LINEAR DIFFERENTIAL EQUATIONS
MODELED AFTER HYPERQUADRICS

Takeshi SASAKI and Masaaki YOSHIDA

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3
Federal Republic of Germany

Permanent addresses:

(T.S.) Kumamoto University
Department of Mathematics
Kumamoto, 860
Japan

(M.Y.) Kyushu University 33
Department of Mathematics
Fukuoka, 812
Japan

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§ 0 Introduction

§§ 0.0 General introduction

In this paper, we study systems of linear partial differential equations in $n\ (\geq 3)$ variables of rank (= the dimention of the solution space) $n+2$. The case $n = 2$ is treated in [SY1] and [SY2].

Here we would like to mention our motivation. Let D be the symmetric domain of type IV of dimension $n\ (\geq 3)$, Γ be a transformation group acting properly discontinuously on D, X be a quotient variety of D under Γ naturally equipped with the structure of orbifold, π be the projection of D onto X and finally let $\varphi = \pi^{-1}: X \to M$ be the developing map of the orbifold X. We think there should be a system of linear differential equations (E) defined on X such that the solution of the system gives rise to the inverse map φ. It is called the uniformizing differential equation of the orbifold X. Since D can be thought of a part of a non-degenerate quadratic hypersurface Q in $\mathbb{C}P^{n+1}$ and since we have the following inclusion relations

$$\mathrm{Aut}(D) \subset \mathrm{Aut}(Q) \subset \mathrm{Aut}(\mathbb{C}P^{n+1}) \subset PGL(n+2)$$

of the groups of complex analytic automorphisms, the system (E) must be of rank $n+2$ and the mapping defined on X by the ratio of $n+2$ linearly independent solutions of (E) has its image in the hyperquadric Q. In this way we encounter equations in n variables of rank $n+2$.

Making a linear change of independent variables $x = (x^1,\dots,x^n)$ if necessary, we may assume that any system in n-variables of rank $n+2$ with the unknown w has the form

$$\text{(EQ)}\qquad \frac{\partial^2 w}{\partial x^i \partial x^j} = g_{ij}\frac{\partial^2 w}{\partial x^1 \partial x^n} + \Sigma_{k=1}^{n} A_{ij}^{k}\frac{\partial w}{\partial x^k} + A_{ij}^{0}\, w \qquad (\ 1 \leq i,j \leq n)$$

where

$$g_{ij} = g_{ji},\ A_{ij}^k = A_{ji}^k,\ A_{ij}^0 = A_{ji}^0,\ g_{1n} = 1,\ A_{1n}^k = A_{1n}^0 = 0\ .$$

This equation is the key to connect the theory of conformal connections, the projective theory of hypersurfaces and the theory of uniformizing differential equations of orbifolds uniformized by symmetic domains of type IV.

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We return to the uniqueness assertion. It suffices to show that the volumes V enclosed by the surfaces indicated increase strictly with r. We divide the volume into two parts, V^+ and V^- as indicated in Figure 7. The volumes V^+ are exactly those that arise from sessile drops on a plane with (nondecreasing) data $\psi(r)$; by the lemma, $\frac{dV^+}{dr} > 0$. As to V^-, we have

$$V^- = \pi r^2 z(r) - 2\pi \int^r \rho z(\rho)\,d\rho$$

and thus

$$\frac{dV^-}{dr} = \pi r^2 z'(r) \geq 0 ,$$

and the uniqueness proof is complete.

We remark finally that if $\mathfrak{C}$ extends to infinity, the existence of symmetric surfaces with prescribed V and γ is easy to prove. For each r we need only seek that surface in the one parameter family determined by v_0 that yields the requisite $\psi(r)$, and it is clear that $V(r)$ varies from 0 to ∞. If $\mathfrak{C}$ is defined only in a bounded set, then by the above monotonicity result for V, there exists a solution for any V not exceeding the value $\hat{V}$ at the "top" of $\mathfrak{C}$.

I wish to thank M. Grüter and R. Gulliver for a number of helpful discussions.

We study geometric meanigs of the equations (EQ) in §2 as explained in §§0.1 and 0.2, and construct an example in §3 as explained in §§0.3. Projective differential geometrical tools which plays an essential role in §2 is briefly reviewed in §1.

§§ 0.1 Introduction for differntial geometers

Let a hypersurface M in P^{n+1} be the image of the map φ: $x \to w(x) = (w^1(x), \dots, w^{n+2}(x))$ where w^j are linearly independent solutions of (EQ). We study in §1, as a preparation, the induced conformal metric II on M and the cubic invariant form III of the embedding $M \subset P^{n+1}$;and fomulate the fundamental theorem of projective hypersurfaces (Theorem 1.3). We show that the coefficients g_{ij} represent the induced conformal metric II and that the coefficients A_{ij}^k and A_{ij}^0 are expressed in terms of II and the cubic invariant III (Theorem 2.1). When M is a quadratic hypersurface, we show that the coefficients A_{ij}^k and A_{ij}^0 are expressed in terms of g_{ij} (Theorem 2.3). Conversely, for a given conformally flat quadratic form g_{ij}, we can associate a differential equation of the form (EQ) with the principal part g_{ij} such that the map φ has its image in a hyperquadric in P^{n+1} (Theorem 2.4).

§§ 0.2 Introduction for topologists

Let X be an n-dimensional orbifold (or simply a manifold) which has a conformally flat structure. As Kuiper ([Kui]) pointed out, there is a conformal map, called the developing map, from the universal cover of X into the model space, hyperquadric in P^{n+1}. We ask

"How can we get the developing map ?"

In this paper, we give an answer. Let $g_{Uij}dx_U^i dx_U^j$ be the conformal structure for coordinate neighborhoods (U, x_U). Consider the system of linear differential equations of rank n+2:

$$(EQ)_U \qquad \frac{\partial^2 w}{\partial x^i \partial x^j} = g_{Uij} \frac{\partial^2 w}{\partial x^1 \partial x^n} + \Sigma_{k=1}^n A_{Uij}^k \frac{\partial w}{\partial x^k} + A_{Uij}^0 w$$

on U. The coefficients A_{Uij}^k and A_{Uij}^0 are determined so that the map φ_U, using n+2 linearly independent solutions w_U^j of $(EQ)_U$:

$$\varphi_U\colon U \ni x \to (w_U^1(x), \dots, w_U^{n+2}(x)) \in P^{n+1}$$

has its image a part of a nondegenerate quadratic hypersurface in P^{n+1}. If V is another chart such that $V \cap U \neq \Phi$ then φ_U and φ_V are projectively related. The developing map of X is given by $\{\varphi_U\}_U$.

§§ 0.3 Introduction for algebraic geometers

Let $M = H_2$ be the Siegel upper half space of genus 2 and $\Gamma(2)$ be the Siegel modular group of level 2. The regular orbit of H_2 modulo $\Gamma(2)$ is known to be the space

$$\Lambda = \{(\lambda^1, \lambda^2, \lambda^3) \in \mathbb{C}^3 \mid \lambda^i \neq 0,\ 1,\ \lambda^j\ (i \neq j)\}$$

Let $\pi : H_2 \to \Lambda$ be the natural projection.

The space Λ can be thought of the parameter space of a family of curves of genus 2:

$$C(\lambda):\ w^3 v^2 = u(u-w)(u-\lambda^1 w)(u-\lambda^2 w)(u-\lambda^3 w)$$

in the projective plane. The periods of $C(\lambda)$ gives a (multi-valued) inverse of π and they satisfy a system of linear differential equations which is sometimes called the Gauss-Manin connection of the fiber space $\cup_\lambda C(\lambda) \to \Lambda$. In this paper, we explicitly write the system of differential equations. In order to do so, we develop a general theory of Gauss-Manin connections related to the n-dimensional symmetric domain of type IV. Notice that the 3-dimensional symmetric domain of type IV is the Siegel space H_2.

§ 1 Review of the projective theory of hypersurfaces

§§ 1.0 Summary

In this section we recall the fundamental formulation of the intrinsic conformal geometry and the projective theory of hypersurfaces, which are necessary in the discussion of systems of linear differential equations in the following sections. Although the fact stated in this section is already known by [Sas], our present version is made in order to clarify and to show up the story of the theory, which may not be easy to grasp in reading [Sas].

To have a better understanding of the theory, we recall first the story of the intrinsic Riemannian geometry, that of hypersurfaces in the euclidean spaces and the fundamental theory connecting them.

— Intrinsic Riemannian geometry — Let M be an n-dimensional manifold equipped with a Riemannian metric. Then there is a unique affine connection compatible with the metric (Levi-Civita connection). The Riemannian curvature tensor is defined by the Levi-Civita connection.

— Hypersurfaces — Let $\iota : M \subset \mathbb{R}^{n+1}$ be an embedding of a manifold M. The induced metric and the second fundamental form are defined on M. The Levi-Civita connection and the Riemannian curvature tensor of the induced metric are defined as above. They are related as follows:

<u>Gauss equation</u>: The Riemannian curvature tensor is expressed in terms of the second fundamental form.

<u>Codazzi-Minardi equation</u>: The covariant derivatives of the second fundamental form and the induced metric are related.

— Fundamental theorem — Let M be a manifold equipped with a Riemannian metric and a quadratic form. They are the induced metric and the second fundamental form defined by some embedding $\iota : M \subset \mathbb{R}^{n+1}$ if they satisfy the Gauss equation and the Codazzi-Minardi equation. The embedding ι is unique up to rigid motions of $\mathbb{R}^{n+1}$.

Now we summarize the story of the intrinsic conformal geometry, that of hypersurfaces in the projective space and the fundamental theorem connecting them.

— Intrinsic conformal geometry — Let M be a manifold equipped with a conformal metric h. Then there is a unique conformal connection π compatible with the conformal metric (the normal conformal connection). The conformal curvature tensor C is defined by the normal conformal connection.

— Hypersurfaces — Let $\iota : M \subset P^{n+1}$ be an embedding of an n-dimensional manifold M. The induced conformal metric h and the form τ (called the invariant of ι) are defined. The normal conformal connection π and the conformal curvature tensor C of the induced metric are defined intrinsically as above. They are related as follows.

<u>Gauss equation</u>: The conformal curvature tensor is expressed in terms of the invariant τ.

<u>Codazzi-Minardi equation</u>: Covariant derivatives of τ and the induced metric h are related.

— Fundamental theorem — Let M be a manifold equipped with a conformal metric h and a form τ. They are the induced conformal metric and the invariant defined by some embedding $\iota : M \subset P^{n+1}$ if they satisfy the Gauss equation and the Codazzi-Minardi equation. The embedding ι is unique up to projective transformations of P^{n+1}.

§§ 1.1 Intrinsic conformal geometry

We recall some facts on the conformal connection. A precise and detailed description can be referred in the book [Kob].

