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Non-autonomous Hamiltonian systems related to higher Hitchin integrals [1]

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Abstract

We describe non-autonomous Hamiltonian systems coming from the Hitchin integrable systems. The Hitchin integrals of motion depend on the $\mathcal{W}$-structures of the basic curve. The parameters of the $\mathcal{W}$-structures play the role of times. In particular, the quadratic integrals dependent on the complex structure ($\mathcal{W}_2$-structure) of the basic curve and times are coordinate on the Teichmüller space. The corresponding flows are the monodromy preserving equations such as the Schlesinger equations, the Painlevé VI equation and their generalizations. The equations corresponding to the highest integrals are monodromy preserving conditions with respect to changing of the $\mathcal{W}_k$-structures ($k > 2$). They are derived by the symplectic reduction from the gauge field theory on the basic curve interacting with $\mathcal{W}_k$-gravity. As by product we obtain the classical Ward identities in this theory.

In memory of Mikhail Saveliev

1 Introduction

Infinite-dimensional symmetries corresponding to the $\mathcal{W}$-algebras play a central role in two-dimensional physics (see [1] for a review). Here we investigate classical Hamiltonian systems incorporated in a gauge theory on a Riemann curve interacting with the $\mathcal{W}$ gravity. Whereas the $\mathcal{W}_2$-gravity has a natural geometric background, there is no satisfactory geometric understanding of the origin of the $\mathcal{W}_k$, $(k > 2)$ theory. There exists a viewpoint on the $\mathcal{W}_k$-gravity as the geometry of certain two-dimensional surfaces embedded in k-dimensional affine space. One of Misha Saveliev's works [2] is an important step in this direction. Here we analyze interrelations between $\mathcal{W}$-geometry and integrable systems. On the later subject Misha Saveliev has had a significant influence.

[1] Contribution in the Proceedings "International Seminar on Integrable systems". In memoriam Mikail V. Saveliev. Bonn, February, 1999

We starting from some subclass of classical completely integrable system with phase flows having the Lax form

$$\partial_s L = [L, M_s], \quad \partial_s = \frac{\partial}{\partial t_s}. \tag{1.1}$$

We assume that
i)L takes value in a complex Lie algebra. We restrict ourself to the case sl$(N, \mathbf{C})$;
ii)$L = L(z)$, where z is a spectral parameter lying on the Riemann curve $\Sigma_{g,n}$ of genus g with n marked points. L can have first order poles in the marked points. Thereby, we exclude in what follows the Stokes phenomena [2].
The integrable systems on curves without marked points was considered by Hitchin [4]. In this case nontrivial equations arise only on the high genus curves ($g > 1$). Up to the recent time there were no explicit examples of such type integrable systems[3]. The generalization of Hitchin approach to the matrices with first order poles in the marked points [7] allowed to include in this scheme some well known completely integrable models like the Toda and the Calogero-Moser systems. The quantum counterpart of these systems are the Knizhnik-Zamolodchikov-Bernard equations for conformal blocks in the WZW theory on the critical level, while the original Hitchin systems correspond to equations for partition functions.

In the Hitchin type systems tr$(L^k) =< L^k >$ being integrated over $\Sigma_{g,n}$ are integrals of motion. Before the integration one should take into account that L are $(1,0)$-forms on $\Sigma_{g,n}$ in some fixed complex structure. To integrate $< L^k >$ one should multiply it on $(-k+1,1)$-differentials $\rho_{k,s} = \rho_{k,s}(z,\bar{z})\partial_z^{k-1} \otimes d\bar{z}$. The index s arises in the following way. The operator $\rho_{k,s}(z,\bar{z})\partial_z^{k-1}$ is defined in local coordinates of the point $(z,\bar{z})$. The fields $\rho_{k,s}(z,\bar{z})\partial_z^{k-1}$ that can be represented as $\bar{\partial}$ derivative do not contribute in the Hamiltonians. In this way $\rho_{k,s}$ can be chosen from $H^1(\Sigma_g, \Gamma^{k-1})$, where Γ are vector fields on $\Sigma_{g,n}$ vanishing in the marked points. Then s enumerates the basis in $H^1(\Sigma_g, \Gamma^{k-1})$.

Represent the differentials as

$$\rho_{k,s} = t_{k,s}\rho^0_{s,k}, \tag{1.2}$$

where $\{\rho^0_{s,k}\}$ is a fixed basis in $H^1(\Sigma_{g,n}, \Gamma^{k-1})$. The integrals of motion take the form

$$H_{k,s} = \frac{1}{k}\int_{\Sigma_g} < L^k > \rho^0_{k,s}, \quad (k = 1, \ldots, N, \ s = 1, \ldots). \tag{1.3}$$

The important class of Hamiltonian equations occurs when $t_{k,s}$ are considered as "times". In this case $H_{k,s}$ become the Hamiltonians of non-autonomous systems, where times are related to the deformations of the internal structure of the base curve $\Sigma_{g,n}$. It turns out that the phase flows of these systems are described by the following modification of the Lax equation (1.1)

$$\partial_a L - \partial M_a + [M_a, L] = 0, \quad a = (k,s), \partial = \frac{\partial}{\partial z}. \tag{1.4}$$

Consider first the quadratic Hamiltonians $H_{2,s}$. In this case $\rho_{2,s} = \mu_s$ are the $(-1,1)$-differentials (the Beltrami differentials). They are $(0,1)$-forms taking values in the vector fields on $\Sigma_{g,n}$. Here we deal with the Lie algebra of vector fields and the group of local diffeomorphisms of $\Sigma_{g,n}$. Roughly speaking the space $H^1(\Sigma_{g,n}, \Gamma)$ can be defined as the space of smooth $(-1,1)$-differentials on $\Sigma_{g,n}$ modulo global diffeomorphisms action. The elements from $H^1(\Sigma_{g,n}, \Gamma)$ play role of deformation parameters of the complex structure on $\Sigma_{g,n}$. In the genus zero case the complex structure is defined by the positions of marked points $t_s = x_s - x_s^0$ and (1.4) leads

[2] This restriction was partly resolved in the recent paper [3].
[3] See, however, [5, 6] for $g = 2$, $L \in sl(2, \mathbf{C})$

to the Schlesinger equation. Another interesting examples including a particular family of the Painlevé VI equation occur when the basic curve is an elliptic curve with marked points [8].

For $k > 2$ the differentials $\rho_{k,s}$ do not generate a Lie algebra. Due to this fact they have not natural geometric description. Along with the dual objects (opers [9]) they generate the so called $\mathcal{W}_k$-geometry of the basic curve $\Sigma_{g,n}$ [10, 11]. $\mathcal{W}_k$-geometry is a generalization of $\mathcal{W}_2$-geometry that coincides with the space of projective structures of $\Sigma_{g,n}$.

The invariant object associated with the Lax form (1.1) of integrable systems is the spectral curve $\mathcal{C}$

$$\mathcal{C}: \ f(\lambda, z) := \det(\lambda + L(z)) = 0.$$

It is N cover of the basic curve $\Sigma_{g,n}$. Then the Prym variety $Prym(\mathcal{C}/\Sigma_g)$ is the Liouvillian torus of the completely integrable system (1.1) [4]. There are different parametrizations of $\mathcal{C}$. The two standard parametrizations are the set of the Hamiltonians $H_{k,s}$ (1.3) and the action variables. The differentials $\rho_{k,s}$ (1.2) provide another parametrization of $\mathcal{C}$ related to the $\mathcal{W}_N$ structure of the basic curve. For small times (1.4) describes an evolution of $\mathcal{C}_t$ near the fixed curve $\mathcal{C}_0$.

The main goal of this paper is to investigate the dynamical systems (1.4) associated with the $\mathcal{W}_k$-geometry. There are two important aspects of this investigation. First, the quantum analog of these systems are higher order Knizhnik-Zamolodchikov-Bernard equations beyond the critical level. It means that conformal blocks in the WZW theory satisfy some analogue of nonstationar Schródinger equations with higher order Casimirs and times $t_{k,s}, k > 2$. We don't aware of some explicit examples of these type of equations. On the other hand, the investigation of the higher $\mathcal{W}$-geometries is interesting by itself. It is by no means an easy problem because apparently these geometries are not of the Klein type - there are no evident group symmetries related to them. The connection of the $\mathcal{W}$-geometries with the integrable systems opens a new way for investigations of the $\mathcal{W}$-geometries. Namely, one can apply tools developed for integrable systems, such as the Whitham quantization method [12] which is well adjusted to the analysis of the perturbations of integrals of motion (1.3).

Our construction is based on the approach developed in [10]. The $\mathcal{W}_k$ structures are described there as a result of Hamiltonian reduction with respect to a maximal parabolic subgroups of $\mathrm{SL}(k, \mathbf{C})$. We generalize this approach in two directions:
- we consider Riemann curves with marked points adding some additional data in the marked points;
- in addition to the $\mathcal{W}$ fields we include gauge fields.

As a result we obtain the classical Ward identities for $\mathcal{W}$-gravity interacting with gauge theory on Riemann curves with marked points. To obtain monodromy preserving flows we exclude the half of $\mathcal{W}$ fields leaving only differentials $\rho_{k,s}$, which play role of "times". This procedure is also based on the Hamiltonian reduction with respect to symmetries generated by the Sugawara type constraints. On this stage the flows are rather trivial, since the systems are free. Finally, the Hamiltonian reduction based on the gauge symmetries leads to nontrivial dynamical systems.

The plan of the paper is follows. In the first sections we revise the derivation of the equations preserving monodromies using the projective structure of basic curves. The similar program is done further in details for the $\mathcal{W}_3$ case. In conclusion we discuss shortly the general case.

