamma_1, \tilde{\mathfrak{B}})$, as required.

$\square$

Let us recall that the simple strata $[\mathfrak{B}, n, q, \gamma_i]$, $i = 1, 2$, intertwine in $\mathcal{K}_{\tilde{\mathfrak{B}}}$. It follows that the strata $[\tilde{\mathfrak{C}}, n, q, \gamma_i \otimes 1]$, $i = 1, 2$, intertwine as well. Indeed, if $k\gamma_1 k^{-1} - \gamma_2 \in rad(\tilde{\mathfrak{B}})^{-q}$, for some $k \in \mathcal{K}_{\tilde{\mathfrak{B}}}$, then $(k \otimes 1)(\gamma_1 \otimes 1)(k \otimes 1)^{-1} - (\gamma_2 \otimes 1) \in rad(\tilde{\mathfrak{B}})^{-q} \otimes 1 \subset rad(\tilde{\mathfrak{C}})^{-q}$ and $k \otimes 1 \in \mathcal{K}_{\tilde{\mathfrak{C}}}$.

Let $[t, P_i]$ be the simple pair attached to $[\tilde{\mathfrak{C}}, n, q, \gamma_i \otimes 1]$, $i = 1, 2$. Recall that P_i is the minimal polynomial of $\gamma_i \otimes 1$ over L (that is, the minimal polynomial of γ_i over K) and $t = [q/e_{\gamma_i \otimes 1}(\tilde{\mathfrak{C}})]$. We check that $e(\tilde{\mathfrak{C}} \mid \mathfrak{o}_L) = e(\tilde{\mathfrak{B}} \mid \mathfrak{o}_K)$. Moreover $e(\tilde{\mathfrak{B}} \mid \mathfrak{o}_K) = e(\tilde{\mathfrak{A}} \mid \mathfrak{o}_F)$, because K/F is unramified. Since $e(\tilde{\mathfrak{A}} \mid \mathfrak{o}_F) = rd$ and $e(L[\gamma_i \otimes 1]/L) = e(F[\gamma_i]/F)$, we get $t = [qe(F[\gamma_i]/F)/rd] = qe(F[\gamma_i]/F)/rd$ (by assumption $rd/e(F[\gamma_i]/F)$ divides q).

(4.3.7) Lemma *For $i = 1, 2$, the pure stratum $[\bar{\mathfrak{B}}, n', q', \gamma_i \otimes 1]$ is a realization of the simple pair $[t, P_i]$.*

Proof. The minimal polynomial of $\gamma_i \otimes 1$ viewed as an element of $\bar{B} = B \otimes_K L$, or as an element of $\tilde{C} = \tilde{B} \otimes_K L$, is the minimal polynomial of γ_i over K (viewed as an element of $B \subset \tilde{B}$). Hence we just have to check that $[q/(\nu e_{\gamma_i \otimes 1}(\bar{\mathfrak{B}}))] = [q/e_{\gamma_i \otimes 1}(\tilde{\mathfrak{C}})]$, for $i = 1, 2$, where $\nu = rd/r'd'$. But this follows from $e(\bar{\mathfrak{B}} \mid \mathfrak{o}_L) = r'd'$ and $e(\tilde{\mathfrak{C}} \mid \mathfrak{o}_L) = rd$.

$\square$

(4.3.8) Corollary *The strata $[\bar{\mathfrak{B}}, n', q', \gamma_i \otimes 1]$, for $i = 1, 2$, are simple and intertwine in $\bar{\mathfrak{B}}^\times$.*

Proof. Indeed, the strata are simple because they are realizations of simple pairs. They intertwine since these simple pairs intertwine when realized as $[\tilde{\mathfrak{C}}, n, q, \gamma_i \otimes 1]$, for $i = 1, 2$, in $\tilde{C}$. Using (4.1.1), we have, moreover, that they intertwine in $\bar{\mathfrak{B}}^\times$.

$\square$

Now the fact that the strata $[\mathfrak{B}, n, q, \gamma_i]$, for $i = 1, 2$, intertwine in $\mathfrak{B}^\times$ follows from the following descent lemma whose proof is postponed to §6.

(4.3.9) Theorem *Let E be a non-archimedean local field, H a central simple E-algebra and $\mathfrak{H}$ a hereditary $\mathfrak{o}_E$-order in H. Let $\bar{E}/E$ be a finite unramified extension and put $\bar{H} = H \otimes_E \bar{E}$, $\bar{\mathfrak{H}} = \mathfrak{H} \otimes_{\mathfrak{o}_E} \mathfrak{o}_{\bar{E}}$. Let $\mathfrak{G}$ (resp. $\bar{\mathfrak{G}}$) be the radical of $\mathfrak{H}$ (resp. $\bar{\mathfrak{H}}$). Finally let $x, y \in H$, $u \in \mathbb{Z}$. Assume that $g(x \otimes 1)g^{-1} - (y \otimes 1) \in \bar{\mathfrak{G}}^u$, for some $g \in \mathcal{K}_{\bar{H}}$ (resp. $g \in \bar{\mathfrak{H}}^\times$). Then there exists $h \in \mathcal{K}_{\mathfrak{H}}$ (resp. $h \in \mathfrak{H}^\times$) such that $hxh^{-1} - y \in \mathfrak{G}^u$.*

We apply the theorem to $H = B$, $\mathfrak{H} = \mathfrak{B}$, $x = \gamma_1$, $y = \gamma_2$ and $u = -q'$. We get that the strata $[\mathfrak{B}, n', q', \gamma_i]$, $i = 1, 2$, intertwine in $\mathfrak{B}^{\times}$ as required. This achieves the proof of proposition (4.1.3) and theorem (4.1.2).