Let M be an n-dimensional complex manifold and $h = (h_{ij})$ be a non-singular symmetric matrix. Define

$$CO(h) = \{ \lambda a \mid a \in GL(n,\mathbb{C}),\ ah^t a = h,\ \lambda \in \mathbb{C}^* \}.$$

Let $L(M)$ be the bundle of complex linear frames on M. A holomorphic principal subbundle P of $L(M)$ with structure group $CO(h)$ is called a <u>holomorphic</u> $CO(h)$-<u>structure</u>. Such subbundles on M are in a natural one-to-one correspondence with the sections $M \to L(M)/CO(h)$. In other words, for such a structure, we associate a conformal covariant tensor field $g = (g_{ij})$ called a <u>conformal metric</u> that is locally written as

$$g_{ij}(x)dx^i dx^j, \qquad \det g_{ij} \neq 0,$$

with respect to a local coordinate system (x^i).(Throughout this paper, if an index occurs twice in a term, once as a superscript and once as a subscript, summation over that index is indicated.)

We consider a non-singular hyperquadric Q^n in P^{n+1} defined in terms of the homogeneous coordinate system $(z^0,\dots,z^{n+1})$ by the equation

$$-2z^0z^{n+1} + h_{ij}z^iz^j = 0.$$

Let Q be the symmetric matrix of degree $n+2$ corresponding to this quadratic form:

$$Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & h & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The group

$$O(Q) = \{g \in GL(n+2) \mid gQ{}^tg = Q\}$$

acts transitively on the hyperquadric. Let H be the isotropy subgroup at ${}^t(0,\dots,0,1)$. It consists of matrices of the form

$$(1.1) \qquad \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix} \qquad \begin{aligned} &\lambda\nu = 1, \quad ah{}^ta = h, \\ &b = \lambda ah{}^tc, \quad \nu = \lambda ch{}^tc/2. \end{aligned}$$

We have a principal bundle $O(Q)$ over $Q^n = O(Q)/H$ with structure group H. The linear isotropy representation of the group H at ${}^t(0,\dots,0,1)$ has a non-trivial kernel consisting of matrices of the form

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ b & \pm I_n & 0 \\ \mu & c & \pm 1 \end{pmatrix}.$$

Denote this kernel by N. Then H/N is isomorphic to $CO(h)$. Thus we have a principal bundle $O(Q)/N$ over the hyperquadric $Q^n = O(Q)/H$ with structure group $CO(h)$.

$$\begin{array}{ccc} & O(Q) & O(Q)/N \\ & \downarrow H & \downarrow H/N \cong CO(h) \\ Q^n = & O(Q)/H & = O(Q)/H \end{array}$$

This is called <u>the canonical conformal structure of the quadric</u>. The associated conformal metric is given as follows. Let $\varphi = -2dz^0dz^{n+1} + h_{ij}dz^idz^j$ be the tensor field on $\mathbb{C}^{n+2} - \{0\}$. Let s be a local section of the bundle $\mathbb{C}^{n+2} - \{0\}$ over P^{n+1}. Although the pull-back $s^*\varphi$ depends on the section s, its restriction to Q^n is defined independently of s up to a multiplicative factor of non-vanishing holomorphic functions. Thus the conformal metric of $s^{*}\varphi|Q^n$ is uniquely defined.

Consider again a $CO(h)$-structure P on a manifold M. Let $P^2(M)$ be the bundle of 2-frames over M with structure group, elements of which are holomorphic 2-frames of $\mathbb{C}^n$ at the origin ([Kob, chapter 4, § 5]). The first prolongation of P, which is a principal subbundle of $P^2(M)$ with structure group H, is denoted by $P^{(1)}$. The correspondence between P and $P^{(1)}$ is known to be bijective ([Kob, chapter 4, §6]). In fact we can recover P from $P^{(1)}$ by putting $P = P^{(1)}/N$. For the hyperquadric Q^n, this bundle $P^{(1)}$ is nothing but the bundle $O(Q) \to O(Q)/H$. The bundle $P^{(1)}$ has Cartan connections ([Kob, chapter 4]). Let $o(Q)$ be the Lie algebra of $O(Q)$. Then a Cartan connection in question is a $o(Q)$-valued 1-form π on $P^{(1)}$ considered as a set of 1-forms (π_i, π_i^j, π^j) by the identification

$$\pi = \begin{pmatrix} \pi^0 & \pi^j & 0 \\ \pi_i & \pi_i^j + \delta_i^j \pi^0 & h_{ik}\pi^k \\ 0 & h^{kj}\pi_k & -\pi^0 \end{pmatrix} \in o(Q)$$

where $\pi^0 = -\frac{1}{n}\Sigma\,\pi_k^k$. The forms π_i^j and π^j are the restriction to $P^{(1)}$ of the components of the canonical form of $P^{(1)}$. They have the property $d\pi^j = \Sigma\,\pi^k \quad \pi_k^j$. The curvature form Π of π is defined by $\Pi = d\pi - \pi \quad \pi$ which is written as

$$\Pi = \begin{pmatrix} \Pi^0 & 0 & 0 \\ \Pi_i^0 & \Pi_i^j & 0 \\ 0 & h^{jk}\Pi_k^0 & -\Pi^0 \end{pmatrix}.$$

There exists a unique Cartan connection, called the <u>normal conformal connection</u>, satisfying the (normalization) condition

$$C^j{}_{ij\ell} = 0$$

where

$$\Pi_i^j - \delta_i^j \Pi^0 = d\pi_i^j - \pi_i^k \wedge \pi_k^j - \pi_i \wedge \pi^j - h_{ik}h^{j\ell}\pi^k \wedge \pi_\ell - \delta_i^j \pi_k \wedge \pi^k$$

$$=: -\frac{1}{2} C^j{}_{ik\ell}\, \pi^k \wedge \pi^\ell .$$

In fact this condition determines the forms π_i uniquely ([Kob, Chapter 4, Theorem 4.2]).

Definition: A CO(h)-structure P (or a conformal metric g) is said to be conformally flat if the normal conformal connection π is integrable, i.e. $\Pi = d\pi - \pi \wedge \pi = 0$.

§§ 1.2 Projective theory of hypersurfaces

Suppose we are given a portion of n-dimensional hypersurface M in the projective space P^{n+1}. Let $i: M \to P^{n+1}$ be the embedding. We assume the map i has a lift, denoted by e_0, to $\mathbb{C}^{n+2}-\{0\}$, the natural covering of P^{n+1}. Let $e_1,\dots,e_n$ be a set of independent tangent vector fields to M along e_0 and choose another vector field e_{n+1} so that $\det(e_0,e_1,\dots,e_n,e_{n+1}) = 1$ with respect to a fixed frame of $\mathbb{C}^{n+2}$. Then the hypersurface M is described by the motion of the vectors e_α $(0 \le \alpha \le n+1)$ which we call a projective moving frame field along M. We introduce the associate Maurer-Cartan form ω by

$$de = \omega e.$$

Here we use abbreviations $e = (e_0,e_1,\dots,e_{n+1})$ and $\omega = (\omega_\alpha^\beta)$. The indices $\alpha,\beta,\dots$ range from 0 to n+1. When we use the indices $i,j,\dots$, these are understood to range from 1 to n. The 1-form ω has values in $s\ell(n+2,\mathbb{C})$. It satisfies the Maurer-Cartan equation:

$$(1.2) \qquad d\omega = \omega \wedge \omega \quad \text{i.e.} \quad d\omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta .$$

First notice that the above choice of a frame implies $\omega_0^{n+1} = 0$, and $\{\omega_0^j \mid 1 \le j \le n\}$ are independent on M. Hereafter we write $\omega^j = \omega_0^j$. Then (1.2) implies

$$0 = d\omega_0^{n+1} = \omega^k \wedge \omega_k^{n+1} ,$$

which allows us to put

$$(1.3) \qquad \omega_i^{n+1} = h_{ik}\omega^k, \qquad\qquad h_{ik} = h_{ki}.$$

We have

$$\omega = (\ \omega_\alpha^\beta\ ) = \begin{pmatrix} \omega_0^0 & \omega^j & 0 \\ \omega_i^0 & \omega_i^j & h_{ik}\omega^k \\ \omega_{n+1}^0 & \omega_{n+1}^j & \omega_{n+1}^{n+1} \end{pmatrix}.$$

Let us define a symmetric quadratic form II on M by

$$(1.4) \qquad II = h_{ij}\omega^i\omega^j.$$

An important property of this form is its invariance in the following sense. Let e' be another projective frame, which, as is easily seen, has a form

$$(1.5) \qquad e' = ge$$
$$g = \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix}$$

where the entries are functions with values in $\lambda, \mu, \nu \in \mathbb{C}$, $a \in GL(n,\mathbb{C})$ and $b, c \in \mathbb{C}^n$. Then the associate Maurer-Cartan form ω' is given by

$$(1.6) \qquad \omega' = (dg + g\omega)g^{-1},$$

and this leads to the identity

$$(1.7) \qquad \lambda\nu h'_{ij} = a_i^k h_{k\ell} a_j^\ell.$$

In particular, the associate quadratic form II' is given by

$$(1.8) \qquad II' = \lambda^2 II.$$

This implies that the conformal class of II is intrinsic on the manifold M. Hence, especially, its rank is independent of the choice of frames. We now assume that the form II is non-degenerate. Notice that the above process defining II shows that it is determined by the second order derivatives of the embedding i. We next derive another invariant which depends on its third derivatives. In order to make the following formulae look simple, we choose a frame so that

(1.9) $\det h_{ij} = 1, \quad \omega_0^0 + \omega_{n+1}^{n+1} = 0.$

This is possible because of the non-degeneracy assumption for II and the transformation rule (1.6). Then the exterior derivation of (1.2) gives

$$(dh_{ij} - h_{ik}\omega_j^k - h_{jk}\omega_i^k) \wedge \omega^j = 0,$$

which enables us to define a symmetric quantity h_{ijk} by

(1.10) $h_{ijk}\omega^k = dh_{ij} - h_{ik}\omega_j^k - h_{jk}\omega_i^k.$

Let us define a symmetric cubic form III on M by

(1.11) $\mathrm{III} = h_{ijk}\omega^i\omega^j\omega^k$

and call this <u>the (Wilczynski-Fubini-Pick) cubic invariant form</u>. Indeed it has the invariance:

(1.12) $\mathrm{III}' = \lambda^2\ \mathrm{III}$

with respect to the frame change (1.5). The role of this form can be seen in

<u>Proposition</u> 1.1. Let M be a connected piece of a hypersurface in P^{n+1}. Assume the quadratic form II is non-degenerate and the cubic invariant form III vanishes everywhere. Then M is contained in a quadratic hypersurface.