2 Projective structures on Riemann curves and symplectic geometry

1. Projective structures.

Let us fix the complex structure on $\Sigma_{g,n}$ by choosing a pair of local coordinates $(z,\bar{z})$ and the corresponding operators $(\partial,\bar{\partial})$. The deformed complex structure can be read off from the solutions of the Beltrami equation

$$(\bar{\partial}+\mu\partial)F=0. \tag{2.1}$$

Locally, $F(z,\bar{z})$ is the diffeomorphism

$$w=F(z,\bar{z}),\ \bar{w}=\bar{F}(z,\bar{z})$$

and the Beltrami differential is

$$\mu(z,\bar{z})=-\frac{\bar{\partial}\bar{F}}{\partial\bar{F}}.$$

It defines the new complex structure operator $\partial_{\bar{w}}:=\bar{\partial}+\mu\partial$. Two Beltrami differentials produce equivalent complex structures if they are related by a global holomorphic diffeomorphisms of $\Sigma_{g,n}$

$$w=z-\epsilon(z,\bar{z}),\ \bar{w}=\bar{z}. \tag{2.2}$$

We assume that the Lie algebra $\mathcal{V}_{g,n}$ of corresponding vector fields on $\Sigma_{g,n}$ is specified by their behavior near the marked points

$$\mathcal{V}_{g,n}=\{\epsilon(z,\bar{z})\partial\mid\epsilon(z,\bar{z})=O(z-x_a)\} \tag{2.3}$$

Under the holomorphic diffeomorphisms μ transforms as $(-1,1)$-differential. The vector fields act on μ as

$$j_\epsilon\mu=-\epsilon\partial\mu+\mu\partial\epsilon+\bar{\partial}\epsilon. \tag{2.4}$$

We specify the dependence of μ on the positions of the marked points in the following way. Let $\mathcal{U}'_a\supset\mathcal{U}_a$ be two vicinities of the marked point x_a such that $\mathcal{U}'_a\cap\mathcal{U}'_b=\emptyset$ for $a\neq b$. Let $\chi_a(z,\bar{z})$ be a smooth function

$$\chi_a(z,\bar{z})=\begin{cases}1, & z\in\mathcal{U}_a\\ 0, & z\in\Sigma_g\setminus\mathcal{U}'_a.\end{cases} \tag{2.5}$$

Introduce times related to the positions of the marked points $t_{2,a}=x_a-x_a^0$. Then μ can be represented as

$$\mu=\sum_{a=1}^{n}t_{2,a}\mu_a^0,\ \ \mu_a^0=\bar{\partial}\epsilon_a(z,\bar{z}),\ \ \epsilon_a(z,\bar{z})=\chi_a(z,\bar{z}),\ \ (t_{2,a}=x_a-x_a^0). \tag{2.6}$$

Let T be the projective connection on $\Sigma_{g,n}$, i.e. T is transformed under the holomorphic diffeomorphisms as $(2,0)$-differential up to the addition of the Schwarzian derivative. Locally it means that

$$j_\epsilon T(z,\bar{z})=-\epsilon\partial T-2T\partial\epsilon-\frac{\kappa^2}{2}\partial^3\epsilon. \tag{2.7}$$

Here κ is a parameter, which later will play role of the "Planck constant" in the Whitham quantization. We assume that T has poles at the marked points $x_a,(a=1,\ldots,n)$ up to the second order:

$$T|_{z\to x_a}\sim\frac{T_{-2,a}}{(z-x_a)^2}+\frac{T_{-1,a}}{(z-x_a)}+\ldots \tag{2.8}$$

Let $\tilde{\mathcal{W}}_2$ be the space of the pairs (T,μ) on $\Sigma_{g,n}$ with the behavior near the marked points defined by (2.6),(2.8).

Definition 2.1 *The space $\mathcal{W}_2$ of projective structure on $\Sigma_{g,n}$ is the subset of $\tilde{\mathcal{W}}_2$ that satisfies the equation*

$$(\bar{\partial}+\mu\partial+2\partial\mu)T=\frac{1}{2}\partial^3\mu, \tag{2.9}$$

with fixed values of $\mathbf{T}_2=(T_{-2,1},\ldots,T_{-2,l})$ in (2.8).

Let ψ be a $(-\frac{1}{2},0)$ differential. Then (2.9) is the compatibility condition for the linear system

$$(\bar{\partial}+\mu\partial-\frac{1}{2}\partial\mu)\psi=0, \tag{2.10}$$

$$(\partial^2-T)\psi=0. \tag{2.11}$$

Consider two linear independent solutions ψ_1,ψ_2 to the system. The projective structure (T,μ) can be equivalently defined by their ratios $F=\psi_1/\psi_2$. In fact, it follows from (2.10) that F satisfies the Beltrami equation (2.9). Therefore, $\mu=\bar{\partial}F/\partial F$. On the other hand, from (2.11) $\psi_1=(\partial F)^{\frac{1}{2}}$ and $T=\mathcal{S}_z(F)$, where $\mathcal{S}_z(F)$ is the Schwarzian derivative of F. Arbitrary linear independent solutions of the system (2.10),(2.11) are obtained from ψ_1,ψ_2 by the SL$(2,\mathbf{C})$ transform. It results in the Möbius transform of F and does not change $T=\mathcal{S}_z(F)$. Thus, the relations of any two independent solutions defines projective structure as well.

2. Symplectic reduction with respect to diffeomorphisms.
The space $\mathcal{W}_2$ is similar to the space of flat connections on $\Sigma_{g,n}$ for some gauge group. It is a symplectic manifold, which can be derived from the affine space of smooth connections via the symplectic reduction. The flatness condition plays the role of the moment constraint equation. The similar procedure can be applied to the space $\tilde{\mathcal{W}}_2$ to obtain $\mathcal{W}_2$. In this case the gauge group is replaced by the group of holomorphic diffeomorphisms of $\Sigma_{g,n}$ (2.2). The space $\tilde{\mathcal{W}}_2$ can be endowed with the symplectic structure

$$\omega=-\kappa^{-1}\int_{\Sigma_g}\delta T\delta\mu. \tag{2.12}$$

The action of the vector fields $\mathcal{V}_{g,n}$ (2.4),(2.7) is the symmetry of ω. The Hamiltonian of this action

$$H_\epsilon=-\kappa^{-1}\int_{\Sigma_g}\epsilon[(\bar{\partial}+\mu\partial+2\partial\mu)T-\frac{\kappa^2}{2}\partial^3\mu]$$

produces the moment map

$$m:\tilde{W}_2\to\mathcal{V}^*_{g,n},\quad m=(\bar{\partial}+\mu\partial+2\partial\mu)T-\frac{\kappa^2}{2}\partial^3\mu, \tag{2.13}$$

where $\mathcal{V}^*_{g,n}$ is the dual space to the algebra $\mathcal{V}_{g,n}$ of vector fields. It is the space of $(2,1)$-forms on $\Sigma_{g,n}$. As it follows from (2.3) in the neighborhoods of marked points elements $y\in\mathcal{V}^*_{g,n}$ take the form

$$y\sim b_{1,a}\partial\delta(x_a)+b_{2,a}\partial^2\delta(x_a)+\ldots. \tag{2.14}$$

Thereby, the terms $T_{-1,a}\delta(x_a)$ that arise in $\bar{\partial}T$ (2.13) from the first order poles of T (2.8) are projected out from the moment map m (2.13). We take

$$m=-\sum_{a=1}^{n}T_{-2,a}\partial\delta(x_a).$$

It follows from (2.8),(2.13) and (2.14) that we put in (2.14) $b_{1,a} = -T_{-2,a}$, $b_{k,a} = 0$, $k > 1$. Thus, the condition (2.9) that distinguish $\mathcal{W}_2$ in $\tilde{\mathcal{W}}_2$ is the moment constraint with respect to the action of the diffeomorphisms.

3."Drinfeld-Sokolov" approach. In [10] another procedure was proposed, which resembles the Drinfeld-Sokolov approach. Shortly, it looks as follows. Consider the affine space $\mathcal{N}_2$ of SL(2, **C**) smooth flat connections on $\Sigma_{g,n}$

$$\mathcal{N}_2 = \{adz + \bar{a}d\bar{z}\},$$

$$F(a,\bar{a}) = \bar{\partial}a - \partial\bar{a} + [a,\bar{a}] = 0. \tag{2.15}$$

The field a has poles in the marked points up to the second order. The space $\mathcal{N}_2$ has the standard symplectic form

$$\omega' = \int_{\Sigma_g} \mathrm{tr}(\delta a \delta \bar{a}). \tag{2.16}$$

The form is invariant under the gauge transform

$$a \to g^{-1}ag + g^{-1}\partial g, \quad \bar{a} \to g^{-1}\bar{a}g + g^{-1}\bar{\partial}g, \quad g \in \mathcal{G} = \mathrm{Map}(\Sigma_{g,n}, \mathrm{SL}(2,\mathbf{C}))$$

We assume that the Lie algebra of the gauge group Lie($\mathcal{G}$) is specialized by the behavior of its matrix element x^{12} near the marked points

$$x^{12}|_{z \to x_a} = O(z - x_a), \quad (x^{JK}) \in \mathrm{Lie}(\mathcal{G}). \tag{2.17}$$

The flatness (2.15) is the moment constraint with respect to the action of the gauge group. The form ω' is degenerated - it vanishes on the orbits of the gauge group, because we only put the moment condition (2.15) and do not fix the gauge. If we do it we come to the finite-dimensional space of flat connections, but it is necessary to leave two fields T and μ. The trick proposed in [10] is to fix the gauge with respect to the Borel subgroup B of the lower triangular matrices. It was proved there that it allows to obtain from the space $\mathcal{N}_2$ the space of projective connections $\mathcal{W}_2$.

The gauge freedom allows to fix a generic matrix a in the form

$$a = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \tag{2.18}$$

where T is a new field satisfying (2.8). The first order poles of T do not contribute in the moment equation (2.15) since they are eaten by the gauge transform (see (2.17)). The moment equation (2.15) allows to express all matrix elements of $\bar{a}$ in terms of T and new field μ

$$\bar{a} = \begin{pmatrix} \frac{1}{2}\partial\mu & -\kappa^{-1}\mu \\ -\kappa^{-1}\mu T + \frac{1}{2}\kappa\partial^2\mu & -\frac{1}{2}\partial\mu \end{pmatrix}. \tag{2.19}$$

The flatness condition becomes trivial for all matrix element except $F_{(2,1)}$. It can be checked that it just coincides with the projectivity condition (2.9).