□

5 Existence of a simple stratum equivalent to a pure one

The aim of this section is to prove the following generalization of [BK2] theorem (2.4.1)(i).

(5.1) Theorem *(i) Let $[\mathfrak{A}, n, q, \beta]$ be a pure stratum and assume that β is $(\mathfrak{A}, \bar{\tau})$-pure for some type of embedding $\bar{\tau}$. Then there exists a simple stratum $[\mathfrak{A}, n, q, \gamma]$ equivalent to $[\mathfrak{A}, n, q, \beta]$; moreover, for any choice of this simple stratum, γ is $(\mathfrak{A}, \bar{\tau})$-pure.*

(ii) If a simple stratum $[\mathfrak{A}, n, q, \gamma]$ is equivalent to a pure stratum $[\mathfrak{A}, n, q, \beta]$ then we have $e(F[\gamma]/F) \mid e(F[\beta]/F)$ and $f(F[\gamma]/F) \mid f(F[\beta]/F)$. Conversely, if among the pure strata equivalent to a given pure stratum $[\mathfrak{A}, n, q, \beta]$, $[\mathfrak{A}, n, q, \gamma]$ has the property that $[F[\gamma] : F]$ is minimal, then it is necessarily simple.

Let us first explain how (ii) follows from (i). With the notation of (ii), we have that $[\tilde{\mathfrak{A}}, n, q, \gamma]$ is simple and equivalent to $[\tilde{\mathfrak{A}}, n, q, \beta]$. According to the split case, the invariants of $F[\gamma]/F$ divide the invariants of $F[\beta]/F$ and this gives the first assertion. Now let $[\mathfrak{A}, n, q, \gamma]$ be a pure stratum equivalent to $[\mathfrak{A}, n, q, \beta]$ with $[F[\gamma] : F]$ minimal. For a contradiction assume that it is not simple. Let $[\mathfrak{A}, n, q, \delta]$ be a simple stratum equivalent to $[\mathfrak{A}, n, q, \gamma]$. Since $[\tilde{\mathfrak{A}}, n, q, \delta]$ is simple and $[\tilde{\mathfrak{A}}, n, q, \gamma]$ is not, we must have ([BK2](2.4.1)) $[F[\delta] : F] < [F[\gamma] : F]$, which gives the contradiction.

We need the following result.

(5.2) Lemma *Let $[\mathfrak{A}, n, q, \gamma_i]$, $i = 1, 2$, be two equivalent simple strata. Then γ_1 and γ_2 have the same type of embedding.*

Proof. The strata $[\tilde{\mathfrak{A}}, n, q, \gamma_i]$, $i = 1, 2$, are simple and equivalent. According to the split case ([BK2] lemma (2.4.12)(i)), the rings of integers $\mathfrak{o}_{F[\gamma_i]}$, $i = 1, 2$ have the same image in $\tilde{\mathfrak{A}}/\tilde{\mathfrak{P}}$, whence in $\mathfrak{A}/\mathfrak{P}$ (seeing $\mathfrak{A}/\mathfrak{P}$ as being canonically embedded in $\tilde{\mathfrak{A}}/\tilde{\mathfrak{P}}$, these images actually lie in $\mathfrak{A}/\mathfrak{P}$). Our result follows then from the definition of the type of embedding and lemma (2.3.6).

□

As a consequence, in theorem (5.1)(i) it suffices to prove the existence of a simple stratum equivalent to $[\mathfrak{A}, n, q, \beta]$ and with the type of embedding $\bar{\tau}$.

Let us then show that we can reduce to the case where $e_\beta(\tilde{\mathfrak{A}}) \mid q$. Let $q_o = e_\beta(\tilde{\mathfrak{A}})[q/e_\beta(\tilde{\mathfrak{A}})]$ and assume that we have constructed a simple stratum $[\mathfrak{A}, n, q_o, \gamma]$ equivalent to $[\mathfrak{A}, n, q_o, \beta]$. We have $\gamma - \beta \in \mathfrak{P}^{-q_o} \subset \mathfrak{P}^{-q}$, since $q_o \leq q$. Moreover, $[\mathfrak{A}, n, q, \gamma]$ is simple. Indeed, we know that this is the case if and only if the invariant $k_o(\gamma, \tilde{\mathfrak{A}})$ is strictly less than $-q$ ([BK2] Def. (1.5.5)). We know ([BK2](1.4.13)) that $k_o(\gamma, \tilde{\mathfrak{A}})$ is divisible by $e_\gamma(\tilde{\mathfrak{A}}) = e(\tilde{\mathfrak{A}} \mid \mathfrak{o}_F)/e(F[\gamma]/F)$, whence by $e_\beta(\tilde{\mathfrak{A}}) = e(\tilde{\mathfrak{A}} \mid \mathfrak{o}_F)/e(F[\beta]/F)$, since $e(F[\gamma]/F)$ divides $e(F[\beta]/F)$. Since $[\mathfrak{A}, n, q_0, \gamma]$ is simple, we have $-q_o > k_o(\gamma, \tilde{\mathfrak{A}})$, i.e. $-q_o \geq k_o(\gamma, \tilde{\mathfrak{A}}) + e_\beta(\tilde{\mathfrak{A}})$ by the previous remark. On the other hand we have $-q > -q_o - e_\beta(\tilde{\mathfrak{A}})$ by definition of q_0, whence $-q > k_o(\gamma, \tilde{\mathfrak{A}})$. This means that $[\tilde{\mathfrak{A}}, n, q, \gamma]$ is simple as required. Therefore

We may and will assume that $e_\beta(\tilde{\mathfrak{A}})$ divides q.

