COMPACT HOMOGENEOUS CR MANIFOLDS

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Abstract. We classify all compact simply connected homogeneous CR manifolds M of codimension one and with non-degenerate Levi form up to CR equivalence. The classification is based on our previous results and on a description of the maximal connected compact group $G(M)$ of automorphisms of M. We characterize also the standard homogeneous CR manifolds as the homogeneous CR manifolds whose group $G(M)$ in not semisimple.

1. Introduction.

In a previous paper ([AS]), we classified all simply connected compact homogeneous CR manifolds $(M = G/L, \mathcal{D}, J)$ of a compact Lie group G up to a G-equivariant isomorphism. Here $(\mathcal{D}, J)$ is a G-invariant CR structure on the homogeneous manifold $M = G/L$, where $\mathcal{D}$ is a codimension one G-invariant distribution on M and J is a complex structure on $\mathcal{D}$, which satisfies the integrability condition (2.1) (see §2.1, below). Note that by the results in [AHR] and [Sp], any simply connected compact homogeneous CR manifold admits a compact transitive group of automorphisms and hence it can be represented in the above form.

In the present paper, we study when two such homogeneous CR manifolds $(M = G/L, \mathcal{D}, J)$ and $(M' = G'/L', \mathcal{D}', J')$ are CR equivalent, that is when there is a diffeomorphism $\phi : M \longrightarrow M'$ such that $\phi_*(\mathcal{D}) \subset \mathcal{D}'$ and $\phi_*(Jv) = J'\phi_*(v)$ for all $v \in \mathcal{D}$.

This question is reduced to the description of maximally compact connected group of automorphisms of a homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$. We give such description using a result by A. L. Onishchik about the maximal compact groups of holomorphic transformations of flag manifolds.

We shortly recall the main results in [AS] about the classification of compact simply connected homogeneous CR manifolds $(M = G/L, \mathcal{D}, J)$ of a compact Lie group G. Such manifolds are subdivided into three natural classes:

a) the *standard homogeneous CR manifolds*, that is homogeneous S^1-bundles over a flag manifold F, with the CR structure induced by an invariant complex structure on F;
b) the *Morimoto-Nagano spaces*, i.e. the sphere bundles $S(N) \subset TN$ of a compact rank one symmetric space $N = G/H$, with the CR structure induced by the natural complex structure of $TN = G^{\mathbb{C}}/H^{\mathbb{C}}$;

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c) the manifolds which admit a non-trivial holomorphic fibration over a flag manifold (F, J_F) with typical fiber $S(S^k)$, where $k = 2, 3, 5, 7, 9$ or 11, respectively; these manifolds are $SU_n/T^1 \cdot SU_{n-2}$, $SU_p \times SU_q/T^1 \cdot U_{p-2} \cdot U_{q-2}$, $SU_n/T^1 \cdot SU_2 \cdot SU_2 \cdot SU_{n-4}$, $SO_{10}/T^1 \cdot SO_6$, $E_6/T^1 \cdot SO_8$, $E_7/T^1 \cdot SO_{10}$.

In this last case, the invariant CR structure is determined by the invariant complex structure J_F on F and by an invariant CR structure on the typical fiber, which depends on one complex parameter.

The only intersection between the classes a), b) and c) is represented by the Morimoto-Nagano spaces $S(\mathbb{C}P^{n-1}) = SU_n/T^1 \cdot SU_{n-2}$, which admit a holomorphic fibration with fiber $S(S^2)$ onto a flag manifold.

First of all, we prove that a non-standard homogeneous CR manifold (i.e. a manifold from class b) or c)) is never CR equivalent to a standard CR manifold. Moreover two non-standard homogeneous CR manifolds $M = G/L$ and $M' = G'/L'$ are CR equivalent only if they are equivalent as homogeneous manifolds, that is only if there exists an isomorphism $\phi : G \to G'$ so that $\phi(L) = L'$.

Moreover, as we proved in [AS], the CR structures of a non-standard homogeneous CR manifold M, with a fixed underlying contact distribution, are naturally parameterized by the points of the unit disc $D \subset \mathbb{C}$. We show here that two CR structures corresponding to $t, t' \in D$ are CR equivalent if and only if $|t| = |t'|$.

Now, let $M = G/L$ be a standard CR manifold and $\pi : M = G/L \to F = G/K$ the associated holomorphic S^1-fibration over a flag manifold.

We prove that any maximal connected compact Lie group A of automorphisms of M, which contains G, preserves the holomorphic fibration $\pi : M \to F$. In particular A acts on the flag manifold F as a group of holomorphic transformations. Conversely, any maximal connected compact group A of holomorphic transformations of the flag manifold $F = G/K$, which contains G, acts on $M = G/L$ as a maximal compact, connected semisimple group of CR transformations.

Therefore the construction of a maximal compact semisimple group of CR transformations of $M = G/L$ reduces to the description of the maximal compact group of holomorphic transformations of the flag manifold F. This problem was solved by Onishchik in [On] (see Theorem 4.1). In particular, he discovered that there exist only few irreducible flag manifolds which admit two different transitive groups of holomorphic transformations, namely $\mathbb{C}P^{2\ell-1}$ $(\ell > 1)$, $Gr_2(\mathbb{R}^7)$ and $Com(\mathbb{R}^{2\ell+2})$ $(\ell > 2)$ (see Table 1).

Using Onishchik's result, we describe the maximal compact semisimple Lie group $A \supset G$ of a given standard CR manifold $M = G/L$ and we prove the following. Let $M = G/L$ and $M' = G'/L'$ be two standard CR manifolds and $M = A/B$ and $M' = A'/B'$ their representations as homogeneous spaces of the maximal compact semisimple Lie groups $A \supset G$ and $A' \supset G'$. Then M is CR equivalent to M' if and only if the homogenous manifolds A/B and A'/B' are equivalent and the associated flag manifolds $F = G/K = A/C$ and $F' = G'/K' = A'/C'$ are A-equivariantly biholomorphic.

In particular, we obtain that if the flag manifold $F = G/K$, associated with a standard CR manifold $M = G/L$, has no factor isomorphic to $\mathbb{C}P^{2\ell-1}$ $(\ell > 1)$, $Gr_2(\mathbb{R}^7)$ or $Com(\mathbb{R}^{2\ell+2})$ $(\ell > 2)$, then G is a maximal connected compact semisimple group of CR transformations of M.

We also prove that a maximal connected compact group A of CR transformations of a compact homogeneous CR manifold M is semisimple if and only if M is non-standard. For a standard CR manifold M, the group A has a 1-dimensional center, which acts trivially on the associated flag manifold. This gives a group characterization of standard CR manifolds.

As a final remark, we would like to stress the fact that, if the Levi form of a compact CR manifold M is positive definite, then the full group of CR transformations is non-compact if and only if $M = S^{2n+1}$ ([We]). We do not know which of the classified compact homogeneous CR manifolds (with indefinite Levi form) has non-compact full group of automorphisms.

Note that in [Ya], Yamaguchi classified homogeneous Levi non-degenerate CR manifolds with sufficiently large group of automorphisms. These manifolds are either compact quadrics or quadrics with some points deleted and all of them have a non-compact group of automorphism.

Also, Fu, Isaev and Krantz ([FIK]) and Zwonek ([Zw]) found examples of non-homogeneous compact CR manifolds of codimension one with non-compact group of automorphisms. In these examples the Levi form is indefinite and degenerates at some points.

2. Preliminaries.

2.1 First definitions.

A CR structure on a manifold M is a pair $(\mathcal{D}, J)$, where $\mathcal{D} \subset TM$ is a distribution on M with a complex structure J, that is a field of endomorphisms $J \in \operatorname{End}\mathcal{D}$, with $J^2 = -1$. A CR structure $(\mathcal{D}, J)$ is called *integrable* if J satisfies the following integrability condition:

$$J([JX,Y] + [X,JY]) \in \mathcal{D} \ ,$$
$$[JX,JY] - [X,Y] - J([JX,Y] + [X,JY]) = 0 \tag{2.1}$$

for any pair of vector fields X, Y in $\mathcal{D}$.

Geometrically this means that the eigendistributions $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$ and $\mathcal{D}^{01} \subset T^{\mathbb{C}}M$ of J, given by the J-eigenspaces in $\mathcal{D}^{\mathbb{C}}$ corresponding to the eigenvalues i and $-i$, are involutive, i.e. the space of their local sections is closed under Lie brackets.

The *codimension* of a CR structure $(\mathcal{D}, J)$ is defined as the codimension of the distribution $\mathcal{D}$. An integrable codimension one CR structure $(\mathcal{D}, J)$ is often called *CR structure of hypersurface type*, because a real hypersurface M of a complex manifold N carries such CR structure.

For a CR structure of hypersurface type, the distribution $\mathcal{D}$ can be locally described as the kernel of a 1-form θ. Such form θ determines an Hermitian metric

$$\mathcal{L}^{\theta}_q \colon \mathcal{D}_q \times \mathcal{D}_q \to \mathbb{R}$$

by the formula

$$\mathcal{L}^{\theta}(v,w) = (d\theta)(v, Jw)$$

for any $v, w \in \mathcal{D}$. This form is called the *Levi form of* $(M, \mathcal{D}, J)$ *associated with the form* θ. Notice that the 1-form θ is defined up to multiplication by a function f everywhere different from zero and that $\mathcal{L}^{f\theta} = f\mathcal{L}^{\theta}$. In particular, the conformal class of a Levi form depends only on the CR structure.