The linear system for $(-\frac{1}{2}, 0)$ differentials ψ

$$(\bar{\partial} + \bar{a})\begin{pmatrix} \psi \\ \partial\psi \end{pmatrix} = 0, \quad (\kappa\partial + a)\begin{pmatrix} \psi \\ \partial\psi \end{pmatrix} = 0$$

is consistent due to (2.15). It is the matrix form of (2.10),(2.11) for the special form of a and $\bar{a}$ (2.18),(2.19).

The original symplectic structure ω' (2.16) is reduced to ω (2.12) on the space of the projective structures $\mathcal{W}_2$.

Though, the diffeomorphisms do not arise in this approach, they are hidden in this construction. To demonstrate it, calculate the commutator of two matrices $\bar{a}_1$, $\bar{a}_2$

$$[\bar{a}_1(\mu_1, T_1), \bar{a}_2(\mu_2, T_2)]_{(1,2)} = \mu_1 \partial \mu_2 - \mu_2 \partial \mu_1.$$

Thus, the commutator of matrix $\bar{a}$ reproduces the commutator of vector fields.

3 Isomonodromic deformations and projective structures

1.$\mathcal{W}_2^N$-structures - definition.

Consider some projective structure on Σ_g defined by the linear system (2.10),(2.11). We generalize it in the following way. Consider the vector $\mathrm{SL}(N, \mathbf{C})$-bundle V over Σ_g. Let $(A, \bar{A})$ be connections in V corresponding to the complex structure we have fixed.

We assume that

- the connection $\bar{A}$ is smooth;
- the connection A has first order poles in the marked points

$$A \sim \frac{A_{-1,a}}{z - x_a} + A_{0,a} + \dots, \; a = 1, \dots, n. \tag{3.1}$$

In addition to this data consider the set of coadjoint orbits of $\mathrm{SL}(N, \mathbf{C})$ in the marked points

$$(\mathcal{O}_1, \dots, \mathcal{O}_n), \quad \mathcal{O}_a = \{(p_a = g_a p_a^0 g_a^{-1} | g_a \in \mathrm{SL}(N, \mathbf{C})\}. \tag{3.2}$$

Here p_a^0 specifies the choice of the orbit $\mathcal{O}_a$. To reconcile this data with the projective structures instead of the operator (2.11) we consider in what follows the matrix operator $(\kappa\partial + A)^2 - T$. Define the matrices

$$\tilde{T} = T - A^2 - \kappa \partial A. \tag{3.3}$$

and

$$f_1 = -\bar{A} + \frac{1}{2}\partial\mu \mathbf{1}_N - \frac{1}{\kappa}\mu A. \tag{3.4}$$

We skip the multiplication on the scalar matrices $\mathbf{1}_N$ in what follows.

Define the space $\tilde{\mathcal{W}}_2^N$ of fields $(T, \mu), (A, \bar{A}), \mathbf{p} = (p_1, \dots, p_l)$ on $\Sigma_{g,n}$, where the behavior of μ, T and A in the neighborhood of marked points satisfies (2.6),(2.8) and (3.1) correspondingly.

Definition 3.1 *$\mathcal{W}_2^N$-structure on Σ_g is the subset of $\tilde{\mathcal{W}}_2^N$ satisfying the following identities:*

$$(\bar{\partial} + \mu\partial + 2\partial\mu)\tilde{T} - \frac{1}{2}\kappa^2\partial^3\mu = [\tilde{T}, \bar{A} + \frac{1}{\kappa}\mu A] + 2\kappa A \partial f_1, \tag{3.5}$$

$$\bar{\partial} A - \kappa \partial \bar{A} + [A, \bar{A}] = 0, \tag{3.6}$$

$$A_{-1,a} = -p_a, \; (a = 1, \dots, n), \tag{3.7}$$

$$T_{-2,a} = \frac{< (p_a^0)^2 >}{N} \; (a = 1, \dots, n). \tag{3.8}$$

In the first relation the right hand side is the contribution of the gauge fields in the classical Ward identity for the gravitational fields (T,μ). The second identity is the standard flatness condition for the gauge fields. The last identity expresses the most singular terms $T_{-2,a}$ of the projective connection T in terms of the second Casimirs parametrizing the orbits $\mathcal{O}_a$.

There is a straightforward generalization of the linear system (2.10),(2.11) defining the projective structure. Let Ψ be the element of the space of sections $\Omega^{(-\frac{1}{2},0)}(\Sigma_g, \mathrm{Aut}V)$.

Proposition 3.1 *The equations (3.5),(3.6), are the compatibility conditions for the linear system*

$$(\bar{\partial} + \mu\partial - \frac{1}{2}\partial\mu + \bar{A} + \frac{1}{\kappa}\mu A)\Psi = 0, \tag{3.9}$$

$$[(\kappa\partial + A)^2 - T]\Psi = 0. \tag{3.10}$$

The space $\mathcal{W}_2^N$ can be endowed with the degenerated symplectic form

$$\omega = \int_{\Sigma_{g,n}} [-\frac{N}{\kappa}\delta T\delta\mu + 2 < \delta A, \delta\bar{A} >] + 4\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a > . \tag{3.11}$$

The last sum in (3.11) is the contribution of symplectic forms on the coadjoint orbits. We demonstrate below that this form is natural and come from a Hamiltonian reduction procedure.

2. Symplectic construction.
Consider the vector bundle $V_{2N} = V_2 \otimes V_N$ over Σ_g with the structure group $G = S(\mathrm{GL}(2,\mathbf{C}) \otimes \mathrm{GL}(N,\mathbf{C})) \sim \mathrm{SL}(2N^2,\mathbf{C})$. In the nondeformed complex structure on $\Sigma_{g,n}$ the connection operators are

$$\kappa\partial - \mathcal{A}, \quad \bar{\partial} - \bar{\mathcal{A}},$$

where $\mathcal{A}, \bar{\mathcal{A}}$ take values in $\mathrm{Lie}(S(\mathrm{GL}(2,\mathbf{C}) \otimes \mathrm{GL}(N,\mathbf{C})))$. The first components of $\mathcal{A}$ and $\bar{\mathcal{A}}$ act on the sections of 1-jets of $\Omega^{(-\frac{1}{2},0)}(\Sigma_{g,n})$, while the second components act on the sections of the bundle $\Omega^{(-\frac{1}{2},0)}(\Sigma_{g,n}, \mathrm{Aut}V)$.

Define the space $\mathcal{K}_2$ as the space of connections $(\mathcal{A}, \bar{\mathcal{A}})$ with coadjoint orbits attached in the marked points

$$(\mathcal{A}, \bar{\mathcal{A}}); (\mathbf{1}_2 \otimes \mathbf{p}) = (\mathbf{1}_2 \otimes p_1, \ldots, \mathbf{1}_2 \otimes p_l)$$

with the following additional restrictions:
- the $\mathcal{A}$ component are gauge equivalent to the special form

$$\mathcal{A} \sim DS \otimes \mathbf{1}_N + \mathbf{1}_2 \otimes (-A), \tag{3.12}$$

where

$$DS = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix};$$

- near the marked points T has the form (2.8);
- near the marked points A has the form (3.2).

Propositions 3.1 follow from the following statement

Proposition 3.2 $\mathcal{W}_2^N$ *is the subset* $\mathcal{K}_2$ *satisfying the flatness condition*

$$[\kappa\partial - \mathcal{A}, \bar{\partial} - \bar{\mathcal{A}}] = 0. \tag{3.13}$$

Proof.
The space $\mathcal{K}_2$ has the standard symplectic form

$$\omega' = \int_{\Sigma_{g,n}} << \delta\mathcal{A}, \delta\bar{\mathcal{A}} >> +4\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a > \tag{3.14}$$

Here $<<,>>$ is the trace in the tensor product, and $<,>$ is the trace in the V_N space.

Introduce the following group of gauge transforms. It is the smooth maps

$$\mathcal{G} = \{\mathrm{Map}(\Sigma_{g,n}, \mathrm{SL}(2N^2, \mathbf{C}))\}$$

with additional restrictions for the maps near the marked points. We formulate them on the Lie algebra Lie($\mathcal{G}$) level. Let $x^{IJ}_{\alpha\beta}$ be the matrix element in Lie($\mathcal{G}$). Here the upper indices I, J are related to the space V_2 and the lower indices α, β to the space V_N. We assume that in the neighborhood of the marked point

$$x^{12}_{\alpha\beta} \sim \delta_{\alpha\beta} O(z - x_a) + (1 - \delta_{\alpha\beta}) O(z - x_a)^2, \tag{3.15}$$

while the other matrix element are continuae in the marked points. It can be checked that these matrices define the Lie subalgebra Lie($\mathcal{G}$) in the Lie algebra of smooth maps.

