Restriction theorem and Poincaré series for U-invariants

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(0.1) Let a connected reductive algebraic group G acts morphically on an irreducible affine variety X. The goal of this paper is to demonstrate in full generality some results on invariants of a maximal unipotent subgroup $U \subset G$, which partially has been known earlier. Our main improvement consists in a transition from a linear action to an arbitrary normal G-variety X. The results obtained have useful applications to the problem of tensor product decompositions. (§3). In a forthcoming paper we shall show how the restriction theorem works in the theory of equivariant embeddings of homogeneous spaces.

The ground field k is algebraically closed and of characteristic 0. Let $k[X]$ be the algebra of regular functions on X. By the Hadžiev-Grosshans theorem the subalgebra $k[X]^U$ of U-invariant functions is finitely generated and we denote $X//U = \operatorname{Spec} k[X]^U$. Moreover, $k[X]^U$ inherits many good properties of $k[X]$, e.g. normality, factoriality, Cohen-Macaulay and rational singularity properties. We concentrate our efforts on the computational aspects and on investigation of the graded structure of $k[X]^U$.

(0.2) The first topic is the *restriction theorem* for U-invariants. The main idea is to reduce the computation of $k[X]^U$ to an action of a smaller group on a smaller variety. For the algebra of G-invariants $k[X]^G$ such a theorem is due to Luna [Lu2, cor.4]. One may consider Luna theorem as a generalization of the classical Chevalley restriction theorem on G-invariants of the adjoint representation. In §1 we shall derive the similar result for U-invariants. Our consiruction goes as follows. Put $B = N_G(U)$ and let B_* be a stabilizer of general position for the B-action on X. Then one can find a regular reductive subgroup $K \subset G$ with maximal unipotent subgroup $U(K) = U \cap K$ and an irreducible component Y of $X^{B_\bullet}$, such that the restriction map $k[X] \to k[Y]$ induces an isomorphism between $k[X]^U$ and an explicitly determined subalgebra in $k[Y]^{U(K)}$ (see 1.8). The analogous theorem for linear actions $(G : V)$ has been proved in [P1, §2]. Here we follow the same way, but with some additions, because: (1) X is not assumed to be factorial and (2) one have to take into account $X^{B_\bullet}$ may be reducible. The proof of the restriction theorem relies upon the following ingredients:

(i) restriction theorem for G-invariants,

(ii) theory of affine horospherical varieties of reductive groups [VP1],

(iii) structure of stabilizers of general position for the actions $(B : X)$, $(U : X)$ [P3].

Inspite of its cumbersome form the theorem 1.8 admits effective applications to linear and prehomogeneous actions (see 1.10 and §3).

As a by-product we found a curious fact that to every action $(G : X)$ one can assigne 2 natural different *stable* G-actions with the same stabilizers of general position (cf. 1.6, 1.7).

(0.3) Let $T \subset N_G(U)$ be a maximal torus of G and $\mathcal{X}(T)_+$ be the semigroup of dominant weights with respect of (T, U). Torus T acts morphically on $X//U$ and this action turns $k[X]^U$ into $\mathcal{X}(T)_+$-graded algebra. In §2 we shall consider the properties of the Poincare series of $\mathcal{X}(T)_+$-graded algebra $k[X]^U$. (It is not assumed here that $k[X]$ is graded.)

The main result of this chapter is a series of the unequalities for the degrees of the Poincare series as a rational function (2.4, 2.10). Most of these unequalities generalize the ones from [B, ch.2], which have been proved for linear representations of G. The method of [B] depends upon the integral form for the Poincare series of U-invariants of linear representations. Our technique is to derive the unequalities from the algebraic properties of X and $X//U$ by using [K],[F],[P2]. Applying this approach we found some unequalities, which have no counterparts in [B]. All results on Poincare series are valid for the factorial varieties with rational singularities.

(0.4) In §3 we shall apply the results obtained to the problem of tensor product desompositions. Let $R(\lambda)$ be the irreducible G-module with highest weight $\lambda \in \mathcal{X}(T)_+$. Consider 2 arbitrary dominant weights λ, μ. It has been found in [L], that in order to find the decomposition of the tensor products $R(n\lambda) \otimes R(m\mu)$, for all $n, m \in \mathbf{N}$ it is sufficient to describe the algebra of U-invariants of the affine double cone $Z(\lambda, \mu)$.

We shall show that under the application of the restriction theorem to a double cone, the subvariety Y (0.2) is a double cone relative to K. Therefore, we get an ability to reduce the decomposition problem to a smaller group K.

It has been shown in [P4] how to compute for any double cone its complexity $\tilde{c} = \tilde{c}(Z(\lambda, \mu))$ with respect of the action of the extended group $\tilde{G} = G \times (k^*)^2$. If $\tilde{c} = 0$ and λ, μ are fundamental weights, then $k[Z(\lambda, \mu)]^U$ is polynomial algebra. In [L] all such pairs together with the degrees and the weights of a homogeneous generators of $k[Z(\lambda, \mu)]^U$ have been found. In §3 we solve the analogous problem for the case $\tilde{c} = 1$.

According to [P4, 2.7] there are 2 serial cases and 2 sporadic cases of fundamental weights with $\tilde{c} = 1$. It is appears, a posteriori, in all these cases $k[Z]^U$ is a *hypersurface* and we explicitely indicate the form of a single relation between the generators. The formulas obtained for the Poincare series are a good illustration to the result of §2 and allows us to make several conclusions on the properties of the action $(U : Z(\lambda, \mu))$.

(0.5) We follow mainly the notations and terminology of [VP2].
G is a reductive algebraic group with fixed subgroups T, U; $T \subset N_G(U)$. If K is a reductive regular (i.e. $T \subset N_G(K)$) subgroup, then $T(K) := T \cap K$, $U(K) := U \cap K$.
$\Sigma(G)$ is the root system of G relative to T.
$U_\alpha \subset U$, $\alpha \in \Sigma(G)$ are the root subgroups.
If $\lambda \in \mathcal{X}(T)_+$, then λ^* is the highest weight of the dual G-module.
Group operation in the character group $\mathcal{X}(T)$ is written additively and $(\ ,\ )$ is a Weyl group invariant scalar product on $\mathcal{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.
L^0, L' are the identity component and commutator subgroup of L respectively.

s.g.p.= stabilizer(s) of general position;

p.g.p.= point(s) of general position;

RS = rational singularities.

If A is a finitely generated integral domain, then $\dim A$ is the Krull dimension of A and QA is its fraction field.
Z_{reg} is the set of non-singular points of an algebraic variety Z.

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§1 Restriction theorem

(1.1) Let G be a connected reductive algebraic group and X be an affine irreducible G-variety. The induced action of G on $k[X]$ determine its decomposition in the direct sum of isotypic components and also the decomposition of the subalgebra of U-invariant functions:

$$k[X]^U = \bigoplus_{\lambda \in \mathcal{X}(T)_+} k[X]^U_\lambda$$

Here $k[X]^U_\lambda = \{f \in k[X]^U \mid t.f = \lambda(t)f \text{ for any } t \in T\}$. Let denote

$$\Gamma(X) = \{\lambda \in \mathcal{X}(T)_+ \mid k[X]^U_\lambda \neq 0\}$$

Since X is irreducible, it is clear that $\Gamma(X)$ is a (finitely generated) subsemigroup of $\mathcal{X}(T)_+$, containing zero. Let

$$\mathcal{T}(X) = \mathbb{Z}\Gamma(X) \cap \mathcal{X}(T)_+ = \{\gamma \in \mathcal{X}(T)_+ \mid \gamma = \lambda - \mu; \lambda, \mu \in \Gamma(X)\}$$

It is obvious, that $\mathcal{T}(X)$ is a finitely generated semigroup, containing $\Gamma(X)$ and

$$\mathbb{Z}\mathcal{T}(X) \cap \mathcal{X}(T)_+ = \mathcal{T}(X) \qquad (1)$$

(1.2) Let $\mathcal{T}$ be a finitely generated subsemigroup of $\mathcal{X}(T)_+$, containing zero and $\lambda_1, ..., \lambda_t$ - be its minimal generator system. If $v_i \in R(\lambda_i)^U \setminus \{0\}$, we define the affine horospherical variety $C(\mathcal{T})$ by the formula

$$C(\mathcal{T}) = \overline{G(v_1 + ... + v_t)} \subset R(\lambda_1) \oplus ... \oplus R(\lambda_t)$$

Put $w = v_1 + ... + v_t$. Clearly $G_w \supset U$. The following properties of $C(\mathcal{T})$ have been proved in [VP1]:

(i) The algebra of regular functions on $C(\mathcal{T})$, considered as a G-module, has the decomposition: $k[C(\mathcal{T})] = \bigoplus_{\lambda \in \mathcal{T}} R(\lambda^*)$, in particular $\Gamma(C(\mathcal{T})) = \mathcal{T}^*$.

(ii) If $\mathbb{Z}\mathcal{T} \cap \mathcal{X}(T)_+ = \mathcal{T}$, then $C(\mathcal{T})$ is normal and $\operatorname{codim}_{C(\mathcal{T})}(C(\mathcal{T}) \setminus Gw) \geq 2$.