The projective description of a hypersurface needs one more invariant. Take a derivation of $\omega_0^0 + \omega_{n+1}^{n+1} = 0$. The we have

$$(h_{ij}\omega_{n+1}^j - \omega^0) \wedge \omega^i = 0,$$

which allows us to define a symmetric quantity L_{ij} by

(1.13) $h_{ij}\omega_{n+1}^j - \omega_i^0 = L_{ij}\omega^j.$

It is possible to show the existence of a projective frame satisfying

(1.14) $\det h_{ij} = 1, \quad \omega_0^0 + \omega_{n+1}^{n+1} = 0 \quad \text{and} \quad \mathrm{trace}_h L\ (= L_{ij}h^{ij}) = 0.$

Now we fix a frame e with this property. Then, at every point p where the frame is defined, the matrix $h = (h_{ij})$ defines a Lie group by (1.1) which we denote by $H(p)$. Analogously the group $O(Q(p))$ and its Lie algebra $o(Q(p))$ are defined. Take another frame e' with the property $h'_{ij} = h_{ij}$ and (1.14). A caluculation shows the frame change g from e to e' belongs to the group $H(p)$ at each p.

We next formulate the fundamental theorem by using the language of conformal geometry. Define a tensorial matrix-valued 1-form τ by

$$(1.15)\quad \tau = \begin{pmatrix} 0 & 0 & 0 \\ (M_{ik} + L_{ik})\omega^k & \frac{1}{2} h_{ik}{}^j\omega^k & 0 \\ -\omega^0_{n+1} & h^{j\ell}M_{\ell k}\omega & 0 \end{pmatrix}$$

where

$$M_{ik} = \frac{-1}{4(n-2)} K_{ik} + \frac{F}{8(n-2)(n-1)} h_{ik} - \frac{1}{2} L_{ik}$$

$$K_{ik} = h_{ipq}h^{pq}{}_k \qquad \text{and} \qquad F = h_{pqr}h^{pqr},$$

and put

$$(1.16)\quad \pi = \omega + \tau.$$

(Here the raising of indices relative to h_{ij} is used. e.g. $h_{ij}{}^k = h_{ijp}h^{pk}$.) Then a computation shows the invariance

$$(1.17)\quad \tau' = g\tau g^{-1}$$

under the frame changes belonging to the group $H(p)$ for each point p; and it is easy to see the form π has its value in the Lie algebra $o(Q(p))$. Let Π be the curvature tensor of π. It has the expression as follows ($\Pi^0 = 0$)

$$\Pi = d\pi - \pi\ \pi = \begin{pmatrix} 0 & 0 & 0 \\ \Pi^0_i & \Pi^j_i & 0 \\ 0 & \Pi^j_{n+1} & 0 \end{pmatrix}$$

where

$$\Pi^0_i = h_{ij}\ \Pi^j_{n+1}.$$

Since Π is a tensorial 2-form, we may put

$$(1.17)\qquad \pi^j_i = -\tfrac{1}{2}\, C^j{}_{ik\ell}\, \omega^k \wedge \omega^\ell \qquad\qquad C^j{}_{ik\ell} + C^j{}_{i\ell k} = 0$$

$$\pi^0_i = -\tfrac{1}{2}\, C_{ik\ell}\, \omega^k \wedge \omega^\ell \qquad\qquad C_{ik\ell} + C_{i\ell k} = 0.$$

The choice of τ has been made by requiring the normalization condition

$$(1.18)\qquad C^j{}_{ij\ell} = 0.$$

With these notations, the following analogue of the Gauss and the Codazzi-Minardi equations holds:

(1.19) (The Gauss equation)

$$C_{ijk\ell} = \tfrac{1}{4}\,(h_{i\ell p} h_{kj}{}^{p} - h_{ikp} h_{j\ell}{}^{p})$$

$$+ \frac{1}{4(n-2)}\,(h_{jk}K_{i\ell} - h_{j\ell}K_{ik} + h_{i\ell}K_{jk} - h_{ik}K_{j\ell})$$

$$+ \frac{1}{4(n-1)(n-2)}\,(h_{ik}h_{j\ell} - h_{i\ell}h_{jk})F$$

$$C_{ik\ell} = f_{ik,\ell} - f_{i\ell,k} + \tfrac{1}{4}\,(h_{i\ell}{}^{j}L_{jk} - h_{ik}{}^{j}h_{j\ell})$$

where $f_{i\ell}$ is the projective analogue of the Schouten tensor defined by

$$f_{i\ell} = -\frac{1}{4(n-2)}\,K_{i\ell} + \frac{F}{8(n-1)(n-2)}\,h_{i\ell}$$

and $f_{i\ell,k}$ is the covariant derivative of $f_{i\ell}$ with respect to π. i.e. $f_{i\ell,k}\,\omega^k = df_{i\ell} - f_{ik}\pi^k_\ell - f_{k\ell}\pi^k_i + 2f_{i\ell}\pi^0_0$.

(1.20)(The Codazzi-Minardi equation)

$$h_{ijk,\ell} - h_{ij\ell,k} = L_{i\ell}h_{jk} - L_{ik}h_{j\ell} + L_{j\ell}h_{ik} - L_{jk}h_{i\ell}$$

$$L_{ij,k} - L_{ik,j} = h_{ij}{}^{\ell} f_{\ell k} - h_{ik}{}^{\ell} f_{\ell j} + 2(h_{ik}\gamma_j - h_{ij}\gamma_k)$$

$$\gamma_{i,j} - \gamma_{j,i} = L_{j\ell}f^{\ell}{}_{i} - L_{i\ell}f^{\ell}{}_{j},$$

where γ_i is defined as $\omega^0_{n+1} = -\gamma_i\omega^i$ and $h_{ijk,\ell}$, $L_{ij,k}$ and $\gamma_{i,j}$ are covariant derivatives of h_{ijk}, L_{ij} and γ_i with respect to π.

Now we choose a frame e so that h be a constant matrix, which we denote by ${}^0h$. This is posible by (1.7). Then the set of projective frames satisfying (1.14) and $h(p) = {}^0h$ becomes a principal bundle denoted by P with the group H, corresponding to ${}^0h$, as a structure group. The 1-forms π and τ corresponding to ge ($g \in H$) can be thought of 1-forms on P in view of (1.6) and (1.17). We denote these forms by $\tilde{\pi}$ and $\tilde{\tau}$. These considerations then show

Proposition 1.2. The pair $(P,\tilde{\pi})$ defines a normal conformal connection defined in §§1.1 on the hypersurface M. The form $\tilde{\tau}$ is the invariant satisfying the relations (1.19) and (1.20).

Conversely we have

Theorem 1.3. Let M be an n (≥ 3) dimensional complex manifold with a normal conformal connection π. Let τ be a tensorial 1-form in the form (1.15). Assume that the covariant derivatives of τ satisfies the relation (1.20) and that the curvature tensor of π is given by (1.19). Then, for a given point p of M, there exists a neighborhood of p which can be embedded as a non-degenerate hypersurface in a projectivee space of dimension $n+1$ so that π and τ become the connection and the invariant induced by this embedding, respectively. This embedding is unique up to projective transformations.

For the proof of this theorem and for the induction of the above formulae, refer [Sas].

For the use in the next section, we review the local expression of π_i^j and π_i^0 in terms of the conformal structure tensor h_{ij}. Let (x^i) be a local coordinate system and choose a frame so that $\omega^i = dx^i$. (This frame is, in general, different from the frames defining the bundle P.) The definition of π in (1.16) is made so that

$$dh_{ij} - h_{ik}\pi_j^k - h_{ik}\pi_i^k = 0.$$

This leads, as usual, to the identity

$$\pi_i^j = \Gamma_{ik}^j\omega^k,$$

where Γ_{ik}^j is the Christoffel symbol of h_{ij}:

$$\Gamma^j_{ik} = \frac{1}{2} h^{j\ell}(h_{i\ell,k} + h_{k\ell,i} - h_{ik,\ell}), \qquad dh_{i\ell} = h_{i\ell,k}\omega^k.$$

Let $R^j{}_{ik\ell}$ be the Riemannian curvature tensor:

$$d\pi^j_i - \pi^k_i \wedge \pi^j_k = -\frac{1}{2} R^j{}_{ik\ell}\omega^k \wedge \omega^\ell.$$

The Ricci and the scalar curvatures are denoted by R_{ij} and by R, respectively:

$$R_{ij} = R^\ell{}_{ij\ell}, \qquad R = h^{ij}R_{ij}.$$

If we put

$$\pi^0_i = S_{ik}\omega^k$$

then the definition (1.17) implies

$$C^j{}_{ik\ell} = R^j{}_{ik\ell} + S_{ik}\delta^j_\ell - S_{i\ell}\delta^j_k + h_{ik}h^{jm}S_{m\ell} - h_{i\ell}h^{jm}S_{mk}.$$

The requirement (1.18) easily shows

$$(1.21)\quad S_{ik} = \frac{1}{n-2}\left(R_{ik} - \frac{R}{2(n-1)} h_{ik} \right)$$

The tensor S_{ij} is called <u>the Schouten tensor</u> relative to the tensor h_{ij}.

§ 2 Local geometric theory of linear differential equations in n-variables of rank n+2

The purpose of this section is to give a geometric interpretation of the system of linear differential equations in n-variables of rank n+2, referring the projective study of hypersurfaces reviewed in §1.

§§ 2.1 Geometry of hypersurfaces defined by linear differential equations

Let us first fix such a differential system. $x = (x^1, \ldots, x^n)$ will denote a coordinate system and subindices attached to functions mean derivatives with respect to these coordinates. e.g. $w_i = \partial w/\partial x^i$, $w_{ij} = \partial^2 w/\partial x^i \partial x^j$. Let us consider n+2 linearly independent functions $w^1, \ldots, w^{n+2}$ in x. They are solutions of linear differential equations

$$\begin{vmatrix} w & w^1 & \cdots & w^{n+2} \\ w_1 & w_1^1 & \cdots & w_1^{n+2} \\ & & & \\ w_n & w_n^1 & \cdots & w_n^{n+2} \\ w_{ij} & w_{ij}^1 & \cdots & w_{ij}^{n+2} \\ w_{k\ell} & w_{k\ell}^1 & \cdots & w_{k\ell}^{n+2} \end{vmatrix} = 0$$

with the unknown w. Since the linear independence of w^i says that

$$\Delta_{ij} = \begin{vmatrix} w^1 & \cdots & w^{n+2} \\ w_1^1 & \cdots & w_1^{n+2} \\ & & \\ w_n^1 & \cdots & w_n^{n+2} \\ w_{ij}^1 & \cdots & w_{ij}^{n+2} \end{vmatrix} \neq 0$$

for some pair (i,j), it loses no generality assuming $\Delta_{1n} \neq 0$. (Change coordinates otherwise.) Then, dividing the equation by Δ_{1n}, we get a system

$$\text{(EQ)} \qquad w_{ij} = g_{ij} w_{1n} + A_{ij}^k w_k + A_{ij}^0 w, \qquad 1 \leq i,j \leq n$$

where

$$(2.1) \qquad A_{ij}^k = A_{ji}^k,\ A_{ij}^0 = A_{ji}^0,\ g_{ij} = g_{ji},\ g_{1n} = 1,\ A_{1n}^k = A_{1n}^0 = 0.$$

The functions w^i satisfy also equations

$$\begin{vmatrix} w & w^1 & \cdots & w^{n+2} \\ w_1 & w_1^1 & \cdots & w_1^{n+2} \\ & & & \\ w_n & w_n^1 & \cdots & w_n^{n+2} \\ w_{1n} & w_{1n}^1 & \cdots & w_{1n}^{n+2} \\ w_{ijk} & w_{ijk}^1 & \cdots & w_{ijk}^{n+2} \end{vmatrix} = 0$$

A part of these equations will be written shortly as

$$(2.2) \qquad w_{1jn} = G_j w_{1n} + B_j^k w_k + B_j^0 w, \qquad 1 \le i,j \le n$$

Notice that these equations are derived from (EQ) by differentiation.