A CR structure $(\mathcal{D}, J)$ of hypersurface type is called *Levi non-degenerate* if it has non-degenerate Levi form or, in other words, *if $\mathcal{D}$ is a contact distribution*. In this case we will call $\mathcal{D}$ *the contact distribution underlying the non-degenerate contact structure* $(\mathcal{D}, J)$.

Let $(M, \mathcal{D})$ and $(M', \mathcal{D}')$ be two contact manifolds with contact distributions $\mathcal{D}$ and $\mathcal{D}'$, respectively. A smooth map $\varphi\colon M \to M'$ is called *contact map* if $\varphi_*(\mathcal{D}) \subset \mathcal{D}'$.

A smooth map $\varphi\colon M \to M'$ of a CR manifold $(M, \mathcal{D}, J)$ into some other CR manifold $(M', \mathcal{D}', J')$ is called *holomorphic map* or *CR map* if

a) $\varphi_*(\mathcal{D}) \subset \mathcal{D}'$;

b) $\varphi_*(Jv) = J'\varphi_*(v)$ for all $v \in \mathcal{D}$.

In particular, we define a *CR transformation* of a CR manifold $(M, \mathcal{D}, J)$ (resp. *contact transformation*) as a transformation $\varphi : M \to M$ such that φ and φ^{-1} are both CR maps (resp. *contact map*). It is known that the group $Aut(M, \mathcal{D}, J)$ of all CR transformations of a Levi non-degenerate CR manifold is a Lie group.

If the opposite is not stated, by CR manifold we will mean *a simply connected Levi non-degenerate CR manifold*.

We will also adopt the following notation. The symbols

$$A = A(M, \mathcal{D}, J), \qquad A^{ss} = A^{ss}(M, \mathcal{D}, J)$$

denote a maximal connected compact subgroup and a maximal connected compact semisimple subgroup of $\mathrm{Aut}(M, \mathcal{D}, J)$, respectively. Recall that any two maximal connected compact subgroups (resp. maximal connected compact semisimple subgroups) are conjugated by an element of $\mathrm{Aut}(M, \mathcal{D}, J)$.

The Lie algebra of a Lie group is always denoted by the corresponding gothic letter. For a subset B of a Lie group G or of a Lie algebra $\mathfrak{g}$, we denote by $C_G(B)$ and $C_{\mathfrak{g}}(B)$ its centralizer in G and $\mathfrak{g}$, respectively.

$Z(G)$ and $Z(\mathfrak{g})$ denote the center of a Lie group G and of a Lie algebra $\mathfrak{g}$, respectively.

For any compact Lie group G and the corresponding Lie algebra $\mathfrak{g}$, the expressions $G = G^{ss} \cdot Z(G)$ and $\mathfrak{g} = \mathfrak{g}^{ss} + Z(\mathfrak{g})$ denote the decomposition into semisimple part plus center of G and $\mathfrak{g}$, respectively.

For a compact Lie group G, we will denote by $\mathcal{B}$ an $\mathrm{Ad}(G)$-invariant scalar product on the Lie algebra $\mathfrak{g}$. For example, if G is simple, $\mathcal{B}$ is a multiple of the Cartan-Killing form of $\mathfrak{g}$. Throughout the paper, any orthogonal decomposition of the Lie algebra $\mathfrak{g}$ has to be understood as orthogonal with respect to the inner product $\mathcal{B}$.

By a homogeneous manifold $M = G/L$, we mean a simply connected homogeneous manifold of a connected Lie group G, which acts almost effectively on M (i.e. with discrete kernel of non-effectivity). It follows that the stability subgroup L is connected.

2.2 First properties of homogeneous CR manifolds of compact Lie groups.

In this section, we recall some elementary facts about infinitesimal description of contact and CR homogeneous manifolds.

Let $M = G/L$ be a homogeneous manifold of a connected compact Lie group G and $\mathfrak{g} = \mathfrak{l} + \mathfrak{l}^\perp$ the associated $\mathcal{B}$-orthogonal decomposition of $\mathfrak{g}$. Recall that $\mathfrak{l}^\perp$ is naturally identified with $T_{eL}G/L$.

Now, let $\mathcal{D} \subset TM = T(G/L)$ be a G-invariant contact distribution and θ a G-invariant contact form, i.e. a 1-form such that $\theta(X) = 0$ for any $X \in \mathcal{D}$. The *Reeb field associated with* θ is the unique vector field ξ^θ on $M = G/L$ such that

$$\theta_p(\xi^\theta) = 1\ , \qquad d\theta_p(\xi^\theta, *)|_{\mathcal{D}} = 0\ , \qquad \forall p \in M\ .$$

By identifying $T_{eL}M$ with $\mathfrak{l}^\perp$, we get a natural Ad_L-invariant decomposition

$$\mathfrak{l}^\perp = \mathbb{R}Z + \mathfrak{m}$$

where $\mathbb{R}Z$ is the 1-dimensional subspace corresponding to $\mathbb{R}\xi^\theta_{eL} \subset T_{eL}M$ and $\mathfrak{m}$ is the codimension one subspace corresponding to the subspace $\mathcal{D}_{eL} \subset T_{eL}M$.

One can check that the decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \tag{2.1}$$

is $\mathcal{B}$-orthogonal and that the element $Z \in \mathfrak{l}^\perp$, defined up to a scaling, generates a closed 1-parametric subgroup of G and has a centralizer $\mathfrak{k} = C_\mathfrak{g}(Z)$, which is equal to $\mathfrak{l} \oplus \mathbb{R}Z$ (see [AS]).

Any element $Z \in \mathfrak{l}^\perp$, which generates a closed 1-parametric subgroup and such that $C_\mathfrak{g}(Z) = \mathfrak{l} + \mathbb{R}Z$, is called *contact element of* G/L. The formula (2.1) establishes a 1-1 correspondence between contact elements Z up to scaling and G-invariant contact distributions $\mathcal{D}$, $\mathcal{D}_{eL} = \mathfrak{m}$.

The adjoint orbit

$$F_Z = \mathrm{Ad}_G\, Z = G/K\ , \qquad K = C_G(Z)$$

is a flag manifold (i.e. a homogeneous manifold of a compact semisimple Lie group G, which is G-isomorphic to an adjoint orbit of G) and it is called *the flag manifold G-associated with the contact manifold* $(M = G/L, \mathcal{D}_Z)$. The S^1-fibration

$$M = G/L \to F_Z = G/K$$

is called *the structural G-fibration* of $(M = G/L, \mathcal{D}_Z)$.

The reader should be aware that, if $\mathcal{B}$ and $\mathcal{B}'$ are two Ad_G-invariant scalar products on $\mathfrak{g}$, then the corresponding contact elements $Z \in \mathfrak{l}^\perp$ and $Z' \in \mathfrak{l}^{\perp\prime}$ associated with a given Reeb vector field ξ^θ, are in general different. Nevertheless, they verify $Z' = Z \mod Z(\mathfrak{l})$. Therefore

$$\mathfrak{k} = C_\mathfrak{g}(Z) = \mathfrak{l} + \mathbb{R}Z = \mathfrak{l} + \mathbb{R}Z' = C_\mathfrak{g}(Z') = \mathfrak{k}'\ .$$

This means that the G-associated flag manifold $F_Z = F'_Z$ is independent on the choice of the invariant inner product $\mathcal{B}$.

In case $G = A^{ss}(M, \mathcal{D}, J)$ is a maximal compact semisimple group of CR transformations of a homogeneous CR manifold, we call F_Z *the flag manifold naturally* *associated with* $(M, \mathcal{D}, J)$ and the structural G-fibration $\pi : M \to F_Z$ is called *the structural fibration associated with* $\mathcal{D}$.

Now, let us choose a flag manifold $F = G/K$. In [AS] it was proved that any homogeneous contact manifold $(M = G/L, \mathcal{D}_Z)$, which have F as G-associated flag manifold, is obtained as follows.

An element $Z \in Z(\mathfrak{k})$ is called $\mathfrak{k}$-regular if:

a) Z generate a closed 1-parametric subgroup of G;
b) $C_{\mathfrak{g}}(Z) = \mathfrak{k}$.

If Z is a $\mathfrak{k}$-regular element, the subalgebra $\mathfrak{l}_Z = \mathfrak{k} \cap (Z)^{\perp}$ generates a closed connected subgroup $L_Z \subset G$. Moreover:

Proposition 2.1. *[AS] Let $F = G/K$ be a flag manifold of a connected, compact semisimple Lie group G. There exists a natural 1-1 correspondence*

$$Z \quad \Leftrightarrow \quad (G/L_Z, \mathcal{D}_Z)$$

between $\mathfrak{k}$-regular elements $Z \in \mathfrak{g}$ (determined up to a scaling) and homogeneous contact manifolds $(G/L, \mathcal{D})$, which have $F = G/K$ as G-associated flag manifold.