The form ω' (3.14) is invariant under the gauge transform

$$\begin{gathered} \bar{\mathcal{A}} \to f^{-1}\bar{\partial} f + f^{-1}\bar{\mathcal{A}} f, \quad f \in \mathcal{G}, \\ \mathcal{A} \to f^{-1}\kappa\partial f + f^{-1}\mathcal{A} f \\ g_a \to g_a f_a, \; p_a \to f_a^{-1} p_a f_a, \; f_a = f(x_a, \bar{x}_a). \end{gathered} \tag{3.16}$$

We identify the dual space (Lie)*($\mathcal{G}$) with matrices living on $\Sigma_{g,n}$ by means of the bilinear form

$$\int_{\Sigma_g} << x, y >>, \; (x \in \mathrm{Lie}(\mathcal{G}), \; y \in \mathrm{Lie}^*(\mathcal{G}).$$

Due to (3.15) the matrix elements $y \in$ (Lie)*($\mathcal{G}$) have the following form in the marked points

$$y^{21}_{\alpha\alpha} \sim \sum_{k\geq 1} b_k \partial^k \delta(x_a), \quad y^{21}_{\alpha\beta} \sim \sum_{k\geq 2} b_{k,\alpha,\beta} \partial^k \delta(x_a), \; (\alpha \neq \beta). \tag{3.17}$$

$$y^{JK}_{\alpha\beta} \sim \sum_{k\geq 0} b_{k,\alpha,\beta} \partial^k \delta(x_a), \quad (J \neq 2, \; K \neq 1). \tag{3.18}$$

The form ω' (3.14) is degenerated on the orbits of the gauge group. The condition (3.12) is similar to the partial gauge fixing. We choose $\mathcal{A}$ in the form

$$\mathcal{A} = \begin{pmatrix} 0 & E \\ \tilde{T} & -2A \end{pmatrix}, \tag{3.19}$$

where $\tilde{T}$ is given by (3.3). This form of $\mathcal{A}$ is the gauge transform of (3.12) by

$$f = \begin{pmatrix} \mathbf{1}_N & 0 \\ A & \mathbf{1}_N \end{pmatrix}.$$

As usual, the curvature tensor $F(\mathcal{A}, \bar{\mathcal{A}})$ is the moment map

$$m : (\mathcal{A}, \bar{\mathcal{A}}, \mathbf{p}) \to (\mathrm{Lie})^*(\mathcal{G}).$$

The flatness (3.13) means that we take $m = 0$. It allows to define $\bar{\mathcal{A}}$. The general solution of (3.13) depends on two fields $\bar{A}$ and μ and takes the form

$$\bar{\mathcal{A}} = \begin{pmatrix} -\bar{A} + \frac{1}{2}\partial\mu - \kappa^{-1}\mu A & -\kappa^{-1}\mu \\ -\kappa^{-1}\mu\tilde{T} - \kappa\partial(\bar{A} - \frac{1}{2}\partial\mu + \kappa^{-1}\mu A) & -\bar{A} + \kappa^{-1}\mu A - \frac{1}{2}\partial\mu \end{pmatrix}, \tag{3.20}$$

where $\bar{A}$ takes value in Lie(SL($N, \mathbf{C}$)). It can be checked immediately that (3.13) becomes the identity for the GL($2, \mathbf{C}$) matrix elements $(1,1)$ and $(1,2)$ and coincides with (3.5) and (3.6) for $(2,1)$ and $(2,2)$ respectively.

Now let us discuss the boundary condition (3.7),(3.8). Since $m = 0$ all coefficients in (3.17) and (3.18) of $F^{2,1}(\mathcal{A}, \bar{\mathcal{A}})$ and $F^{2,2}(\mathcal{A}, \bar{\mathcal{A}})$ vanish. The expression $\bar{\partial}\tilde{T}$ in the matrix element $(2,1)$ has the terms proportional to $\partial\delta(x_a)$. Their cancellation lead to (3.8). In the similar way, using (3.18) we obtain (3.7) from $\bar{\partial}A$ in the matrix element $(2,2)$.$\square$

The flatness of these connections is equivalent to the compatibility of the linear system

$$(\kappa\partial - \mathcal{A})\begin{pmatrix} \psi \\ \kappa\partial\psi \end{pmatrix} = 0, \quad (\bar{\partial} - \bar{\mathcal{A}})\begin{pmatrix} \psi \\ \kappa\partial\psi \end{pmatrix} = 0.$$

Substituting in this system $\mathcal{A}$ (3.19) and $\bar{\mathcal{A}}$ (3.20) we obtain the linear system (3.9), (3.10) in Proposition 3.1.

The reduced symplectic form is read off from (3.14),(3.19) and (3.20). It coincides with ω (3.11).

3. Isomonodromic deformations.
To obtain nontrivial dynamical systems with Hamiltonians we modified our previous procedure. Instead of the gauge group $\mathcal{G}$ defined in (3.16) consider its parabolic subgroup

$$\mathcal{G}^P = \left\{\Sigma_{g,n} \to \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \otimes \mathrm{SL}(N, \mathbf{C})\right\},$$

Due to this choice there are no moment constraints coming from the matrix elements $(2,1)$ of GL($2, \mathbf{C}$). Therefore, the conditions (3.5),(3.8) are absent in this case. In this way we come to the manifold $\mathcal{K}_2^N$ with the constraints (3.6),(3.7) and the symplectic form (3.11). Evidently $\mathcal{W}_2^N \subset \mathcal{K}_2^N \subset \mathcal{K}_2$.

Consider the following transformations of the fields in $\mathcal{K}_2^N$

$$\mu \to \mu + \xi, \quad \bar{A} \to \bar{A} + \frac{1}{\kappa}A\xi, \quad T \to T, \quad A \to A, \tag{3.21}$$

where $\xi \in \Omega^{(-1,1)}(\Sigma_g)$. It is the symmetry of the form ω (3.11). The transformations (3.21) are generated by the Hamiltonian

$$\mathcal{H}_\xi = \int_{\Sigma_{g,n}} \xi(T - \frac{< A^2 >}{N}).$$

We put $\mathcal{H}_\xi = 0$. It means, that the Sugawara relation

$$T = \frac{1}{N} < A^2 > \tag{3.22}$$

is the moment constraint with respect to the symmetry (3.21). We don't fix the gauge and thereby come to the space with fields

$$\mathcal{P}_2^N = \{A, \bar{A}, \mathbf{p}, \mu\} \sim \mathcal{K}_2^N/(T = \frac{1}{N} < A^2 >).$$

It is a bundle over the space of times $\{\mu\}$. It follows from (3.11) that on this space ω take the form

$$\frac{1}{2}\omega = \int_{\Sigma_g} < \delta A, \delta \bar{A} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1} \delta g_a > -\frac{1}{2\kappa} \int_{\Sigma_g} \delta < A^2 > \delta\mu. \tag{3.23}$$

Since we did not fix the gauge of this transformation and preserve the field μ the form ω (3.23) is degenerated on $\mathcal{P}_2^N$. If one replaces

$$\bar{A}' = \bar{A} - \frac{1}{\kappa}\mu A, \tag{3.24}$$

(3.23) takes the canonical form

$$\frac{1}{2}\omega = \int_{\Sigma_g} < \delta A, \delta \bar{A}' > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1} \delta g_a > . \tag{3.25}$$

Now we pass to the finite-dimensional space of equivalent complex structures $\mathcal{T}_{g,n}$ (the Teichmüller space). The tangent space to the Teichmüller space is $H^1(\Sigma_g, \Gamma)$. Note, that only elements of $H^1(\Sigma_g, \Gamma)$ contribute in the second integral in (3.23). According to the Riemann-Roch theorem $H^1(\Sigma_g, \Gamma)$ has dimension

$$l_2 = \dim H^1(\Sigma_g, \Gamma) = 3g - 3 + n. \tag{3.26}$$

We fix a reference point $\mu^0 = (\mu_1^0, \ldots, \mu_{l_2}^0)$. Then

$$\mu = \sum_{s=1}^{l_2} t_s \mu_s^0. \tag{3.27}$$

defines a new complex structure and $\mathbf{t} = (t_1, \ldots, t_{l_2})$ are coordinates of the tangent vector to μ^0 in $H^1(\Sigma_g, \Gamma)$.

Expanding μ in the basis (3.27) we rewrite ω as

$$\frac{1}{2}\omega = \omega^0 - \frac{1}{\kappa}\sum_s \delta H_s \delta t_s, \quad H_s = H_s(A, \mathbf{t}) = \frac{1}{2}\int_{\Sigma_g} < A^2 > \mu_s^0, \tag{3.28}$$

where

$$\omega^0 = \int_{\Sigma_g} < \delta A, \delta \bar{A} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1} \delta g_a >, \quad (\mathbf{t} = (t_1, \ldots, t_l), \ \partial_s = \partial/\partial t_s).$$

We keep the same notations for the space $\mathcal{P}_2^N$

$$\begin{array}{c} \mathcal{P}_2^N \\ \downarrow \\ \mathcal{T}_g \end{array} \sim \mathcal{R} = \{A, \bar{A}, \mathbf{p}\}.$$

$\mathcal{P}_2^N$ is the extended phase space. The form ω^0 is nondegenerated on the fibers. In this situation the equation of motion for any function F on $\mathcal{P}_2^N$ takes the form [13]

$$\kappa\frac{dF}{dt_s} = \kappa\frac{\partial F}{\partial t_s} + \{H_s, F\}_{\omega^0}.$$

In addition, there are the consistency conditions for the Hamiltonians (*the Whitham equations*)

$$\kappa\partial_s H_r - \kappa\partial_r H_s + \{H_r, H_s\} = 0. \tag{3.29}$$

They are satisfied since the Hamiltonians (3.28) commute. On this stage the equations of motion are trivial.

$$\partial_s A = 0, \quad \partial_s \bar{A} = \frac{1}{2} A\mu_s^0 \quad (s = 1, \ldots, l). \tag{3.30}$$

We call this system the *Hierarchy of the Isomonodromic Deformations* (HID). This notion will be justified below.