(iii) $C(\mathcal{T})$ is factorial iff $\mathcal{T}$ is generated by the fundamental weights.

Since $C(\mathcal{T})$ is a spherical variety, then normality of $C(\mathcal{T})$ implies RS-property. Following [VP1] we shall say $C(\mathcal{T})$ is *$\mathcal{S}$-variety of G.*

If X is an irreducible G-variety and $\mathcal{T} \subset \mathcal{X}(T)_+$ is an arbitrary finitely generated semigroup, then clearly $k[X]^U_{\mathcal{T}} := \bigoplus_{\lambda \in \mathcal{T}} k[X]^U_\lambda$ is a subalgebra of $k[X]^U$. Consider the diagonal action $(G : X \times C(\mathcal{T}))$.

Proposition. $k[X \times C(\mathcal{T})]^G \cong k[X]^U_{\mathcal{T}}$. More exactly, this isomorphism is of the form:

$$(f(\cdot,\cdot) \in k[X \times C(\mathcal{T})]^G) \mapsto (\overline{f}(\cdot) = f(\cdot, w) \in k[X]^U_{\mathcal{T}}) \qquad (2)$$

Proof. See [P1,§2]. □

Corollary. If $\mathcal{T} \supset \Gamma(X)$, then $k[X \times C(\mathcal{T})]^G \cong k[X]^U$. □.

(1.3) Let Z be an irreducible affine normal G-variety. Put $Z//G := \mathrm{Spec}k[Z]^G$ and let $\pi : Z \to Z//G$ be the quotient morphism. For $\xi \in Z//G$ we denote by the $T(\xi)$ the single closed G-orbit in $\pi^{-1}(\xi)$. Then there is an open subset $(Z//G)_{pr} \subset Z//G$, such that for any $\xi, \eta \in (Z//G)_{pr}$ the orbits $T(\xi)$ and $T(\eta)$ are G-isomorphic [Lu1]. The points from $Z_{pr} := \bigcup_{\xi \in (Z//G)_{pr}} T(\xi)$ will be referred as the *principal* ones (in Z). Recall, that action $(G : Z)$ is called *stable* if $\overline{Z}_{pr} = Z$.

Let $z \in Z_{pr}$ and $S := G_z$. Put $C = \overline{Z^S \cap Z_{pr}}$. This is a closed $N_G(S)$-invariant subset of Z^S. Moreover, C coincide with the union of all irreducible components of Z^S, containing principal points. The restriction theorem for G-invariants asserts, that $k[Z]^G \cong k[C]^{N_G(S)}$ or equivalently, $Z//G \cong C//N_G(S)$ [Lu2, cor.4]. We need a slight modification of this result.

Proposition. Let $C = \cup_i C_i$ be the irreducible decomposition of C. Then
(i) $N_G(S)$ acts transitively on the set $\{C_i\}$;
(ii) If $N_1 = \{n \in N_G(S) \mid nC_1 \subset C_1\}$, then $N_G(S)^0 \subset N_1$ and $C_1//N_1 \cong C//N_G(S)$;
(iii) If the action $(G : Z)$ is stable and $Y \subset Z^S$ is a closed irreducible $N_G(S)$-invariant subvariety, then the following conditions are equivalent:
(a) $Y = C_i$ for some i;
(b) $\overline{GY} = Z$.

Proof. Parts (i),(ii) easily follow from next results, that has been established in [Lu2]:

- The morphism $Z^S//N_G(S) \to Z//G$ is finite , in particular, $\pi(C_i)$ is closed in $Z//G$ for each i;
- If $z \in Z^S \cap Z_{pr}$, then $Gz \cap Z^S = N_G(S)z$.

By applying these assertions we find if C_1 is an irreducible component of C with $\pi(C_1) = Z//G$, then $N_G(S)C_1 = C$ and moreover, $C_1//N_1 \to C//N_G(S)$ is a surjective birational morphism. Therefore this is an isomorphism, because $C//N_G(S) \cong Z//G$ is normal.

(iii) If $Y = C_1$, then by (i) $\pi(Y) = Z//G$. Stable actions possess the following nice property: if $D \subset Z$ is a closed G-invariant subset, then $\pi(D) \neq Z//G$ [Po]. Therefore (a) implies (b). The opposite implication follows from the fact that the action $(N_G(S) : C)$ is always stable [Lu2]. □

The components C_i will be referred as the *principal* ones of Z^S.

(1.4) Let $\theta \in \mathrm{Aut}G$ be an involution, such that $\theta(t) = t^{-1}$ for any $t \in T$. Then $\theta(U_\alpha) = U_{-\alpha}$, $\alpha \in \Sigma$. By X^* we denote the G-variety , which is isomorphic to X as an abstract variety, but provided with the twisted G-action. If $x \in X$ and $i : X \to X^*$ is the isomorphism, then put $x^* = i(x)$. The twisted action is defined by $(g, x^*) \mapsto (\theta(g)x)^*$, $g \in G, x \in X$. Consider the diagonal action $(G : X \times X^*)$. It has been proved in [P3, §1] that there exists a point $z = (x, x^*) \in X \times X^*$ such that

(a) $U_* := U_x$ is a s.g.p. for (U, X);

(b) $B_* := B_x$ is a s.g.p. for (B, X);

(c) $S := G_z$ is a s.g.p. for $(G, X \times X^*)$;

(d) there exists $t \in T$ such that $Z_G(t)' \subset S \subset Z_G(t)$;

(e) $U_* = U \cap S$, $B_* = B \cap S$.

It follows from (d),(e) that S is a reductive regular subgroup of G, U_* is a maximal unipotent subgroup of S and B_*^0 is a Borel subgroup of S^0. Moreover it has been shown in [loc. cit] that

$$\mathcal{T}(X) = \{\omega \in \mathcal{X}(T)_+ \mid \omega|_{B_* \cap T} = 1\} \tag{3}$$

$$U \cap S = \prod_{\alpha \in \Sigma_+, (\alpha, \mathcal{T}(X)) = 0} U_\alpha \tag{4}$$

Proposition. (i) $X^{B_*} = X^S$, (ii) $\overline{U.X^S} = X$.

Proof. (i) Since X is affine and B_*^0 is a Borel subgroup of S^0, we have $X^{B_*^0} = X^{S^0}$. It follows from (d),(e) that $S = B_* S^0$, hence the assertion.

(ii) By definition of s.g.p. we have $\overline{BX^{B_*}} = X$. But (d) and (e) imply $T \subset N_G(B_*)$, therefore $TX^{B_*} \subset X^{B_*}$. □

Definition-Remark. S.g.p. S is determined up to conjugacy in G. The subgroup S, satisfying the properties (d),(e),(3),(4) will be called the *canonical s.g.p.* and the points $x \in X$, $z = (x, x^*)$ with the properties (a),(b),(c) will be called the *canonical p.g.p.* Clearly, the choice of these canonical objects depends on the choice of T, U.

(1.5) It follows from (d) that

$$N_G(S)^0 = S^0 K \tag{5}$$

where $K \subset G$ is a connected reductive group such that $\theta(K) = K$ and $\mid K \cap S^0 \mid \leq \infty$. If we shall identify Lie$T$ and LieT^* via Weyl group invariant scalar product, then (3) and (4) show that the root system of K is $\Sigma \cap \langle \mathcal{T}(X) \rangle$. Here $\langle \mathcal{T}(X) \rangle$ is the subgroup of $\mathcal{X}(T)$, generated by $\mathcal{T}(X)$. Moreover, rk$K =$ rk$G -$ rk$S =$ rk$\langle \mathcal{T}(X) \rangle$ and Lie$T(K) = \langle \mathcal{T}(X) \rangle \otimes_{\mathbf{Z}} k$.

(1.6) Here we shall prove the assertion closely related with the ones from [P1],[P3]. We use the notation from (1.4).

Proposition. (i) The action $(G, X \times X^*)$ is stable; moreover, one can find a canonical p.g.p. $z = (x, x^*)$ with $x \in X_{reg}$ and $Gz = \overline{Gz}$.

(ii) There is an irreducible component Y of X^S, such that $Y \times Y^*$ is a principal component of $(X \times X^*)^S$.

Proof. (i) All assertions of such sort for G-varieties of the form $X \times X^*$ are proved by using the inductive process, which has been described in [P1] for linear actions (i.e. $X = V$ is a G-module) and in [P3] for arbitrary actions. Recall the idea of this procedure.

We argue by induction on semisimple rank of G, srk$G =$ rkG'. If G is a torus, then all the statements of proposition are obvious. Let srk$G > 0$. Then there is a proper parabolic subgroup $P \subset G$, such that $\theta(P) \cap P =: L$ is a Levi subgroup, and L-invariant closed subvariety $W \subset X$ with the properties:

(α) if N_+ is the unipotent radical of P ($N_+ \subset U$), then $\overline{N_+ W} = X$ (hence, $\overline{\theta(N_+) W^*} = X^*$);

(β) there is an open L-invariant affine subset $\hat{W} \subset W$, such that $N_+ \times \hat{W} \cong N_+ \hat{W}$.