Let K/F be the unramified part of $F[\beta]/F$, B the centralizer of K in A, $\mathfrak{B} = B \cap \mathfrak{A}$ and $\mathfrak{Q} = rad(\mathfrak{B})$. With the arguments and notation of (4.3.4), we have that $\beta\mathfrak{B} = \mathfrak{Q}^{-n'}$, where $n' = nr'd'/rd$, that $q' = qr'd'/rd$ is an integer and that $\mathfrak{Q}^{-q'} \subset \mathfrak{P}^{-q}$. Since $K[\beta] = F[\beta]$ is a field satisfying $K[\beta]^\times \subset \mathcal{K} \cap B \subset \mathcal{K}_\mathfrak{B}$, the stratum $[\mathfrak{B}, n', q', \beta]$ is pure.

Let L/K be an unramified extension which splits B. Set $\bar{B} = B \otimes_K L$ and $\bar{\mathfrak{B}} = \mathfrak{B} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L \subset \bar{B}$. Identify $\bar{B}$ with $\mathrm{End}_L W$ for some L-space W and put $\tilde{B} = \mathrm{End}_F W$. Let $\tilde{\mathfrak{B}}$ be the hereditary $\mathfrak{o}_F$-order of $\tilde{B}$ attached to the same lattice chain as $\bar{\mathfrak{B}}$. Note that $K[\beta]/K$ is totally ramified of degree $e(K[\beta]/K) = e(F[\beta]/F)$. As in (4.3.7), the stratum $[\bar{\mathfrak{B}}, n', q', \beta \otimes 1]$ is pure. By applying (4.2.3), it is equivalent to a simple stratum $[\bar{\mathfrak{B}}, n', q', \gamma]$ such that $[\tilde{\mathfrak{B}}, n', q', \gamma]$ is simple as well. The integer $e(L[\gamma]/L)$ divides $e(L[\beta \otimes 1]/L) = e(K[\beta]/K)$ and $f(L[\gamma]/L) = 1$.

(5.3) Lemma *The minimal polynomial μ_L of γ over L and its minimal polynomial μ_K over K coincide. Moreover, $e(K[\gamma]/K) \mid e(K[\beta]/K)$ and $f(K[\gamma]/K) = 1$.*

Proof. Let $\hat{\mathfrak{B}}$ be the hereditary $\mathfrak{o}_K$-order of $\mathrm{End}_K W$ attached to the same lattice chain as $\bar{\mathfrak{B}}$. We can form the equivalent pure strata $[\hat{\mathfrak{B}}, n', q', \beta \otimes 1]$ and $[\hat{\mathfrak{B}}, n', q', \gamma]$. By [BH] theorem (2.4), since $[\tilde{\mathfrak{B}}, n', q', \gamma]$ is simple so is $[\hat{\mathfrak{B}}, n', q', \gamma]$. This proves the second assertion of the lemma.

We can decompose $L[\gamma]/L$ in two towers. The first one is $L[\gamma]/L/K$, with $L[\gamma]/L$ totally ramified and L/K unramified. The second one is $L[\gamma]/K[\gamma]/K$, with $K[\gamma]/K$ totally ramified and $L[\gamma]/K[\gamma]$ unramified, since this extension is generated over $K[\gamma]$ by the roots of unity prime to $p = char(k_F)$ in $L^\times$. This automatically implies that $e(L[\gamma]/K) = [L[\gamma] : L] = [K[\gamma] : K]$. Therefore μ_L and μ_K have the same degree and, since $\mu_L \mid \mu_K$, we must have $\mu_L = \mu_K$.

□

Thanks to the previous lemma, we can apply Corollary (3.16)(ii), to get a $\mathfrak{B}$-pure element γ_o in B with minimal polynomial $\mu_K = \mu_L$. The two elements of $\bar{B}$, $\gamma_o \otimes 1$ and γ, have the same minimal polynomial over L (namely $\mu_K = \mu_L$) and are both $\bar{\mathfrak{B}}$-pure. By [BH] lemma (1.6), they are conjugate under $\bar{\mathfrak{B}}^\times$: there exists $k \in \bar{\mathfrak{B}}^\times$ such that $k(\gamma_o \otimes 1)k^{-1} - \beta \otimes 1 \in rad(\bar{\mathfrak{B}})^{-q'}$. We can now apply theorem (4.3.9) to get an element l of $\mathfrak{B}^\times$ such that $l\gamma_o l^{-1} - \beta \in \mathfrak{Q}^{-q'}$. Write $\gamma_1 = l\gamma_o l^{-1}$.

(5.4) Lemma *The stratum* $[\mathfrak{A}, n, q, \gamma_1]$ *is simple and equivalent to* $[\mathfrak{A}, n, q, \beta]$.

Proof. We need to prove that the stratum $[\mathfrak{A}, n, q, \gamma_1]$ is pure. We first note that the stratum $[\mathfrak{B}, n', q', \gamma_1]$ is simple. Indeed the stratum $[\hat{\mathfrak{B}}, n', q', \gamma_1 \otimes 1]$ is simple because $[\hat{\mathfrak{B}}, n', q', \gamma]$ is simple and γ, $\gamma_1 \otimes 1$ have the same minimal polynomial over K. Write $B = \mathrm{End}_{D'} V'$, for some K-division algebra D' and some right D'-module V', and $\mathfrak{B} = \mathrm{End}^0_{\mathfrak{o}_{D'}}(Y_k)$ for some $\mathfrak{o}_{D'}$-lattice chain $(Y_k)_{k\in\mathbb{Z}}$ in V'. Consider the stratum $[\mathfrak{C}, n', q', \gamma_1]$, where $\mathfrak{C} = \mathrm{End}^0_{\mathfrak{o}_K}(Y_k) \subset \mathrm{End}_K V'$ ($\mathfrak{C}$ is attached to $\mathfrak{B}$ in the same way that $\tilde{\mathfrak{A}}$ is attached to $\mathfrak{A}$). We easily check that this stratum is a realization of the simple pair of $[\hat{\mathfrak{B}}, n', q', \gamma]$. Hence $[\mathfrak{B}, n', q', \gamma_1]$ is simple by definition.