The degenerated symplectic form (3.28) is invariant under the gauge transformations $\tilde{\mathcal{G}} = \mathrm{Map}(\Sigma_{g,n}\mathrm{SL}(N, \mathbf{C}))$

$$A \to f^{-1}Af + f^{-1}\kappa\partial f, \ \bar{A} \to f^{-1}\bar{A}f + f^{-1}(\bar{\partial} + \mu\partial)f, \ p_a \to f^{-1}(x_a)p_a f(x_a), \ g_a \to g_a f(x_a). \tag{3.31}$$

The flatness condition

$$(\bar{\partial} + \partial\mu)A - \kappa\partial\bar{A} + [\bar{A}, A] = 0. \tag{3.32}$$

is the moment constraint generating this symmetry. It allows to consider the linear consistent system on Σ_g

$$(\kappa\partial + A)\Psi = 0, \tag{3.33}$$

$$(\bar{\partial} + \sum_s t_s \mu_s^0 \partial + \bar{A})\Psi = 0, \tag{3.34}$$

where $\Psi \in \Omega^0(\Sigma_{g,n}, V)$. The monodromy matrix $Y \in \mathrm{Rep}(\pi_1(\Sigma_{g,n})) \to \mathrm{SL}(N, \mathbf{C})$ transforms solutions as

$$\Psi \to \Psi Y.$$

The isomonodromy of (3.33),(3.34) is the independence of Y on the deformations of the complex structure of $\Sigma_{g,n}$.

Proposition 3.3 *The linear system (3.33),(3.34) has the property of the isomonodromic deformations with respect to the "times" t_s iff the equations of motion (3.30) are satisfied. In other words, (3.30) are monodromy preserving equations.*

Proof
It follows from (3.30) that ∂_s commute with $(\kappa\partial + A)$ and $(\bar{\partial} + \sum_s t_s\mu_s^0\partial + \bar{A})$. Thus, in addition to (3.33),(3.34) one has consistent equations

$$\partial_s \Psi = 0, \quad (s = 1, \ldots, l). \tag{3.35}$$

Then it follows from (3.35) that

$$\partial_s Y = 0, \quad (s = 1, \ldots, l). \tag{3.36}$$

Assume now that the monodromy of the linear system (3.33),(3.34) is the time independent (3.36). Then (3.35) is fulfilled. The first equation of motion (3.30) follows from (3.33) and (3.35) and the second from (3.34) and (3.35). □

The equations (3.30) are not very interesting, because they describe a free motion. They become nontrivial after the Hamiltonian reduction with respect to the gauge transformations (3.31). The flatness (3.23) is the moment equation. Consider the gauge fixing assuming that $\bar{L}$ parametrizes the orbits of the gauge group $\tilde{\mathcal{G}}$ in the phase space $\mathcal{R}$

$$\bar{L} = f^{-1}\bar{A}f + f^{-1}(\bar{\partial} + \mu\partial)f. \tag{3.37}$$

It allows partly to fix f. At the same time the gauge transformation defines

$$L = f^{-1}Af + f^{-1}\kappa\partial f.$$

We keep the same notations for the transformed matrices p_a. Substituting these two expressions in the moment equation we obtain

$$(\bar{\partial} + \partial\mu)L - \kappa\partial\bar{L} + [\bar{L}, L] = 0, \tag{3.38}$$

where according to (3.1),(3.7) $\mathrm{Res}L|_{z=x_a} = p_a$. The gauge fixing (3.37) and the moment constraint (3.38) kill almost all degrees of freedom. The fibers $\mathcal{R}^{red} = \{L, \bar{L}, \mathbf{p}\}$ become finite-dimensional, as well as the bundle $\mathcal{P}_2^{red,N}$. The form ω (3.28) on $\mathcal{P}_2^{red,N}$ is

$$\omega^0 = \int_{\Sigma_{g,n}} < \delta L, \delta\bar{L} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a >, \quad H_s = H_s(L, \mathbf{t}) = \frac{1}{2}\int_{\Sigma_{g,n}} < L^2 > \mu_s^0. \tag{3.39}$$

But now the system is no long free because due to (3.38) L depends on $\bar{L}$ and $\mathbf{p}$. The equations of motion (3.30) take the form

$$\kappa\partial_s L - \kappa\partial M_s + [M_s, L] = 0, \quad M_s = \partial_s f f^{-1}, \tag{3.40}$$

$$\kappa\partial_s\bar{L} - (\bar{\partial} + \mu\partial)M_s + [M_s, \bar{L}] = L\mu_s^0. \tag{3.41}$$

The equation (3.40) is the analog of the Lax equation. The essential difference is the differentiation with respect to the spectral parameter ∂. The last equation determines the matrices M_s. These equations are nontrivial and for the genus $g = 0, 1$ reproduce the Schlesinger system, Elliptic Schlesinger system, multicomponent generalization of the Painlevé VI equation [8]. The equations (3.40), (3.41) along with (3.38) are consistency conditions for the linear system

$$(\kappa\partial + L)\Psi = 0, \tag{3.42}$$

$$(\bar{\partial} + \sum_s t_s\mu_s^0\partial + \bar{L})\Psi = 0, \tag{3.43}$$

$$(\partial_s + M_s)\Psi = 0, \quad (s = 1, \ldots, l_2). \tag{3.44}$$

As in Proposition 3.3 the equations (3.44) provides the isomonodromy property of the system (3.42), (3.43) with respect to variations of the times t_s.

4.Scaling limit.

Consider the limit $\kappa \to 0$. The value $\kappa = 0$ is called critical. The symplectic form ω (3.28) is singular in this limit. Let us replace the times

$$t_s \to t_s^0 + \kappa t_s,$$

and assume that the times t_s^0, $(s = 1, \ldots)$ are fixed. After this rescaling the form (3.28) become regular. The rescaling procedure means that we blow up a vicinity of the fixed point $\mu_s^{(0)}$ in $\mathcal{T}_{g,n}$ and the whole dynamic is developed in this vicinity. This fixed point is defined by the complex coordinates

$$w_0 = z - \sum_s t_s^0\epsilon_s(z, \bar{z}), \quad \bar{w}_0 = \bar{z}, \quad (\partial_{\bar{w}_0} = \bar{\partial} + \sum_s t_s^0\mu_s^0). \tag{3.45}$$

For $\kappa = 0$ the connection A is transformed into the one-form Φ (the Higgs field) $\kappa\partial + A \to \Phi$, (see (3.31)). Let $L^0 = \lim_{\kappa\to 0} L$, $\bar{L}^0 = \lim_{\kappa\to 0} \bar{L}$. Then we obtain the autonomous Hamiltonian systems with the form

$$\omega^0 = \int_{\Sigma_{g,n}} < \delta L^0, \delta \bar{L}^0 > + 2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1} \delta g_a >$$

and the commuting quadratic integrals (3.29). The phase space $\mathcal{R}^{red}$ is the cotangent bundle to the moduli of stable holomorphic SL$(N, \mathbf{C})$-bundles over $\Sigma_{g,n}$. These systems are completely integrable for the SL$(2, \mathbf{C})$ bundles [7].

The corresponding set of linear equations has the following form. The level κ can be considered as the Planck constant (see (3.42)). We consider the quasi-classical regime

$$\Psi = \phi \exp \frac{\mathcal{S}}{\kappa},$$

where ϕ is a group-valued function and $\mathcal{S}$ is a scalar phase. Assume that

$$\frac{\partial}{\partial \bar{w}_0}\mathcal{S} = 0, \quad \frac{\partial}{\partial t_s}\mathcal{S} = 0.$$

In the quasi-classical limit we set

$$\partial \mathcal{S} = \lambda.$$

Then instead of (3.42), (3.43), (3.44) we obtain

$$(\lambda + L^0)\Psi = 0,$$

$$(\bar{\partial}_{\bar{w}_0} + \lambda \sum_s t_s \mu_s^0 + \bar{L}^0)\Psi = 0,$$

$$(\partial_s + M_s^0)\Psi = 0, \quad (s = 1, \ldots, l_2).$$

Note, that the consistency conditions for the first and the last equations are the standard Lax equations (1.1).

4 $\mathcal{W}_3$ and Isomonodromic deformations

1. $\mathcal{W}_3$-structure.

Define the space $\tilde{\mathcal{W}}_3$ of fields $(W, \rho), (T, \mu)$ on $\Sigma_{g,n}$, where (T, μ) are the same as in the case of the projective structures, ρ is the $(-2, 1)$-differential and a variation of W is $(3, 0)$-differential [10, 11]. We assume that behavior of ρ near the marked points has the form (compare with (2.6))

$$\rho|_{z\to x_a} \sim (t_{3,a,0} + t_{3,a,1}(z - x_a^0))\bar{\partial}\chi_a(z, \bar{z}). \tag{4.1}$$

The dual field W has poles in the marked points

$$W|_{z\to x_a} \sim \frac{W_{-3,a}}{(z - x_a)^3} + \frac{W_{-2,a}}{(z - x_a)^2} + \frac{W_{-1,a}}{(z - x_a)} + \ldots \tag{4.2}$$

Definition 4.1 *The space of $\mathcal{W}_3$-structure on $\Sigma_{g,n}$ is the subset of fields in $\tilde{\mathcal{W}}_3$ that satisfy the equations*

$$\kappa^2\partial^3\mu - (\bar{\partial} + \mu\partial + 2\partial\mu)T = \frac{2}{3}\kappa^2\partial^2(\partial^2 - \frac{1}{\kappa^2}T)\rho + \frac{1}{\kappa}\partial(W\rho) - \frac{1}{\kappa}W(\mu - \partial\rho) \tag{4.3}$$

$$(\bar{\partial} + \rho\partial^2 + (\mu + 2\partial\rho)\partial + 3\partial\mu +)W = \kappa^3\partial[(\partial^2 - \frac{1}{\kappa^2}T)(\partial\mu - \frac{2}{3}(\partial^2 - \frac{1}{\kappa^2}T)\rho)]. \tag{4.4}$$

with fixed values of

$$\mathbf{T}_2 = (T_{-2,1}, \dots, T_{-2,n}) \tag{4.5}$$

in (2.7) and

$$\mathbf{W}_2 = (W_{-3,1}, \dots, W_{-3,n}) \tag{4.6}$$

in (4.2).

Proposition 4.1 *The equations (4.3),(4.4), are the compatibility conditions for the linear system*

$$(\kappa^3\partial^3 + \kappa T\partial + W)\Psi = 0, \tag{4.7}$$

$$(\bar{\partial} + \frac{2}{3}(\partial^2 - \frac{1}{\kappa^2}T)\rho - \partial\mu + (\mu - \partial\rho)\partial - \rho\partial^2)\Psi = 0, \tag{4.8}$$

where Ψ is a section of $\Omega^{(-1,0)}(\Sigma_{g,n})$.

