Explicit bases for the cohomology groups of the Hilbert scheme of points on a surface

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1. Introduction

Let S be a smooth projective surface over the complex numbers, n a positive integer. The Hilbert scheme $Hilb^n_{\mathbb{C}}(S)$ (which we will usually denote by $S^{[n]}$) parametrizes 0-dimensional subschemes of S of length n; it is a smooth, $2n$-dimensional projective variety, and a resolution of singularities of the n-fold symmetric product $S^{(n)}$.

The cohomology and Chow ring of $S^{[n]}$ have long been objects of investigation. For $S = \mathbb{P}^2$, the Betti numbers and a cell decomposition of $S^{[n]}$ have been computed in [ES1], [ES2]; a more "geometrical" basis of the homology has been constructed in [MS].

For general S, the Betti numbers of $S^{[n]}$ have been computed in [G], the Hodge numbers and the Hodge structure in [GS] (the Hodge numbers also independently in [Ch]). To be precise, the following is proven in [GS]:

Theorem (Göttsche-Soergel). *For any $k \in \mathbb{N}$ there is an isomorphism of Hodge structures*

$$H^k(S^{[n]}, \mathbb{C}) = \bigoplus_{\mathbf{b}} H^{k-(n-l(\mathbf{b}))}(S^{(\mathbf{b})}, \mathbb{C}),$$

where $\mathbf{b}$ runs over the partitions of n, $l(\mathbf{b})$ is the length of the partition $\mathbf{b}$ and $S^{(\mathbf{b})}$ is a product of symmetric products of S (see definition 2.1).

The aim of this paper is the explicit construction of this isomorphism. The starting point is the idea that, given a partition $(b_1, \dots, b_r)$ of n and for $i = 1, \dots, r$ cycles Γ_i on S, we can define a cycle on $S^{[n]}$ by taking schemes with support $\sum b_i P_i$ with $P_i \in \Gamma_i$; the homology classes so defined, modulo the "obvious" equivalences, should give a basis for the homology of $S^{[n]}$. We learned about this conjecture by Lothar Göttsche during the work on [FG], where such a basis is explicitly constructed for $n = 3$.

We start the paper by formalising the previous idea, i.e. by giving a precise definition of the morphism described above. We then define another possible such morphism, which is not anymore defined in terms of S alone, but depends on the choice of a very ample line bundle L on S (this is analogous to the construction in [MS]); we then compute intersection products between the classes so constructed, deduce that they are linearly independent and conclude that they are bases by comparing their cardinality with the Betti numbers of $S^{[n]}$.

Key words and phrases. Hilbert scheme of points, enumerative geometry.

We therefore construct several isomorphisms as in theorem 1.1 (see remark 2.2). We would like to note that this paper is inspired by the results of [GS], but it is logically dependent only on [G], and in particular provides an independent and more elementary proof (corollary 3.4) of [GS]'s description of the Hodge structure of $S^{[n]}$ in the case where S is projective.

Acknowledgements. As we already mentioned, this paper's starting point is a conjecture of Lothar Göttsche, essentially the isomorphism (2.4.1) in 2.4. Many subsequent discussions with him helped the author to clarify her ideas in a substantial way. He communicated to us lemma 3.3, in particular against our previous belief that the intersection multiplicity should always be one. The author is also thankful to Wolfgang Soergel for a discussion of the results in [GS].

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2. Statement of the main theorems

In this paper S will be a fixed smooth, complex projective surface, n a positive integer. We will denote the n-th symmetric product of S by $S^{(n)}$ and the Hilbert scheme $Hilb^n_{\mathbb{C}}(S)$ by $S^{[n]}$. We will also fix a very ample line bundle L on S.

For any topological space X we will denote the cohomology (resp. homology) groups with $\mathbb{Q}$ coefficients by $H^\bullet(X)$ (resp. $H_\bullet(X)$); cupproduct in cohomology will be denoted by a dot. A (k-)cycle on a real C^∞ manifold X will be a C^∞ map from a (possibly disconnected) compact oriented C^∞ k-dimensional manifold to X; to a k-cycle Γ one can naturally associate a homology class $\bar{\gamma}$ in $H_k(X)$, and we will say that Γ represents $\bar{\gamma}$. If X is a compact complex manifold, we will also say that Γ represents the cohomology class γ, Poincaré dual to $\bar{\gamma}$. The support of a cycle will be the image of the map.