The system (EQ) when $n=2$ was first treated by E.J. Wilczynski in his memoirs [Wil] in the beginning of this century. The reformulation of this case is given by the authors in [SY 1] in view of the moving frame method. In this paper we treat this system when $n \geq 3$, aiming at making the geometric meaning of the coefficients clear.

Let us consider the equation (EQ) with (2.1) of rank $n+2$ which satisfies

$$(2.3) \qquad \Delta = \det g_{ij} \neq 0.$$

We fix a vector $w = (w^1,\ldots,w^{n+2})$ made of linearly independent solutions, which defines a local embedding of the x-space into the projective space of dimension $n+1$. We call this embedding <u>the projective solution</u> of (EQ), which is unique up to $PGL(n+2,\mathbb{C})$. By abuse of language we sometimes consider w as the embedded hypersurface. Put

$$(2.4) \qquad e^\theta = \det({}^t w, {}^t w_1, \ldots, {}^t w_n, {}^t w_{1n}),$$

which we call <u>the normalization factor</u> of the system (EQ). The function θ is independent of the choice of w up to additive constant. Define a set of vectors $e = {}^t(e_0,\ldots,e_{n+1})$ by

$$(2.5) \qquad e_0 = w, \quad e_i = w_i, \quad e_{n+1} = e^{-\theta} w_{1n}.$$

Then this is a projective frame along w in the sense explained in §1. The system (EQ) and the equations (2.2) can be written in a Pfaffian form

$$(2.6)\qquad de = \omega e$$

where

$$\omega = (\omega_\alpha^\beta) = \begin{pmatrix} 0 & dx^j & 0 \\ A_{ik}^0 dx^k & A_{ik}^j dx^k & e^\theta g_{ik} dx^k \\ e^{-\theta} B_k^0 dx^k & e^{-\theta} B_k^j dx^k & (G_k - \theta_k) dx^k \end{pmatrix}$$

is the Maurer-Cartan form of the frame e. The result of §1 says that the tensr $h_{ij} = e^\theta g_{ij}$ defines the induced conformal metric of the hypersurface. Then the process of normalization in §1 can be applied to the above frame e. A suitable choice of a transformation g in the form

$$(2.7)\quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda I_n & 0 \\ \mu & c & \lambda^{-n} \end{pmatrix}$$

suffices for this normalization. Namely, writing

$$\omega' = dg\, g^{-1} + g\omega g^{-1},$$

which is a coframe of the transformed frame $e' = ge$, the element g is determined so that

$$(2.8)\quad \det h'_{ij} = 1, \quad \omega'^{0}_{0} + \omega'^{n+1}_{n+1} = 0 \quad \text{and} \quad \operatorname{trace} L'_{ij} = 0.$$

(see Proposition 1.2). Then by (1.16) ω' is decomposed into the sum of the connection form π associated with h'_{ij} and the tensorial invariant form τ of the embedding w. Reversing this process, we have

$$(2.9)\quad \omega = dh\, h^{-1} + h(\pi - \tau)h^{-1}$$

for $h = g^{-1}$. The point here is that the right hand side is known to have a geometrically invariant meaning. And, consequently, the coefficients of the system (EQ) is written in terms of the invariants of the hypersurface w, which, in the following, we write down explicitly. Since $h'_{ij} = \lambda^{n+2}h_{ij} = \lambda^{n+2}e^{\theta}g_{ij}$, the component λ is determined by

$$(2.10) \qquad \lambda = (e^{n\theta}\Delta)^{-1/n(n+2)}.$$

Other components c and μ may be computed following the normalization process. In the present case, however, they can be determined also by the requirements $A^k_{1n} = A^0_{1n} = 0$. We prove

Theorem 2.1. Let the equation (EQ) of rank n+2 ($n \geq 3$) with (2.1) and (2.3) be given. If the normalization factor satisfies

$$(2.11) \qquad \det\,(e^{\theta}g_{ij}) = 1$$

then the coeefients $A^{\cdot}_{ik}$ are given by

$$A^j_{ik} = (\Gamma^j_{ik} - g_{ik}\Gamma^j_{1n}) - \frac{1}{2}\,(h^j{}_{ik} - g_{ik}h^j{}_{1n})$$

$$A^0_{ik} = (S_{ik} - g_{ik}S_{1n}) - \{M_{ik} + L_{ik} - g_{ik}(M_{1n} + L_{1n})\}.$$

Here Γ^j_{ik} and S_{ik} are the Christoffel symbol and the Schouten tensr of the tensor $e^{\theta}g_{ij}$, defined in §§1.2. The $h^j{}_{ik}$, L_{ik} and M_{ik} are components of the form τ defined in §§1.2 with respect to the normalized frame ge.

Proof: Put $h_{ij} = e^{\theta}g_{ij}$. Since λ is chosen to be 1, we have $h'_{ij} = h_{ij}$. Recall that the discussion in §1 shows that π and τ have the following form

$$\pi = \begin{pmatrix} 0 & \pi^j & 0 \\ \pi^0_i & \pi^j_i & \pi^{n+1}_i \\ 0 & \pi^j_{n+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & dx^j & 0 \\ S_{ik}dx^k & \Gamma^j_{ik}dx^k & h_{ik}dx^k \\ 0 & h^{jl}S_{lk}dx^k & 0 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & 0 & 0 \\ \tau^0_i & \tau^j_i & 0 \\ \tau^0_{n+1} & \tau^j_{n+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ (M_{ik}+L_{ik})dx^k & \frac{1}{2}\,h^j{}_{ik}dx^k & 0 \\ -\,\omega^0_{n+1} & h^{jl}M_{lk}dx^k & 0 \end{pmatrix}$$

Note that $\pi_0^0 = \pi_{n+1}^{n+1} = 0$ because $\omega'^{0}_{0} = 0$, which is deduced by $\omega_0^0 = 0$ and by the choice of g ($\lambda = 1$). Insert these expression into (2.9) to yield

$$(2.12)\quad \omega = \begin{pmatrix} 0 & \pi^j & 0 \\ \pi_i^0 - \tau_i^0 + \mu\pi_i^{n+1} & \pi_i^j - \tau_i^j + c^j\pi_i^{n+1} & \pi_i^{n+1} \\ -\tau_{n+1}^0 - d\mu & \pi_{n+1}^j - \tau_{n+1}^j - dc^j - \mu\pi^j & c^i\pi_i^{n+1} \\ -c^i(\pi_i^0 - \tau_i^0 + \mu\pi_i^{n+1}) & -c^i(\pi_i^j - \tau_i^j + c^j\pi_i^{n+1}) & \end{pmatrix}$$

Hence

$$(2.13)\quad A_{ik}^j = \Gamma_{ik}^j - \frac{1}{2} h^j{}_{ik} + c^j h_{ik}$$

$$A_{ik}^0 = S_{ik} - (M_{ik} + L_{ik}) + \mu h_{ik}.$$

The requirments $A_{1n}^k = A_{1n}^0 = 0$ are satisfied when

$$(2.14)\quad c^j = -e^{-\theta}(\Gamma_{1n}^j - \frac{1}{2} h^j{}_{1n}),\quad \mu = -e^{-\theta}(S_{1n} - M_{1n} - L_{1n}).$$

With these equalities inserted in (2.13), we have the formulae (2.11) and complete the proof. □

Remark: From (2.12) and (2.14) follow also the formulae for G_j and B_j in (2.2).

If the equation (EQ) does not satisfy the condition (2.11), then by multiplying a suitable function to the unknown w, one can transform (EQ), without changing the hypersurface w nor the coefficients g_{ij}, into the one satisfying the condition . The other coefficients are obtained by the following lemma, of which proof is a straightforward computation.

Lemma 2.2 : Let the system (EQ) be given with the normalization factor e^θ. If the unknown w is transformed into a new unknown by $w = e^{-\alpha}u$, then the system is subject to the change

$$(2.15)\quad u_{ik} = g_{ik}u_{1k} + P_{ik}^j u_j + P_{ik}^0 u,$$

where

$$(2.16)\quad P_{ik}^j = A_{ik}^j + \alpha_i\delta_k^j + \alpha_k\delta_i^j - g_{ik}(\alpha_1\delta_n^j + \alpha_n\delta_1^j)$$

$$P_{ik}^0 = A_{ik}^0 + (\alpha_{ik} - \alpha_i\alpha_j) + A_{ik}^j\alpha_j - g_{ik}(\alpha_{1n} - \alpha_1\alpha_n)$$

The new normalization factor is $e^{\theta+(n+2)\alpha}$.

§§ 2.2 Linear differntial equations defining maps into hyperquadrics

Definition: The system (EQ) is said to satisfy the quadric condition if the image of w is contained in a certain quadratic hypersurface, i.e. if the cubic invariant form III vanishes identically (Proposition 1.1).

Since the quadratic hypersurface is conformally flat, the invariant τ vanishes under the quadric condition; and the connection form π itself is flat. This fact can be also seen directly from the formulae (1.19) and (1.20). Therefore we have a corollary to Theorem 2.1.

Theorem 2.3. Let the equation (EQ) of rank n+2 ($n \geq 3$) satisfying (2.1), (2.3) and (2.11) be given. If it satisfies the quadric condition then the coefficients $A^{\cdot}_{ik}$ are expressed as rational functions in g_{ij} and their derivatives:

$$(2.17) \qquad A^j_{ik} = \Gamma^j_{ik} - g_{ik}\Gamma^j_{1n}$$

$$A^0_{ik} = S_{ik} - g_{ik}S_{1n}.$$

Here Γ^j_{ik} and S_{ik} are the Christoffel symbol and the Schouten tensor of $e^\theta g_{ij}$.

Converse of this theorem holds.

Theorem 2.4. Let g_{ij} ($g_{1n} = 1$) be a non-degenerate symmetric tensor which represents a conformally flat structure. Define θ so that $\det(e^\theta g_{ij}) = 1$; and define quantities A^k_{ij} and A^0_{ij} by (2.17) according to the tensor $e^\theta g_{ij}$. Then the equation

$$w_{ij} = g_{ij}w_{1n} + A^k_{ij}w_k + A^0_{ij}w$$

is of rank n+2 and satisfies the quadric condition. Its normalization factor is e^θ.