For a given flag manifold G/K and a $\mathfrak{k}$-regular element $Z \in \mathfrak{g}$, we say that $(G/L_Z, \mathcal{D}_Z)$ is *the contact homogeneous manifold associated with the pair* $(G/K, Z)$.

Any G-invariant CR structure $(\mathcal{D}_Z, J)$ on G/L, with underlying contact distribution $\mathcal{D}_Z$ and with associated decomposition (2.1), is uniquely associated with a complex subspace $\mathfrak{m}^{10}$ of $\mathfrak{m}^{\mathbb{C}}$ such that

i) $\mathfrak{m}^{10} \cap \overline{\mathfrak{m}^{10}} = \{0\}$ and $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \overline{\mathfrak{m}^{10}}$;
ii) $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

In fact, the decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01} = \mathfrak{m}^{10} + \overline{\mathfrak{m}^{10}}$ defines a complex structure $J : \mathfrak{m} \to \mathfrak{m}$ which corresponds to the complex structure $J : \mathcal{D} \to \mathcal{D}$ of the distribution $\mathcal{D}$.

We call $\mathfrak{m}^{10}$ *the holomorphic subspace G-associated with* $(\mathcal{D}, J)$, since the associated complex sub-distribution $T^{10}M \subset \mathcal{D}^{\mathbb{C}}$ is the eigenspace distribution of J with eigenvalue i.

Note that if we denote by $\mathfrak{k} = C_{\mathfrak{g}}(Z)$ and if the subspace $\mathfrak{m}^{10}$ is $\mathrm{ad}_{\mathfrak{k}}$-invariant, then $\mathfrak{m}^{10}$ defines also an invariant complex structure J_F on the flag manifold $F_Z = G/K$, since we may identify $T^{10}_o F = \mathfrak{m}^{10}$ where $o = eK$ (see [AS] for more details).

Definition 2.2. Let $(M = G/L, \mathcal{D}, J)$ be a homogeneous CR manifold of a connected compact semisimple Lie group G with contact element $Z \in \mathfrak{g}$ and holomorphic subspace $\mathfrak{m}^{10} \subset \mathfrak{g}^{\mathbb{C}}$.

We say that the CR structure on M is *G-standard* if

$$[Z, \mathfrak{m}^{10}] \subset \mathfrak{m}^{10} .$$

If $G = A^{ss}(M, \mathcal{D}, J)$, a G-standard CR structure will be called *standard*.

Note that a CR structure $(\mathcal{D}, J)$ is G-standard if and only if the holomorphic subspace $\mathfrak{m}^{10}$ is Ad_K-invariant, where $K = C_G(Z_{\mathfrak{g}})$. In this case we will denote the invariant complex structure defined by the subspace $\mathfrak{m}^{10}$ on F be J_F, and we will consider the associated flag manifold $F = G/K$ as a complex homogeneous manifold with complex structure J_F.

Then the canonical fibration

$$\pi : M = G/L \to F = G/K$$

is a holomorphic fibration with respect to the CR structure $(\mathcal{D}, J)$ on M and the complex structure J_F (which may be considered as a codimension zero CR structure). This is a characteristic property of standard CR structures (see [AS]).

2.3 Compact homogeneous CR manifolds as homogeneous manifolds of compact semisimple Lie groups.

It is known that if $(M, \mathcal{D}, J)$ is compact and homogeneous, then any maximal compact subgroup $A(M, \mathcal{D}, I) \subset \mathrm{Aut}(M, \mathcal{D}, I)$ acts transitively on M (see Thm. 1.1 [Sp]; see also [AHR]). This result together with Cor. 3.2 in [AS] can be used to prove the following more precise result.

Proposition 2.3. *Let $(M, \mathcal{D}, J)$ be a compact homogeneous CR manifold. Then a maximal connected compact semisimple subgroup $A^{ss}(M, \mathcal{D}, J)$ acts transitively on M.*

Proof. By the above remarks, we may represent the manifold M as the homogeneous manifold $M = G/L$, where $G = A(M, \mathcal{D}, J)$ is a maximal connected compact subgroup of $\mathrm{Aut}(M, \mathcal{D}, J)$. By Corollary 3.2 in [AS], if G is not semisimple, $G = G^{ss} \cdot Z(G)$ where the center $Z(G)$ has dimension one. Moreover, $\dim(\mathfrak{l} \cap \mathfrak{g}^{ss}) = \dim \mathfrak{l} - 1$. Hence, the semisimple part G^{ss} acts transitively on M because the G^{ss}-orbit of $o = eL$ has dimension $\dim G^{ss} \cdot o = \dim \mathfrak{g}^{ss} - \dim(\mathfrak{l} \cap \mathfrak{g}^{ss}) = \dim G/L$. $\square$

3. Characterization of G-standard CR manifolds.

Let $(M = G/L, \mathcal{D}, J)$ be a homogeneous CR manifold of a connected compact semisimple Lie group G. We will prove that the property of being G-standard does not depend on the group G.

Let $A \subset \mathrm{Aut}(M, \mathcal{D})$ be a connected semisimple group of contact automorphisms, which contains G. Then $M = A/B = G/L$ where $L = G \cap B$ and we have the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathfrak{g}} + \mathfrak{m}_{\mathfrak{g}}\ , \qquad \mathfrak{a} = \mathfrak{b} + \mathbb{R}Z_{\mathfrak{a}} + \mathfrak{m}_{\mathfrak{a}}$$

associated with the invariant contact structure $\mathcal{D}$, where $Z_{\mathfrak{g}}$ and $Z_{\mathfrak{a}}$ are the corresponding contact elements.

If A is a group of CR transformations, then we denote by $\mathfrak{m}_{\mathfrak{g}}^{10} \subset \mathfrak{m}_{\mathfrak{g}}^{\mathbb{C}}$ and $\mathfrak{m}_{\mathfrak{a}}^{10} \subset \mathfrak{m}_{\mathfrak{a}}^{\mathbb{C}}$ the holomorphic subspaces of the CR manifolds $(G/L, \mathcal{D}, J)$ and $(A/B, \mathcal{D}, J)$.

Lemma 3.1. *Let $(M = G/L, \mathcal{D})$ be a homogeneous contact manifold of a compact connected semisimple Lie group G and $A \supset G$ a compact connected semisimple subgroup of $Aut(M, \mathcal{D})$ so that $M = A/B = G/L$ and $L = G \cap B$. Then:*

1. *if $\pi : \mathfrak{a} \to \mathfrak{m}_{\mathfrak{a}}$ is the projection parallel to $\mathfrak{b} + \mathbb{R}Z_{\mathfrak{a}}$, then $\pi|_{\mathfrak{m}_{\mathfrak{g}}} : \mathfrak{m}_{\mathfrak{g}} \to \mathfrak{m}_{\mathfrak{a}}$ is an isomorphism;*
2. *$C_{\mathfrak{g}}(Z_{\mathfrak{a}}) = C_{\mathfrak{g}}(Z_{\mathfrak{g}}) = \mathfrak{l} + \mathbb{R}Z_{\mathfrak{g}}$.*

Moreover, if $(\mathcal{D}, J)$ is a G-invariant CR structure on G/L with underlying contact structure $\mathcal{D}$ and if $A \subset Aut(M, \mathcal{D}, J)$, then:

3. *the isomorphism $\pi|_{\mathfrak{m}_{\mathfrak{g}}} : \mathfrak{m}_{\mathfrak{g}} \to \mathfrak{m}_{\mathfrak{a}}$ commutes with the complex structures on $\mathfrak{m}_{\mathfrak{g}}$ and $\mathfrak{m}_{\mathfrak{a}}$ induced by the CR structure $(\mathcal{D}, J)$ and*
$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{g}}^{10} \subset \mathfrak{b}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{a}}^{10} \ ;$$
4. *$[Z_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}^{10}] \subset \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{g}}^{10}$ if and only if $[Z_{\mathfrak{a}}, \mathfrak{m}_{\mathfrak{a}}^{10}] \subset \mathfrak{b}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{a}}^{10}$.*

Proof. We denote by $\mathcal{B}$ an invariant scalar product on $\mathfrak{a}$ and we may assume that the invariant scalar product on $\mathfrak{g}$ is the restriction of $\mathcal{B}$ to $\mathfrak{g} \subset \mathfrak{a}$.

Then (1) follows immediately from the fact that $(\mathfrak{l} + \mathfrak{m}_{\mathfrak{g}})/\mathfrak{b} \simeq (\mathfrak{b} + \mathfrak{m}_{\mathfrak{a}})/\mathfrak{b} \simeq \mathcal{D}_o$ (here $o = eL = eB$).