A *partition* of n of length r will be an r-tuple $\mathrm{b} = (b_1, \dots, b_r)$ of positive integers such that $\sum b_j = n$, and the sequence b_j is non-decreasing. The symmetric group on r letters will be denoted by $\mathfrak{S}_r$.

Definition 2.1 ([GS]). To each partition b of n, we associate a projective variety $S^{(\mathrm{b})}$ with only quotient singularities as follows: for $i = 1, \dots, n$, let a_i be the number of b_j's which are equal to i, and define $S^{(\mathrm{b})}$ to be the product of the $S^{(a_i)}$. Notice that $S^{(\mathrm{b})}$ is a rational homology manifold of dimension $2r$, and there is a natural map $\pi_{\mathrm{b}} : S^r \to S^{(\mathrm{b})}$ induced by the map $S^r \to S^{(a_i)}$ given by $(P_1, \dots, P_r) \mapsto (\sum_{b_j = i} P_j)$. In particular the (co)homology groups of $S^{(\mathrm{b})}$ are known, and the mapping $\pi_{\mathrm{b}*} : H^\bullet(S^r) \to H^\bullet(S^{(\mathrm{b})})$ is surjective.

In fact, let $\mathfrak{S}_{\mathrm{b}}$ be the subgroup of $\mathfrak{S}_r$ defined by the condition

$$\mathfrak{S}_{\mathrm{b}} = \{\sigma \in \mathfrak{S}_r \mid b_i = b_{\sigma(i)} \ \forall i = 1, \dots, r\};$$

the variety $S^{(\mathbf{b})}$ is the quotient of S^r by $\mathfrak{S}_{\mathbf{b}}$, where $\mathfrak{S}_r$ acts on S^r in the obvious way. Therefore $\pi_{\mathbf{b}*}$ induces an isomorphism between $H^*(S^r)^{\mathfrak{S}_{\mathbf{b}}}$ and $H^*(S^{(\mathbf{b})})$; moreover, for any $\alpha \in H^*(S^r)$,

$$\pi_{\mathbf{b}}^*\pi_{\mathbf{b}*}(\alpha) = \sum_{\sigma\in\mathfrak{S}_{\mathbf{b}}} \sigma^*(\alpha).$$

In particular, if α and α' are elements of $H^*(S^r)$, by projection formula

$$\pi_{\mathbf{b}*}\alpha \cdot \pi_{\mathbf{b}*}\alpha' = \pi_{\mathbf{b}*}\sum_{\sigma\in\mathfrak{S}_{\mathbf{b}}} \alpha\cdot\sigma^*(\alpha').$$

Fix a partition $\mathbf{b}$ of n, and let $W_{\mathbf{b}}$ be the $(2r+4n)$-dimensional projective manifold $S^r \times S^{[b_1]} \times \cdots \times S^{[b_r]} \times S^{[n]}$; a point of $W_{\mathbf{b}}$ is a tuple (P_i, Z_i, Z) where P_i are points in S, Z_i are length b_i subschemes of S, and Z is a length n subscheme of S (and the index i runs from 1 to r).

Let $Z_{\mathbf{b}}$ be the subvariety of $W_{\mathbf{b}}$ defined as the closure of the locally closed set

$$\{(P_i, Z_i, Z) \mid P_i \in Z_i,\ Z_i \subset Z,\ Z_i \cap Z_j = \emptyset \text{ if } i \neq j\}.$$

Let $\zeta_{\mathbf{b}}$ be the cohomology class of $Z_{\mathbf{b}}$ in $H^*(W_{\mathbf{b}})$.

Let $Pt_{\mathbf{b}}$ be the subvariety of $W_{\mathbf{b}}$ of the tuples (P_i, Z_i, Z) such that each of the Z_i is concentrated in a point; let $pt_{\mathbf{b}}$ be the corresponding cohomology class in $H^*(W_{\mathbf{b}})$.