Proof: Put $h_{ij} = e^\theta g_{ij}$. Since by assumption h_{ij} is conformally flat, the associated normal conformal connection π is integrable. Apply Theorem 1.3 by putting $\tau = 0$. The Gauss and the Codazzi-Minardi equations are trivially satisfied (all the terms are zero) so that there is an unique embedding $w = (w^1, \ldots, w^{n+2})$ of x-space into

P^{n+1} such that the induced conformal metric is h_{ij} and the invariant form τ is zero. Let

$$(\#) \qquad w_{ij} = g'_{ij}w_{1n} + A'^{k}_{ij}w_k + A'^{0}_{ij}w$$

be the system with the projective solution w and with the normalization factor e^{θ}. The argument in §§2.1 tells us that the surface w has the induced conformal metric $e^{\theta}g'_{ij}$. Therefore we have $g'_{ij} = g_{ij}$. Since (#) is of rank n+2, Theorem 2.2 asserts that $A'^{k}_{ij} = A^{k}_{ij}$ and $A'^{0}_{ij} = A^{0}_{ij}$. This completes the proof. □

We can formulate this in a more symmetric way:

Theorem 2.5. Let σ_{ij} be a non-degenerate symmetic tensor which represents a conformally flat structure. The the system

$$\sigma_{ij}(w_{k\ell} - \Gamma^{p}_{k\ell}w_p - \frac{1}{n-2}R_{k\ell}w) = \sigma_{k\ell}(w_{ij} - \Gamma^{p}_{ij}w_p - \frac{1}{n-2}R_{ij}w)$$

is of rank n+2 and satisfies the quadric condition. Here Γ^{p}_{ij} and R_{ij} stand for the Christoffel symbol and the Ricci tensor with respect to σ_{ij}.

Proof: Assume $e^{\eta} := \sigma_{1n} \neq 0$ and put $g_{ij} = e^{-\eta}\sigma_{ij}$ and $\det(g_{ij}) = e^{-2n\rho}$. Define $h_{ij} = e^{2\rho}g_{ij}$ so that $\det(h_{ij}) = 1$. We have only to combine Theorem 2.4 and Lemma 2.2 as well as the transformation formulae of the Christoffel symbol and the Ricci tensor for h_{ij} into those for σ_{ij}:

$$\Gamma^{j}_{ik}(\sigma) = \Gamma^{j}_{ik}(h) + \alpha_i\delta^{j}_{k} + \alpha_k\delta^{j}_{i} - h_{ik}h^{jp}\alpha_p$$

$$R_{ik}(\sigma) = R_{ik}(h) - (n-2)(\alpha_{ik} - \alpha_i\alpha_k - \alpha_j\Gamma^{j}_{ik}(h))$$

$$- \{\Delta_h\alpha + (n-2)h^{jp}\alpha_j\alpha_p\}h_{ik}$$

where $\alpha = \frac{1}{2}\eta - \rho$ and Δ_h is the Laplacian of h_{ij} (see [Gol, p. 115]). □

§ 3 Uniformizing equation of a Siegel modular orbifold

§§ 3.1 Statement of the result

The domain

$$D = \{(z^1,\dots,z^n) \in \mathbb{C}^n \mid (\mathrm{Im}\, z^1)(\mathrm{Im}\, z^n) - \Sigma_{j=2}^{n-1}(\mathrm{Im}\, z^j)^2 > 0,\ \mathrm{Im}\, z^1 > 0\}$$

is called the symmetric domain of type IV of dimension n (≥ 2). If $n = 2$, D is biholomorphically equivalent to the product $H \times H$ where H is the upper half plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}\, \tau > 0\}$. If $n = 3$, D is biholomorphically equivalent to the Siegel upper half space $\mathbb{H}_2$ of genus 2:

$$\left\{ \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^2 & \tau^3 \end{pmatrix} \,\middle|\, (\mathrm{Im}\, \tau^1)(\mathrm{Im}\, \tau^3) - (\mathrm{Im}\, \tau^2)^2 > 0,\ \mathrm{Im}\, \tau^1 > 0 \right\}$$

Let Q be an $(n+2)$ by $(n+2)$ symmetric matrix given by

$$(t^0,\dots,t^{n+1})Q\,{}^t(t^0,\dots,t^{n+1}) = -\,t^0t^{n+1} + t^1t^n - \Sigma_{j=2}^{n-1}(t^j)^2$$

and let Q^n be the quadratic hypersurface of P^{n+1} defined by Q. Then the domain D can be considered as a connected component of the open subset

$$\{(t^0,\dots,t^{n+1}) \in Q^n \mid (t^0,\dots,t^{n+1})Q\,{}^t(t^0,\dots,t^{n+1}) = 0\}$$

of Q^n through the embedding:

$$(z^1,\dots,z^n) \to (t^0,\dots,t^{n+1}) = (1, z^1,\dots,z^n, z^1z^n - \Sigma_{j=2}^{n-1}(z^j)^2).$$

The group $\mathrm{Aut}(D)$ of analytic automorphisms of D is a subgroup of $\{X \in GL(n+2,\mathbb{R}) \mid XQ\,{}^tX = Q\}/\pm$ of index two via the embedding $D \subset Q^n \subset P^{n+1}$. The restriction of the canonical conformal structure of Q^n on D is represented by

$$\omega = dz^1dz^n + dz^ndz^1 - 2\Sigma_{j=2}^{n-1}(dz^j)^2$$

Let $\Gamma \subset \mathrm{Aut}(D)$ be a properly discontinuous transformation group acting on D, and let D' be the maximal open subset of D on which

Γ acts freely. Denote the quotient manifold D'/Γ by X and the natural projection $D' \to X$ by π. Since $\mathrm{Aut}(D)$ acts conformally on D, there is a holomorphic conformal structure $\pi_*\omega$ on X. Let $g_{ij}dx^i dx^j$ ($g_{1n} = 1$) be a conformal metric representing $\pi_*\omega$ on a chart, with local coordinates x^i, of X. There is a linear differential equation (UDE) called the uniformizing equation of X in the form (EQ) with the principal part g_{ij} such that the projective solution gives the inverse of π. When $n \geq 3$, Theorem 2.2 tells us that g_{ij} determine the remaining coefficients of (UDE). Therefore if one knows g_{ij} as functions of x^i, one can know the equation (UDE).

Indeed this is the case for the Siegel modular group $\Gamma(2)$ of level 2 acting on the Siegel upper half space H_2, equipped with the canonical conformal structure

$$\omega = d\tau^1 d\tau^3 + d\tau^3 d\tau^1 - 2(d\tau^2)^2.$$

<u>Theorem</u> 3.1 The regular orbit of H_2 under $\Gamma(2)$ is isomorphic to the space $X = \{(x^1,x^2,x^3) \in \mathbb{C}^3 \mid x^i \neq 0,\ 1,\ x^j\ (i \neq j)\}$. The image $\pi_*\omega$ is a form on X conformal to $(x^1-x^2)x^3(x^3-1)(dx^1dx^2 + dx^2dx^1) + (x^2-x^3)x^1(x^1-1)(dx^2dx^3 + dx^3dx^2) + (x^3-x^1)x^2(x^2-1)(dx^3dx^1 + dx^1dx^3)$. The uniformizing equation (UDE) on X is given as follows:

$$(3.1)\qquad w_{ii} + \Big\{ \frac{1}{x^i} + \frac{1}{x^i-1} + \frac{1}{2}\Big(\frac{1}{x^i-x^j} + \frac{1}{x^i-x^k} \Big)\Big\} w_i - \frac{x^j(x^j-1)}{2x^i(x^i-1)(x^i-x^j)} w_j - \frac{x^k(x^k-1)}{2x^i(x^i-1)(x^i-x^k)} w_k + \frac{1}{x^i(x^i-1)} w = 0,$$

$$(x^k-x^i)x^j(x^j-1)\Big\{2w_{ij} + \Big(\frac{1}{x^i-x^k} + \frac{1}{x^j-x^i} \Big) w_i + \Big(\frac{1}{x^i-x^k} + \frac{1}{x^i-x^j} \Big) w_j + \frac{1}{(x^k-x^i)(x^k-x^j)} w \Big\}$$

$$= (x^i-x^j)x^k(x^k-1)\Big\{2w_{ik} + \Big(\frac{1}{x^k-x^j} + \frac{1}{x^k-x^i} \Big) w_i + \Big(\frac{1}{x^i-x^k} + \frac{1}{x^i-x^j} \Big) w_k + \frac{1}{(x^i-x^j)(x^j-x^k)} w \Big\}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

In the next subsection (Proposition 3.4), we shall express $\pi_*\omega$ in terms of the x-coordinate. Once it is done, the equation (UDE) is derived as follows: We apply Theorem 2.5 to the tensor σ_{ij} of the form

$$(\sigma_{ij}) = \begin{pmatrix} 0 & E_3 & E_2 \\ E_3 & 0 & E_1 \\ E_2 & E_1 & 0 \end{pmatrix},$$

which is assumed to be conformally flat and $E_1E_2E_3 \neq 0$. Put

$$W_{ij} = w_{ij} - \Gamma^k_{ij}w_k - R_{ij}w.$$

Then the system is

$$\begin{aligned} (3.2) \qquad & W_{ii} = 0 \\ & E_jW_{ij} - E_kW_{ik} = 0, \end{aligned}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

The actual computation will be sketched. The inverse matrix of σ is given by

$$(2E_1E_2E_3)^{-1} \begin{pmatrix} -E_1^2 & E_1E_2 & E_1E_3 \\ E_2E_1 & -E_2^2 & E_2E_3 \\ E_3E_1 & E_3E_2 & -E_3^2 \end{pmatrix}$$

<u>Lemma</u> 3.2. The Christoffel symbols of σ are given by

$$\Gamma^i_{ii} = \tfrac{1}{2}(\log E_jE_k)_i, \qquad \Gamma^j_{ii} = \frac{E_j}{2E_i}(\log \frac{E_j}{E_k})_i,$$

$$\Gamma^i_{ij} = \frac{A_k}{4E_j}, \qquad \Gamma^k_{ij} = \frac{-A_kE_k}{4E_iE_j},$$

where $A_i = (E_j)_j + (E_k)_k - (E_i)_i$ and (i,j,k) is as above.