Here we need a generalization of a lemma of Zink that we are going to state for the algebra A.

(5.5) Lemma *Let* $[\mathfrak{A}, n, q, \delta]$ *be a simple stratum equivalent to a pure one* $[\mathfrak{A}, n, q, \alpha]$. *Then we have* $F[\delta]^\times(1 + \mathfrak{P}) \subset F[\alpha]^\times(1 + \mathfrak{P})$.

Proof. By [Z2](1.6.1), this lemma is true when A is split. Therefore the equivalence $[\tilde{\mathfrak{A}}, n, q, \delta] \sim [\tilde{\mathfrak{A}}, n, q, \alpha]$ gives $F[\delta]^\times(1 + \tilde{\mathfrak{P}}) \subset F[\alpha]^\times(1 + \tilde{\mathfrak{P}})$. We conclude by taking the intersection with A.

□

Since $F[\gamma_1] \subset K[\gamma_1]$, $F[\gamma_1]$ is necessarily a field. By applying the previous lemma to the equivalence $[\mathfrak{B}, n', q', \gamma_1] \sim [\mathfrak{B}, n', q', \beta]$, we have that $F[\gamma_1]^\times(1+\mathfrak{Q}) \subset K[\gamma_1]^\times(1+\mathfrak{Q}) \subset K[\beta]^\times(1+\mathfrak{Q})$. Moreover $K[\beta]^\times = F[\beta]^\times$ is contained in $\mathcal{K}$, and so is $1 + \mathfrak{Q}$, since it is contained in $\mathfrak{B}^\times$. Therefore $F[\gamma_1]^\times(1 + \mathfrak{Q}) \subset \mathcal{K}$, i.e. $F[\gamma_1]^\times \subset \mathcal{K}$. Since both β and γ_1 have $\mathfrak{B}$-valuation $-n'$, we have $\gamma_1\beta^{-1} \in \mathfrak{B}^\times \subset \mathfrak{A}^\times$. Hence $\nu_{\mathfrak{A}}(\gamma_1) = \nu_{\mathfrak{A}}(\beta) = -n'$. We have proved that $[\mathfrak{A}, n, q, \gamma_1]$ is pure. Since $\mathfrak{Q}^{-q'} \subset \mathfrak{P}^{-q}$, this pure stratum is, moreover, equivalent to $[\mathfrak{A}, n, q, \beta]$.

To prove that $[\mathfrak{A}, n, q, \gamma_1]$ is simple, let us check that $[\tilde{\mathfrak{A}}, n, q, \gamma_1]$ is a realization of the simple pair attached to $[\tilde{\mathfrak{B}}, n', q', \gamma]$. First, the minimal polynomial μ_F of γ over F is uniquely determined by its minimal polynomial μ_K over K. Indeed, we have $\mu_K \mid \mu_F$ and μ_F is the unique irreducible polynomial of $F[X]$ with leading coefficient 1 which is a multiple of μ_K: μ_F is the product of the (distinct) elements of the set $\{\mu_K^\sigma \mid \sigma \in Gal(K/F)\}$. A similar remark for γ_1 proves that γ and γ_1 have the same minimal polynomial over F. It remains to prove that $[q'/e_\gamma(\tilde{\mathfrak{B}})] = [q/e_{\gamma_1}(\tilde{\mathfrak{A}})]$, but this follows from $q' = r'd'/rd$. This completes the proof of lemma (5.4).

□

Finally we must check that γ_1 is $(\mathfrak{A}, \bar{\tau})$-pure. For this it is enough to show that the unramified part $F[\gamma_1]^{unr}$ of $F[\gamma_1]$ is contained in the unramified part K of $F[\beta]$. By construction $K[\gamma_1]/K$ is totally ramified and K is therefore the maximal unramified part of $K[\gamma_1]/F$. Now, since $F \subset F[\gamma_1] \subset K[\gamma_1]$, we must have $F[\gamma_1]^{unr} \subset K$, as required. This complete the proof of (5.1).

□

6 A descent lemma for the intertwining

We are going to prove theorem (4.3.9). Our proof uses classical ideas from Galois cohomology. This type of argument has already been used by Henniart in his thesis.

We give up the notation used to state theorem (4.3.9). In particular, $\mathfrak{A}$ is a given hereditary $\mathfrak{o}_F$-order in our algebra A, $\mathfrak{P}$ denotes its radical. Let L/F be any finite unramified extension and put $\bar{A} = A \otimes_F L$, $\bar{\mathfrak{A}} = \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$. We have $\bar{\mathfrak{P}}^k = \mathfrak{P}^k \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$, $k \in \mathbb{Z}$, where $\bar{\mathfrak{P}} = rad(\bar{\mathfrak{A}})$. As usual, $\mathcal{K}$ denotes the normalizer of $\mathfrak{A}$ in $A^\times$ and $\bar{\mathcal{K}}$ denotes the normalizer of $\bar{\mathfrak{A}}$ in $\bar{A}^\times$.

Let $x, y \in A$, $m \in \mathbb{Z}$ and $k \in \bar{\mathcal{K}}$ be such that $k(x \otimes 1)k^{-1} - (y \otimes 1) \in \bar{\mathfrak{P}}^m$. We have to prove that there exists $l \in \mathcal{K}$ such that $lxl^{-1} - y \in \mathfrak{P}^m$. Moreover we have to check that one can choose l in $\mathfrak{A}^\times$ whenever k is in $\bar{\mathfrak{A}}^\times$.