Choose a pencil $\mathcal{P}$ of curves C_t in the linear system $|L|$, with a finite number of base points and with smooth general element. Say that a subscheme Z of S is $\mathcal{P}$-linear if there is a curve C in the pencil such that Z is contained in C (as a subscheme, not just as a set of points!).

Let $Al_{\mathbf{b}}$ be the subvariety of $W_{\mathbf{b}}$ of the tuples (P_i, Z_i, Z) where each of the Z_i's is $\mathcal{P}$-linear, and let $al_{\mathbf{b}}$ be the corresponding cohomology class in $H^*(W_{\mathbf{b}})$.

Remark 2.2. The class $al_{\mathbf{b}}$ does not depend on the choice of $\mathcal{P}$; on the other hand it does depend on the choice of the line bundle L that we fixed at the beginning. By varying L we can obtain different bases of $H^*(S^{[n]})$.

Denote the natural projection from $W_{\mathbf{b}}$ to S^r (resp. $S^{[n]}$) by $p_{\mathbf{b}}$ (resp. $q_{\mathbf{b}}$).

Definition 2.3. The additive homomorphism

$$q_{\mathbf{b}*}(pt_{\mathbf{b}} \cdot \zeta_{\mathbf{b}} \cdot p_{\mathbf{b}}^*(\cdot)) : H^*(S^r, \mathbb{Q}) \to H^*(S^{[n]}, \mathbb{Q})$$

has degree $2(n-r)$ and factors via the pushforward $\pi_{\mathbf{b}*} : H^*(S^r) \to H^*(S^{(\mathbf{b})})$. We denote the induced degree $2(n-r)$ homomorphism from $H^*(S^{(\mathbf{b})})$ to $H^*(S^{[n]})$ by $\varphi(\mathbf{b},\cdot)$. By replacing $pt_{\mathbf{b}}$ by $al_{\mathbf{b}}$ we get the definition of $\psi(\mathbf{b},\cdot)$.

Theorem 2.4. *Let $P(n)$ be the set of all partitions of n. Then the graded homomorphisms of graded $\mathbb{Q}$-vector spaces*

$$\oplus\varphi(\mathbf{b},\cdot) : \bigoplus_{\mathbf{b}\in P(n)} H^*(S^{(\mathbf{b})}, \mathbb{Q}) \to H^*(S^{[n]}, \mathbb{Q}) \tag{2.4.1}$$

$$\oplus\psi(\mathbf{b},\cdot) : \bigoplus_{\mathbf{b}\in P(n)} H^*(S^{(\mathbf{b})}, \mathbb{Q}) \to H^*(S^{[n]}, \mathbb{Q}) \tag{2.4.2}$$

are isomorphisms.

It is convenient to restate the theorem in terms of bases of the cohomology. Fix a homogeneous basis $A = \{\alpha\}$ of $H^*(S,\mathbb{Q})$, with $\alpha \in H^{d(\alpha)}(S,\mathbb{Q})$; fix a total ordering $<$ of A.

Let $M(k,n)$ be the set of the data $(\mathbf{b},\gamma)$ where $\mathbf{b}$ is a length r partition of n, and γ is a function from $\{1,\ldots,r\} \to A$ such that if $b_i = b_j$ then $\gamma(i) < \gamma(j)$, and such that $k - \sum d(\gamma(i)) = 2\sum(b_i - 1) = 2(n-r)$. We identify γ with the cohomology class

$$\pi_1^*(\gamma(1)) \cdot \ldots \cdot \pi_r^*(\gamma(r))$$

in the $(k+2r-2n)$-th cohomology group of S^r (where $\pi_i : S^r \to S$ is the projection onto the i-th factor).

For fixed $\mathbf{b}$, the classes γ such that $(\mathbf{b},\gamma) \in M(k,n)$ induce via $\pi_{\mathbf{b}*}$ a basis for $H^{k-2(n-r)}(S^{(\mathbf{b})},\mathbb{Q})$. In particular from [G] it follows that the cardinality of $M(k,n)$ is equal to the dimension of $H^k(S^{[n]},\mathbb{Q})$. We can now restate theorem 2.4 in terms of bases.