To show (2), note that $C_{\mathfrak{a}}(Z_{\mathfrak{a}}) = \mathfrak{b} + \mathbb{R}Z_{\mathfrak{a}}$ and $C_{\mathfrak{g}}(Z_{\mathfrak{g}}) = \mathfrak{l} + \mathbb{R}Z_{\mathfrak{g}}$. Hence, it is sufficient to check that
$$(\mathfrak{b} + \mathbb{R}Z_{\mathfrak{a}}) \cap \mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathfrak{g}} \ .$$
But this follows immediately from the fact that $X \in \mathfrak{b} + \mathbb{R}Z_{\mathfrak{a}}$ (resp. $X \in \mathfrak{l} + \mathbb{R}Z_{\mathfrak{g}}$) if and only if the value at o of the corresponding vector field $\hat{X}$ is proportional with the Reeb vector $\xi_o^{\theta} \in T_oM$ at o.

For (3), observe that the first claim follows directly from the definitions. The second claim is proved by the fact that $\mathfrak{m}_{\mathfrak{g}}^{10} = \mathfrak{m}_{\mathfrak{a}}^{10} \mod \mathfrak{b}^{\mathbb{C}}$.

Let us now prove (4). From the proof of (2), after rescaling $Z_{\mathfrak{g}}$, we may assume that
$$Z_{\mathfrak{a}} = Z_{\mathfrak{g}} + W$$
for some $W \in \mathfrak{b}$. Moreover, by (1) and (3), any element $X_{\mathfrak{a}}^{10} \in \mathfrak{m}_{\mathfrak{a}}^{10}$ can be written as
$$X_{\mathfrak{a}}^{10} = X_{\mathfrak{g}}^{10} + X_{\mathfrak{b}}$$
where $X_{\mathfrak{g}}^{10} \in \mathfrak{m}_{\mathfrak{g}}^{10}$ and $X_{\mathfrak{b}} \in \mathfrak{b}^{\mathbb{C}}$.

Assume that $[Z_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}^{10}] \subset \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{g}}^{10}$. Then for any $X_{\mathfrak{a}}^{10} = X_{\mathfrak{g}}^{10} + X_{\mathfrak{b}} \in \mathfrak{m}_{\mathfrak{a}}^{10}$ we have
$$[Z_{\mathfrak{a}}, X_{\mathfrak{a}}^{10}] = [Z_{\mathfrak{g}} + W, X_{\mathfrak{a}}^{10}] \cong [Z_{\mathfrak{g}}, X_{\mathfrak{a}}^{10}](\mod (\mathfrak{b}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{a}}^{10})) \cong$$
$$\cong [Z_{\mathfrak{g}}, X_{\mathfrak{g}}^{10}] + [Z_{\mathfrak{g}}, X_{\mathfrak{b}}] \ (\mod (\mathfrak{b}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{a}}^{10})) =$$
$$= [Z_{\mathfrak{g}}, X_{\mathfrak{g}}^{10}] + [Z_{\mathfrak{a}}, X_{\mathfrak{b}}] - [W, X_{\mathfrak{b}}] \mod (\mathfrak{b}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{a}}^{10}) =$$
$$= [Z_{\mathfrak{g}}, X_{\mathfrak{g}}^{10}] \mod (\mathfrak{b}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{a}}^{10}) =$$
$$= 0 \mod (\mathfrak{b}^{\mathbb{C}} + \mathfrak{m}_{\mathfrak{a}}^{10})$$
by assumption and (3).

Conversely, assume that $[Z_{\mathfrak{a}}, \mathfrak{m}^{10}_{\mathfrak{a}}] \subset \mathfrak{b}^{\mathbb{C}} + \mathfrak{m}^{10}_{\mathfrak{a}}$. Then

$$[Z_{\mathfrak{g}}, \mathfrak{m}^{10}_{\mathfrak{g}}] \subset [Z_{\mathfrak{g}}, \mathfrak{b}^{\mathbb{C}} + \mathfrak{m}^{10}_{\mathfrak{a}}] \cap \mathfrak{g}^{\mathbb{C}} = [Z_{\mathfrak{a}} - W, \mathfrak{b}^{\mathbb{C}} + \mathfrak{m}^{10}_{\mathfrak{a}}] \cap \mathfrak{g}^{\mathbb{C}} \subset$$

$$\subset (\mathfrak{b}^{\mathbb{C}} + \mathfrak{m}^{10}_{\mathfrak{a}}) \cap \mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{\mathfrak{g}}$$

by assumption and (3). □

Lemma 3.1 implies the following proposition.

Proposition 3.2. *Let $(M = G/L, \mathcal{D}, J)$ be a homogeneous contact manifold and $A \subset Aut(M, \mathcal{D})$ a connected compact semisimple group of contact transformations which contains G, so that $M = A/B = G/L$. Denote by $Z_{\mathfrak{g}}$ and $Z_{\mathfrak{a}}$ contact elements in $\mathfrak{g}$ and $\mathfrak{a}$ associated with the contact distribution $\mathcal{D}$. Then:*

a) *the G-associated flag manifold $F_{Z_{\mathfrak{g}}} = G/K = G/C_G(Z_{\mathfrak{g}})$ is equivalent (as G-manifold) to the A-associated flag manifold $F_{Z_{\mathfrak{a}}} = A/C = A/C_A(Z_{\mathfrak{a}})$.*

b) *if A is a group of CR transformations, then $(M = G/L = A/B, \mathcal{D}, J)$ is a G-standard CR manifold if and only if it is an A-standard CR manifold.*

Proof. a) It follows from the fact that the orbit $G \cdot o \subset A/C_A(Z_{\mathfrak{a}})$ coincides with $A/C_A(Z_{\mathfrak{a}}) = F_{Z_{\mathfrak{a}}}$ and it is equal to $G/C_G(Z_{\mathfrak{a}}) = G/C_G(Z_{\mathfrak{g}})$, by Lemma 3.1 (2).

b) follows directly from definitions and Lemma 3.1 (4). □

We have the following corollary, which is the main result of this section.

Corollary 3.3. *For any connected semisimple Lie group G acting transitively on a compact CR manifold $(M, \mathcal{D}, J)$, the G-associated flag manifold is G-equivalent with the naturally associated flag manifold F of M.*

Moreover $(M, \mathcal{D}, J)$ is G-standard if and only if it is standard.

4. Maximal compact semisimple groups of automorphisms.

4.1 Inclusion relations between transitive groups of transformations of a flag manifold.

First of all we quote the following result by A. L. Onishchik, which describes the inclusion relations between compact semisimple transitive groups of holomorphic transformations of a flag manifold.

Theorem 4.1. *[On] Let $F = G/K$ be a flag manifold of a compact semisimple Lie group $G = G_1 \times \cdots \times G_p$, where G_i are the simple factors of G, and let*

$$F = G/K = G_1/K_1 \times \cdots \times G_p/K_p$$

be the corresponding decomposition of the flag manifold F.

Let A be a compact semisimple Lie group of transformations of F, which contains G and preserves some complex structure J on F. Then A is of the form

$$A = A_1 \times \cdots \times A_p$$

where each $A_i \supseteq G_i$ acts only on $F_i = G_i/K_i = A_i/C_i$ and either $A_i = G_i$ or the pair $(A_i/C_i, G_i/K_i)$ is one of those contained in the following table:

Table 1

n^o	$F = A/C = G/K$	A/C	G/K
I	$\mathbb{C}P^{2\ell-1}$ $\ell > 1$	$\frac{SU_{2\ell}}{U_{2\ell-1}}$	$\frac{Sp_\ell}{Sp_{\ell-1}\cdot T^1}$
II	$Gr_2(\mathbb{R}^7)$	$\frac{SO_7}{SO_5\cdot SO_2}$	$\frac{G_2}{U_2}$
III	$Com(\mathbb{R}^{2\ell+2})$ $\ell > 2$	$\frac{SO_{2\ell+2}}{U_{\ell+1}}$	$\frac{SO_{2\ell+1}}{U_\ell}$

In row II, the manifold G_2/U_2 is equal to $\mathrm{Ad}_{G_2}(H_\alpha) = G_2/C_{G_2}(H_\alpha)$, where H_α is the dual vector of a long root of G_2.

In the following, we will call *Onishchik pair* any pair $(A/C, G/K)$ of homogeneous spaces from a row of Table 1.

As a direct corollary of Onishchik's result, we get the following theorem.

Theorem 4.2. *Let $F = G/K$ be a flag manifold and A a connected compact group of transformations of F which contains G and preserves some complex structure on F. Then any G-invariant complex structure on F is A-invariant.*

Proof. By Theorem 4.1, it is sufficient to consider only the three cases of Table 1. In all such cases, A/C is an irreducible Hermitian symmetric manifold and hence it admits a unique (up to a sign) invariant complex structure. It is also easy to check that also $M = G/K$, where M is $Sp_\ell/Sp_{\ell-1}\cdot T^1$, $SO_{2\ell+1}/U_\ell$ or $G_2/U_2 = G_2/C_{G_2}(H_\alpha)$, with α long root, admits only one (up to a sign) invariant complex structure. In fact, any invariant complex structures on a flag manifold G/K (considered up to conjugation) corresponds to a black-white Dynkin diagram of the Lie group G, where the subdiagram formed by the white nodes is equal to the Dynkin diagram of the semisimple part K' of K (see e.g. [Al], [AP]); for the three manifolds $M = G/K$ above, there is only one black-white Dynkin diagram. □

Remark 4.3. From Theorem 4.2, it follows immediately that if $F = G/K = G_1/K_2 \times \dots G_p/K_p$ is the decomposition into irreducible factors of a flag manifold with invariant complex structure (F, J_F), then the maximal group A of holomorphic transformations is $A = A_1 \times \dots \times A_p$, where $A_i = \tilde{A}$ if there exists an Onishchik pair of the form $(\tilde{A}/\tilde{C}, G_i/K_i)$ and it is $A_i = G_i$ otherwise.