The powers $\bar{\mathfrak{P}}^k$, $k \in \mathbb{Z}$ are stable under the action of $\Gamma = Gal(L/F)$, so is the group $\bar{\mathcal{K}}$, for if $a \in \bar{A}^\times$ satisfies $a\bar{\mathfrak{A}}a^{-1} = \bar{\mathfrak{A}}$, we have $a^s\bar{\mathfrak{A}}^s a^{-s} = \bar{\mathfrak{A}}^s$, i.e. $a^s\bar{\mathfrak{A}}a^{-s} = \bar{\mathfrak{A}}$. In particular, for all $s \in \Gamma$, $k^s(x \otimes 1)(k^s)^{-1} - (y \otimes 1) \in \bar{\mathfrak{P}}^m$, therefore $k^s(x \otimes 1)(k^s)^{-1} \equiv k(x \otimes 1)k^{-1} \bmod \bar{\mathfrak{P}}^m$ and it follows that $(k^{-1}k^s)(x \otimes 1)(k^{-1}k^s)^{-1} - (x \otimes 1) \in \bar{\mathfrak{P}}^m$, since $\mathcal{K}$ normalizes $\bar{\mathfrak{P}}^m$. For all s, k and k^s have the same $\bar{\mathfrak{A}}$-valuation and $k^{-1}k^s$ is therefore in $\bar{\mathfrak{A}}^\times$.

Put $I = I^0 = \{u \in \bar{\mathfrak{A}}^\times \mid u(x \otimes 1)u^{-1} - (x \otimes 1) \in \bar{\mathfrak{P}}^m\}$. This set is a group: this is the normalizer of the coset $(x \otimes 1) + \bar{\mathfrak{P}}^m$ under the action by conjugation of $\bar{\mathfrak{A}}^\times$ on $\bar{A}/\bar{\mathfrak{P}}^m$. Moreover, we easily check that this is a Γ-set, whose Γ-action is compatible with the group structure. The map $s \mapsto k^{-1}k^s$ is then a 1-cocycle of Γ in I.

(6.1) Lemma *We have $H^1(\Gamma, I) = 1$ (non abelian cohomology).*

Proof. Put $I^n = I \cap (1 + \bar{\mathfrak{P}}^n)$, $n \geq 1$. Then, for all $n \geq 1$, I^n is a Γ-stable normal subgroup of I. By a classical filtration argument, we are reduced to proving that for all $n \geq 0$, $H^1(\Gamma, I^n/I^{n+1}) = 1$. For all $n \geq 0$, put $\bar{\mathcal{J}}^n = \{h \in \bar{\mathfrak{P}}^n \ / \ h(x \otimes 1) - (x \otimes 1)h \in \bar{\mathfrak{P}}^m\}$. The $\bar{\mathcal{J}}^n$ are $\mathfrak{o}_L$-lattices in $\bar{A}$. The identity $(hg)(x \otimes 1) - (x \otimes 1)(hg) = h(g(x \otimes 1) - (x \otimes 1)g) + (h(x \otimes 1) - (x \otimes 1)h)g$ implies that, for $n, n' \geq 0$, we have $\bar{\mathcal{J}}^n\bar{\mathcal{J}}^{n'} \subset \bar{\mathcal{J}}^{n+n'}$. It follows that $\bar{\mathcal{J}}^0$ is a ring for which $\bar{\mathcal{J}}^1$ is a two sided ideal;

moreover I^0 is the multiplicative group $(\bar{\mathcal{J}}^0)^\times$. We have a group homomorphism $I^0 \longrightarrow (\bar{\mathcal{J}}^0/\bar{\mathcal{J}}^1)^\times$, given by $(x \otimes 1) \mapsto (x \otimes 1) + \bar{\mathcal{J}}^1$. It is onto, for an element of $\bar{\mathcal{J}}^0$ is invertible if and only if it is invertible mod $\bar{\mathcal{J}}^1$. Its kernel is I^1, whence we have an isomorphism: $I^0/I^1 \simeq (\bar{\mathcal{J}}^0/\bar{\mathcal{J}}^1)^\times$.

The $\bar{\mathcal{J}}^n$ are stable under Γ and we have $\bar{\mathcal{J}}^n = \mathcal{J}^n \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$, where $\mathcal{J}^n = \bar{\mathcal{J}}^{n\ \Gamma} = \{h \in \mathfrak{P}^n \ / \ hx - xh \in \mathfrak{P}^m\}$. The $\mathcal{J}^n$ enjoy properties similar to those of the $\bar{\mathcal{J}}^n$. We have $\mathfrak{p}_L \bar{\mathcal{J}}^n \subset \bar{\mathcal{J}}^{n+1}$ and the quotients $\bar{\mathcal{J}}^n/\bar{\mathcal{J}}^{n+1}$ are k_L-vector spaces. Similarly the $\mathcal{J}^n/\mathcal{J}^{n+1}$ are k_F-vector spaces. Since $\mathfrak{o}_L$ is a flat $\mathfrak{o}_F$-module, we have isomorphisms of Γ-modules: $\bar{\mathcal{J}}^n/\bar{\mathcal{J}}^{n+1} \simeq \mathcal{J}^n/\mathcal{J}^{n+1} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L \simeq \mathcal{J}^n/\mathcal{J}^{n+1} \otimes_{k_F} k_L$.