Theorem 2.4'. *The map φ (resp. ψ) of definition 2.3 induces a bijection between $M(k,n)$ and a basis $\mathcal{B}_k$ (resp. $\mathcal{B}'_k$) of $H^k(S^{[n]},\mathbb{Q})$.*

3. Proof of theorem 2.4

We will prove the theorem by showing that the intersection pairing between $\mathcal{B}_k$ and $\mathcal{B}'_{4n-k}$ is nondegenerate. In order to do so, we have to give a partial ordering on the elements of our bases; we will then prove that the intersection matrix is block triangular and that all the blocks on the diagonal have nonzero determinant.

Definition 3.1. A 5-sequence is a sequence of 5 nonnegative integers $l = (l_0,\ldots,l_4)$. The dual of a 5-sequence l is the 5-sequence $\hat{l} = (l_4,\ldots,l_0)$.

We view the 5-sequences as ordered in a reverse lexicographical way, i.e., $l > l'$ if $l_4 > l'_4$, or $l_4 = l'_4$ and $l_3 > l'_3$, etc. To each $(\mathbf{b},\gamma) \in M(k,n)$ we can associate a 5-sequence $l(\mathbf{b},\gamma)$ by letting $l_j = \#\{i | d(\gamma(i)) = j\}$.

Theorem 3.2. *Let $(\mathbf{b},\gamma) \in M(k,n)$ (resp. $(\mathbf{b}',\gamma') \in M(4n-k,n)$). Then*

1. *If $l(\mathbf{b},\gamma) < \hat{l}(\mathbf{b}',\gamma')$, then $\varphi(\mathbf{b},\gamma)\cdot\psi(\mathbf{b}',\gamma') = 0$;*
2. *If $l(\mathbf{b},\gamma) = \hat{l}(\mathbf{b}',\gamma')$, then $\varphi(\mathbf{b},\gamma)\cdot\psi(\mathbf{b}',\gamma') = 0$ unless $\mathbf{b} = \mathbf{b}'$;*
3. *$\varphi(\mathbf{b},\gamma)\cdot\psi(\mathbf{b},\gamma')$ is equal to $(\prod b_i^2)$ times $\pi_{\mathbf{b}*}\gamma \cdot \pi_{\mathbf{b}*}\gamma'$ in $H^{4r}(S^{(\mathbf{b})},\mathbb{Q})$.*

Proof. To compute the product of $\varphi(\mathbf{b},\gamma)$ with $\psi(\mathbf{b}',\gamma')$ we can compute the intersection of their product in $H^*(S^{[n]} \times S^{[n]})$ with the class of the diagonal. This we can in turn pullback to $W_{\mathbf{b}} \times W_{\mathbf{b}'}$; by definition of φ and ψ the required product is equal to

$$\pi_{W_{\mathbf{b}}}^*(al_{\mathbf{b}} \cdot \zeta_{\mathbf{b}} \cdot p_{\mathbf{b}}^*(\gamma)) \cdot \pi_{W_{\mathbf{b}'}}^*(pt_{\mathbf{b}'} \cdot \zeta_{\mathbf{b}'} \cdot p_{\mathbf{b}'}^*(\gamma')) \cdot \delta \tag{3.2.1}$$

in $H^{4r+8n}(W_{\mathbf{b}} \times W_{\mathbf{b}'})$ where δ is the pullback of the class of the diagonal in $S^{[n]} \times S^{[n]}$.

To prove the theorem we will represent each of these classes by a cycle, and then prove that in cases (1) and (2) the supports of the cycles don't intersect; in case (3) the intersection is a finite set, and we will compute the intersection multiplicities.

We represent (the dual of) $pt_{\mathbf{b}}$ by $Pt_{\mathbf{b}}$, $al_{\mathbf{b}'}$ by $Al_{\mathbf{b}'}$ (for a fixed generically chosen pencil $\mathcal{P}$), $\zeta_{\mathbf{b}}$ by $Z_{\mathbf{b}}$ and similarly for $\mathbf{b}'$; for each i we choose a representative cycle Γ_i of $\gamma(i)$ (resp. Γ_i' of $\gamma'(i)$) such that the cycles so chosen meet transversally and are in generic position with respect to the pencil $\mathcal{P}$ (i.e., we require that these cycles are transversal to both the base locus of $\mathcal{P}$ and the union of its singular elements). We represent δ by the inverse image of the diagonal; more generally, we represent the pullback of a class in a product by the chosen representative times the other factor(s).