The space $\mathcal{W}_3$ can be endowed with the symplectic form

$$\omega = \int_{\Sigma_{g,n}} (\delta T\delta\mu + \delta W\delta\rho). \tag{4.9}$$

As for the projective connections the space $\mathcal{W}_3$ can be derived from the space $\mathcal{N}_3$ of flat $\mathrm{SL}(3,\mathbf{C})$ connections

$$\mathcal{N}_3 = \{a, \bar{a} | F(a, \bar{a}) = 0\}$$

over $\Sigma_{g,n}$ with the symplectic form (2.16). In fact, (4.9) is induced by (2.16). The flatness constraints generate the gauge symmetry of $\mathcal{N}_3$. We assume that the matrix elements of the Lie algebra of the gauge group are continuous in the marked points and, moreover,

$$x^{13}|_{z\to x_a} = O(z - x_a)^2, \quad x^{23}|_{z\to x_a} = O(z - x_a). \tag{4.10}$$

(see (2.17)). In fact, (4.10) defines a subalgebra in the Lie algebra of continuous gauge transformations.

The partial gauge fixing with respect to the parabolic subgroup [10]

$$P = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \tag{4.11}$$

allows to pick up a special form of connections

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W & T & 0 \end{pmatrix} \tag{4.12}$$

with the prescribed behavior of W and T in neighborhoods of the marked points. Then the exact form of the matrix $\bar{a}$ as well as (4.3), (4.4) are extracted from the flatness condition. The defining properties (4.10) of the gauge algebra allows to fix the coefficients of highest poles (4.5),(4.6). They define the value of the moment $m = F(a, \bar{a})$. We generalize this procedure below.

The flatness is the compatibility conditions for the linear system

$$(\kappa\partial - a)\begin{pmatrix}\psi\\ \kappa\partial\psi\\ \kappa^2\partial^2\psi\end{pmatrix} = 0, \quad (\bar{\partial} - \bar{a})\begin{pmatrix}\psi\\ \kappa\partial\psi\\ \kappa^2\partial^2\psi\end{pmatrix} = 0,$$

where $\psi \in \Omega^{(-1,0)}(\Sigma_{g,n})$. For the special form of a (4.12) and $\bar{a}$ it leads to (4.7),(4.8) in Proposition 4.1.

2. $\mathcal{W}_3^N$ structures.
As in **3.1** consider SL$(N, \mathbf{C})$-bundle V over $\Sigma_{g,n}$ with a fixed complex structure. Let $(A, \bar{A})$ be connections in V. We generalize the $\mathcal{W}_3$ structure (4.3),(4.4) taking into account the gauge degrees of freedom.

Define the space

$$\tilde{\mathcal{W}}_3^N = \tilde{\mathcal{W}}_2^N \cup (W, \rho),$$

where ρ and W satisfy (4.1),(4.2). Introduce the following matrices

$$\tilde{T} = T - 3(A^2 + \kappa\partial A), \tag{4.13}$$

$$\tilde{W} = W + TA - A^3 - \kappa(A\partial A + \partial A^2) - \kappa^2\partial^2 A. \tag{4.14}$$

Definition 4.2 *$\mathcal{W}_3^N$ structure on $\Sigma_{g,n}$ is the subset of fields in $\tilde{\mathcal{W}}_3^N$ satisfying the following identities:*

$$\bar{\partial}\tilde{W} - \kappa\partial f_5 + \tilde{W}f_1 + \tilde{T}f_3 - 3Af_5 - f_7\tilde{W} = 0, \tag{4.15}$$

$$\bar{\partial}\tilde{T} - \kappa\partial f_6 + \tilde{W}f_2 + \tilde{T}f_4 - 3Af_6 - f_7\tilde{T} = 0, \tag{4.16}$$

$$\bar{\partial}A - \kappa\partial\bar{A} + [\bar{A}, A] = 0. \tag{4.17}$$

$$A_{-1,a} = p_a, \ (a = 1, \dots, n), \tag{4.18}$$

$$T_{-2,a} = \frac{3 < (p_a^0)^2 >}{N} \ (a = 1, \dots, n). \tag{4.19}$$

$$W_{-3,a} = \frac{< (p_a^0)^3 > -3\kappa < (p_a^0)^2 >}{N}, \ (a = 1, \dots, n). \tag{4.20}$$

The matrix coefficients f_j have the form

$$f_1 = -\frac{2}{3}(\partial^2 - \frac{1}{\kappa^2}T)\rho + \partial\mu - \bar{A} + \frac{2}{\kappa}A\partial\rho - \frac{1}{\kappa}\mu A - \frac{2}{\kappa^2}A^2\rho, \tag{4.21}$$

$$f_2 = -\frac{1}{\kappa}(\mu - \partial\rho) - \frac{3}{\kappa^2}A\rho, \tag{4.22}$$

$$f_3 = \kappa\partial f_1 - \frac{1}{\kappa^2}\tilde{W}\rho = \tag{4.23}$$

$$-\frac{2}{3}\kappa\partial(\partial^2 - \frac{1}{\kappa^2}T)\rho + \kappa\partial^2\mu - \kappa\partial\bar{A} + 2\partial(A\partial\rho) - \frac{1}{\kappa^2}\tilde{W}\rho - \partial(\mu A + \frac{2}{\kappa}A^2\rho),$$

$$f_4 = \kappa\partial f_2 + f_1 - \frac{1}{\kappa^2}\tilde{T}\rho = \tag{4.24}$$
$$\frac{1}{3}(\partial^2 - \frac{1}{\kappa^2}T)\rho - \frac{1}{\kappa}A\partial\rho - \bar{A} + \frac{1}{\kappa^2}A^2\rho - \frac{1}{\kappa}\mu A,$$
$$f_5 = -\frac{\mu}{\kappa}\tilde{W} + \kappa\partial f_3 = \tag{4.25}$$
$$-\frac{2}{3}\kappa^2\partial^2(\partial^2 - \frac{1}{\kappa^2}T)\rho + \kappa^2\partial^3\mu - \kappa^2\partial^2\bar{A} + 2\kappa\partial^2(A\partial\rho) - \frac{1}{\kappa}\partial(\tilde{W}\rho) - \frac{\mu}{\kappa}\tilde{W} - \kappa\partial^2(\mu A + \frac{2}{\kappa}A^2\rho),$$
$$f_6 = -\frac{\mu}{\kappa}\tilde{T} + \kappa\partial f_4 + f_3 = \tag{4.26}$$
$$-\frac{2}{3}\kappa\partial(\partial^2 - \frac{1}{\kappa^2}T)\rho + \kappa\partial^2\mu - \kappa\partial\bar{A} + \partial(A\partial\rho) - \frac{1}{\kappa^2}\tilde{W}\rho - \frac{\mu}{\kappa}\tilde{T} + \frac{3}{\kappa}\partial(A^2)\rho - 2\partial(\mu A + \frac{2}{\kappa}A^2\rho),$$
$$f_7 = \frac{3}{\kappa}\mu A - \partial\mu + f_4 = \tag{4.27}$$
$$\frac{2}{\kappa}\mu A - \partial\mu + \frac{1}{3}(\partial^2 - \frac{1}{\kappa^2}T)\rho - \frac{1}{\kappa}A\partial\rho - \bar{A} + \frac{1}{\kappa^2}A^2\rho.$$

Note that (4.15),(4.16) is reduced to the standard $\mathcal{W}_3$ structure if $A = \bar{A} = 0$, while (4.17) is the flatness of the bundle V.

Proposition 4.2 *The equations (4.15),(4.16), (4.17) are the compatibility conditions for the linear system*
$$(\kappa^3\partial^3 + 3A\kappa^2\partial^2 + \kappa\tilde{T}\partial + \tilde{W})\Psi = 0, \tag{4.28}$$
$$(\bar{\partial} - f_1 - f_2\kappa\partial - \rho\partial^2)\Psi = 0, \tag{4.29}$$
where and Ψ is a section of $\Omega^{(-1,0)}(\Sigma_{g,n}, \mathrm{Aut}V)$.

Again $\mathcal{W}_3^N$ has a natural symplectic form
$$\omega = \int_{\Sigma_{g,n}} [-\frac{N}{\kappa}\delta T\delta\mu - \frac{N}{\kappa^2}\delta W\delta\rho + 3 < \delta A, \delta\bar{A} >] + 6\pi i\sum_{a=1}^{n}\delta < p_a, g_a^{-1}\delta g_a > . \tag{4.30}$$

The proof of Proposition and the derivation of ω (4.30) is based on the same procedure that we already used in the $\mathcal{W}_2^N$ case. We consider $\kappa\partial + \mathcal{A}$, $\bar{\partial} + \bar{\mathcal{A}}$ connections in the $(S(\mathrm{GL}(3,\mathbf{C}) \otimes \mathrm{GL}(N,\mathbf{C})))$ vector bundle. The first component of $\mathcal{A}, \bar{\mathcal{A}}$ acts on the sections of 3-jets of $\Omega^{(-1,0)}(\Sigma_{g,n})$, while the second component acts on the sections of the bundle $\Omega^{(-1,0)}(\Sigma_{g,n}, \mathrm{Aut}V)$. We define the space $\mathcal{K}_3$ of connections $(\mathcal{A}, \bar{\mathcal{A}})$ and the set of coadjoint orbits (3.2):
$$(\mathcal{A}, \bar{\mathcal{A}}, \mathbf{1}_3 \otimes \mathbf{p}).$$

We assume that $\mathcal{A}$ satisfies additional restrictions:
- the $\mathcal{A}$ component are gauge equivalent to the special form
$$\mathcal{A} \sim DS \otimes \mathbf{1}_N + \mathbf{1}_3 \otimes (-A), \tag{4.31}$$

where
$$DS = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W & T & 0 \end{pmatrix}$$

- near the marked points W has the form (4.2);
- near the marked points T has the form (2.8);
- near the marked points A has the form (3.2).