4.2 Homogeneous manifolds with non-standard CR structures.

By Corollary 3.3, a homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$ of a compact semisimple Lie group G is not G-standard if and only if it is non-standard. We recall the classification of such non-standard CR manifolds in the following Theorem (see [AS]).

Theorem 4.4. *[AS] Let $(M = G/L, \mathcal{D}_Z, J)$ be a simply connected non-standard CR manifold of a compact connected Lie group G and $F = G/K$ the associated flag manifold. Then the triple (G, L, K) is one of those given in Table 2.*

n^o	G	L	K
1	$SU_2 \times SU_2'$	T^1	$T^1 \times T^{1\prime}$
2	$Spin_7$	SU_3	$T^1 \cdot SU_3$
3	F_4	$Spin_7$	$T^1 \cdot SO_7$
4	SU_2	$\{e\}$	T^1
5	$SO_{2n+1}\ {}_{n>1}$	SO_{2n-1}	$T^1 \cdot SO_{2n-1}$
6	$SO_{2n}\ {}_{n>2}$	SO_{2n-2}	$T^1 \cdot SO_{2n-2}$
7	Sp_n	$Sp_1 \cdot Sp_{n-2}$	$T^1 \cdot Sp_1 \cdot Sp_{n-2}$
8	SU_n	$T^1 \cdot SU_{n-2}$	$T^1 \cdot U_{n-2}$
9	$SU_p \times SU_q'$ ${}_{p+q>4}$	$T^1 \cdot U_{p-2} \cdot U_{q-2}'$	$(T^1 \cdot U_{p-2}) \cdot (T^{1\prime} \cdot U_{q-2}')$
10	$SU_n\ {}_{n>4}$	$T^1 \cdot (SU_2 \times SU_2) \cdot SU_{n-4}$	$T^1 \cdot (SU_2 \times SU_2) \cdot U_{n-4}$
11	SO_{10}	$T^1 \cdot SO_6$	$T^2 \cdot SO_6$
12	E_6	$T^1 \cdot SO_8$	$T^2 \cdot SO_8$
13	E_7	$T^1 \cdot SO_{10}$	$T^2 \cdot SO_{10}$

Table 2

4.3 Maximal compact semisimple groups of automorphisms of non-standard CR manifolds.

Using Onishchik's theorem (Theorem 4.1), Proposition 3.2 and Theorem 4.4, we can now describe the maximal compact semisimple group of CR transformations for any compact homogeneous non-standard CR manifold. The answer is quite simple.

Theorem 4.5. *Let $(M = G/L, \mathcal{D}, J)$ be a non-standard homogeneous CR manifold of a semisimple connected compact Lie group G. Then G is a maximal connected semisimple compact Lie group of automorphisms.*

Proof. Assume that there exists a compact semisimple group A of automorphism of M which properly contains G, so that $M = G/L = A/B$, with $L = G \cap B$. By Proposition 3.2 (a), the A-associated flag manifold $F = A/C$ is equivalent to the G-associated flag manifold G/K and hence, by Theorem 4.1, one of the factors of $F = A/C$ must be a member of an Onishchik pair.

At the same time, $(M = A/B, \mathcal{D}, J)$ is non-standard and hence the groups A, B and C must be in a row of Table 2.

Comparing Table 1 and Table 2, we find that there is only one possibility for $F = A/C$, that is $F = \frac{SO_7}{SO_5 \cdot SO_2}$. It follows that $G/K = \frac{G_2}{U_2}$. But this is impossible because, by Table 2, there is no non-standard homogeneous CR manifold with $G = G_2$. $\square$

4.4 Maximal compact semisimple groups of automorphisms of standard CR manifolds.

It remains to describe the maximal connected, compact semisimple group of CR transformations of a standard CR manifold.

For this, we first need the following Lemma, where we denote by $\mathcal{B}_{\mathfrak{a}}$ (resp. $\mathcal{B}_{\mathfrak{g}}$) an invariant scalar product on $\mathfrak{a}$ (resp. $\mathfrak{g}$).

Lemma 4.6. *Let $F = A/C = G/K$ be a flag manifold with $A \supsetneq G$, and $(M = G/L, \mathcal{D})$ and $(M' = A/B, \mathcal{D}')$ be two homogeneous contact manifolds, having F as associated flag manifold. Denote by $Z_{\mathfrak{g}} \in \mathfrak{g}$ and $Z_{\mathfrak{a}} \in \mathfrak{a}$ the contact elements associated with $\mathcal{D}$ and $\mathcal{D}'$, respectively.*

Then the following two properties are equivalent:

i) *G acts transitively on $M' = A/B$ and the homogeneous manifold $M' = A/B = G/L'$ is G-equivalent to $M = G/L$;*

ii)
$$Z(\mathfrak{k}) \cap Z_{\mathfrak{g}}^{\perp_{\mathfrak{g}}} = Z(\mathfrak{k}) \cap Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}} , \tag{4.1}$$
where $Z_{\mathfrak{g}}^{\perp_{\mathfrak{g}}} = \{ X \in \mathfrak{g} \ : \mathcal{B}_{\mathfrak{g}}(X, Z_{\mathfrak{g}}) = 0 \}$ *and* $Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}} = \{ X \in \mathfrak{a} \ : \mathcal{B}_{\mathfrak{a}}(X, Z_{\mathfrak{g}}) = 0 \}$.

Furthermore, if this is the case, then under the identification $M = A/B = G/L$, where $L = B \cap G$, the invariant contact structures $\mathcal{D}$ and $\mathcal{D}'$ coincide.

Proof. Let $F = G/K = G_1/K_1 \times \dots G_p/K_p$ be the decomposition of F into irreducible factors. By Theorem 4.1,
$$F = A/C = A_1/C_1 \times \cdots \times A_p/G_p$$
where for each i, either $A_i/C_i = G_i/K_i$ or $(A_i/C_i, G_i/K_i)$ is an Onishchik pair.

Note that G acts transitively on $M' = A/B = G/L'$ and that G/L' is equivalent to G/L if and only if $\mathfrak{l} = \mathfrak{l}' = \mathfrak{b} \cap \mathfrak{g}$. On the other hand, since $\mathfrak{l} = \mathfrak{k} \cap Z_{\mathfrak{g}}^{\perp_{\mathfrak{g}}} = \mathfrak{k}^{ss} + Z(\mathfrak{k}) \cap Z_{\mathfrak{g}}^{\perp_{\mathfrak{g}}}$ and $\mathfrak{b} = \mathfrak{c} \cap Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}} = \mathfrak{c}^{ss} + Z(\mathfrak{c}) \cap Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}}$, we also have that
$$\mathfrak{b} \cap \mathfrak{g} = (\mathfrak{c} \cap \mathfrak{g}) \cap Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}} = \mathfrak{k} \cap Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}} = \mathfrak{k}^{ss} + Z(\mathfrak{k}) \cap Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}} .$$
Hence $\mathfrak{l} = \mathfrak{l}' = \mathfrak{b} \cap \mathfrak{g}$ if and only if (4.1) holds.

Assume now that (4.1) holds and that $\mathcal{D}$ and $\mathcal{D}'$ are not equal. Then $M = G/L = A/B$ admits two invariant contact structure and hence it is a special contact manifold according to Def. 3.4 in [AS]. Then, by Thm. 3.6 of [AS], G and A are both simple. Theorem 4.1 implies that $(A/C, G/K)$ is an Onishchik pair and, in particular, that the center $Z(\mathfrak{k})$ is one dimensional. By Proposition 2.1, this implies that there exists only one homogeneous contact manifold $(G/L, \mathcal{D})$ with associated flag manifold $F = G/K$; this is in contradiction with the fact that $(G/L, \mathcal{D})$ and $(G/L, \mathcal{D}')$ are two distinct invariant contact homogeneous manifolds with associated F. $\square$

Now we can prove the main theorem of this subsection.

Theorem 4.7. *Let $(M = G/L, \mathcal{D}, J)$ be a standard homogeneous CR manifold of a semisimple connected compact Lie group G. Let also $F = G/C_G(Z) = G/K$ be the associated flag manifold and J_F the complex structure induced by the projection $\pi : G/L \to G/K$.*

Then the maximal connected compact semisimple group $A^{ss}(M, \mathcal{D}, J)$ which contains G acts on $(F = G/K, J_F)$ as a maximal compact group of holomorphic transformations.

Proof. By Proposition 3.2, any connected compact semisimple group, which contains G, acts naturally on F as a group of holomorphic transformations. Hence it is sufficient to prove that a maximal connected semisimple group A of holomorphic transformations of (F, J_F), which contains G, acts on $M = G/L$ as a group of CR transformations.