When $n = 0$, we get $(\bar{\mathcal{J}}^0/\bar{\mathcal{J}}^1)^\times \simeq [(\mathcal{J}^0/\mathcal{J}^1) \otimes_{k_F} k_L]^\times$, where $\mathcal{J}^0/\mathcal{J}^1$ is a finitely generated k_F-algebra. The triviality of $H^1(\Gamma, I/I^1)$ follows then from [S] chap. X, §1, ex. 2: if E/K is a finite Galois extension and H a finitely generated K-algebra, then $H^1(Gal(E/K), [H \otimes E]^\times) = 1$.

If $n > 1$, the map $I^n/I^{n+1} \longrightarrow \bar{\mathcal{J}}^n/\bar{\mathcal{J}}^{n+1}$, induced by $1+u \mapsto u$, $u \in \mathcal{J}^n$, is a group homomorphism since $\bar{\mathcal{J}}^n\bar{\mathcal{J}}^n \subset \bar{\mathcal{J}}^{n+1}$. We easily check that this is an isomorphism. Moreover, we have $I^n/I^{n+1} \simeq (\mathcal{J}^n/\mathcal{J}^{n+1}) \otimes_{k_F} k_L$ and the triviality of $H^1(\Gamma, I^n/I^{n+1})$ follows from the triviality of $H^1(\Gamma, k_L)$.

Let us complete the proof of the theorem. By lemma (6.1), the cocycle $s \mapsto k^{-1}k^s$ is a coboundary: there exists $k_0 \in I$ such that $k^{-1}k^s = k_0^{-1}k_0^s$ for all $s \in \Gamma$. This means that $kk_0^{-1} \in \bar{\mathcal{K}}^\Gamma = \mathcal{K}$. Consequently we have $(kk_0^{-1})(x \otimes 1)(kk_0^{-1})^{-1} = k(k_0^{-1}(x \otimes 1)k_0)k^{-1} \equiv k(x \otimes 1)k^{-1}$ $(mod\ \bar{\mathfrak{P}}^m)$, since $k_0 \in I$. Therefore $(kk_0^{-1})(x \otimes 1)(kk_0^{-1})^{-1} \equiv y \otimes 1$ $(mod\ \bar{\mathfrak{P}}^m)$. By putting $l \otimes 1 = kk_0^{-1}$, we have that $lxl^{-1} - y \in \bar{\mathfrak{P}}^m \cap A = \mathfrak{P}^m$, as required. Of course, if $k \in \bar{\mathfrak{A}}^\times$ then $l \in \mathfrak{A}^\times$, since k_0 lies in $I \subset \bar{\mathfrak{A}}^\times$.

□

7 Comparison of pure elements for different algebras

Let $\mathfrak{S}$ be a central F-division algebra of dimension $(Nd)^2$, where N is the reduced degree and d the index of our central simple F-algebra A. Each element $x \in \mathfrak{S}^\times$ generates a field extension and is pure for the unique hereditary $\mathfrak{o}_F$-order $\mathfrak{o}_\mathfrak{S}$ in $\mathfrak{S}$, which has normalizer $\mathfrak{S}^\times$. Moreover in terms of (3.1) there is only one type of embedding and all non-zero elements are pure for this type. We have a canonical bijection

(7.1) $Ad_{\mathfrak{S}^\times} \backslash \mathfrak{S} \leftrightarrow F^{irr}_{Nd}[T]$,

where $F^{irr}_{Nd}[T]$ is the set of all irreducible polynomials of degree dividing Nd and the map is given by associating to each (pure) element its minimal polynomial over F. On the other hand, for a given hereditary order $\mathfrak{A}$ with period r in A and an associated type of embedding $\bar{\tau}$, (3.15) leads to an injection

(7.2) $Ad_{\mathcal{K}_\mathfrak{A}} \backslash A(\mathfrak{A}, \bar{\tau}) \hookrightarrow F^{irr}_{Nd}[T]$.

Combining (7.1) and (7.2), this leads to an injection

(7.3) $Ad_{\mathcal{K}_{\mathfrak{A}}}\backslash A(\mathfrak{A},\bar{\tau}) \hookrightarrow Ad_{\mathfrak{S}^\times}\backslash\mathfrak{S}$.

(7.4) Definition *(i) Let $\omega_{\mathfrak{A}} : A \to \mathbb{Z}\cup\{\infty\}$ be the pseudo valuation (cf. [Be]) associated to the hereditary order $\mathfrak{A}$; hence, for $x \in A\backslash\{0\}$, we put $\omega_{\mathfrak{A}}(x) := max\{j \in \mathbb{Z} \mid x \in \mathfrak{P}^{tj}\}$, where t is the smallest positive power of $\mathfrak{P}$ which is a principal ideal of $\mathfrak{A}$, and $\omega_{\mathfrak{A}}(0) = \infty$.*

(ii) Let $\underline{x}, \underline{y} \in Ad_{\mathcal{K}_{\mathfrak{A}}}\backslash A(\mathfrak{A},\bar{\tau})$, then write

$$\nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x},\underline{y}) := \tfrac{t}{rd} max\{\omega_{\mathfrak{A}}(x-y) \mid x \in \underline{x}, y \in \underline{y}\}.$$

(iii) Let $\underline{\beta}, \underline{\gamma} \in Ad_{\mathfrak{S}^\times}\backslash\mathfrak{S}$, then write

$$\nu_F^{\mathfrak{S}}(\underline{\beta},\underline{\gamma}) := \tfrac{1}{Nd} max\{\nu_{\mathfrak{S}}(\beta-\gamma) \mid \beta \in \underline{\beta}, \gamma \in \underline{\gamma}\}.$$

(7.5) Proposition *For all $\underline{x}, \underline{y} \in Ad_{\mathcal{K}_{\mathfrak{A}}}\backslash A(\mathfrak{A},\bar{\tau})$ we have $\nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x},\underline{y}) = \nu_F^{\mathfrak{S}}(\bar{x},\bar{y})$, where $\bar{x}$, $\bar{y}$ are the images of $\underline{x}$, $\underline{y}$ under the injection (7.3).*

Proof.