Let $\bar{\Gamma}_i$ be the support of Γ_i, and similarly for Γ_i', and let $(P_i, Z_i, Z, P_i', Z_i', Z)$ be a point in the intersection (necessarily $Z = Z'$ since the point lies in the inverse image of the diagonal).

We must have: $P_i \in \bar{\Gamma}_i$; $P_i' \in \bar{\Gamma}_i'$; $\operatorname{Supp} Z_i = P_i$; $\operatorname{Supp} Z = \cup P_i$; $P_i \in Z_i' \subset Z$ and Z_i' $\mathcal{P}$-linear.

Let $l = (l_0, \dots, l_4)$ be the 5-length of $(\mathbf{b}, \gamma)$, and $l' = (l_0', \dots, l_4')$ be the 5-length of $(\mathbf{b}', \gamma')$. Let r be the length of $\mathbf{b}$, let $I = \{1, \dots, r\}$ and $I_j = \{i \in I | d(\gamma(i)) = j\}$; make analogous definitions for $(\mathbf{b}', \gamma')$.

We start by proving that if $l_4' > l_0$ then the intersection is empty. In fact for any point in the intersection, P_k' must coincide with (at least) one of the P_i's; hence there must be an application $\eta : \{1, \dots, r'\} \to \{1, \dots, r\}$ such that $P_k' = P_{\eta(k)}$. This in turn implies that for each $i \in I$ we have

$$P_i \in \bar{\Gamma}_i \cap \bigcap_{\eta(k)=i} \bar{\Gamma}_k'.$$

By transversality this implies that the restriction of η to I_4' must be injective and have image contained in I_0, hence $l_4' \leq l_0$.

We now repeat the same argument 4 times to prove (1). For instance, assume that $l_4 = l_0'$. Then $I_4 = \eta(I_0')$, hence the inverse image of an element of I_4 via η must be one point. Then (applying transversality again) the restriction of η to I_1' must be injective and have image contained in I_3, and so on; this finishes the proof of (1).

If l is the dual of l', the previous argument shows that we can associate to each point of the intersection a bijection $\eta : I' \to I$, defined by requiring $P_{\eta(i)} P_i' \in \bar{\Gamma}_i' \cap \bar{\Gamma}_{\eta(i)}$ (this implies that $b_i' = 4 - b_{\eta(i)}$). By transversality this proves that the P_i's are all distinct, hence that η is unique.

Assume now $l = l''$, and fix a point $(P_i, Z_i, Z, P_i', Z_i', Z)$ in the intersection, with induced bijection η, and let $\bar{\eta} = \eta^{-1}$. By assumption, the points P_i lie on different, smooth curves C_i of the pencil. Hence, Z is a union of disjoint subschemes Z_i (of length b_i) supported on P_i. Now the subscheme $Z'_{\bar{\eta}(i)}$ must be $\mathcal{P}$-linear and contain P_i, hence it must lie on the curve C_i; on the other hand its support is contained in the union of the P_i's, so in fact it must be supported in P_i. Hence we have $Z'_{\bar{\eta}(i)}$ contained in Z_i, and therefore $b'_{\bar{\eta}(i)} \leq b_i$. On the other hand the sum of all the b_i's and of all the (b_i')'s is the same, namely n; so we must have $b'_{\bar{\eta}(i)} = b_i$; this proves (2).