We first show that A acts on $(M = G/L, \mathcal{D})$ as a group of contact transformations.

We may assume that $\mathcal{B} = \mathcal{B}_{\mathfrak{g}}$ is the Cartan-Killing form of $\mathfrak{g}$.
Let

$$F = G/K = G_1/K_1 \times \cdots \times G_p/K_p \ , \quad Z_{\mathfrak{g}} = Z_1 + \cdots + Z_p \ , \quad Z_i \in \mathfrak{g}_i$$

be the decomposition of F into irreducible factors and the associated decomposition of the contact element. We may also assume that G is simply connected and that $\exp(Z_{\mathfrak{g}}) = \exp(Z_1) \cdot \ldots \cdot \exp(Z_p) = e$ where e is the identity element of G. Clearly, this implies that $\exp(Z_i) = e \in G_i$ for any i.

Recall that, since $Z_{\mathfrak{g}} \in Z(\mathfrak{k})$ and $\mathfrak{l} = \mathfrak{k} \cap (\mathbb{R}Z)^{\perp}$, $\mathfrak{g}$ has the following $\mathcal{B}_{\mathfrak{g}}$-orthogonal decomposition:

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} = [(\mathfrak{k}_1^{ss} + \mathfrak{k}_2^{ss} + \cdots + \mathfrak{k}_p^{ss}) + Z(\mathfrak{k}) \cap (\mathbb{R}Z_{\mathfrak{g}})^{\perp}] + \mathbb{R}Z_{\mathfrak{g}} + (\mathfrak{m}_1 + \cdots + \mathfrak{m}_p)$$

with $\mathfrak{m}_i = \mathfrak{m} \cap \mathfrak{g}_i$.

By Remark 4.3, F can be decomposed into

$$F = G/K = A/C = A_1/C_1 \times \cdots \times A_p/C_p$$

where either $A_i/C_i = G_i/K_i$ or $(A_i/C_i, G_i/K_i)$ is an Onishchik pair. In case $(A_i/C_i, G_i/K_i)$ is an Onishchik pair, $\dim Z(\mathfrak{c}_i) = \dim Z(\mathfrak{k}_i) = 1$. We choose a generator $E^{\mathfrak{c}_i}$ for $Z(\mathfrak{c}_i)$ and a generator $E^{\mathfrak{k}_i}$ for $Z(\mathfrak{k}_i)$, which verify the following property: for any $X \in \mathbb{R}E^{\mathfrak{c}_i}$ and any $Y \in \mathbb{R}E^{\mathfrak{k}_i}$, $\exp(X) = e$ and $\exp(Y) = e$ if and only if $X \in \mathbb{Z}E^{\mathfrak{c}_i}$ and $Y \in \mathbb{Z}E^{\mathfrak{k}_i}$, respectively.

We fix now an Ad_A-invariant scalar product $\mathcal{B}_{\mathfrak{a}}$ on $\mathfrak{a}$. For each simple algebra $\mathfrak{a}_i$, we assume that $\mathcal{B}_{\mathfrak{a}}|_{\mathfrak{a}_i}$ is a multiple of the Cartan-Killing form of $\mathfrak{a}_i$ determined with the following rules: if $A_i = G_i$, we assume that $\mathcal{B}_{\mathfrak{a}}|_{\mathfrak{a}_i}$ is the Cartan-Killing form without rescaling (note that in this case, $\mathcal{B}_{\mathfrak{a}}|_{\mathfrak{a}_i} = \mathcal{B}_{\mathfrak{g}}|_{\mathfrak{g}_i}$); if $(A_i/C_i, G_i/K_i)$ is an Onishchik pair, we assume that $\mathcal{B}_{\mathfrak{a}}|_{\mathfrak{a}_i}$ is the multiple of the Cartan-Killing form which verifies $\mathcal{B}_{\mathfrak{a}}(E^{\mathfrak{c}_i}, E^{\mathfrak{c}_i}) = -1$.

We now consider the element $Z_{\mathfrak{a}} = Z'_1 + \ldots Z'_p \in Z(\mathfrak{c}) \subset \mathfrak{a}$ defined as follows: if $(A_i/C_i, G_i/K_i)$ is an Onishchik pair and Z_i is of the form $Z_i = \lambda_i E^{\mathfrak{k}_i}$, we set

$$Z'_i = \lambda_i \frac{\mathcal{B}_{\mathfrak{g}}(E^{\mathfrak{k}_i}, E^{\mathfrak{k}_i})}{\mathcal{B}_{\mathfrak{a}}(E^{\mathfrak{k}_i}, E^{\mathfrak{c}_i})} E^{\mathfrak{c}_i} = \frac{\mathcal{B}_{\mathfrak{g}}(Z_i, Z_i)}{\mathcal{B}_{\mathfrak{a}}(Z_i, E^{\mathfrak{c}_i})} E^{\mathfrak{c}_i} \ ; \tag{4.2}$$

in all other cases, we set $Z'_i = Z_i$.

It is an immediate consequence of definitions that $\mathcal{B}_{\mathfrak{a}}(Z_i, Z'_i) = \mathcal{B}_{\mathfrak{g}}(Z_i, Z_i)$ for any i.

We claim that:

a) $Z_{\mathfrak{a}}$ is a $\mathfrak{c}$-regular element, that is $C_{\mathfrak{a}}(Z_{\mathfrak{a}}) = \mathfrak{c}$ and $Z_{\mathfrak{a}}$ generates a closed subgroup of A;

b) $Z(\mathfrak{k}) \cap Z_{\mathfrak{g}}^{\perp_{\mathfrak{g}}} = Z(\mathfrak{k}) \cap Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}}$ where $Z_{\mathfrak{g}}^{\perp_{\mathfrak{g}}} = \{\ X \in \mathfrak{g}\ : \mathcal{B}_{\mathfrak{g}}(X, Z_{\mathfrak{g}}) = 0\ \}$ and $Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}} = \{\ X \in \mathfrak{a}\ : \mathcal{B}_{\mathfrak{a}}(X, Z_{\mathfrak{a}}) = 0\ \}$.

Assume for the moment that a) and b) are true. Then by Proposition 2.1, there exists a unique homogeneous contact manifold $(A/B, \mathcal{D}_{Z_{\mathfrak{a}}})$ with contact structure $\mathcal{D}_{Z_{\mathfrak{a}}}$ with contact element $Z_{\mathfrak{a}}$ and with $F = A/C = G/K$ as associated flag manifold; moreover, by Lemma 4.6, $(M = G/L, \mathcal{D})$ is equal to $(A/B, \mathcal{D}_{Z_{\mathfrak{a}}})$ and hence A acts on M as a group of contact transformations, as we needed to prove.

To show a), observe that

$$C_{\mathfrak{a}}(Z_{\mathfrak{a}}) = C_{\mathfrak{a}_1}(Z'_1) + C_{\mathfrak{a}_2}(Z'_2) + \cdots + C_{\mathfrak{a}_p}(Z'_p)\ .$$

It is clear that $C_{\mathfrak{a}_i} = \mathfrak{c}_i$ for any i and hence that $C_{\mathfrak{a}}(Z_{\mathfrak{a}}) = \mathfrak{c}$. So, we only need to check that $Z_{\mathfrak{a}}$ generates a closed subgroup. For this, consider the following facts:

(1) if $A_i = G_i$, then $\exp(Z'_i) = \exp(Z_i) = e \in G_i$;

(2) if $(A_i/C_i, G_i/K_i)$ is an Onishchik pair, then the elements $E^{\mathfrak{c}_i}$ and $E^{\mathfrak{k}_i}$ are defined in such a way that from the fact

$$\exp(Z_i) = \exp(\lambda_i E^{\mathfrak{k}_i}) = e \in A_i$$

it follows that $\lambda_i \in \mathbb{Z}$;

(3) since $\exp(\mathbb{R}Z_i)$ is closed in G_i for any i and since $\mathcal{B}_{\mathfrak{g}}|_{\mathfrak{g}_i}$ is the Cartan-Killing form of $\mathfrak{g}_i$, we have that $\mathcal{B}_{\mathfrak{g}}(Z_i, Z_i) \in \mathbb{Q}$ for any i;

(4) for any Onishchik pair $(A_i/C_i, G_i/K_i)$, consider the projection $\pi : C_i \to Z(C_i) = \exp(\mathbb{R}E^{\mathfrak{c}_i}) = C/C^{ss}$; the differential $\pi_* : \mathfrak{c}_i \to Z(\mathfrak{c}_i)$ is equal to the orthogonal projection

$$\pi_* : \mathfrak{c}_i \longrightarrow Z(\mathfrak{c}_i)\ , \qquad \pi_*(X) = \mathcal{B}_{\mathfrak{a}}(X, E^{\mathfrak{c}_i})E^{\mathfrak{c}_i}\ ;$$

so

$$e = \pi(\exp(E^{\mathfrak{k}_i})) = \exp(\pi_*(E^{\mathfrak{k}_i})) = \exp(\mathcal{B}_{\mathfrak{a}}(E^{\mathfrak{k}_i}, E^{\mathfrak{c}_i})E^{\mathfrak{c}_i})$$

and hence $\mathcal{B}_{\mathfrak{a}}(E^{\mathfrak{k}_i}, E^{\mathfrak{c}_i}) \in \mathbb{Z}$; from (2) this implies that $\mathcal{B}_{\mathfrak{a}}(Z_i, E^{\mathfrak{c}_i}) = \lambda_i \mathcal{B}_{\mathfrak{a}}(E^{\mathfrak{k}_i}, E^{\mathfrak{c}_i}) \in \mathbb{Z}$.