Let $\tilde{\mathfrak{A}}$ be the hereditary order in $\tilde{A}$, as in the beginning of (4.1). Fix a simple A-module V and an $\mathfrak{o}_D$-lattice chain $\{X_j\}_{j\in\mathbb{Z}}$ in V, such that $\mathfrak{A} = End^0_{\mathfrak{o}_D}\{X_j\}$ and $\tilde{\mathfrak{A}} = End^0_{\mathfrak{o}_D}\{X_j\}$. Let t be as in (7.4). Put $\tilde{\mathfrak{A}}_1 := \mathrm{End}^0_{\mathfrak{o}_F}\{X_{tj}\}_{j\in\mathbb{Z}}$; then $\tilde{\mathfrak{A}}_1$ is a canonical principal order of $\tilde{\mathfrak{A}}$ as in [G](1.2) with period $\frac{rd}{t}$.

We also have a natural injection

(7.6) $Ad_{\mathcal{K}_{\tilde{\mathfrak{A}}_1}}\backslash\tilde{A}(\tilde{\mathfrak{A}}_1) \hookrightarrow F^{irr}_{Nd}[T]$;

hence, using (7.1) an injection

(7.7) $Ad_{\mathcal{K}_{\tilde{\mathfrak{A}}_1}}\backslash A(\tilde{\mathfrak{A}}_1) \hookrightarrow Ad_{\mathfrak{S}^\times}\backslash\mathfrak{S}$,

where $Ad_{\mathcal{K}_{\tilde{\mathfrak{A}}_1}}\backslash A(\tilde{\mathfrak{A}}_1)$ denotes the set of conjugacy classes of $\tilde{\mathfrak{A}}_1$-pure elements. We note that we only have one type of embedding with respect to $\tilde{\mathfrak{A}}_1$, because $\tilde{A}$ is a split algebra. For $\underline{\mu}, \underline{\eta} \in Ad_{\mathcal{K}_{\tilde{A}_1}}\backslash\tilde{A}(\tilde{\mathfrak{A}}_1)$, we put $\nu_F^{\tilde{A}(\tilde{\mathfrak{A}}_1)}(\underline{\mu},\underline{\eta}) := \frac{t}{rd} max\{\nu_{\tilde{\mathfrak{A}}_1}(\mu-\eta) \mid \mu \in \underline{\mu}, \eta \in \underline{\eta}\}$.

To describe the image of (7.2), let λ vary over all embedding indices such that (λ) is of type $\bar{\tau}$. Then an irreducible polynomial p over F belongs to the image, if and only if for $E := F[T]/(p(T))$, there is an embedding $i : E \hookrightarrow A$, with $i(E)^\times \subset \mathcal{K}_{\mathfrak{A}}$, such that $(i(E),\mathfrak{A})$ is an embedding which belongs to the equivalence class of (λ), whence the extension E/F have to fulfill the relations of (2.4.4)(a)-(c). Clearly, if f is the degree of λ, we have $\langle\lambda_1^1,\dots,\lambda_1^r,\dots,\lambda_f^1,\dots,\lambda_f^r\rangle f \mid \frac{Nt}{r}$. Moreover, if $C := C_A(E_d)$, $\mathfrak{C} := \mathfrak{A}\cap C$ and $\mathcal{K}_{\mathfrak{C}} = \langle z\rangle\mathfrak{C}^\times$, then in the language of (2.4.4), we have $\frac{r'd'}{t(\lambda)v(\lambda)} = \frac{\nu_{\mathfrak{C}}(\pi_F)}{\nu_{\mathfrak{C}}(z^{v(\lambda)})} = \frac{\nu_{\mathfrak{A}}(\pi_F)}{\nu_{\mathfrak{A}}(z^{v(\lambda)})} = \frac{rd}{\nu_{\mathfrak{A}}(z^{v(\lambda)})}$ divides $\frac{rd}{t}$.

But the image of (7.6) consists of all irreducible polynomials p over F such that, for $K := F[T]/(p(T))$, we have $f(K/F) \mid \frac{Nt}{r}$ and $e(K/F) \mid \frac{rd}{t}$, whence it contains the image of (7.2). Therefore, the image of (7.3) is contained in the image of (7.7) and [Z2](theorem 1.8(i)(ii)) applies. We deduce that $\nu_F^{\mathfrak{S}}(\bar{x}, \bar{y}) = \nu_F^{\tilde{A}(\tilde{\mathfrak{A}}_1)}(\tilde{x}, \tilde{y})$, where $\tilde{x}, \tilde{y}$ are the preimages of $\bar{x}, \bar{y}$ under the map (7.7).

(7.8) *Therefore, it is enough to show that* $\nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x}, \underline{y}) = \nu_F^{\tilde{A}(\tilde{\mathfrak{A}}_1)}(\tilde{x}, \tilde{y})$.

We can view $A \simeq \mathrm{End}_{\mathrm{D}}(V)$ as being canonically embedded in $\tilde{A} \simeq \mathrm{End}_F(V)$. With this convention, we have

(7.9) $\mathcal{K}_{\mathfrak{A}} \subset \mathcal{K}_{\tilde{\mathfrak{A}}} \subset \mathcal{K}_{\tilde{\mathfrak{A}}_1}$.