This construction allow us to make an inductive step from $(G : X \times X^*)$ to $(L : W \times W^*)$, because srk$G >$ srk$L$. To prove (i) it is necessary to show that stability of $(L : W \times W^*)$ implies the one of $(G : X \times X^*)$ and if $z \in W \times W^*$ is a canonical p.g.p. with $Lz = \overline{Lz}$, then $Gz = \overline{Gz}$.

If X is a G-module, then it has been shown in [P1] that $(L : W \times W^*)$ is "almost" the slice-representation at the point $p = (v_M, v_M^*)$, where $v_M \in X$ is a lowest weight vector (then, by the way, $\theta(P)$ is the stabilizer of the line $\langle v_M \rangle \in \mathbf{P}X$). Therefore for the linear actions part (i) follows from the Luna slice-theorem [Lu1]. In order to use this idea in general case one can take a G-equivariant embedding $X \hookrightarrow V$, where V is a G-module. Then X^* has a closed embedding in the dual G-module V^*. The obstacle is that it may happen X does not contain lowest weight vectors from V. To overcome it, one can replace X by the cone $CX = \overline{k^* X} \subset V$, which certainly contains lowest weight vectors from V. In this situation the inductive step for $(G :$

$CX \times (CX)^*$) coincide already with taking of the ètale slice. Finally, it is easy to see, that stability of action $(G : CX \times (CX)^*)$ is equivalent to that of $(G : X \times X^*)$.

(ii) According to (1.4)Proposition , there is an irreducible component Y of X^S, such that $\overline{UY} = X$. Hence, $\overline{\theta(U)Y^*} = X^*$. Therefore , $Y \times Y^*$ is an irreducible component of $(X \times X^*)^S$, such that $\overline{G(Y \times Y^*)} = X \times X^*$. Hence, $Y \times Y^*$ is principal by (1.3)Proposition(iii). □

In the sequel the component Y of X^S with $\overline{UY} = X$ also will be called the *princiral* one.

(1.7) Let X be an irreducible normal G-variety and $\mathcal{T}(X)$ be the semigroup, defined in 1.1. Let S be the canonical s.g.p. for $(G : X \times X^*)$ and $Y \subset X^S$ be a principal component. Put $X(\mathcal{T}) = X \times C(\mathcal{T}(X))$ and consider the diagonal action $(G : X(\mathcal{T}))$. Recall, that w is a special point from $C(\mathcal{T}(X))$ (see 1.2).

Proposition. (i) S is a s.g.p. for $(G : X(\mathcal{T}))$;

(ii) The action $(G : X(\mathcal{T}))$ is stable;

(iii) Subvariety $C_1 = Y \times \overline{N_G(S)^0 w}$ is a principal component of $X(\mathcal{T})^S$ and $N(C_1) = N_G(S)^0$.

Proof. (i) Let $x \in Y$ be a canonical s.g.p. such that for $z = (x, x^*)$ we have $S = G_z$. We shall show, that $p := (x, w)$ is a p.g.p. for $(G : X(\mathcal{T}))$ and $G_p = S$. It follows from the definition of w and (3),(4) that $G_w = SU$. Since $G_x \cap U = U_x \subset S$ and $G_x \supset G_z = S$ (1.4), we have $G_p = G_w \cap G_x = SU \cap G_x = S$. Hence, the stabilizers of almost all points from $Y \times \{w\}$ is equal to S. Since $\overline{UY} = X$ and $\overline{\theta(B)w} = C(\mathcal{T}(X))$, we have

$$\overline{G(Y \times \{w\})} = X(\mathcal{T}) \tag{6}$$

i.e. S is a s.g.p.

(ii) If the action $(G : X(\mathcal{T}))$ is not stable, then it follows from the reductivity of s.g.p. that one can find a G-invaiant divisor $D \subset X(\mathcal{T})$ with the isomorphism $k[X(\mathcal{T})]^G \cong k[D]^G$, which is induced by the restriction of the regular functions [Po]. We shall show it is not possible.

Let $D \subset X(\mathcal{T})$ be a G-invariant divisor. Consider the set $D_w := D \cap (X \times \{w\})$. It follows from (1) and 1.2(ii) that $D_w \neq \emptyset$, therefore D_w is a U-invariant divisor in $X \times \{w\} \cong X$. Let $\Im \subset k[X]$ be the defining ideal of D_w. By the Lie-Kolchin theorem $\Im$ contains a U-invariant function $\bar{f}$. If $f \in k[X]^G$ is the function, corresponding to $\bar{f}$ under (2) then, clearly, $f\mid_D = 0$. Hence, $k[X(\mathcal{T})]^G \neq k[D]^G$.

(iii) C_1 is a closed irreductble $N_G(S)^0$-invariant subvariety of $X(\mathcal{T})^S$. It follows from (6) that $\overline{GC_1} = X(\mathcal{T})$. Therefore 1.3(iii) implies C_1 is a principal component of $X(\mathcal{T})^S$. The proof of the statement on $N(C_1)$ is similar to the one for a linear case, which has been presented in [P1, §2]. □

At last, after all preliminaries we are able to formulate and prove the restriction theorem. We use the notations from 1.1, 1.4, 1.5.

(1.8) Theorem. Let G be a connected reductive algebraic group and X be an irreducible affine normal G-variety. If S is a canonical s.g.p. for the action $(G : X \times X^*)$ and $Y \subset X^S$ is a principal component, then the restriction of regular functions $k[X] \to k[Y]$ induces the isomorphism

$$k[X]^U \cong k[Y]^{U(K)}_{\mathcal{T}(X)}$$

Remark. According to 1.5 one may consider $\mathcal{T}(X)$ as a subsemigroup of $\mathcal{X}(T(K))_+$. Therefore, notation $k[Y]^{U(K)}_{\mathcal{T}(X)}$ make sense. In practice it is necessary to express the generators of $\mathcal{T}(X)$ through the fundamental weights of K (see examples in 1.10, 3.4).

Proof. According to 1.3(iii), 1.7(iii) we have the isomorphism, induced by the restriction

$$k[X \times C(\mathcal{T}(X))]^G \cong k[Y \times \overline{N_G(S)^0 w}]^{N_G(S)^0}$$

Since $N_G(S)^0 = S^0K$, $S \subset G_w$, $Y \subset X^S$ one can see, that last algebra coincides with $k[Y \times \overline{Kw}]^K$. The construction from 1.2 and the definition of K shows $\overline{Kw}$ is the horospherical $\mathcal{T}(X)$-variety of K, when $\mathcal{T}(X)$ is being considered as a subsemigroup of $\mathcal{X}(T(K))_+$. Applying (1.2)Proposition we obtain the chain of isomorphisms:

$$k[X]^U \cong k[X \times \overline{Gw}]^G \cong k[Y \times \overline{Kw}]^K \cong k[Y]^{U(K)}_{\mathcal{T}(X)}$$

And it follows from (2) and Luna theorem (see 1.3) that the resulting isomorphism is induced by the restriction. □

Corollary. $k[X]^G \cong k[Y]^K$.

Proof. This follows from the fact that algebra of G-invariants is the homogeneous subalgebra of $k[X]^U$, corresponding to the weight $0 \in \mathcal{T}(X)$. And also for K. □

(1.9) It is worth to mention that $\dim k[Y]^{U(K)}_{\mathcal{T}(X)} = \dim k[Y]^{U(K)}$, though the algebras itself not necesserily coincide.

Indeed, it follows from 1.5 that s.g.p. for $(K : Y \times Y^*)$ is finite. Therefore 1.4 implies that $(U(K) : Y)$ has trivial s.g.p., i.e. $\dim k[Y]^{U(K)} = \dim Y - \dim U(K)$. On the other hand, it follows from 1.3, 1.6(ii) that $\dim(Y \times Y^*)//K = \dim(X \times X^*)//G$, i.e. $2\dim Y - \dim K = 2\dim X - \dim G + \dim S$. Since K, G are reductive and $\mathrm{rk}K + \mathrm{rk}S = \mathrm{rk}G$, we have $2\dim Y - 2\dim U(K) = 2\dim X - 2\dim U + 2\dim U(S) = 2\dim k[X]^U$. Therefore $\dim k[Y]^{U(K)}_{\mathcal{T}(X)} = \dim k[X]^U = \dim Y - \dim U(K)$.