¿From (1), (3) and (4) and formula (4.2), it follows immediately that there exists an integer N such that $\exp(NZ_{\mathfrak{a}}) = e \in A$ and hence that $\exp(\mathbb{R}Z_{\mathfrak{a}})$ is closed.

To prove b), consider an orthonormal basis $B_i = (Z_i, Y^i_2, \ldots, Y^i_{p_i})$ for each abelian algebra $Z(\mathfrak{k}_i)$, with first element equal to Z_i. It follows that an element

$$X = x^i Z_i + \sum_{j=2}^{p_i} y^j_i Y^i_j \in Z(\mathfrak{k})$$

is an element of $Z(\mathfrak{k}) \cap Z_{\mathfrak{g}}^{\perp_{\mathfrak{g}}}$ if and only if

$$x^1 \mathcal{B}_{\mathfrak{g}}(Z_1, Z_1) + x^2 \mathcal{B}_{\mathfrak{g}}(Z_2, Z_2) + \cdots + x^p \mathcal{B}_{\mathfrak{g}}(Z_p, Z_p) = 0 \ . \tag{4.3}$$

The same element is in $Z(\mathfrak{k}) \cap Z_{\mathfrak{a}}^{\perp_{\mathfrak{a}}}$ if and only if

$$x^1 \mathcal{B}_{\mathfrak{a}}(Z_1, Z_1') + x^2 \mathcal{B}_{\mathfrak{a}}(Z_2, Z_2') + \cdots + x^p \mathcal{B}_{\mathfrak{a}}(Z_p', Z_p') = 0 \tag{4.4}$$

Since $\mathcal{B}_{\mathfrak{a}}(Z_i, Z_i') = \mathcal{B}_{\mathfrak{g}}(Z_i, Z_i)$ for any i, equations (4.3) and (4.4) coincide.

It remains to check that A acts on $M = G/L$ as a group of CR transformations of $(\mathcal{D}, J)$. Since the CR structure is standard and A preserves the contact structure $\mathcal{D}$, the above claim is immediately proved by recalling that A is a group of holomorphic transformations for (F, J_F). $\square$

5. Maximal compact groups of automorphisms of a homogeneous CR manifold.

In the previous section, we showed how to reconstruct a maximal compact connected semisimple group of CR transformations $A^{ss} = A^{ss}(M, \mathcal{D}, J)$ of a compact homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$. Now we want to show how to determine a maximal compact connected group $A = A(M, \mathcal{D}, J)$ which contains A^{ss}.

Theorem 5.1. *Let $(M, \mathcal{D}, J)$ be a simply connected compact homogeneous CR manifold .*

(a) *If $(M, \mathcal{D}, J)$ is non-standard, then $A = A^{ss}$.*
(b) *If $(M, \mathcal{D}, J)$ is standard, then $A = A^{ss} \times T^1$.*

Proof. (a) was proved in [AS], Proposition 4.6.

(b) Assume now that $(M, \mathcal{D}, J)$ is standard and identify M with the homogeneous manifold A^{ss}/B, with $A^{ss} = A^{ss}(M, \mathcal{D}, J)$. Let

$$\mathfrak{a}^{ss} = \mathfrak{b} + \mathbb{R}Z + \mathfrak{m}$$

be the associated decomposition. Without loss of generality we may assume that the contact element Z is so that $\exp(Z) = e$ and has $\mathcal{B}$-norm equal to 1. Let also $\mathfrak{m}^{10} \subset \mathfrak{m}^{\mathbb{C}}$ the associated holomorphic subspace (recall that it is $\mathrm{ad}(\mathfrak{b} + \mathbb{R}Z)$-invariant). By Corollary 3.2 in [AS], the center $Z(A)$ of $A = A(M, \mathcal{D}, J)$ has dimension 0 or 1. Hence to prove (b) it is sufficient to define a CR action on M of the group $A = A^{ss} \times T^1$. This can be done by constructing a homogeneous CR manifold $(\tilde{M} = A/\tilde{B}, \tilde{\mathcal{D}}, \tilde{J})$ and then show that it is A^{ss}-diffeomorphic and CR-equivalent to $(M = A^{ss}/B, \mathcal{D}, J)$.

Let $\tilde{B}$ be the connected subgroup of $A = A^{ss} \times T^1$ generated by the subalgebra

$$\tilde{\mathfrak{b}} = \mathfrak{b} + \mathbb{R}(Z - \xi) \subset \mathfrak{a} = \mathfrak{a}^{ss} \oplus \mathbb{R}\xi$$

where ξ is a generator of T^1 such that $\exp(\xi) = e$. We may also assume that the $\mathcal{B}$-norm of ξ is equal to 1.

One can check easily that $\tilde{Z} = Z + \xi \in \tilde{\mathfrak{b}}^{\perp}$ and that it is a contact element. Let $\tilde{D} = \mathcal{D}_{\tilde{Z}}$ be the corresponding contact structure on $\tilde{M} = A/\tilde{B}$. Note that the subspace $\mathfrak{m}^{10}$ defines an invariant CR structure $(\tilde{\mathcal{D}}, \tilde{J})$ on $\tilde{M}$ and that the subgroup A^{ss} of $A = A^{ss} \times T^1$ acts transitively on $\tilde{M} = A/\tilde{B}$ with stabilizer $B = A^{ss} \cap \tilde{B}$. Therefore, $\tilde{M} = A/\tilde{B} = A^{ss}/B = M$. It is quite simple to check that the homogeneous CR manifold $(\tilde{M} = A/\tilde{B}, \tilde{\mathcal{D}}, \tilde{J})$ is CR equivalent to $(M = A^{ss}/B = \tilde{M}, \mathcal{D}, J)$. $\square$

The theorem implies the following characterization of standard CR structures:

Corollary 5.2. *A compact homogeneous CR manifold is standard if and only if a maximal connected compact group of automorphisms has 1-dimensional center.*

6. Equivalences of homogeneous compact CR manifolds.

The goal of this section is to determine when two simply connected homogeneous compact CR manifolds $(M = G/L, \mathcal{D}, J)$, $(M' = G'/L', \mathcal{D}', J')$ are CR diffeomorphic. We will give our results considering the cases of standard and non-standard CR manifolds separately.

6.1 The case of non-standard CR manifolds.

Remark that a non-standard CR manifold can not be CR diffeomorphic to a standard (see, for instance, Corollary 4.2). Moreover, two distinct non-standard CR manifolds G/L in Table 2 are not CR diffeomorphic, because in the non-standard case, G coincides with a maximal connected compact semisimple group $A^{ss}(M, \mathcal{D}, J)$ of automorphisms of $M = G/L$. Finally, it is known (see [AS]) that invariant CR structures $(\mathcal{D}, J)$ (considered up to sign of J) on non standard homogeneous CR manifold $M = G/L$ with fixed contact structure $\mathcal{D}$ are naturally parameterized by points of the unite disc $D \subset \mathbb{C}$.

It remains to find out when two of such CR structures are CR equivalent. The answer is given in the following proposition.

Proposition 6.1. *Let $(M = A/B, \mathcal{D})$ be a homogeneous contact manifold of a connected compact semisimple Lie group A, which admits an A-invariant non-standard CR structure $(\mathcal{D}, J)$ (see Table 2). Let also $(\mathcal{D}, J_t)$ and $(\mathcal{D}, J_{t'})$ be two invariant CR structure on $M = A/B$ of the family of non-standard CR structures parametrized by the points $t \in D \setminus \{0\} \subset \mathbb{C}$ of the punctured unit disc in $\mathbb{C}$, as described in Cor. 5.2 and Prop. 6.3 and 6.4 in [AS].*

Then $(\mathcal{D}, J_t)$ is CR equivalent to $(\mathcal{D}, J_{t'})$ (up to sign of $J_{t'}$) if and only if $|t| = |t'|$.

Proof. Let
$$\phi : (M = A/B, \mathcal{D}, J_t) \to (M = A/B, \mathcal{D}, J_{t'})$$
be a CR diffeomorphism. By Theorems 4.4 and 5.1, A is a maximal compact group of automorphisms in $Aut(M, \mathcal{D}, J_t)$ and in $Aut(M, \mathcal{D}, J_{t'})$. Therefore ϕ transforms A into $A' = \phi \circ A \circ \phi^{-1}$, which is a maximal compact group of automorphisms of $(M, \mathcal{D}, J_{t'})$. Since any two maximal compact subgroups of $Aut(M, \mathcal{D}, J_{t'})$ are conjugated, without loss of generality we may assume that $A' = A$. Hence ϕ induces

a Lie group automorphism of A which preserves the isotropy B, the contact element Z associated with $\mathcal{D}$ (up to a scaling) and transforms the holomorphic subspace $\mathfrak{m}_t^{10}$ into $\mathfrak{m}_{t'}^{10}$. This means that $(\mathcal{D}, J_t)$ and $(\mathcal{D}, J_{t'})$, with $t, t' \in D \subset \mathbb{C}$, are CR equivalent if and only if there exists a Lie automorphism $\phi : \mathfrak{a}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}$ such that

i) $\phi(\mathfrak{a}) = \mathfrak{a}$ and $\phi(\mathfrak{b}) = \mathfrak{b}$;
ii) $\phi(Z) \in \mathbb{R}Z$;
iii) $\phi(\mathfrak{m}_t^{10}) = \mathfrak{m}_{t'}^{10}$.