<u>Proof</u>: Here the summation rule is <u>not</u> applied to the indices i,j and k. By definition, we have

$$\begin{aligned} \Gamma^i_{ii} &= \tfrac{1}{2}\,\Sigma_\ell\, \sigma^{i\ell}(2\sigma_{i\ell,i} - \sigma_{ii,\ell}) \\ &= \Sigma_\ell\, \sigma^{i\ell}\sigma_{i\ell,i} \qquad\qquad (\sigma_{ii} = 0) \end{aligned}$$

$$= \sigma^{ij}\sigma_{ij,i} + \sigma^{ik}\sigma_{ik,i}$$

$$= \frac{1}{2E_k}(E_k)_i + \frac{1}{2E_j}(E_j)_i = \frac{1}{2}(\log E_jE_k)_i.$$

Hence the first equality. Others are similarly obtained. □

As for the Ricci tensor recall the definition:

$$R_{ij} = \Sigma_\ell R^\ell{}_{i\ell j}$$

$$= \Sigma_\ell \Gamma^\ell_{i\ell,j} - \Sigma_\ell \Gamma^\ell_{ij,\ell} + \Sigma_{\ell,m}\Gamma^m_{i\ell}\Gamma^\ell_{mj} - \Sigma_{\ell,m}\Gamma^m_{ij}\Gamma^\ell_{\ell m}$$

<u>Lemma</u> 3.3: The Ricci tensor is given as follows

$$R_{ii} = \frac{1}{2}\ \{\ (\log E_i)_{ii} - \frac{E_j}{E_i}(\log \frac{E_j}{E_k})_{ij} - \frac{E_k}{E_i}(\log \frac{E_k}{E_j})_{ik}\ \}$$

$$+ \frac{1}{4}(\log E_i)_i\ (\log \frac{E_i}{E_jE_k})_i$$

$$+ \frac{1}{4}(\log \frac{E_j}{E_k})_i\ \{\ \frac{E_j}{E_i}(\log \frac{E_i}{E_jE_k})_j - \frac{E_k}{E_i}(\log \frac{E_i}{E_jE_k})_k\ \}$$

$$E_jR_{ij} - E_kR_{ik} = \frac{1}{2}\ \{\ E_j(\log E_k)_{ij} - E_k(\log E_j)_{ik}\}$$

$$- \frac{1}{4}(\log E_i)_i(E_j-E_k) + \frac{1}{4}(\log \frac{E_j}{E_k})_i\{E_j(\log \frac{E_i}{E_k})_j + E_k(\log \frac{E_i}{E_j})_k\}.$$

<u>Proof</u>: We show the first identity only. R_{ii} is the sum of three parts:

$$R_{ii} = \Sigma_\ell \Gamma^\ell_{i\ell,i} - \Sigma_\ell \Gamma^\ell_{ii,\ell} + \Sigma_{\ell,m}(\ \Gamma^m_{i\ell}\Gamma^\ell_{mi} - \Gamma^m_{ii}\Gamma^\ell_{\ell m}\ ).$$

Lemma 3.2 shows

$$\Sigma_\ell \Gamma^\ell_{i\ell} = \Gamma^i_{ii} + \Gamma^j_{ij} + \Gamma^k_{ik}$$

$$= \frac{1}{2}(\log E_jE_k)_i + \frac{A_k}{4E_i} + \frac{A_j}{4E_i} = \frac{1}{2}(\log E_iE_jE_k)_i.$$

Here note that $A_j + A_k = 2(E_i)_i$. The second term is

$$\Sigma_\ell \Gamma^\ell_{ii,\ell} = \Gamma^i_{ii,i} + \Gamma^j_{ii,j} + \Gamma^k_{ii,k}$$

$$= \frac{1}{2}(\log E_jE_k)_{ii} + \{\frac{E_j}{2E_i}(\log \frac{E_j}{E_k})_i\}_j + \{\frac{E_k}{2E_i}(\log \frac{E_k}{E_j})_i\}_k.$$

The third term is computed as follows:

$$\Sigma_{\ell,m}(\Gamma^m_{i\ell}\Gamma^\ell_{mi} - \Gamma^m_{ii}\Gamma^\ell_{\ell m})$$

$$= \Sigma_m(\Gamma^m_{ij}\Gamma^j_{mi} - \Gamma^m_{ii}\Gamma^j_{jm} + \Gamma^m_{ik}\Gamma^k_{mi} - \Gamma^m_{ii}\Gamma^k_{km})$$

$$= \Gamma^j_{ii}(\Gamma^i_{ij} - \Gamma^j_{jj} - \Gamma^k_{kj}) + \Gamma^k_{ii}(\Gamma^i_{ik} - \Gamma^j_{jk} - \Gamma^k_{kk})$$

$$- \Gamma^i_{ii}(\Gamma^j_{ji} + \Gamma^k_{ki}) + \{(\Gamma^j_{ij})^2 + 2\Gamma^j_{ik}\Gamma^k_{ij} + (\Gamma^k_{ik})^2\}.$$

The first bracket is equal to $\frac{A_k - A_i}{4E_j} - \frac{1}{2}(\log E_iE_k)_j$. Notice that $A_k - A_i = 2\{(E_i)_i - (E_k)_k\}$. Hence the sum of the first two terms is

$$\frac{1}{4E_i}(\log \frac{E_j}{E_k})_i\{ E_k(\log \frac{E_iE_j}{E_k})_k - E_j(\log \frac{E_iE_k}{E_j})_j \}.$$

The third term is, in view of $\Gamma^j_{ji} + \Gamma^k_{ki} = \frac{1}{2}(\log E_i)_i$, equal to

$$- \frac{1}{4}(\log E_jE_k)_i(\log E_i)_i.$$

The last term is

$$(\frac{A_k}{4E_i})^2 + (\frac{A_j}{4E_i})^2 + 2 \frac{- A_jE_j}{4E_iE_k} \frac{- A_kE_k}{4E_iE_j}$$

$$= \frac{1}{16} E_i^{-2}(A_j + A_k)^2 = \frac{1}{4}\{(\log E_i)_i\}^2.$$

Summing up these, we have

$$R_{ii} = \frac{1}{2}(\log E_iE_jE_k)_{ii} - \frac{1}{2}(\log E_jE_k)_{ii}$$

$$- \{ \frac{E_j}{2E_i}(\log \frac{E_j}{E_k})_i \}_j - \{ \frac{E_k}{2E_i}(\log \frac{E_k}{E_j})_i \}_k$$

$$+ \frac{1}{4E_i}(\log \frac{E_j}{E_k})_i\{ E_k(\log \frac{E_iE_j}{E_k})_k - E_j(\log \frac{E_iE_k}{E_j})_j \}$$

$$- \frac{1}{4}(\log E_j E_k)_i (\log E_i)_i + \frac{1}{4}\{(\log E_i)_i\}^2,$$

which implies the first equality of the lemma. □

Proof of (3.1): We choose a conformal class σ given by

$$(3.3) \qquad E_i^{-1} = -(x^i - x^j)(x^i - x^k)x^j(x^j - 1)x^k(x^k - 1).$$

which is conformal to $\pi_*\omega$. Inserting these into the identities in Lemma 3.2 and 3.3, the system (3.2) becomes the system (3.1). □

§§ 3.2 The conformal structure on a modular variety

The real symplectic group $Sp(2,\mathbb{R})$ is by definition,

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(4,\mathbb{R}) \,\middle|\, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} {}^t\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}$$

where I_k stands for the k by k identity matrix. The group of analytic automorphisms of H_2 is given by $Sp(2,\mathbb{R})/\pm$ by the action

$$\tau = \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^2 & \tau^3 \end{pmatrix} \quad \to \quad (A\tau + B)(C\tau + D)^{-1}$$

Let us consider the following two discrete subgroups of $Sp(2,\mathbb{R})$:

$\Gamma = GL(4,\mathbf{Z}) \cap Sp(2,\mathbb{R})$: the full modular group

$\Gamma(2) = \{X \in \Gamma \mid X \equiv I_4 \mod 2\}$: the principal congruence subgroup of level two.

The group $\Gamma(2)$ is a normal subgroup of Γ such that

$$\Gamma/\Gamma(2) \cong (\Gamma/\pm)/(\Gamma(2)/\pm) \cong S_6$$

where S_6 is the symmetric group on six letters. The transformation

$$\iota : \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^2 & \tau^3 \end{pmatrix} \to \begin{pmatrix} \tau^1 & -\tau^2 \\ -\tau^2 & \tau^3 \end{pmatrix}$$

($\iota \in \Gamma(2)$) fixes the hyperplane $F_0 \subset H_2$ given by $\tau^2 = 0$. The set F of fixed points of $\Gamma(2)$ on H_2 is given by $F = \Gamma F_0$.

Consider the space

$$\Xi = \{\xi = (\xi^1, \dots, \xi^6) \in (P^1)^6 \mid \xi^i \neq \xi^j \ (i \neq j)\}.$$

The group $PGL(2,\mathbb{C})$ and the symmetric group S_6 act on Ξ as follows:

$$\begin{array}{llll} \gamma: & (\xi^1, \dots, \xi^6) \to & (\gamma\xi^1, \dots, \gamma\xi^6) & \gamma \in PGL(2,\mathbb{C}) \\ \sigma: & (\xi^1, \dots, \xi^6) \to & (\xi^{\sigma(1)}, \dots, \xi^{\sigma(6)}) & \sigma \in S_6 \end{array}$$

For $\xi \in \Xi$, we consider a non-singular plane curve

$$C(\xi): \ w^4v^2 = (u - \xi^1 w)\dots(u - \xi^6 w)$$

of genus two in P^2 with a homogeneous coordinate system (u,v,w). Two curves $C(\xi)$ and $C(\xi')$ are biholomorphically equivalent if and only if $\xi = g\xi'$ for some $g \in S_6 \times PGL(2,\mathbb{C})$. The space Ξ modulo $PGL(2,\mathbb{C})$ is isomorphic to the space

$$\Lambda = \{(\lambda^1,\lambda^2,\lambda^3) \in \mathbb{C}^3 \mid \lambda^i \neq 0,1,\lambda^j \ (i \neq j)\},$$

which parameterizes plane curves in Rosenhein normal form

$$C(\lambda): \ w^3v^2 = u(u - w)(u - \lambda^1 w)(u - \lambda^2 w)(u - \lambda^3 w).$$

Notice that the group $Aut(\Lambda)$ of automorphisms is isomorphic to S_6. We consider the curve $C(\lambda)$ for a fixed $\lambda \in \Lambda$. We take a basis of the homology group $H_1(C(\lambda),\mathbf{Z})$ so that the corresponding four by four intersection matrix takes the canonical form, i.e. $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Then we take two linearly independent differentials of the first kind on $C(\lambda)$ such that the period matrix takes the form (τ, I_2). This is always possible and we get a point τ of H_2. Notice that the choice of the basis of $H_1(C(\lambda),\mathbf{Z})$ is not unique but, once it is chosen, the choice of two differentials is unique. Notice also that $\tau \in H_2 - F$, since the Jacobian variety $\mathbb{C}^2/(\tau,I_2)\mathbf{Z}^4$ of the curve $C(\lambda)$ can not be the product of two elliptic curves. Now we let $\lambda \in \Lambda$ vary and let the basis of $H_1(C(\lambda),\mathbf{Z})$ depend continuously on λ. Then the correspondence $\lambda \to \tau(\lambda)$ gives a multivalued map

$$\varphi: \Lambda \to H_2 - F,$$

which turns out to be an inverse map of the natural projection

$$\pi : H_2 - F \to (H_2 - F)/\Gamma(2) \cong \Lambda.$$

Notice that $(H_2 - F)/\Gamma \cong \Lambda/S_6$. The isomorphism $(H_2 - F)/\Gamma(2) \cong \Lambda$ can be explicitly given as follows.