(1.10) Example. Let G be the simple group of type F_4, and $\tilde{\varphi}_i$, $\tilde{\alpha}_i$, $i = 1, ..., 4$ - be its fundamental weights and simple roots respectively. We use the numeration and the notations from [VO]. Let apply the theorem 1.8 to the representation $X = R(\tilde{\varphi}_1)$. Here $S^0 = S$ is a simple group of type A_2. It follows from 1.4 that the set of simple roots of S is a subset of the simple roots of F_4. Since $\tilde{\varphi}_1 \in \Gamma(R(\tilde{\varphi}_1))$ it follows from (4) that $\tilde{\alpha}_3, \tilde{\alpha}_4$ are the simple roots of S. Therefore, $\mathcal{T}(R(\tilde{\varphi}_1))$ is generated by $\tilde{\varphi}_1, \tilde{\varphi}_2$. Simple calculations show $\Sigma(\mathrm{F}_4) \cap \langle\tilde{\varphi}_1, \tilde{\varphi}_2\rangle$ forms the root system of A_2. Hence $N_{\mathrm{F}_4}(S)^0 \cong S \times \mathrm{A}_2$ and $K = \mathrm{A}_2$. The simple roots of A_2 are $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ and $\alpha_2 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$. If φ_1, φ_2 are the fundamental weights of A_2, then $\varphi_1 = \frac{1}{3}(2\alpha_1 + \alpha_2)$, $\varphi_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$. Since $\tilde{\varphi}_1 = \varepsilon_1$, $\tilde{\varphi}_2 = \frac{1}{2}(3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$, we have

$$\tilde{\varphi}_1 = \varphi_1 + \varphi_2,\ \tilde{\varphi}_2 = 3\varphi_1$$

Thus, $\mathcal{T} := \mathcal{T}(R(\tilde{\varphi}_1)$ as a subset in $\mathcal{X}(T(\mathrm{A}_2))$ is generated by $\varphi_1 + \varphi_2$, $3\varphi_1$. Consideration of the weights shows that $R(\tilde{\varphi}_1)^S = R(\varphi_1 + \varphi_2)$ as A_2-modules. Therefore, by 1.8 we have the natural isomorphism

$$k[R(\tilde{\varphi}_1)]^{U(\mathrm{F}_4)} \cong k[R(\varphi_1 + \varphi_2)]^{U(\mathrm{A}_2)}_{\mathcal{T}} \tag{7}$$

Since $3\varphi_2 \in \Gamma(R(\varphi_1 + \varphi_2))$ and $3\varphi_2 \notin \mathcal{T}$, we have $k[R(\varphi_1 + \varphi_2)]^{U(\mathrm{A}_2)} \neq k[R(\varphi_1 + \varphi_2)]^{U(\mathrm{A}_2)}_{\mathcal{T}}$. It is known, that $k[R(\tilde{\varphi}_1)]^{U(\mathrm{F}_4)}$ is free [B], and it is easy to check that $k[R(\varphi_1 + \varphi_2)]^{U(\mathrm{A}_2)}$ is a hypersurface.

A generic F_4-orbit in $R(\tilde{\varphi}_1)$ is closed and isomorphic to $\mathrm{F}_4/\mathrm{D}_4$ (here $\Sigma(\mathrm{D}_4) \subset \Sigma(\mathrm{F}_4)$ is the set of the long roots) and generic A_2-orbit in $R(\varphi_1 + \varphi_2)$ is closed and isomorphic to $\mathrm{A}_2/T(\mathrm{A}_2)$. Therefore, it follows from (7) and (1.8)Corollary, that

$$k[\mathrm{F}_4/\mathrm{D}_4]^{U(\mathrm{F}_4)} \cong k[\mathrm{A}_2/T(\mathrm{A}_2)]^{U(\mathrm{A}_2)}_{\mathcal{T}} \tag{8}$$

Also it is not difficult to prove using (8), that if $\mathcal{O} \subset R(\tilde{\varphi}_1)$ is a generic F_4-orbit, then $\mathcal{O} \cap R(\tilde{\varphi}_1)^S$ is irreducible , i.e. $(F_4/D_4)^S \cong A_2/T(A_2)$.

§2 Degrees of the Poincare series of algebra of U-invariants

(2.1) In this chapter G be a connected simply-connected semisimple group, $l = \mathrm{rk}G$; T, U, B - are the same as in §1. Let $\Pi = \{\alpha_1, ..., \alpha_l\}$ - be the set of simple roots and $\varphi_1, ..., \varphi_l$ - be the fundamental weights of G relative to (T, U).

Let G acts on irreducible affine variety X and $X//U := \mathrm{Spec}k[X]^U$. The natural action $(T : X//U)$ defines $\Gamma(X)$-grading on $k[X]^U$:

$$k[X//U] = \bigoplus_{\lambda \in \Gamma(X)} k[X]^U_\lambda$$

Since $\Gamma(X) \subset \mathcal{X}(T)_+ \cong \mathbf{N}^l$, we can define Poincare series of $k[X]^U$ by

$$F_{X//U}(\underline{t}) = \sum_{n_1, ..., n_l} \mathrm{dim}k[X]^U_\lambda t_1^{n_1} \cdots t_l^{n_l} \tag{9}$$

Here $(\underline{t}) = (t_1, ..., t_l)$, $\lambda = \sum_{i=1}^l n_i \varphi_i$. It is well-known , that $F_{X//U}(\underline{t})$ is the Taylor expansion of a rational function on t_i. Let $F_{X//U}(\underline{t}) = P(\underline{t})/Q(\underline{t})$, where P, Q are polynomials and let $\deg_i P$, $\deg_i Q$ are their degrees on variable t_i. The integers $b_i = \deg_i Q - \deg_i P$ will be called the *degrees* of Poincare series.

If $k[X//U]$ is Gorenstein, then b_i has the following description (see e.g. [St2, ch.1, §12]):

- ratiolnal function $F_{X//U}(\underline{t})$ satisfy the following functional equation

$$F_{X//U}(\underline{t}^{-1}) = (-1)^{\mathrm{dim}X//U} \underline{t}^{\underline{b}} F_{X//U}(\underline{t}) \tag{10}$$

  Here $(\underline{t}^{-1}) = (t_1^{-1}, ..., t_l^{-1})$, $(\underline{b}) = (b_1, ..., b_l)$.

- Let $\Omega(X//U)$ be the canonical module of $k[X//U]$, $\Omega(X//U)$ has a natural $\mathbf{Z}^l$-grading and if $\omega_{X//U} \in \Omega(X//U)$ is a homogeneous generator, then

$$\deg \omega_{X//U} = (b_1, ..., b_l) \tag{11}$$

(2.2) Consider the affine variety $C(\mathcal{X}(T)_+)$. We shall write $\mathcal{X}_+$ instead of $\mathcal{X}(T)_+$ in the sequel. Let $w \in C(\mathcal{X}_+)$ be the point with $G_w = U$. The semigroup $\mathcal{X}_+$ is generated by the fundamental weights, therefore $C(\mathcal{X}_+)$ is factorial and

$$C(\mathcal{X}_+) \subset R(\varphi_1 \oplus ... \oplus R(\varphi_l) =: V$$

Clearly $C(\mathcal{X}_+)$ is a cone in V, equipped with the natural action of $(k^*)^l$. Each multiplicative group k^* acts by homoteties on its own fundamental G-module. This action invert $k[C(\mathcal{X}_+)]$ into $\mathbf{N}^l$-graded algebra in such a way, that if $R(\lambda) \subset k[C(\mathcal{X}_+)]$, $\lambda = \sum n_i\varphi_i^*$, then the functions from $R(\lambda)$ have multidegree $(n_1, ..., n_l)$.

It follows from [P2, Th.4.2], that algebra $k[C(\mathcal{X}_+)]$ is Gorenstein and its Poincare series satisfy the equation:

$$F_{C(\mathcal{X}_+)}(\underline{t}^{-1}) = (-1)^{\dim B} t_1^2 ... t_l^2 F_{C(\mathcal{X}_+)}(\underline{t}) \tag{12}$$

Therefore, if $\omega_C \in \Omega(C(\mathcal{X}_+))$ is a homogeneous generator, then

$$\deg \omega_C = (2, ..., 2) \tag{13}$$

Put $Z = X \times C(\mathcal{X}_+)$ and consider the action $((k^*)^l : Z)$, where $(k^*)^l$ acts trivially on X and the action on $C(\mathcal{X}_+)$ was introduced before. Clearly, this action commutes with the diagonal action $(G : Z)$ and induces the $(k^*)^l$-action on $Z//G$. By Hadžiev-Grosshans theorem (cf. also 1.2) $Z//G \cong X//U$. Moreover, the action $((k^*)^l : Z//G)$ is inverted in $(T : X//U)$ under this isomorphism. This follows from (2) and the description of $\mathbf{N}^l$-grading on $k[C(\mathcal{X}_+)]$.

Remark. Recall, that $\mathbf{N}^l$-graded algebra $A = \bigoplus_{\alpha \in \mathbf{N}^l} A_\alpha$ is called *connected*, if $A_0 = k$. Clearly, $k[Z//G] \cong k[X//U]$ is not always connected, because $k[X//U]_0 = k[X]^G$. A sufficient condition of connectivity is the existence of an open G-orbit in X.

(2.3) Let $G(i)$ be the simple 3-dimensional subgroup in G, generated by U_{λ_i} and $U_{\lambda_{-i}}$ and put $T_i = (G(i) \cap T)^0$. Being a subgroup in T, T_i acts on $X//U$. The construction of T_i shows, that $\varphi_j |_{T_i} = 1$ if $i \neq j$. Therefore, if $F_i(t)$ is the Poincare series of invariants of action $(T_i : X//U)$, then $F_i(t) = F_{X//U}(.., 1, t, 1, ..)$. If $X//U$ is Gorenstein, then (10) shows

$$F_i(t^{-1}) = (-1)^d t^{b_i} F_i(t) \tag{14}$$

Let U_* be the canonical s.g.p. for $(U : X)$. By (4) U_* is generated by some U_α, $\alpha \in \Pi$.