First we consider an inner automorphism ϕ which satisfies i), ii) and iii). In this case, we may assume that it is of the form $\phi = \mathrm{Ad}_{\exp(sZ)}$. Indeed, ϕ verifies i) and ii) if and only if it is of the form $\phi = \mathrm{Ad}_{\exp(sZ)} \cdot \mathrm{Ad}_{\exp(X)}$, for some $X \in \mathfrak{b}$ and $s \in \mathbb{R}$. Since $\mathfrak{m}^{10}$ is $\mathrm{Ad}_{\mathfrak{b}}$-invariant, we may neglect the factor $\mathrm{Ad}_{\exp(X)}$, which preserves $\mathfrak{m}_t^{10}$.

Using the explicit description of Z and of $\mathfrak{m}_t^{10}$ in Cor. 5.2 and Prop. 6.3 and 6.4 in [AS], the reader can easily check that $\mathrm{Ad}_{\exp(sZ)}$ acts on the space $\mathfrak{m}_t^{10}$ by transforming it into the space $\mathfrak{m}_{t'}^{10}$, where $t' = e^{iCs}t$ for some constant $C \neq 0$ depending only on the Lie algebra $\mathfrak{a}$. This shows that there exists an inner automorphism which verifies i), ii) and iii) if and only if $|t| = |t'|$.

It remains to consider the case when ϕ is an outer automorphism. Composing it with an inner automorphism, we may always assume that it preserves a Cartan subalgebra $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{b}) + \mathbb{R}Z \subset \mathfrak{a}$.

By Cor. 5.2, Prop. 6.3 or of Prop. 6.4 in [AS], we know that in all cases of Table 2, there exists either one or two pairs of equivalent $\mathfrak{b}$-moduli in $\mathfrak{m}^{\mathbb{C}}$. Assume for simplicity that there is only one pair $(\mathfrak{m}_1, \mathfrak{m}_2)$ of equivalent $\mathfrak{b}$-moduli. We may also assume that each $\mathfrak{m}_i$, $i = 1, 2$, is also a $(\mathfrak{b} + \mathbb{R}Z)$-module with highest weight α_i, where α_i is a root. Then $\mathfrak{m}_t^{10}$ and $\mathfrak{m}_{t'}^{10}$ are two irreducible $\mathfrak{b}$-moduli with highest weight vectors $E_{\alpha_1} + tE_{\alpha_2}$ and $E_{\alpha_1} + t'E_{\alpha_2}$, respectively, where E_{α_i} is the root vector with root α_i in the Chevalley normalization.

Since ϕ preserves the root system of $(\mathfrak{a}, \mathfrak{h})$, either ϕ preserves the moduli $\mathfrak{m}_i$ or interchange them. In the first case $\phi(\mathfrak{m}_t^{10}) = \mathfrak{m}_t^{10}$ and in the second case $\phi(\mathfrak{m}_t^{10}) = \mathfrak{m}_{1/t}^{10}$. Similar arguments show that the same conclusion holds also when there exists two pairs of equivalent $\mathfrak{b}$-moduli in $\mathfrak{m}^{\mathbb{C}}$.

Since $1/t \notin D$, it follows that an outer automorphism verifies i), ii) and iii) if and only if $t = t'$. $\square$

6.2 The case of standard CR manifolds.

Now we consider the standard homogeneous CR manifolds. Enlarging the group G of automorphisms of a standard homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$, we may always assume that $G = A^{ss}(M, \mathcal{D}, J)$.

Assume that $(M = G/L, \mathcal{D}, J)$ and $(M' = G'/L', \mathcal{D}', J')$ are standard and CR diffeomorphic. Then, using a CR diffeomorphism, we may identify M' with M and $G' = A^{ss}(M', \mathcal{D}', J')$ with a transitive subgroup of $\mathrm{Aut}(M, \mathcal{D}, J)$ which is conjugated to G. Using this conjugation, we may also identify G with G' and L with L'. Therefore, the problem reduces to the description of all pairs of standard invariant CR structure $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ on the same homogeneous manifold $M = G/L$ that are CR equivalent.

The following proposition gives a necessary condition for two invariant standard CR structures on a given homogeneous manifold $M = G/L$ to be CR equivalent.

Proposition 6.2. *Assume that $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ are two invariant standard CR structures on a homogenous CR manifold $M = G/L$. If they are CR equivalent then the associated flag manifolds $F = G/K$ and $F' = G/K'$ are biholomorphic with respect to the invariant complex structures J_F and $J_{F'}$, induced by J and J', respectively.*

Proof. By Corollary 3.3, we may assume that G is equal to a maximal semisimple groups of CR transformations of M. Therefore, if we denote by Z and Z' two contact elements associated with $\mathcal{D}$ and $\mathcal{D}'$, then $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ are CR equivalent only if there is a Lie automorphism $\phi : \mathfrak{g} = Lie(G) \to \mathfrak{g} = Lie(G)$ with the following properties:

$$\text{a) } \phi(\mathfrak{l}) = \mathfrak{l} \; ; \qquad \text{b) } \phi(\mathbb{R}Z) = \mathbb{R}Z' \; ; \qquad \text{c) } \phi(\mathfrak{m}^{10}) = \mathfrak{m}^{10\prime} \; .$$

Since a) and b) imply that $\phi(\mathfrak{k}) = \mathfrak{k}'$, the automorphism ϕ induces a G-equivariant biholomorphic map between F and F', with respect to the complex structures associated with $\mathfrak{m}^{10}$ and $\mathfrak{m}^{10\prime}$. □

Now we describe all invariant CR structures $(M = G/L, \mathcal{D}, J)$ with given associated flag manifold $(F = G/K, J_F)$ up to CR equivalence. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$ and R_K and R the root systems of $\mathfrak{k}^{\mathbb{C}}$ and of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}$. Fix a basis Π_K of R_K. It is known that there exists a 1-1 correspondence between invariant complex structures J_F on $F = G/K$ and bases Π of R containing Π_K. We call such a root system R and such a basis Π *a root system and a basis adapted to the flag manifold*

Recall that any adapted basis can be represented by Dynkin graph with black and white nodes, where the black nodes are associated with the simple roots in $\Pi' = \Pi \setminus \Pi_K$.

The complex structure which is associated with an adapted basis Π of the above kind is determined by the holomorphic subspace

$$\mathfrak{m}^{10} = \mathfrak{g}(\Pi \setminus \Pi_K) \; ,$$

where we use the notation $\mathfrak{g}(S)$, $S \subset R$, to denote the subalgebra generated by the root vectors E_α, $\alpha \in S$.

Now, let us fix a flag manifold $(F = G/K, J_F)$ with an invariant complex structure J_F. If Π is the corresponding adapted basis, we will denote by $\pi_1, \ldots, \pi_m$ the fundamental weights, which corresponds to the 'black' simple roots $\Pi' = \Pi \setminus \Pi_K = \{\alpha_1, \ldots, \alpha_m\}$. We call $\pi_1, \ldots \pi_m$ the *black weights associated with the adapted basis* Π.

Theorem 6.3. *Let $(F = G/K, J_F)$ be a flag manifold and let $\pi_1, \ldots \pi_m$ the black weights associated with the adapted basis Π corresponding to J_F. Then there exists a 1-1 correspondence between standard homogeneous CR manifolds $(M = G/L, \mathcal{D}, J)$ up to CR equivalence and the set of all m-tuples $\bar{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}$ such that the p_i's have no common divisor.*

Any such m-tuple $\bar{p}$ corresponds to the following homogeneous CR manifold $(M = G/L, \mathcal{D}_Z, J)$:

(1) *the subgroup L is the connected subgroup of G, generated by the subalgebra*

$$\mathfrak{l} = [\mathfrak{k}, \mathfrak{k}] + Z(\mathfrak{k}) \cap \ker \theta \; , \quad \text{where} \quad \theta = p_1\pi_1 + \cdots + p_m\pi_m \; ;$$

(2) *the contact structure* $\mathcal{D}$ *is defined by the contact element* $Z = \mathcal{B}^{-1}\theta$*;*
(3) *the CR structure* J *is the one associated with the holomorphic subspace* $\mathfrak{m}^{10} = \mathfrak{g}(\Pi')$*, where* $\Pi' = \{\alpha_1, \dots, \alpha_m\}$ *are the 'black' roots of the adapted basis* Π.

Proof. It follows immediately from Proposition 6.2 and Proposition 2.1. □

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