Assume now $bb = \mathbf{b}'$, and identify I with I'. The previous argument shows that, for each point in the intersection, the induced permutation η is in $\mathfrak{S}_{\mathbf{b}}$. Therefore if we project the intersection on the first factor S^r, the image is precisely the union over $\sigma \in \Sigma_{\mathbf{b}}$ of

$$(\Gamma_1 \times \dots \times \Gamma_r) \cap (\Gamma'_{\sigma(1)} \times \dots \times \Gamma'_{\sigma(r)}),$$

that is the intersection of cycles in S^r representing γ and $\sigma^*(\gamma')$. So it is enough to prove that the intersection living over a given point $(P_1, \dots, P_r)$ has multplicity $\prod b - i^2$; assume without loss of generality that the permutation η associated to $(P_1, \dots, P_r)$ is the identity. Note that the projection is set-theoretically bijective.

Locally near (P_i, Z_i, Z) the natural projection of $W_{\mathbf{b}}$ to $S^r \times \prod S^{[b_i]}$ is an isomorphism. The inverse image of the diagonal in $S^{[n]} \times S^{[n]}$ maps (locally) via this isomorphism to the product of the diagonals in $S^{[b_i]} \times S^{[b_i]}$. So near (P_i, Z_i, P_i', Z_i') the intersection is locally a product of the intersection in $S \times S \times S^{[b_i]}$ of Z_1, Z_2, Pt_{b_i}, Al_{b_i} and $p^*(\Gamma_i \times \Gamma_i')$, where these subvarieties are defined in the statement of the following lemma; applying the lemma concludes the proof. □

Lemma 3.3 (Göttsche). *Let $X = S \times S \times S^{[d]}$, with projections p on $S \times S$ and q on $S^{[d]}$. Define subvarieties Z_1 and Z_2 of X by*

$$Z_i = \{(P_1, P_2, Z) | P_i \in Z\}.$$

Let Pt_d (resp. Al_d) be the subvarieties of X of triples (P_1, P_2, Z) where the subscheme Z is supported in a point (resp. $\mathcal{P}$-linear). Then

$$p_*([Z_1] \cdot [Z_2] \cdot [Al_d] \cdot [Pt_d]) = d^2\, [\Delta_S].$$

Proof. The cohomology class which we want to prove is $d^2\,[\Delta_S]$ can be represented by a 4-cycle supported on

$$p(Z_1 \cap Z_2 \cap Al_d \cap Pt_d) = \Delta_S.$$

As Δ_S is a 2-dimensional variety, the cycle in question must be a multiple of the class $[\Delta_S]$. To find out which multiple, we compute its intersection with the class $P \times S$, where $P \in S$ is a generic point. This is in turn equivalent to computing the length of the (zero-dimensional) subscheme intersection of Z_1, Z_2, Al_d, Pt_d and $p^{-1}(\{P\} \times S)$.

Let C be the unique curve of the pencil $\mathcal{P}$ containing P; we can assume that C is smooth. The intersection is set theoretically one point, the triple (P, P, Z), where Z is the (unique) subscheme of C supported in P of length d. We can chose local coordinates (x, y) near P on S such that P is $(0, 0)$ and the curves of the pencil $\mathcal{P}$ have equations $y =$ constant; hence Z is defined by the ideal (y, x^d). It is well-known that there are local coordinates $(a_0, \dots, a_{d-1}, b_0, \dots, b_{d-1})$ on $S^{[d]}$ near Z such that (a_i, b_j) corresponds to the subscheme defined by the ideal

$$(x^d + a_{d-1}x^{d-1} + \ldots + a_0, y + b_{d-1}x^{d-1} + \ldots + b_0).$$

It is more convenient to choose other coordinates $(\delta, c_0, \dots, c_{d-2}, b_0, \dots, b_{d-1})$ defining the subscheme

$$((x+\delta)^d + c_{d-2}(x+\delta)^{d-2} + \ldots + c_0, y + b_{d-1}x^{d-1} + \ldots + b_0).$$