(2.4) Theorem. Let X be an irreducible affine factorial RS-variety and G acts on X. Then

(i) $0 \le b_i \le 2$;

(ii) (a) if $U_{\alpha_i} \subset U_*$, then $b_i = 0$,
(b) if $U_{\alpha_i} \not\subset U_*$, then $b_i > 0$;

(iii) the following conditions are equivalent:

(a) $b_i = 2$ for all i;

(b) if $D = \{x \in X \mid \dim U_x > 0\}$, then $\mathrm{codim}_X D > 1$.

Remarks. 1. The assumptions of the theorem can be partially weakened. Part (ii) is true if $X//U$ is an RS-variety only.
2. In the proof of (i),(iii) we heavily relies upon the methods of [K]. The results from [K] may be interprete as a statements on the properties of the Poincare series and the canonical module of a **N**-graded connected k-algebra. All arguments from [K] are carried on the multigraded case without changes. But the connectivity is a more delicate point. In the proof of $(a) \Rightarrow (b)$ in corollary 4 in [K] the connectivity of graded algebra under consideration is essentially used. Since $k[X//U]$ is not necessarily connected, we shall use additional arguments, which are related with 1.4.

(2.5) *Proof of the theorem.*

Since $C(\mathcal{X}_+)$ is factorial and has RS-property, the same assertion is true for Z. Therefore, $Z//G = X//U$ is also factorial RS-variety. In particular, $X//U$ is Gorenstein.

(ii)(a) If $U_{\alpha_i} \subset U_*$, then by (4) $(\alpha_i, \Gamma(X)) = 0$. Therefore, if $\lambda = \sum n_j \varphi_j \in \Gamma(X)$, then $n_i = 0$. Hence $F_{X//U}$ do not depends of t_i and $b_i = 0$.
(b) If $U_{\alpha_i} \not\subset U_*$, then $(\alpha_i, \Gamma(X)) \neq 0$, i.e. there is $\lambda \in \Gamma(X)$ with $n_i \neq 0$. Therefore $\lambda|_{T_i} \neq const$ and thus, T_i acts non-trivially on $X//U$. Since $X//U$ has RS-property, then 2.3 and [F, Satz 3.1] show that $b_i > 0$.

(i) The unequality $b_i \geq 0$ already proved in (ii). To prove the other part we can consider the action $((k^*)^l : Z//G)$ instead of $(T : X//U)$. Let $\pi_G : Z \to Z//G$ be the quotient map and $m = \max_{z \in Z} \dim Gz$. Let apply theorems 1,2 from [K] to $(G : Z)$. They assert, in particular, that there exists a G-equivariant injection of degree 0 of graded modules:

$$\bar{\gamma} : \Omega(Z) \to \wedge^m \mathcal{G}^* \otimes \pi_G^* \Omega(Z//G) \qquad (15)$$

Here $\mathcal{G} = \mathrm{Lie}G$. Since $\Omega(Z) = \Omega(X) \otimes_k \Omega(C(\mathcal{X}_+))$, then by (13) the generator

$\omega_Z \in \Omega(Z)$ is of degree $(2,..,2)$. So far as $\bar{\gamma}$ is an injection, then it follows from (11) that $b_i \leq 2$ for every i.

(iii) Proof of (i) shows (a) is equivalent to

(a') $\deg\omega_{Z//G} = \deg\omega_Z$.

For the open orbit $Gw \subset C(\mathcal{X}_+)$ we have $G_w = U$ and $\mathrm{codim}(C(\mathcal{X}_+) \setminus Gw) \geq 2$ (see 1.2(ii)). Therefore (b) is equivalent to

(b') if $D' = \{z \in Z \mid \dim G_z > 0\}$, then $\mathrm{codim}_Z D' > 1$.

We shall prove the equivalence of (a') and (b'). The injection (15) means, that there exists $c \in \wedge^m \mathcal{G}^* \otimes k[Z]$, such that $\bar{\gamma}(\omega_Z) = c \otimes \omega_{Z//G}$. Then c may be considered as a G-equivariant map

$$c: \ Z \to \wedge^m \mathcal{G}^* \tag{16}$$

It has been shown in [K] that if $\dim Gz = m$ and $z \in Z_{reg}$, then $c(z)$ are the Plücker coordinates of the subspace $\mathcal{G}_z^{\perp} \subset \mathcal{G}^*$.

$(a') \Rightarrow (b')$. If $\deg\omega_{Z//G} = \deg\omega_Z$, then $\deg c = 0$. That is, $c \in \wedge^m \mathcal{G}^* \otimes k[Z]_0$. So far as $k[Z]_0 = k[X]$, then (16) is completed to the following commutative diagram:

$$\begin{tikzcd} X \times C(\mathcal{X}_+) = Z \arrow[r, "c"] \arrow[dr] & \wedge^m \mathcal{G}^* \\ & X \arrow[u] \end{tikzcd} \tag{17}$$

Let $z = (x, w')$ be a generic point, $x \in X_{reg}$, $w' \in Gw$. Since c depends on x only, we have $\mathcal{G}_z$ do not depends on w'. But this is possible only if $\dim\mathcal{G}_z = 0$, i.e. $m = \dim G$. Hence we have already proved $\mathrm{codim}_Z D' > 0$. Since $\wedge^m \mathcal{G}^* \cong k$, we have $c \in k[X]^G \subset k[Z]^G$. If $\mathrm{codim}_Z D' = 1$, then formulas (6),(7),(12) from [K] give us $D' = \mathbf{V}(c)$. Hence, $D' = \tilde{D} \times C(\mathcal{X}_+)$, where $\tilde{D}$ is a G-invariant divisor in X. For $z = (x, w)$ we have $G_z = U_x$. Therefore, next property is satisfied for $\tilde{D}$: for every $x \in \tilde{D}$ we have $\dim U_x > 0$. But lemma 2.7 below shows it is impolssible. Hence $\mathrm{codim}_Z D' > 1$ (and c is constant).

$(b') \Rightarrow (a')$. If D' is a subvariety of Z, then $m = \dim G$ and $c \in \wedge^m \mathcal{G}^* \otimes k[Z] \cong k[Z]$. Since $\mathrm{codim}_Z D' > 1$ and Z is normal, then c does not vanish on Z. Therefore c is constant, $\deg c = 0$ and $\deg\omega_{Z//G} = \deg\omega_Z$. □

(2.6) Remark. The properties of c, which were used in 2.5, are derived from the existence of a commutative diagram with good properties, containing $\bar{\gamma}$ [K]. The diagram itself exists only over the open subset Z_{reg}. But $\mathrm{codim}_Z(Z \setminus Z_{reg}) > 1$ and this is sufficient for our purposes and for the proof of corollary 4 in [K].

(2.7) Lemma. Let G be a connected semisimple group, acting on an irreducible affine variety X and U_* is a s.g.p. for $(U : X)$. If $D = \{x \in X \mid \dim U_x > \dim U_*\}$, then D is not a G-invariant principal divisor.

Proof. If $D = \mathbf{V}(f)$, $f \in k[X]^G$, then it is easy to see that $\Gamma(X) = \Gamma(D)$, $\mathcal{T}(X) = \mathcal{T}(D)$. Therefore, by (4) s.g.p. for $(U : X)$, $(U : D)$ have to coincide. □

(2.8) Corollary. Under the conditions of 2.4, if there exists $x \in X$ with $\dim G_x = 0$, then $b_i = 2$ for every i.

Proof. By the condition we have $\mathrm{codim}_X D > 0$. Assume D_1 is an irreducible component of D of codimension 1. Since X is factorial, it follows from 2.7, that D_1 is not G-invariant. Therefore $\overline{GD_1} = X$ and hence, $\dim G_y > 0$ for any $y \in \overline{GD_1}$. A contradiction! Hence, $\mathrm{codim}_X D \geq 2$ and we can apply now 2.4(iii). □

(2.9) At the rest of §2 we assume that X is an irreducible conical variety, i.e. $k[X]$ is a connected $\mathbf{N}^s$-graded algebra. Let $F_X(\underline{\lambda})$ be the Poincare series of $k[X]$, $(\underline{\lambda}) = (\lambda_1, \ldots, \lambda_s)$. If $F_X(\underline{\lambda}) = P(\underline{\lambda})/Q(\underline{\lambda})$, then we put $q_i(X) = \deg_i Q - \deg_i P$, $\underline{q}(X) = (q_1(X), \ldots, q_s(X))$. If X is Gorenstein, then by [St2, Ch.1, §12] we have:

$$F_X(\underline{\lambda}^{-1}) = (-1)^{\dim X} \underline{\lambda}^{\underline{q}(X)} F_X(\underline{\lambda})$$

In this case $k[X]^U$ is a connected $\mathbf{N}^{l+s}$-graded algebra and if $X//U$ is Gorenstein, then

$$F_{X//U}(\underline{\lambda}^{-1}, \underline{t}^{-1}) = (-1)^{\dim X//U} \underline{\lambda}^{\underline{q}(X//U)} \underline{t}^{\underline{b}} F_{X//U}(\underline{\lambda}, \underline{t}), \tag{18}$$

where $\underline{q}(X//U) = (q_1(X//U), \ldots, q_s(X//U))$.