We define sixteen theta constants $\theta_{g'g''h'h''}(\tau)$ for g', g'', h', h'' = 0,1 by

$$\theta_{g'g''h'h''}(\tau) = \Sigma_{p',p''\in\mathbf{Z}} \exp \pi i\{(p'+\tfrac{g'}{2})^2\tau^1$$

$$+ 2(p'+\tfrac{g'}{2})(p''+\tfrac{g''}{2})\tau^2 + (p''+\tfrac{g''}{2})^2\tau^3 + (p'+\tfrac{g'}{2})h' + (p''+\tfrac{g''}{2})h''\}$$

which are holomorphic functions in $\tau = \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^2 & \tau^3 \end{pmatrix} \in H_2$. In terms of theta constants the natural map $\pi: \tau \to (\lambda^1, \lambda^2, \lambda^3)$ can be expressed by

$$\lambda^1 = \left(\frac{\theta_{1100}(\tau)}{\theta_{0100}(\tau)}\right)^2 \left(\frac{\theta_{1000}(\tau)}{\theta_{0000}(\tau)}\right)^2$$

$$\lambda^2 = \left(\frac{\theta_{1100}(\tau)}{\theta_{0100}(\tau)}\right)^2 \left(\frac{\theta_{1001}(\tau)}{\theta_{0001}(\tau)}\right)^2$$

$$\lambda^3 = \left(\frac{\theta_{1000}(\tau)}{\theta_{0000}(\tau)}\right)^2 \left(\frac{\theta_{1001}(\tau)}{\theta_{0001}(\tau)}\right)^2$$

We have chosen the above expression from $6! = \#\,\mathrm{Aut}(\Lambda)$ possibilities. We want an explicit expression of the form

$$\pi_*(d\tau^1 d\tau^3 + d\tau^3 d\tau^1 - 2(d\tau^2)^2)$$

in terms of coordinates λ^1, λ^2 and λ^3.

<u>Proposition</u> 3.4: The quadratic form $\pi_*(d\tau^1 d\tau^3 + d\tau^3 d\tau^1 - 2(d\tau^2)^2)$ is a form on Λ conformal to $(\lambda^1 - \lambda^2)\lambda^3(\lambda^3 - 1)(d\lambda^1 d\lambda^2 + d\lambda^2 d\lambda^1)$ $+ (\lambda^2-\lambda^3)\lambda^1(\lambda^1-1)(d\lambda^2 d\lambda^3 + d\lambda^3 d\lambda^2) + (\lambda^3-\lambda^1)\lambda^2(\lambda^2-1)(d\lambda^3 d\lambda^1 + d\lambda^1 d\lambda^3)$, whose discriminant is $\lambda^1\lambda^2\lambda^3(\lambda^1-1)(\lambda^2-1)(\lambda^3-1)(\lambda^1-\lambda^2)(\lambda^2-\lambda^3)(\lambda^3-\lambda^1)$.

The rest of this section is devoted to the proof of the proposition We put $q = \exp \pi i\tau^3$ and study expansions of three lambdas λ^1, λ^2 and λ^3 in q. Put

$$\lambda^1 = \lambda_0^1 + \lambda_1^1 q + O(q^2), \qquad \lambda^2 = \lambda_0^2 + \lambda_1^2 q + O(q^2),$$

$$\lambda^3 = \lambda_0^3 + \lambda_1^3 q + O(q^2),$$

where $O(q^k)$ stands for a holomorphic function or a form divisible by q^k, then we have

<u>Lemma</u> 3.5 : $\lambda_0^1 = \lambda_0^2 = \left(\dfrac{\Sigma_{p\in\mathbf{Z}}\exp \pi i\{(p+\frac{1}{2})^2\tau^1 + (p+\frac{1}{2})\tau^2\}}{\Sigma_{p\in\mathbf{Z}}\exp \pi i(p^2\tau^1 + p\tau^2)}\right)^2 \left(\dfrac{\theta_{10}(\tau^1)}{\theta_{00}(\tau^1)}\right)^2$

$$\lambda_1^1 = -\lambda_1^2$$

$$= 4\lambda_0^1 \left\{ \frac{\Sigma_{p\in\mathbf{Z}}\exp \pi i\{(p+\frac{1}{2})^2\tau^1 + 2(p+\frac{1}{2})\tau^2\}}{\theta_{10}(\tau^1)} - \frac{\Sigma_{p\in\mathbf{Z}}\exp \pi i(p^2\tau^1 + 2p\tau^2)}{\theta_{00}(\tau^1)} \right\}$$

$$\lambda_0^3 = \lambda(\tau^1), \qquad \lambda_1^3 = 0.$$

where $\theta_{gh}(\omega)$ $(g,h = 0,1)$ are elliptic theta constants:

$$\theta_{gh}(\omega) = \Sigma_{p\in\mathbf{Z}}\exp \pi i\{(p+\tfrac{g}{2})^2\omega + (p+\tfrac{g}{2})h\} \qquad (\omega \in \mathbb{H})$$

and $\lambda(\omega)$ is the lambda function defined by $\lambda(\omega) = \left(\dfrac{\theta_{10}(\omega)}{\theta_{00}(\omega)}\right)^4$.

<u>Proof</u>: We have

$$\theta_{0000}(\tau) = \Sigma_{p,n\in\mathbf{Z}}\exp \pi i(p^2\tau^1 + 2pn\tau^2)q^{n^2}$$

$$= \theta_{00}(\tau^1) + \theta_{0000}^{(1)}q + O(q^2)$$

$$\theta_{0001}(\tau) = \Sigma_{p,n\in\mathbf{Z}}\exp \pi i(p^2\tau^1 + 2pn\tau^2 + n)q^{n^2}$$

$$= \theta_{00}(\tau^1) + \theta_{0001}^{(1)}q + O(q^2)$$

$$\theta_{1000}(\tau) = \Sigma_{p,n\in\mathbf{Z}}\exp \pi i\{(p+\tfrac{1}{2})^2\tau^1 + 2(p+\tfrac{1}{2})n\tau^2\}q^{n^2}$$

$$= \theta_{10}(\tau^1) + \theta^{(1)}_{1000} q + O(q^2)$$

$$\theta_{1001}(\tau) = \Sigma_{p,n\in\mathbf{Z}} \exp \pi i\{(p+\tfrac{1}{2})^2\tau^1 + 2(p+\tfrac{1}{2})n\tau^2 + n\} q^{n^2} \quad .$$

$$= \theta_{10}(\tau^1) + \theta^{(1)}_{1001} q + O(q^2)$$

$$\theta_{0100}(\tau) = \Sigma_{p,n\in\mathbf{Z}} \exp \pi i\{p^2\tau^1 + 2p(n+\tfrac{1}{2})\tau^2\} q^{(n+\frac{1}{2})^2}$$

$$= q^{1/4}\{\theta^{(0)}_{0100} \qquad + O(q^2)\}$$

$$\theta_{1100}(\tau) = \Sigma_{p,n\in\mathbf{Z}} \exp \pi i\{(p+\tfrac{1}{2})^2\tau^1 + 2(p+\tfrac{1}{2})(n+\tfrac{1}{2})\tau^2\} q^{(n+\frac{1}{2})^2}$$

$$= q^{1/4}\{\theta^{(0)}_{1100} \qquad + O(q^2)\},$$

where

$$\theta^{(1)}_{0000} = -\theta^{(1)}_{0001} = 2\Sigma_{p\in\mathbf{Z}} \exp \pi i(p^2\tau^1 + 2p\tau^2)$$

$$\theta^{(1)}_{1000} = -\theta^{(1)}_{1001} = 2\Sigma_{p\in\mathbf{Z}} \exp \pi i\{(p+\tfrac{1}{2})^2\tau^1 + 2(p+\tfrac{1}{2})\tau^2\}$$

$$\theta^{(0)}_{0100} = 2\,\Sigma_{p\in\mathbf{Z}} \exp \pi i(p^2\tau^1 + p\tau^2)$$

$$\theta^{(0)}_{1000} = 2\,\Sigma_{p\in\mathbf{Z}} \exp \pi i\{(p+\tfrac{1}{2})^2\tau^1 + (p+\tfrac{1}{2})\tau^2\}.$$

The following identity

$$\left(\frac{a_0 + a_1 q + O(q^2)}{b_0 + b_1 q + O(q^2)}\right)^2 \left(\frac{c_0 + c_1 q + O(q^2)}{d_0 + d_1 q + O(q^2)}\right)^2$$

$$= \left(\frac{a_0 c_0}{b_0 d_0}\right)^2 + 2\,\frac{a_0 c_0}{(b_0 d_0)^2}\Big\{ \frac{a_0}{d_0}(c_1 d_0 - c_0 d_1) + \frac{c_0}{b_0}(a_1 b_0 - a_0 b_1)\Big\} q + O(q^2)$$

leads to

$$\lambda^1 = \left(\frac{\theta^{(0)}_{1100}\ \theta_{10}(\tau^1)}{\theta^{(0)}_{0100}\ \theta_{00}(\tau^1)}\right)^2 + 2\,\frac{\theta^{(0)}_{1100}\ \theta_{10}(\tau^1)}{\{\theta^{(0)}_{0100}\ \theta_{00}(\tau^1)\}^2}$$

$$\times\{ \frac{\theta^{(0)}_{1100}}{\theta_{00}(\tau_1)}(\theta^{(1)}_{1000}\theta_{00}(\tau^1) - \theta_{10}(\tau^1)\theta^{(1)}_{0000})\}q + O(q^2)$$

$$\lambda^2 = \left(\frac{\theta^{(0)}_{1100}\ \theta_{10}(\tau^1)}{\theta^{(0)}_{0100}\ \theta_{00}(\tau^1)}\right)^2 + 2\ \frac{\theta^{(0)}_{1100}\ \theta_{10}(\tau^1)}{\{\theta^{(0)}_{0100}\ \theta_{00}(\tau^1)\}^2}$$

$$\times\{ \frac{\theta^{(0)}_{1100}}{\theta_{00}(\tau^1)}(\theta^{(1)}_{1001}\theta_{00}(\tau^1) - \theta_{10}(\tau^1)\theta^{(1)}_{0001})\}q + O(q^2)$$

$$\lambda^3 = \left(\frac{\theta_{10}(\tau^1_1)}{\theta_{00}(\tau^1)}\right)^4 + 2\ \frac{\theta_{10}(\tau^1_1)^2}{\theta_{00}(\tau_1)^4}\quad \frac{\theta_{10}(\tau^1_1)}{\theta_{00}(\tau^1)}$$

$$\times\{\theta^{(1)}_{1001}\theta_{00}(\tau^1) - \theta_{10}(\tau^1)\theta^{(1)}_{0001}$$

$$+ \theta^{(1)}_{1000}\theta_{00}(\tau^1) - \theta_{10}(\tau^1)\theta^{(1)}_{0000}\}q + O(q^2).$$

Since we have $\theta^{(1)}_{1000} = -\theta^{(1)}_{1001}$ and $\theta^{(1)}_{0000} = -\theta^{(1)}_{0001}$, the lemma is proved. □

Corollary 3.6: $\lambda^1 - \lambda^2 = qh(\tau^1,\tau^2,q)$, $\det(\frac{\partial(\lambda^1,\lambda^2,\lambda^3)}{\partial(\tau^1,\tau^2_1,\tau^3_2)}) = qf(\tau^1,\tau^2,q)$, where h and f are holomorphic functions in τ^1,τ^2 and q which are not divisible by q.