Putting everything together we get local coordinates on X

$$(x_1, y_1, x_2, y_2, \delta, c_0, \dots, c_{d-2}, b_0, \dots, b_{d-1}).$$

In this coordinates the equations of Z_1, Z_2, Al_d, Pt_d and $p^{-1}(\{P\} \times S)$ are (in order, remembering that the first two are defined by two equations each, the third and the fourth by $(d-1)$ each and the fifth by two equations again):

$$
\begin{aligned}
&(x_1+\delta)^d + c_{d-2}(x_1+\delta)^{d-2} + \ldots + c_0 = 0\\
&y_1 + b_{d-1}x_1^{d-1} + \ldots + b_0 = 0\\
&(x_2+\delta)^d + c_{d-2}(x_2+\delta)^{d-2} + \ldots + c_0 = 0\\
&y_2 + b_{d-1}x_2^{d-1} + \ldots + b_0 = 0\\
&b_1 = \ldots = b_{d-1} = 0\\
&c_0 = \ldots = c_{d-2} = 0\\
&x_1 = y_1 = 0
\end{aligned}
$$

So the ideal of the intersection is generated by $(b_i, c_j, x_1, y_1, y_2, \delta^d, (x_2+\delta)^d)$, which has length d^2. □

Proof of theorem 2.4. We prove that the intersection pairing between $\mathcal{B}_k$ and $\mathcal{B}'_{4n-k}$ is nondegenerate for any given k; this implies that the elements of $\mathcal{B}_k$ are linearly independent, hence as they are precisely $b_k(S^{[n]})$ by [G] they are a basis (and the same for $\mathcal{B}'_k$).

Order the elements of $\mathcal{B}_k$ compatibly with the ordering on the 5-lengths and the elements of $\mathcal{B}'_{4n-k}$ compatibly with the ordering of the duals of the 5-lengths; then (by theorem 3.2 (1)) the intersection matrix is block triangular, with the blocks being the intersection matrices between elements such that the 5-length of the first is dual to the 5-length of the second.

By 3.2 (2), each of this blocks is in turn block diagonal, the subblocks being the intersection matrices of elements with a given partition: these subblocks are nondegenerate because of 3.2 (3). □

Corollary 3.4. *The morphism*

$$\varphi \otimes \mathbb{C} : \bigoplus_{\mathfrak{b} \in P(n)} H^\bullet(S^{(\mathfrak{b})}, \mathbb{C}) \to H^\bullet(S^{[n]}, \mathbb{C})$$

induces an isomorphism of Hodge structures, and the same for ψ.

Proof. The morphism $H^\bullet(S^r, \mathbb{C}) \to H^\bullet(S^{[n]}, \mathbb{C})$ inducing $\varphi \otimes \mathbb{C}$ is given by composing pull back to $W_\mathfrak{b}$ (which is a homomorphism of Hodge structures of type $(0,0)$) with intersection with the class $\zeta_\mathfrak{b} \cdot pt_\mathfrak{b}$ (which is a homomorphism of Hodge structures of type $(-2r,-2r)$) and finally with the pushforward to $S^{[n]}$ (which is a homomorphism of Hodge structures of type $(n+r, n+r)$). So it is a homomorphism of Hodge structures of type $(n-r, n-r)$; as it is an isomorphism of vector spaces, it is also an isomorphism of Hodge structures. □

Remark 3.5. The morphisms (2.4.1) in theorem 2.4 can also be defined if S is any complex surface and $S^{[n]}$ is the Douady space representing 0-dimensional subschemes of length n. It seems natural to conjecture that it is always an isomorphism. If the surface is a deformation of an algebraic surface, this follows easily from the algebraic case. However we have no idea of how to do the general case (in which the Betti numbers are also unknown).

References

[Ch] J. Cheah, *The Hodge numbers of the Hilbert scheme of points of a smooth projective variety*, preprint (1993).

[Ch2] ———, Thesis, 1994.

[ES1] G. Ellingsrud, S. A. Strømme, *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. **87** (1987), 343–352.

[ES2] ———, *On a cell decomposition of the Hilbert scheme of points in the plane*, Invent. Math. **91** (1988), 365–370.

[FG] B. Fantechi, L. Göttsche, *The cohomology ring of the Hilbert scheme of 3 points on a smooth projective variety*, J. reine angew. Math. **439** (1993), 147–158.

[G] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), 193–207.

[GS] L. Göttsche, W. Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. **296** (1993), 235–245.

[MS] R. Mallavibarrena, I. Sols, *Bases for the homology groups of the Hilbert scheme of points in the plane*, Comp. Math. **74** (1990), 169–202.

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