(2.10) Theorem. Let X be an affine conical factorial RS-variety and $(G : X)$. Then in addition to the statements of theorem 2.4 we have:

(i) $0 \leq q_i(X//U) \leq q_i(X)$ for every i;

(ii) the following conditions are equivalent:

(a) $\underline{q}(X//U) = \underline{q}(X)$,

(b) $b_i = 2$ for every i;

(iii) if there exists $x \in X$ with $\dim G_x = 0$, then $\underline{q}(X//U) = \underline{q}(X)$.

Proof. (i) $q_i(X)$ is responsible for some **N**-specialization of $\mathbf{N}^s$-grading on $k[X]$. If this specialization leads to trivial grading on $k[X//U]$, then clearly, $q_i(X//U) = 0$, otherwise, it follows from [F, Satz 3.1], that $q_i(X//U) > 0$. The right hand unequality follows from [K, cor.3], applied to $\pi_U : X \to X//U$.

(ii) So far as $k[X]$ is connected, then applying [K, cor.4] to π_U we have: $\underline{q}(X//U) = \underline{q}(X) \Leftrightarrow \operatorname{codim}_X D > 1$, where D is defined in 2.4. Comparing with 2.4(ii), we obtain the assertion.

(iii) This directly follows from 2.8 and 2.4(iii). □

Remark. The proof of 2.10 may be obtained by applying [K] either to $\pi_U : X \to X//U$ or to $\pi_G : Z \to Z//G$.

(2.11) In conclusion of this chapter I formulate a conjecture, which has been observed in particular case in [P1, §3].
Let X be the same as in 2.10, $k[X]$ is **N**-graded, $(G : X)$. Let $(G : X^*)$ be the twisted action, defined in 1.4. The algebras $k[X \times X^*]$ and $k[X \times X^*]^G$ are $\mathbf{N}^2$-graded. Let (q, q^*) be the bi-degree of the Poincare series of $k[X \times X^*]^G$. Since $(X^*)^* = X$, we have $q = q^*$. Let $q(X//U) = \underline{q}(X//U)$ to be defined by (18).

Conjecture. $q = q^* = q(X//U)$.
It is not difficult to prove that if there is $x \in X$ with $\dim G_x = 0$, then $q = q^* = q(X)$. Therefore, comparing with 2.10(ii) we see the conjecture is true for "almost all" actions.

§3 U-invariants of affine double cones

(3.1) In this chapter we shall apply the preceding results to the decomposition problem of tensor products of irreducible representations of semisimple groups. Let $\lambda, \mu \in \mathcal{X}(T)_+$. We wish to solve the following

Problem 1. For any $n, m \in \mathbf{N}$ to find the decomposition of the representation $R(n\lambda) \otimes R(m\mu)$ into the irreducible ones.
The right approach to this problem looks as follows. Consider the affine variety $Z(\lambda, \mu) = C(\lambda^*) \times C(\mu^*)$. According to 1.2 $C(\lambda^*)$ is a cone in $R(\lambda^*)$ and $k[C(\lambda^*)]_n \cong R(n\lambda)$. Therefore, $Z(\lambda, \mu)$ is called a *double cone* and clearly,

$$k[Z(\lambda, \mu)]_{n,m} = R(n\lambda) \otimes R(m\mu) \tag{19}$$

Hence, the first problem is equivalent to

Problem 2. To find an explicit formula for the Poincare series of $\mathbf{N}^2 \times \mathcal{X}_+$-graded algebra $k[Z(\lambda,\mu)]^U$.

The second problem in its turn, is closely related with

Problem 3. To find the minimal generator system and relations in $k[Z(\lambda,\mu)]^U$. If $k[Z(\lambda,\mu)]^U$ appears to be a complete intersection, then the solution of the last problem allows us to resolve also the previous ones. At least, the solution of the problem 3 is sufficient for a weakened version of the problem 1, when we are interested only in representations appearing in tensor product decompositions, but not in their multiplicities.

(3.2) Trying to solve the third problem it is natural , at first, to apply the restriction theorem to a double cone. In order to make it, it is necessary to know how to do 2 points:

(a) to find s.g.p. S and its canonical embedding in G,

(b) to find a principal component $Y \subset Z(\lambda,\mu)^S$.

The solution of part (a) is contained in [P4]. It has been shown there, that finding of S is reduced to the search of s.g.p. of a special linear action of the form $(H : V + V^*)$; besides that in [P4] the canonical embedding of S^0 in G has been indicated for pairs of the *fundamental* weights. The solution of part (b) will be presented in 3.3. We shall show the principal component Y is a double cone of the group $K \cong N_G(S)^0/S^0$. Therefore, the problem 1 for G is reduced to the one for K.

Let $v_{-\lambda}$ be a lowest weight vector in $R(\lambda^*)$ relative to U. Let $f_\lambda \in k[R(\lambda^*)]_1^U$ be the single U-invariant of the weight λ and degree 1. Clearly, $f_\lambda(v_{-\lambda}) \neq 0$. So far as $v_{-\lambda} \in C(\lambda^*)$, we have

$$0 \neq f_\lambda \mid_{C(\lambda^*)} = \bar{f}_\lambda \in k[C(\lambda^*)]_1^U \tag{20}$$

(3.3) Theorem. Let G be a connected semisimple group and $X = Z(\lambda,\mu)$. If S be the canonical s.g.p. for action $(G : X \times X^*)$, then $\overline{Kv_{-\lambda}} \times \overline{Kv_{-\mu}}$ is the single principal irreducible component of X^S. In particular,

$$k[Z(\lambda,\mu)]^U \cong k[\overline{Kv_{-\lambda}} \times \overline{Kv_{-\mu}}]^{U(K)}_{T(X)}.$$

Proof. Recall, that $N_G(S)^0 = KS^0$. Clearly, $X^S = C(\lambda^*)^S \times C(\mu^*)^S$. Let us investigate the irreducible components of $C(\lambda^*)^S$. It is well-known, that

$$C(\lambda^*) = \mathcal{O}(\lambda^*) \cup \{0\} \tag{21}$$

where $\mathcal{O}(\lambda^*)$ is the G-orbit of the highest (or, equivalently, lowest) weight vectors. Since $\mathcal{O}(\lambda^*)$ is smooth and S is reductive, we have $\mathcal{O}(\lambda^*)^S$ is smooth also [Lu 1]. Therefore, if $y \in \mathcal{O}(\lambda^*)^S$, then

$$T_y(\mathcal{O}(\lambda^*)^S) = (\mathcal{G}/\mathcal{G}_y)^S = \mathcal{G}^S/\mathcal{G}_y^S = T_y(N_G(S)^0 y).$$

That is, the orbit $N_G(S)^0 y = Ky$ is dense in some irreducible component of $\mathcal{O}(\lambda^*)^S$. So far as it is true for any $y \in \mathcal{O}(\lambda^*)^S$, each irreducible component of $\mathcal{O}(\lambda^*)^S$ is a K-orbit. Let $\mathcal{O}(\lambda^*)^S = \bigsqcup_i \mathcal{O}(\lambda^*)_i^S$. It follows from (21) that for every i

$$\overline{\mathcal{O}(\lambda^*)_i^S} = \mathcal{O}(\lambda^*)_i^S \cup \{0\} \tag{22}$$

Therefore, every irreducible componet of $\mathcal{O}(\lambda^*)^S$ is the closure of the K-orbit of highest weight vectors in an irreducible K-module. Thus, all irreducible components of X^S are double cones of K. It remains to pick out the principal ones.

It follows from (20), that $\lambda, \mu \in \Gamma(X)$. Therefore, by (3) and (4), if S is a canonical s.g.p., then $S \subset G_{v_{-\lambda}}$, $S \subset G_{v_{-\mu}}$, i.e. $v_{-\lambda} \in C(\lambda^*)^S$, $v_{-\mu} \in C(\mu^*)^S$. Thus, $\overline{Kv_{-\lambda}} \times \overline{Kv_{-\mu}}$ is an irreducible component of X^S. Now we shall show, that the other irreducible components can not be principal. Let $Kv_{-\lambda} = \mathcal{O}(\lambda^*)_1^S$, $Kv_{-\mu} = \mathcal{O}(\mu^*)_1^S$. It follows from (22), that for every i subspace $\langle \mathcal{O}(\lambda^*)_1^S \rangle \subset R(\lambda^*)$ is an irreducible K-module. Therefore, $\langle \mathcal{O}(\lambda^*)_1^S \rangle \cap \langle Kv_{-\lambda} \rangle = \{0\}$ if $i > 1$. So far as $f_\lambda(v) = 0$ for any weight vector $v \in R(\lambda^*)$, which is distinct from $v_{-\lambda}$ and K is a regular subgroup of G, then it follows from (20), that $\bar{f}_\lambda \mid_{\mathcal{O}(\lambda^*)_i^S} \equiv \{0\}$ if $i > 1$. Since the restriction map have to be injective on U-invariants for principal components, we see the only possibility for principal component is $\overline{Kv_{-\lambda}} \times \overline{Kv_{-\mu}}$. □

(3.4) At the rest of the paper we shall use the *multiplicative* notation for irreducible representations in tensor products, i.e. we shall write $2\lambda\mu^2 + \nu^3$ instead of $2R(\lambda + 2\mu) + R(3\nu)$.