Let us choose $x \in \underline{x}$, $y \in \underline{y}$ and assume first that $\nu_{\mathfrak{A}}(x) \neq \nu_{\mathfrak{A}}(y)$, then $\nu_{\tilde{\mathfrak{A}}_1}(x) \neq \nu_{\tilde{\mathfrak{A}}_1}(y)$ and $\nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x}, \underline{y}) = \frac{t}{rd} min\{\omega_{\mathfrak{A}}(x), \omega_{\mathfrak{A}}(y)\} = \frac{t}{rd} min\{\nu_{\tilde{\mathfrak{A}}_1}(x), \nu_{\tilde{\mathfrak{A}}_1}(y)\} = \nu_F^{\tilde{A}(\tilde{\mathfrak{A}}_1)}(\tilde{x}, \tilde{y})$. Hence, our result follows in this case.

So let us assume that $\omega_{\mathfrak{A}}(x) = \omega_{\mathfrak{A}}(y) =: -n$; hence $\nu_{\tilde{\mathfrak{A}}_1}(x) = \nu_{\tilde{\mathfrak{A}}_1}(y) = -n$. Then $\nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x}, \underline{y})$, $\nu_F^{\mathfrak{A}(\tilde{\mathfrak{A}}_1)}(\tilde{x}, \tilde{y}) \geq \frac{-nt}{rd}$.

Clearly we are done if $\nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x}, \underline{y}) = \frac{-nt}{rd} = \nu_F^{\tilde{\mathfrak{A}}_1}(\tilde{x}, \tilde{y})$.

Assume first that $\nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x}, \underline{y}) = \frac{-qt}{rd} > \frac{-nt}{rd}$, then the strata $[\mathfrak{A}, nt, qt, x], [\mathfrak{A}, nt, qt, y]$ are pure and intertwine in $\mathcal{K}_{\mathfrak{A}}$; hence by (7.9) and

(7.10) $\mathfrak{P}^{-qt} \subset \tilde{\mathfrak{P}}^{-qt} \subset \tilde{\mathfrak{P}}_1^{-q}$

(where $\tilde{\mathfrak{P}}_1$ is the Jacobson radical of $\tilde{\mathfrak{A}}_1$), the strata $[\tilde{\mathfrak{A}}_1, n, q, x], [\tilde{\mathfrak{A}}_1, n, q, y]$ intertwine in $\mathcal{K}_{\tilde{\mathfrak{A}}_1}$ as well. Therefore $\nu_F^{\mathfrak{A}(\tilde{\mathfrak{A}}_1)}(\tilde{x}, \tilde{y}) \geq \nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x}, \underline{y})$ in this case.

On the other hand, if $\nu_F^{\tilde{A}(\tilde{\mathfrak{A}}_1)}(\tilde{x}, \tilde{y}) = \frac{-qt}{rd} > \frac{-nt}{rd}$, then the strata $[\tilde{\mathfrak{A}}_1, n, q, x], [\tilde{\mathfrak{A}}_1, n, q, y]$ are pure and intertwine in $\mathcal{K}_{\tilde{\mathfrak{A}}_1}$. The strata $[\mathfrak{A}, nt, qt, x], [\mathfrak{A}, nt, qt, y]$ are pure as well and, by (5.1), are equivalent to simple strata $[\mathfrak{A}, nt, qt, x'], [\mathfrak{A}, nt, qt, y']$ respectively; hence, because of (7.10), the strata $[\tilde{\mathfrak{A}}_1, n, q, x], [\tilde{\mathfrak{A}}_1, n, q, y]$ are equivalent to $[\tilde{\mathfrak{A}}_1, n, q, x'], [\tilde{\mathfrak{A}}_1, n, q, y']$ respectively. By definition, the strata $[\tilde{\mathfrak{A}}, nt, qt, x'], [\tilde{\mathfrak{A}}, nt, qt, y']$ are simple. Hence the strata $[\tilde{\mathfrak{A}}_1, n, q, x'], [\tilde{\mathfrak{A}}_1, n, q, y']$ are also simple, because it is easily seen that they are realizations of the same simple pair. Clearly the strata $[\tilde{\mathfrak{A}}_1, n, q, x'], [\tilde{\mathfrak{A}}_1, n, q, y']$ intertwine in $\mathcal{K}_{\tilde{\mathfrak{A}}_1}$ and using the same argument again, by (4.2.1), the strata $[\tilde{\mathfrak{A}}, nt, qt, x'], [\tilde{\mathfrak{A}}, nt, qt, y']$ intertwine; hence, by (4.1.3), the strata $[\mathfrak{A}, nt, qt, x'], [\mathfrak{A}, nt, qt, y']$ intertwine in $\mathcal{K}_{\mathfrak{A}}$. We deduce that the strata $[\mathfrak{A}, nt, qt, x], [\mathfrak{A}, nt, qt, y]$ intertwine in $\mathcal{K}_{\mathfrak{A}}$ as well and therefore $\nu_F^{A(\mathfrak{A},\bar{\tau})}(\underline{x}, \underline{y}) \geq \nu_F^{\mathfrak{A}(\tilde{\mathfrak{A}}_1)}(\tilde{x}, \tilde{y})$ in this case.

Altogether we obtain the equality (7.8), as required.

□

Concluding remark: With the last proposition (7.5) and [Z2](theorem 1.8) we can also compare our metric on pure elements of (7.4)(i) and the polynomial metric of [Z2](definition 1.8). Moreover, using corollary 3.16 and imitating the procedure of [Z2](1.11), we can construct a fundamental domain for $A(\mathfrak{A}, \bar{\tau})$ as in the remark following [Z2](definition/theorem 1.8).

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Université de Poitiers
Mathématiques, SP2MI
Boulevard 3 - Téléport 2 - BP 179
86960 Futuroscope Cedex
broussou@wallis.univ-poitiers.fr

Humboldt-Universität
Reine Mathematik
Unter den Linden 6
10099
Germany
grabitz@mathematik.hu-berlin.de