Proof: The first assertion is obvious. The second follows from the calculations below.

$$\frac{\partial\lambda}{\partial\tau} = \begin{pmatrix} \frac{\partial\lambda^1_0}{\partial\tau^1} + \frac{\partial\lambda^1_1}{\partial\tau^1}q & \frac{\partial\lambda^1_0}{\partial\tau^2} + \frac{\partial\lambda^1_1}{\partial\tau^2}q & \pi i\lambda^1_1 q \\ \frac{\partial\lambda^2_0}{\partial\tau^1} + \frac{\partial\lambda^2_1}{\partial\tau^1}q & \frac{\partial\lambda^2_0}{\partial\tau^2} + \frac{\partial\lambda^2_1}{\partial\tau^2}q & \pi i\lambda^2_1 q \\ \frac{\partial\lambda^3_0}{\partial\tau^1} & 0 & 0 \end{pmatrix} + O(q^2)$$

$$\det(\frac{\partial\lambda}{\partial\tau}) = \pi i\ \frac{\partial\lambda^3_0}{\partial\tau^1}\ (-\lambda^1_1\ \frac{\partial\lambda^2_0}{\partial\tau^2} + \lambda^2_1\ \frac{\partial\lambda^1_0}{\partial\tau^2})q + O(q^2)$$

$$= -2\pi i\lambda_1^1 \frac{\partial\lambda_0^3}{\partial\tau^1}\frac{\partial\lambda_0^1}{\partial\tau^2} q + O(q^2) \qquad \square$$

<u>Lemma</u> 3.7:
$$qf(\tau^1,\tau^2,q)d\tau^1 = 2\pi i\lambda_1^1 \frac{\partial\lambda_0^1}{\partial\tau^3} q\, d\lambda^3 + O(q^2)$$
$$qf(\tau^1,\tau^2,q)d\tau^2 = O(q)$$
$$qf(\tau^1,\tau^2,q)d\tau^3 = \frac{\partial\lambda_0^3}{\partial\tau^1}\frac{\partial\lambda_0^2}{\partial\tau^2}(d\lambda^1 - d\lambda^2) + O(q^2)$$

<u>Proof</u>: This follows from the expression of $qf(\tau^1,\tau^2,q)\frac{\partial\tau}{\partial\lambda}$:

$$\begin{pmatrix} O(q^2) & O(q^2) & 2\pi i\lambda_1^1 \frac{\partial\lambda_0^1}{\partial\tau^3} q + O(q^2) \\ O(q) & O(q) & O(q) \\ \frac{\partial\lambda_0^3}{\partial\tau^1}\frac{\partial\lambda_0^2}{\partial\tau^3} + O(q), & -\frac{\partial\lambda_0^3}{\partial\tau^1}\frac{\partial\lambda_0^1}{\partial\tau^3} + O(q), & -\frac{\partial\lambda_0^1}{\partial\tau^3}\frac{\partial\lambda_0^2}{\partial\tau^1} + \frac{\partial\lambda_0^2}{\partial\tau^3}\frac{\partial\lambda_0^1}{\partial\tau^1} + O(q) \end{pmatrix} \square$$

Let $U \subset \Lambda$ be an open subset of $\mathbb{C}^3$ such that the closure $\bar{U}$ in $\mathbb{C}^3$ has the property: $\bar{U} \cap \{(\lambda^1,\lambda^2,\lambda^3) \in \mathbb{C}^3 \mid \lambda^1 \neq \lambda^2\} = \bar{U} \cap \Lambda$.

<u>Corollary</u> 3.8. The quadratic form $\pi_*(d\tau^1 d\tau^2 + d\tau^2 d\tau^1 - 2(d\tau^3)^2)$ is conformally equivalent on U to a quadratic form with the following local expression around $U \cap \Lambda$:

$$\frac{1}{\lambda^1-\lambda^2}(d\lambda^2 d\lambda^3 + d\lambda^3 d\lambda^2) - \frac{1}{\lambda^1-\lambda^2}(d\lambda^1 d\lambda^3 + d\lambda^3 d\lambda^1)$$

$+$ (holomorphic quadratic form in λ),

whose discriminant has double pole along $\{\lambda^1 = \lambda^2\}$.

This implies that that the holomorphic conformal structure $\pi_*(d\tau^1 d\tau^2 + d\tau^2 d\tau^1 - 2(d\tau^3)^2)$ on Λ can be extended to a meromorphic conformal structure η on $P^3 \supset \Lambda$. We can put

$$\eta = \Sigma_{i,j=1}^3 a_{ij}(\lambda) d\lambda^i d\lambda^j \qquad a_{ij} = a_{ji}$$

$$a_{ij}(\lambda) = \frac{p_{ij}(\lambda)}{D(\lambda)} \qquad p_{ij} \in \mathbb{C}[\lambda^1,\lambda^2,\lambda^3]$$

$$D(\lambda) = \lambda^1\lambda^2\lambda^3(\lambda^1-1)(\lambda^2-1)(\lambda^3-1)(\lambda^1-\lambda^2)(\lambda^2-\lambda^3)(\lambda^3-\lambda^1).$$

We can assume

$$\det(a_{ij}(\lambda)) = D(\lambda)^{-2}$$

since η should be a holomorphic nondegenerate quadratic form on Λ. Since $\Gamma/\Gamma(2) \cong S_6$ acts on Λ as the group of automorphisms of Λ, the conformal structure of Λ represented by η is invariant under the action of S_6. In particular it is invariant under the transformation:

$$\sigma: \quad \mu^1 = \frac{\lambda^1}{\lambda^3}, \qquad \mu^2 = \frac{\lambda^2}{\lambda^3}, \qquad \mu^3 = \frac{1}{\lambda^3}.$$

Put

$$\sigma^*\eta = \Sigma\, b_{ij}(z)\, d\mu^i d\mu^j \qquad b_{ij} = b_{ji},$$

then we have

$$b_{ij}(\mu) = \frac{a_{ij}(\lambda)}{(\mu^3)^2} \quad (i,j = 1,2)$$

$$b_{i3}(\mu) = -\frac{\mu^j a_{ij}(\lambda) + a_{i3}(\lambda) + \mu^i a_{ii}(\lambda)}{(\mu^3)^3} \qquad \{i,j\} = \{1,2\}$$

$$b_{33}(\mu) = \frac{\mu^1\mu^2 a_{12}(\lambda) + \Sigma_{k=1}^{2}\{\mu^k a_{k3}(\lambda) + (\mu^k)^2 a_{kk}(\lambda)\} + a_{33}(\lambda)}{(\mu^3)^4}$$

Since

$$D(\lambda) = -(\mu^3)^{-10} D(\mu) \quad \text{and} \quad \det\left(\frac{\partial\lambda}{\partial\mu}\right) = -(\mu^3)^{-4},$$

we have

$$\det(b_{ij}(\mu)) = \det(a_{ij}(\lambda))\,(\mu^3)^{-8}$$

$$= D(\lambda)^{-2}\,(\mu^3)^{-8} = (\mu^3)^{12}\, D(\mu)^{-2}.$$

Therefore multiplying a conformal factor $(\mu^3)^{-4}$ to $g^*\eta$, we should have

$$a_{ij}(\mu) = (\mu^3)^{-4}\, b_{ij}(\mu). \qquad (i,j = 1,2,3)$$

In prticular if $i,j = 1,2$ then

$$a_{ij}(\mu) = (\mu^3)^{-4}\, b_{ij}(\mu) = (\mu^3)^{-6}\, a_{ij}(\lambda)$$
$$= (\mu^3)^{-6}\, p_{ij}(\lambda)\, D(\lambda)^{-1}$$

so

$$(3.4)\quad p_{ij}(\mu^1,\mu^2,\mu^3) = -\,(\mu^3)^4\, p_{ij}\Big(\frac{\mu^1}{\mu^3},\ \frac{\mu^2}{\mu^3},\ \frac{1}{\mu^3}\Big).$$

This implies in particular that the total degree $\deg(p_{ij})$ of p_{ij} $(i,j = 1,2)$ is at most four. Since the form η is invariant under permutations of λ^1,λ^2 and λ^3, we conclude that $\deg(p_{ij}) \leq 4$ $(i,j = 1,2,3)$. On the other hand, by Corollary 3.8, $p_{12}(\lambda)$ and $p_{kk}(\lambda)$ $(k = 1,2,3)$ are divisible by $\lambda^1-\lambda^2$. By using symmetricity with respect to λ^1,λ^2 and λ^3 again, we have the following expressions

$$p_{kk}(\lambda) = (\lambda^1-\lambda^2)(\lambda^2-\lambda^3)(\lambda^3-\lambda^1)q_{kk}$$

$$p_{ij}(\lambda) = (\lambda^i-\lambda^j)r_{ij} \qquad (i \neq j)$$

where q_{kk} and r_{ij} are polynomials with $\deg(q_{kk}) \leq 1$ and $\deg(r_{ij}) \leq 3$. The first expression satisfies (3.4) if and only if q_{kk} is identically zero. Thus we have

$$a_{kk}(\lambda) = 0 \qquad (k = 1,2,3)$$

and that the determinant of the matrix $(a_{ij}(\lambda))$ can be computed as follows:

$$\det(a_{ij}(\lambda)) = 2\,\frac{p_{12}(\lambda)p_{23}(\lambda)p_{31}(\lambda)}{D(y)^3}.$$

This expression with the identity: $\det(a_{ij}(\lambda)) = D(\lambda)^{-2}$ imply that $\deg(p_{ij}) = 3$ $(1 \leq i \neq j \leq 3)$ and $p_{12}(\lambda)p_{23}(\lambda)p_{31}(\lambda) = D(\lambda)/2$. Substitute the expression

$$p_{12}(y) = (\lambda^1-\lambda^2)r_{12} \qquad \deg(r_{12}) = 2$$

into (3.2), we have

$$p_{12}(\lambda) = (\text{const.})(\lambda^1-\lambda^2)\lambda^3(\lambda^3-1).$$

By using symmetricity of η with respect to the λ's and the identity $D(\lambda^1,\lambda^3,\lambda^2) = - D(\lambda^1,\lambda^2,\lambda^3)$, we conclude that η is expressed, up to a multiplicative constant, by

$$\frac{(\lambda^1-\lambda^2)\lambda^3(\lambda^3-1)}{D(\lambda)}(d\lambda^1 d\lambda^2 + d\lambda^2 d\lambda^1)$$
$$+\frac{(\lambda^2-\lambda^3)\lambda^1(\lambda^1-1)}{D(\lambda)}(d\lambda^2 d\lambda^3 + d\lambda^3 d\lambda^2)$$
$$+\frac{(\lambda^3-\lambda^1)\lambda^2(\lambda^2-1)}{D(\lambda)}(d\lambda^3 d\lambda^1 + d\lambda^1 d\lambda^3).$$

This completes the proof of Proposition 3.4.

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