Now we shall elaborate a concrete example. Let $G = \mathrm{E}_7$ and $\tilde{\varphi}_i, \tilde{\alpha}_i$, $i = 1, \dots, 7$ - be the fundamental weights and the simple roots with the numeration from [VO]. It is always $\mu = \mu^*$ for E_7.

Let $X = Z(\tilde{\varphi}_1, \tilde{\varphi}_2)$. It has been shown in [P4, 2.8] that here $S^0 = \mathrm{A}_1$ and the simple root $\tilde{\alpha}_4$ is the positive root of A_1 under the canonical embedding. Furthermore, applying [P4, Thm.3(v)] one can check that $S = S^0$. Therefore, it follows from (3) that semigroup $\mathcal{T}(X)$ is generated by

$\tilde\varphi_1, \tilde\varphi_2, \tilde\varphi_3, \tilde\varphi_5, \tilde\varphi_6, \tilde\varphi_7$. Let us find K. By 1.5 the root system of K is $\Sigma(\mathrm{E}_7) \cap \langle \mathcal{T}(X) \rangle$. A straightforward calculations show this is the root system D_6, i.e. $K = \mathrm{D}_6$. More exactly, let α_i, φ_i, $i = 1, \dots, 6$ - be the simple roots and the fundamental weights of D_6. Then $\alpha_1 = \tilde\alpha_4 + \tilde\alpha_5 + \tilde\alpha_6$, $\alpha_2 = \tilde\alpha_6$, $\alpha_3 = \tilde\alpha_3 + \tilde\alpha_4 + \tilde\alpha_5$, $\alpha_4 = \tilde\alpha_2$, $\alpha_5 = \tilde\alpha_3 + \tilde\alpha_4 + \tilde\alpha_7$, $\alpha_6 = \tilde\alpha_1$. By using the Cartan matrices and their inverses [VO, table 2] it is possible to write $\{\tilde\varphi_i\}$ through $\{\tilde\alpha_i\}$ and $\{\alpha_i\}$ through $\{\varphi_i\}$. As a result we shall find the expressions of $\{\tilde\varphi_i\}$ through $\{\varphi_i\}$. Here is it:

$$\tilde\varphi_1 = \varphi_6,\ \tilde\varphi_2 = \varphi_4,\ \tilde\varphi_3 = \varphi_3 + \varphi_5,\ \tilde\varphi_5 = \varphi_1 + \varphi_3,\ \tilde\varphi_6 = \varphi_2,\ \tilde\varphi_7 = \varphi_1 + \varphi_5 \quad (23)$$

Therefore $\overline{Kv_{-\tilde\varphi_1}} = C(\varphi_6^*) = C(\varphi_6)$, $\overline{Kv_{-\tilde\varphi_2}} = C(\varphi_4^*) = C(\varphi_4)$. Thus, according to the theorem 3.3 we obtain the natural isomorphism

$$k[Z(\tilde\varphi_1, \tilde\varphi_2)]^{U(\mathrm{E}_7)} = k[Z(\varphi_6, \varphi_4)]_{\mathcal{T}}^{U(\mathrm{D}_6)} \quad (24)$$

where $\mathcal{T}$ is generated by φ_6, φ_4, $\varphi_3 + \varphi_5$, $\varphi_1 + \varphi_3$, φ_2, $\varphi_1 + \varphi_5$.

(3.5) Our problem is reduced to the description of $\mathbf{N}^2 \times \mathcal{X}(\mathrm{D}_6)_+$-graded algebra $A := k[Z(\varphi_6, \varphi_4)]^{U(\mathrm{D}_6)}$. If $(n, m, \mu) \in \mathbf{N}^2 \times \mathcal{X}(\mathrm{D}_6)_+$, then clearly

$$A_{n,m,\mu} \neq 0 \Longleftrightarrow \tilde\varphi_6^n \otimes \tilde\varphi_4^m \supset \mu \quad (25)$$

For $a \in A_{n,m,\mu}$ we shall say, that a is of the *degree* (n, m) and of the *weight* μ. If $\dim A_{n,m,\mu} = 1$, then $(\overline{n, m, \mu})$ denote a non-zero vector of this subspace. By using [VO, table 5] we find:

$$\varphi_6 \otimes \varphi_4 = \varphi_4\varphi_6 + \varphi_2\varphi_6 + \varphi_6 + \varphi_3\varphi_5 + \varphi_1\varphi_5$$

$$\varphi_6^2 \otimes \varphi_4 = \varphi_6^2\varphi_4 + \varphi_2\varphi_6^2 + \varphi_6^2 + \varphi_3\varphi_5\varphi_6 + \varphi_1\varphi_5\varphi_6 + \varphi_4 + \varphi_2\varphi_4 + \varphi_1\varphi_3 + \varphi_2$$

This formulas give us that the functions

$$(\overline{1,0,\varphi_6}), (\overline{0,1,\varphi_4}), (\overline{1,1,\varphi_2\varphi_6}), (\overline{1,1,\varphi_6}), (\overline{1,1,\varphi_3\varphi_5}),$$

$$(\overline{1,1,\varphi_1\varphi_5}), (\overline{2,1,\varphi_4}), (\overline{2,1,\varphi_2\varphi_4}), (\overline{2,1,\varphi_1\varphi_3}), (\overline{2,1,\varphi_2}) \quad (26)$$

are contained in the minimal generator system of A.

Comparing the dimensions we are convinced, that there are now new generators in degrees $(1, 2)$ and $(2, 2)$. Hence, the other generators of A (if they do exist) must have the degrees $> (2, 2)$.

(3.6) Let B be the homogeneous subalgebra in A, generated by the functions (26). Our goal is to prove that $A = B$. First of all we explicitly formulate the known facts on A.

(α) By 1.2(iii) $k[Z(\varphi_6, \varphi_4)]$ is factorial and therefore A also is factorial.

(β) It follows from [P2, §3] that $q(C(\varphi_6)) = 10$, $q(C(\varphi_4)) = 7$. Therefore $\overline{q}(A) \leq (10, 7)$.

(γ) There is a natural action of 8-dimensional torus $T_8 = T(\mathrm{D}_6) \times (k^*)^2$ on Spec$A$. (Here $(k^*)^2$ is the torus, defining the $\mathbf{N}^2$-grading on $k[Z(\varphi_6, \varphi_4)]$.) It has been shown in [P4, ch.2] that $\tilde{c} := \mathrm{trdeg}(QA)^{T_8} = 1$. Considering that $Z(\varphi_6, \varphi_4)$ is a rational variety, we see that $(QA)^{T_8}$ is a rational function field.

(δ) So far as A is factorial and list (26) contains all the generators of degree $\leq (2, 2)$, then the functions from (26) are prime elements of A.

(3.7) Now we pass to the consideration of B. Clearly B is a T_8-invariant subalgebra of A. At first we show that first 9 generators from (26) are algebraically independent. The existence of an algebraic dependence means, that there is a linear combination of monomials, which is equal identitically to zero. That is, it is necessary to find 2 different groups of monomials, in such a way that their products lie in the same homogeneous subspace of B. A straightforward calculation show that all relations of such sort should be a consequenses of the following one: $(\overline{1,1,\varphi_2\varphi_6})(\overline{2,1,\varphi_4}) = \xi(\overline{1,0,\varphi_6})(\overline{0,1,\varphi_4})(\overline{2,1,\varphi_2})$, $\xi \in k^*$. But existence of the relation of this form contradicts to factoriality of A and 3.6(δ). Hence dim$B = 9$ and there is a single relation, connecting all the functions (26). The monomials $\bar{p} = (\overline{1,1,\varphi_2\varphi_6})(\overline{2,1,\varphi_4})$, $\bar{q} = (\overline{1,0,\varphi_6})(\overline{0,1,\varphi_4})(\overline{2,1,\varphi_2})$, $\bar{r} = (\overline{1,1,\varphi_6})(\overline{2,1,\varphi_2\varphi_4})$ lie in $B_{3,2,\varphi_2\varphi_4\varphi_6}$. If we assume that dim$B_{3,2,\varphi_2\varphi_4\varphi_6} = 3$, then the direct calculation shows dim$B_{3,2} >$ dim$A_{3,2} = \dim(\varphi_6^3 \otimes \varphi_4^2)$. It is impossible! The conditions dim$B_{3,2,\varphi_2\varphi_4\varphi_6} = 1$ contradicts to 3.6(δ). Hence, the dimension under consideration is equal 2 and the single relation is of the form

$$\bar{p} + \xi_0\bar{q} + \tau_0\bar{r} = 0, \quad \xi_0, \tau_0 \in k^* \tag{27}$$

(3.8) It follows from (27) that $const \neq \bar{p}/\bar{q} \in (QB)^{T_8} \subset (QA)^{T_8}$. Since $\bar{p}, \bar{q}$ are monomials on prime elements of A, A is factorial and $\tilde{c} = 1$, then it is easy to prove that $\bar{p}/\bar{q}$ generates $(QA)^{T_8}$. In particular, $(QA)^{T_8} = (QB)^{T_8}$. Let us observe, $\bar{p} + \xi\bar{q}$, $\xi \neq \xi_0$ is undecomposable element in A. Actually, it is undecomposable in B, but this element is of degree $(3, 2)$ and $A_{n,m} = B_{n,m}$ if $(n, m) \leq (2, 2)$. Therefore the undecomposability in B implies that in A.