Proposition 4.1 follows from the following statement

Proposition 4.3 *$\mathcal{W}_3^N$ is the subset of $\mathcal{K}_3$, satisfying the flatness condition*

$$\bar{\partial}\mathcal{A} - \kappa\partial\bar{\mathcal{A}} + [\mathcal{A}, \bar{\mathcal{A}}] = 0. \tag{4.32}$$

Proof.
The symplectic form ω' on $\mathcal{K}_3$ has the similar form as in the $\mathcal{K}_2$ case:

$$\omega' = \int_{\Sigma_{g,n}} << \delta\mathcal{A}, \delta\bar{\mathcal{A}} >> +6\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a > \tag{4.33}$$

The gauge group symmetry of ω' is

$$\mathcal{G} = \{\mathrm{Map}(\Sigma_{g,n} \to \mathrm{SL}(3N^2, \mathbf{C}))\}.$$

The action of $\mathcal{G}$ is the same as in (3.16). For the matrix elements of Lie($\mathcal{G}$) we assume that they are continuous in the marked points with additional restrictions for the matrix elements $x^{13}_{\alpha\beta}$,$x^{23}_{\alpha\beta}$

$$x^{13}_{\alpha\beta} \sim \delta_{\alpha\beta}O(z - x_a)^2 + (1 - \delta_{\alpha\beta})O(z - x_a)^3,$$
$$x^{23}_{\alpha\beta} \sim \delta_{\alpha\beta}O(z - x_a) + (1 - \delta_{\alpha\beta})O(z - x_a)^2.$$

Then for the dual space $y \in$Lie$^*(\mathcal{G})$ we have

$$y^{31}_{\alpha\alpha} \sim \sum_{k\geq 2} b_k\partial^k\delta(x_a), \quad y^{31}_{\alpha\beta} \sim \sum_{k\geq 3} b_{k,\alpha,\beta}\partial^k\delta(x_a), \ (\alpha \neq \beta). \tag{4.34}$$

$$y^{32}_{\alpha\alpha} \sim \sum_{k\geq 1} b_k\partial^k\delta(x_a), \quad y^{32}_{\alpha\beta} \sim \sum_{k\geq 2} b_{k,\alpha,\beta}\partial^k\delta(x_a), \ (\alpha \neq \beta). \tag{4.35}$$

$$y^{JK}_{\alpha\beta} \sim \sum_{k\geq 0} b_{k,\alpha,\beta}\partial^k\delta(x_a), \quad (J \neq 3, \ K \neq 1, 2). \tag{4.36}$$

We choose $\mathcal{A}$ in the form

$$\mathcal{A} = \begin{pmatrix} 0 & E & 0 \\ 0 & 0 & E \\ \tilde{W} & \tilde{T} & -3A \end{pmatrix} \tag{4.37}$$

Substituting this form of $\mathcal{A}$ in (4.32) we find $\bar{\mathcal{A}}$. Solutions $\bar{\mathcal{A}}$ are parametrized by the fields μ, ρ and the matrix $\bar{A}$. Then

$$\bar{\mathcal{A}} = \begin{pmatrix} f_1 & f_2 & -\rho/\kappa^2 \\ f_3 & f_4 & -\mu/\kappa \\ f_5 & f_6 & f_7 \end{pmatrix}.$$

We set $f_1 + f_4 + f_7 = -3\bar{A}$. This condition along with algebraic equation for the blocks (J, K), $J = 1, 2$, $K = 1, 2, 3$ of (4.32) allows to find $f_1, \ldots, f_7$ (4.21)-(4.27). The differential identities arising in the last row $J = 3, K = 1, 2, 3$ lead to (4.15),(4.16),(4.17). The behavior of the most singular terms near the marked points (4.18),(4.19),(4.20) follows from the special form of Lie$^*(\mathcal{G})$ (4.35), (4.34),(4.36).

Moreover, ω' (4.33) gives ω (4.30) on $\mathcal{W}_3^N$ space.

Due to the special form of $\mathcal{A}$ and $\bar{\mathcal{A}}$ the compatible system of differential equations

$$(\kappa\partial - \mathcal{A})\begin{pmatrix} \Psi \\ \kappa\partial\Psi \\ \kappa^2\partial^2\Psi \end{pmatrix} = 0, \quad (\bar{\partial} - \bar{\mathcal{A}})\begin{pmatrix} \Psi \\ \kappa\partial\Psi \\ \kappa^2\partial^2\Psi \end{pmatrix} = 0.$$

is equivalent to (4.28),(4.29). $\Box$

3.$\mathcal{W}_3^N$ and isomonodromic deformations.
To get a nontrivial phase flow we should get rid of from the restrictive constraints (4.15), (4.16), (4.19) and (4.20). The remaining constraints generate the gauge subgroup

$$\mathcal{G}^P = \{\mathrm{Map}(\Sigma_{g,n} \to P)\}\,,$$

where P is the parabolic subgroup (4.11). Thus, we come to the space $\mathcal{K}_3^N \subset \mathcal{K}_3$ of the fields $(W, \rho, T, \mu, A, \bar{A}, p_a)$ with constraints (4.17),(4.18).

The form ω (4.30) is invariant under the following transformations

$$A \to A,\ \bar{A} \to \bar{A} + \frac{1}{\kappa}\xi_2 A + \frac{1}{\kappa^2}\xi_3 A^2,\ T \to T,\ \mu \to \mu + \xi_2,\ \rho \to \rho + \xi_3, \tag{4.38}$$

where $\xi_j \in \Omega^{(1-j,1)}(\Sigma_{g,n})$ with the same behavior near the marked points as μ and ρ. The moment constraints generated by these transformations are

$$T = \frac{3}{2N} < A^2 >,\ \ W = \frac{1}{N} < A^3 > .$$

Substituting T and W in ω (4.30) we obtain

$$\frac{1}{3}\omega = \int_{\Sigma_{g,n}} < \delta A, \delta \bar{A} > + 2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a > \tag{4.39}$$

$$-\frac{1}{\kappa}\int_{\Sigma_{g,n}} < \delta A, A > \delta\mu - \frac{1}{\kappa^2}\int_{\Sigma_{g,n}} < \delta A, A^2 > \delta\rho.$$

Since we don't fix the gauge of the transformations (4.38), the form ω becomes degenerated. If one replace

$$\bar{A} \to \bar{A}' = \bar{A} - \frac{1}{\kappa}\mu A - \frac{1}{2\kappa^2}\rho A^2$$

ω takes the canonical form

$$\frac{1}{3}\omega = \int_{\Sigma_{g,n}} < \delta A, \delta \bar{A}' > + 2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a > .$$

It is invariant under the gauge transformations

$$A \to f A f^{-1} + f\kappa\partial f^{-1},\ \ \bar{A}' \to f\bar{A}' f^{-1} + f\bar{\partial} f^{-1}. \tag{4.40}$$

The moment equation resulting from this symmetry is

$$F(A, \bar{A}') := \bar{\partial} A - \kappa\partial\bar{A} + [\bar{A}, A] = 0,$$

or in the original variables

$$(\bar{\partial} + \partial\mu + \frac{1}{2\kappa}\partial\rho A)A - \kappa\partial\bar{A} + [\bar{A}, A] = 0. \tag{4.41}$$

In this case the constraints become nonlinear.

Now instead of infinite-dimensional space of smooth differentials ρ consider the finite dimensional space $H^1(\Sigma_{g,n},\Gamma^2)$. It has dimension

$$l_3 = \dim H^1(\Sigma_{g,n},\Gamma^2) = 5g-5+2n. \tag{4.42}$$

Expanding ρ in the basis of $H^1(\Sigma_{g,n},\Gamma^2)$ we obtain $\rho=\sum_{s=1}^{l_3} t_{3,s}\rho_s^0$. Then ω (4.39) gives

$$\frac{1}{3}\omega = \omega^0 - \frac{1}{\kappa}\sum_{s=1}^{l_2}\delta H_{2,s}\delta t_{2,s} - \frac{1}{2\kappa^2}\sum_{s=1}^{l_3}\delta H_{3,s}\delta t_{3,s}, \tag{4.43}$$

$$H_{2,s} = \frac{1}{2}\int_{\Sigma_{g,n}} < A^2 > \mu_s^{(0)},\ H_{3,s} = \frac{1}{3}\int_{\Sigma_{g,n}} < A^3 > \rho_s^{(0)},$$

where

$$\omega^0 = \int_{\Sigma_{g,n}} < \delta A, \delta\bar{A} > +2\pi i\sum_{a=1}^{n}\delta < p_a, g_a^{-1}\delta g_a > .$$

The equations of motion defining by ω are

$$\partial_r A = 0, \quad (r=(k,s),\ k=2,3,\ \ \partial_r = \frac{\partial}{\partial t_{k,s}}), \tag{4.44}$$

$$\partial_r\bar{A}' = 0. \tag{4.45}$$

The solutions describe the free motion

$$\bar{A} = \bar{A}_0 + \frac{1}{\kappa}A_0\sum_s t_{2,s} + \frac{1}{2\kappa^2}A_0^2\sum_s t_{3,s}. \tag{4.46}$$

Making use of the gauge symmetry we represent the fields as

$$L = fAf^{-1} + f\kappa\partial f^{-1},$$

$$\bar{L}' = f\bar{A}'f^{-1} + f\bar{\partial}f^{-1},$$

where

$$M_r = f^{-1}\partial_r f.$$

Thus we come to the finite-dimensional bundle

$$\mathcal{P}_3^{red,N} = \{L,\bar{L},\mathbf{p},\mu,\rho\}.$$

The bundle $\mathcal{P}_3^N$ has the same fibers $\mathcal{R}^{red}$ as the bundle $\mathcal{P}_2^{red,N}$.

The equation (4.41) takes the form

$$(\bar{\partial} + \partial\mu + \frac{1}{2\kappa}\partial\rho L)L - \kappa\partial\bar{L} + [\bar{L},L] = 0, \quad L|_{z\to x_a} = \frac{p_a}{z-x_a}, \tag{4.47}$$

where

$$\bar{L} = \bar{L}' + \frac{1}{\kappa}\mu L + \frac{1}{2\kappa^2}\rho L^2.$$

The equation of motion (4.44), on the reduced phase space $(L,\bar{L}')$ takes the form of the Lax equation

$$\partial_r L - \kappa\partial M_r + [M_r, L] = 0. \tag{4.48}$$

The second equation (4.45) allows to define the matrices M_r in the Lax equation

$$\partial_r \bar{L}' - \bar{\partial} M_r + [M_r, \bar{L}'] = 0. \tag{4.49}$$

The equations (4.48), (4.49) with the flatness condition (4.47) are the compatibility conditions of the linear system

$$(\kappa\partial + L)\Psi = 0, \tag{4.50}$$

$$(\bar{\partial} + \bar{L}')\Psi = 0, \tag{4.51}$$

$$(\kappa\partial_r + M_r)\Psi = 0. \tag{4.52}$$

The last equation means that the monodromy matrix Y of the linear system (4.50), (4.51) on $\Sigma_{g,n}$ is independent with respect to the $\mathcal{W}_3$ moduli μ_r, ρ_r.

On the critical level $\kappa = 0$ the systems pass into the Hitchin systems with quadratic and cubic commuting integrals.

Consider a rational curve $\Sigma_{0,n}$ with n fixed marked points $x_1^0, \ldots, x_n^0$. According to (3.26) and (4.42) we have $8n - 8$ times. Since in this case $\bar{L} = 0$, (4.47) takes the form

$$[\bar{\partial} + \partial \sum_{a=1}^{n} t_{2,a} \bar{\partial}\chi_a(z,\bar{z}) + \frac{1}{2\kappa}\partial \sum_{a=1}^{n}(t_{3,a,0} + t_{3,a,1}(z - x_a^0))\bar{\partial}\chi_a(z,\bar{z})L]L = 0.$$

We remind that the times $t_{2,a}$ are related to the positions of the marked points ($t_{2,a} = x_a - x_a^0$). If the t_3 times vanish then solution of this equation

$$L = \sum_{a=1}^{n} \frac{p_a}{w - x_a}, \quad (w = z - \sum_{a=1}^{n} t_{2,a}\chi_a(z,\bar{z}))$$

is the L operator for the Schlesinger system. For generic points ($t_{3,a} \neq 0$) the equation is nonlinear and its solutions are unknown.

5 Conclusion

The generalization of the previous analysis on the arbitrary $\mathcal{W}_k^N$ structures is straightforward, though the explicit calculations on intermediate steps become rather long. In this section we present some general formulas.

Let $W_j, j = 2, \ldots, k$ be the set of j-differentials on $\Sigma_{g,n}$ and $\rho_j, j = 2, \ldots, k$ are the dual objects, ($W_2 = T, \rho_2 = \mu$). In addition we have the gauge data $(A, \bar{A})$ and the coadjoint orbits in the marked points $p_a, (a = 1, \ldots, n)$ with the following behaviour

$$W_j|_{z\to x_a} \sim \frac{W_{-j,a}}{(z - x_a)^j} + \frac{W_{-j+1,a}}{(z - x_a)^{j-1}} + + \ldots,$$

$$\rho_j|_{z\to x_a} \sim (t_{j,a,0} + t_{j,a,1}(z - x_a^0) + \ldots + t_{j,a,j-2}(z - x_a)^{j-2})\bar{\partial}\chi_a(z,\bar{z}),$$

and A has poles of the first order (3.1) with residues (3.7). The highest order coefficients $W_{-j,a}$ are the linear combinations of corresponding Casimirs up to the order j

$$W_{-j,a} \sim \frac{1}{N}(< (p_a^0)^j > + \ldots).$$

Let Ψ be a section of $\Omega^{(-\frac{k-1}{2},0)}(\Sigma_{g,n}, \mathrm{Aut}V)$. Then the $\mathcal{W}_k^N$ structure on $\Sigma_{g,n}$ is the set of fields

$$(W_j, \rho_j), \ j = 2, \ldots, k, (A, \bar{A}), p_a, a = 1, \ldots, n,$$

providing the consistency of the following linear system

$$(\kappa^k\partial^k + kA\kappa^{k-1}\partial^{k-1} + \ldots + \tilde{W}_k)\Psi = 0,$$

$$(\bar{\partial} + \alpha_k\partial^{k-1} + \ldots + \alpha_1)\Psi = 0.$$

Here

$$\tilde{W}_k = W_k + AW_{k-1} + A^2W_{k-2} + \ldots + A^{k-2}W_2 - A^k,$$

and other coefficients can be read off from the flatness of the connections $\mathcal{A}, \bar{\mathcal{A}}$ in the $\mathrm{SL}(kN, \mathbf{C})$ bundle with

$$\mathcal{A} = DS \otimes \mathbf{I}_N + \mathbf{I}_k \otimes (-A)$$

$$DS = \begin{pmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdots & & \cdot \\ 0 & \cdot & \cdot & \cdots & 0 & 1 \\ W_k & W_{k-1} & \cdot & \cdots & W_2 & 0 \end{pmatrix}.$$

The symplectic form on $\mathcal{W}_k^N$ has the universal structure

$$\omega = \int_{\Sigma_{g,n}} [k < \delta A, \delta\bar{A} > -N\sum_{j=2}^{k}\frac{1}{\kappa^{j-1}}\delta W_j\delta\rho_j] + 2k\pi i\sum_{a=1}^{n}\delta < p_a, g_a^{-1}\delta g_a > . \qquad (5.1)$$

We drop out details of calculations. Essentially they are the same as in the case $\mathcal{W}_3^N$.

Again, the constraints on the fields $W_j, \rho_j, j = 2, \ldots, k$ can be discarded by the restriction to the parabolic subgroup of the gauge transformations. The form acquire the following Abelian symmetry

$$\rho_j \to \rho_j + \xi_j,\ (j = 2, \ldots, k),\quad \bar{A} \to \bar{A} + \sum_{j=2}^{k}\frac{A^{j-1}\xi_j}{\kappa^{j-1}},$$

where $\xi_j \in \Omega^{(-j,1)}(\Sigma_{g,n})$. This symmetry is generated by the constraints

$$W_j = \frac{k < A^j >}{Nj}.$$

Substituting this expression in (5.1) we obtain the symplectic form

$$\frac{1}{k}\omega = \int_{\Sigma_{g,n}} < \delta A, \delta\bar{A} > +2\pi i\sum_{a=1}^{n}\delta < p_a, g_a^{-1}\delta g_a > -\sum_{j=2}^{k}\frac{1}{\kappa^{j-1}}\int_{\Sigma_{g,n}} < A^{j-1}\delta A > \delta\rho_j. \qquad (5.2)$$

Now we restrict the fields ρ_j to the spaces $H^1(\Sigma_{g,n}, \Gamma^{j-1})$. Let

$$\rho_j = \sum_{s=1}^{l_j} t_{j,s}\rho_{j,s}^0, \qquad (l_j = \dim H^1(\Sigma_{g,n}, \Gamma^{j-1}) = (2j-1)(g-1) + (j-1)n)$$

be the expansion of ρ_j in the basis of $H^1(\Sigma_{g,n}, \Gamma^{j-1})$. Then we come to the following form

$$\frac{1}{k}\omega = \omega^0 - \sum_{j=2}^{k}\frac{1}{\kappa^{j-1}}\sum_{s=1}^{l_j}\delta H_{j,s}\delta t_{j,s}, \qquad (5.3)$$

where

$$\omega^0 = \int_{\Sigma_{g,n}} < \delta A, \delta \bar{A} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1} \delta g_a >, \quad H_{j,s} = \frac{1}{j} \int_{\Sigma_{g,n}} < A^j > \rho^0_{j,s}.$$

In terms of the field

$$\bar{A}' = \bar{A} - \sum_{j=2}^{k} \frac{1}{\kappa^{j-1}} A^{j-1} \rho_j$$

it takes the canonical form

$$\frac{1}{k}\omega = \int_{\Sigma_g} [< \delta A, \delta \bar{A}' > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1} \delta g_a > .$$

It is a free system with solutions

$$A = A_0 = const, \quad \bar{A}(\mathbf{t}) = \bar{A}_0 + \sum_{j=2}^{k} \frac{A^{j-1}}{\kappa^{j-1}} \sum_{s=1}^{l_j} t_{j,s} \rho^0_{j,s},$$

After the Hamiltonians reduction with respect to the gauge symmetry (4.40) we come to the extended phase space

$$\mathcal{P}_k^N = \{L, \bar{L}, \mathbf{p}, \rho_j, j = 2, \ldots, k\}.$$

The equations of motion are the same as in the $\mathcal{P}_3^N$ case (4.48),(4.49), where

$$\bar{L}' = \bar{L} - \sum_{j=2}^{k} \frac{1}{\kappa^{j-1}} \rho_j L^{j-1}.$$

On the critical level we obtain the Hitchin systems with integrals of order $j = 2, \ldots, k$. In the case $k < N$ the number of integrals is less then the dimension of the configuration space. For $k \geq N$ the systems are completely integrable though the integrals of order $j > N$ are not independent. Thus, the distinguish case is $k = N$.

It is interesting to consider the limit $k \to \infty$. Because the differential operators of an arbitrary order generate a Lie algebra, $\mathcal{W}$ structure acquire the group-theoretical background as in the $\mathcal{W}_2$ case. As we argued above the simultaneous limit $k \to \infty$, $N \to \infty$ is distinguish and can lead to the interesting field theories.

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