(3.9) Assume $B \neq A$. Then an additional generator should to be in A, except the ones in (26). Let $d \in A_{n,m,\omega}$ be an extra-generator. The values n, m, ω can not be arbitrary. It follows from (25) that $n\varphi_6 + m\varphi_4 - \omega$ have to lie in the root lattice of D_6. Let $M \subset \mathbf{Z}^8 = \mathbf{Z}^2 \times \mathcal{X}(T(D_6))$ be the free abelian group, generated by the weights and the degrees of the functions (26) (i.e. M is generated by $(1,0,\varphi_6), \ldots, (2,1,\varphi_2)$). A straightforward calculation give us that if $n\varphi_6 + m\varphi_4 - \omega$ lies in the root lattice, then $(n, m, \omega) \in M$. This means, that there exist monomials m_1, m_2 of functions (26), such that dm_1 and m_2 lie in the same homogeneous subspace of A. Hence, $dm_1/m_2 \in (QA)^{T_6}$, i.e. $dm_1/m_2 \in k(\bar{p}/\bar{q})$. After a simple transformation this leads to a relation of the form

$$dm_1 \prod_i (\bar{p} - \gamma_i \bar{q}) = m_2 \prod_j (\bar{p} - \delta_j \bar{q}) \bar{q}^t, \; \gamma_i, \delta_j \in k, \; t \in \mathbf{Z}$$

So far as all undecomposable factors on the right-hand side lie in B, then condition $d \notin B$ contradicts to the factoriality of A. Hence $A = B$. Thus, the following result is proved.

(3.10) Theorem. The algebra $A = k[Z(\varphi_6, \varphi_4)]^{U(D_6)}$ is generated by the functions (26). This is a hypersurface in $\mathbf{A}^{10}$, defined by the equation (27). The Poincare series $F_A(z_1, z_2, t_1, \ldots, t_6)$ of A is of the form:

numerator: $1 - z_1^2 z_2 t_2 t_4 t_6$,

denominator: $(1 - z_1 t_6)(1 - z_2 t_4)(1 - z_1 z_2 t_2 t_6)(1 - z_1 z_2 t_6)(1 - z_1 z_2 t_3 t_5)(1 - z_1 z_2 t_1 t_5)(1 - z_1^2 z_2 t_4)(1 - z_1^2 z_2 t_2 t_4)(1 - z_1^2 z_2 t_1 t_3)(1 - z_1^2 z_2 t_2)$. □

This formulas show us that $\underline{q}(A) = (10, 7) = \underline{q}(Z(\varphi_6, \varphi_4))$ and all $b_i = 2$. The coefficient in $z_1^n z_2^m$ in the Taylor expansion of F_A give us the decomposition of $\varphi_6^n \otimes \varphi_4^m$.

(3.11) It is turns out, a posteriori, that the weights of all generators of A lie in $\mathcal{T}$ (see 3.4). Therefore it follows from (23),(24)

Theorem. Algebra $k[Z(\tilde{\varphi}_1, \tilde{\varphi}_2)]^{U(E_7)}$ is naturally isomorphic to A. Weight correspondences for the generators are given by (23). The single relation between the functions from the minimal generator system is of multi-degree $(3, 2, \tilde{\varphi}_1 \tilde{\varphi}_2 \tilde{\varphi}_6)$. □

In this case $b_1 = b_2 = b_6 = 2$, $b_3 = b_5 = b_7 = 1$, $b_4 = 0$ and $\underline{q}(Z(\tilde{\varphi}_1, \tilde{\varphi}_2)//U(E_7)) = (10, 7)$. It follows from [P2, 3.7], that $\underline{q}(Z(\tilde{\varphi}_1, \tilde{\varphi}_2)) = (18, 13)$.

Remark. The author takes this opportunity to make some corrections to [P2]. In [P2, 3.7] the right values for E_7 are $a_2 = 13, a_7 = 14$.

(3.12) The example elaborated in 3.4-3.11 has the following origin. In [P4] all the pairs of the fundamental weights of simple algebraic groups with the property $\tilde{c}(Z(\varphi_i, \varphi_j)) = 1$ has been found. The list of this weights is rather short:

- 2 serial cases $(\mathrm{B}_m;\ \varphi_2, \varphi_m)$, $(\mathrm{C}_m;\ \varphi_2, \varphi_m)$, $m \geq 3$;
- 2 sporadic cases $(\mathrm{D}_6;\ \varphi_4, \varphi_6)$, $(\mathrm{E}_7;\ \varphi_1, \varphi_2)$.

If $\tilde{c}(Z(\varphi_i, \varphi_j)) = 0$, then $k[Z(\varphi_i, \varphi_j)]^{U(G)}$ is a polynomial algebra and all these cases were described in [Li]. Thus, we have done the following step: the generators and the relations for sporadic cases with $\tilde{c} = 1$ are found.

(3.13) The same method as before, allows us to determine the generators and the relations for the serial cases with $\tilde{c} = 1$. The results obtained are gathered in the table. It is found, a posteriori, that everywhere in the table $k[Z(\varphi_i, \varphi_j)]^{U(G)}$ is a hypersurface and the single relation is of the form $\bar{p} + \bar{q} + \bar{r} = 0$, where $\bar{p}, \bar{q}, \bar{r}$ are all monomials of generators, having the degree and the weight, indicated in the column "Relation".

Table

Group	Weights	Generators	Relation
C_m	(φ_2, φ_m) $m \geq 4$	$(1,0,\varphi_2), (0,1,\varphi_m), (1,1,\varphi_{m-2}),$ $(1,1,\varphi_1\varphi_{m-1}), (2,1,\varphi_1\varphi_{m-1}),$ $(2,1,\varphi_m), (2,1,\varphi_1^2\varphi_m), (2,2,\varphi_{m-1}^2)$	$(4,3,\varphi_1^2\varphi_{m-1}^2\varphi_m)$
	$m = 3$	$(1,0,\varphi_2), (0,1,\varphi_3), (1,1,\varphi_1),$ $(1,1,\varphi_1\varphi_2), (2,1,\varphi_3),$ $(2,1,\varphi_1^2\varphi_3), (2,2,\varphi_2^2)$	
B_m	(φ_2, φ_m) $m \geq 4$	$(1,0,\varphi_2), (0,1,\varphi_m), (1,1,\varphi_1\varphi_m),$ $(1,1,\varphi_m), (1,2,\varphi_1\varphi_{m-1}),$ $(1,2,\varphi_{m-1}), (1,2,\varphi_{m-2}), (2,2,\varphi_1\varphi_{m-1})$	$(2,3,\varphi_1\varphi_{m-1}\varphi_m)$
	$m = 3$	$(1,0,\varphi_2), (0,1,\varphi_3), (1,1,\varphi_1\varphi_3),$ $(1,1,\varphi_3), (1,2,\varphi_1),$ $(1,2,\varphi_2), (1,2,\varphi_1\varphi_2)$	

Let us give some hints on application of the restriction theorem (see 1.8) here. For C_m the cases $m = 3, 4$ are basic, because they have not non-trivial reduction for U-invariants ($S = \{e\}$ for $m = 3$ and $S = \{\text{center of } \mathrm{C}_m\}$ for $m = 4$). If $m \geq 5$, then $S = \mathbf{Z}_2 \times \mathrm{A}_{m-5}$. The canonical embedding $S^0 \subset G$ has been indicated in [P4]: the set of simple roots of S^0 is $\{\alpha_3, \dots, \alpha_{m-3}\}$. The embedding of $\mathbf{Z}_2$ looks as follows: if $\langle b \rangle = \mathbf{Z}_2$, $b \in T(\mathrm{C}_m)$, then $\varphi_1(b) =$

$\varphi_{m-1}(b) = -1$, $\varphi_i(b) = 1$, $i \neq 1, m-1$. Then one can compute without difficulties, that $K = \mathrm{C}_4 \times \mathrm{A}_1$, if $m = 5, 6$ and $K = \mathrm{C}_4$, if $m > 6$. As a matter of fact, in all these cases we have a reduction to the basic case $m = 4$. The situation for B_m is similar.

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