THE NORMALIZED CURVE SHORTENING FLOW AND HOMOTHETIC SOLUTIONS

U. Abresch and J. Langer

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D - 5300 Bonn 3

Sonderforschungsbereich 40
Theoretische Mathematik
Beringstraße 4
D - 5300 Bonn 1

MPI/SFB 84-64

THE NORMALIZED CURVE SHORTENING FLOW AND HOMOTHETIC SOLUTIONS

The curve shortening problem, by now widely known, is to understand the evolution of regular closed curves $\gamma : \mathbb{R}/\mathbb{Z} \longrightarrow M$ moving according to the curvature normal vector: $\frac{\partial\gamma}{\partial t} = kN = -$ "the L^2 gradient of arc length". One motivation for this problem has been the view expressed in this connection by C. Croke, H. Gluck, W. Ziller, and others, that it would be desirable to improve on some complicated and ad hoc constructions that have been used in the theory of closed goedesics to iteratively shorten curves.

As a test case it has been a goal to prove the conjecture that kN generates a flow on the space of simple closed curves in the plane, preserving embeddedness and making any such curve circular asymptotically as length approaches zero. However, the evolution equation for the curvature of γ_t turns out to be quite subtle, and the conjecture is not yet settled. Indeed, in the non-simple case one generally expects singular behavior, and part of the intrinsic interest of the problem lies in the fact that the global condition of embeddedness is apparently recognized by the "near-sighted" equation.

What is known thus far is that the conjecture is true for convex curves, that simple curves do in fact remain simple (provided curvature stays bounded), and that short time solutions to the equations exist in full generality; these results are due to M. Gage and R. Hamilton (see [G1], [G2], [G-H]).

In Section 1 of the present paper it is shown that by adding on a reparametrizing tangential vector bT to kN (thus leaving the flow geometrically unchanged) one obtains an apparently nicer evolution equation for the curvature of $\gamma : \mathbb{R}/\mathbb{Z} \longrightarrow M^2$ (see Theorem 1), which readily yields useful information (see, e.g., Propositions 1.7, 1.9).

In particular, one is led quite directly to a very tractable equation (2.3) for homothetic solutions in the plane, i.e., curves which evolve simply by scaling. The equation is integrable by quadrature, and essentially this fact makes it possible to give a complete classification of all closed homothetic solutions in $\mathbb{R}^2$ (Theorem 2), the main result of Section 2. In agreement with the above conjecture, the circle is the only embedded closed homothetic solution, but there are infinitely many others which are not embedded.

Part of the significance of the homothetic solutions is that they represent the possible asymptotic limits for the curve shortening flow in an arbitrary 2-manifold; this statement is made precise in Section 3 (see Theorem 3). Section 3 concludes with some non-convergence results for certain trajectories of the curve shortening flow.

Another interesting aspect of the homothetic solutions is that there is reason to regard them as possible comparison solutions for the flow. This point is discussed in Section 4, where the existence of all closed homothetic solutions is explained heuristically by relating these curves to multiply covered circles, on the one hand, and to singular limits, on the other. This discussion is based on computations in the spirit of linear stability analysis.

An appendix contains some technical results which are crucial to the complete classification of closed homothetic solutions.

Finally, it should be noted that the authors have heard that E. Calabi has also obtained some results regarding homothetic solutions of the curve shortening flow (though apparently not for closed curves).

1. The Normalized Flow

Our notation regarding trajectories of the curve shortening flow will be as follows. We will denote by $\gamma : [0,t_0) \times \mathbb{R}/\mathbb{Z} \longrightarrow M$ a smooth one-parameter family of regular closed curves in a Riemannian manifold M. The circle $\mathbb{R}/\mathbb{Z}$ will be parametrized by $\sigma \in [0,1]$ (or by $s \in [0,1]$ in case of arclength parametrization). We will write $\gamma_t(\sigma) = \gamma(t,\sigma)$ and, for fixed t, denote the speed, curvature, unit tangent and unit normal vectors of γ_t by $\alpha = |\frac{\partial\gamma}{\partial\sigma}|$, k, T, and N, respectively. Derivatives with respect to σ (or s) will often be denoted by primes and derivatives with respect to t by dots.

Now let us suppose γ as above happens to be generated from some initial curve γ_0 by the curve shortening flow, i.e., according to $\dot{\gamma} = kN$. Then we can always reparametrize in the σ variable (smoothly in t) to obtain a new γ for which the curves γ_t all have constant speed, i.e., α depends only on t. Of course, this new γ no longer satisfies $\dot{\gamma} = kN$; rather, it evolves according to an equation of the form

$$(1.1) \qquad \dot{\gamma} = W = bT + kN ,$$

where $b : [0,t_0) \times \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}$ is some smooth function.

Proposition 1.1 If γ satisfies equation (1.1) then each γ_t has constant speed if and only if γ_0 has constant speed and b satisfies

(1.2) $$\frac{b'}{\alpha} = k^2 - \int_0^1 k^2 d\sigma \quad .$$

<u>Proof</u>: Equation (1.1) and the Frenet equations yield

$$\frac{\partial}{\partial t} \alpha^2 = 2\langle \nabla_W \gamma', \gamma' \rangle = 2\langle \nabla_\gamma W, \gamma' \rangle = 2\alpha(b' - \alpha k^2) \quad .$$

dividing by α^2 and differentiating with respect to σ, one obtains

(*) $$\frac{\partial^2}{\partial t \partial \sigma} \ln\alpha = \frac{\partial}{\partial \sigma} \left(\frac{b'}{\alpha} - k^2\right)$$

which proves both directions: constant speed parametrization implies that $\frac{\partial}{\partial \sigma} \ln\alpha$ vanishes and hence that $\frac{b'}{\alpha} - k^2$ is constant along each γ_t. Now the periodicity requirement on b determines the constant and gives 1.2. Conversely, by 1.2 the right hand side of (*) vanishes and therefore the condition $\frac{\partial}{\partial \sigma} \ln\alpha = 0$ is preserved under the flow.

□

Since equation (1.2) determines b up to an additive constant we may fix b, henceforth, by the requirement

(1.3) $$0 = av(b) = \int_0^1 b(\sigma) d\sigma \quad .$$

We wish to further modify the original flow by replacing the variable t by a new time parameter τ according to

(1.4) $$\frac{dt}{d\tau} = \alpha^2 \quad .$$

This change makes the flow equivariant with respect to dilations and the advantages will become evident below; for the moment we note only that for a circle in $\mathbb{R}^2$ to shrink to a point requires finite t but <u>infinite</u> τ . This follows from the general fact that $\alpha(t)$ - i.e., the length of a curve γ_t - decreases according to

$$\frac{d\alpha}{dt} = b'-\alpha k^2 = -\alpha\int_0^1 k^2 d\sigma \quad . \tag{1.5}$$

The <u>normalized curve shortening flow</u> can now be defined by

$$\frac{\partial\gamma}{\partial\tau} = \alpha^2(bT + kN) \quad , \tag{1.6}$$

where b satisfies (1.2), (1.3). The resulting one-parameter families of constant speed closed curves will be denoted $\gamma(\tau,\sigma)$.

We consider now how normalization affects the evolution of curvature (we restrict ourselves here to the simpler two-dimensional case - the only case we will need).

<u>Proposition 1.2</u> If M^2 has curvature $R_0(p)$ and $\gamma(\tau,\sigma)$ is a trajectory of the normalized curve shortening flow then the curvature of γ satisfies the evolution equation

$$\frac{\partial k}{\partial\tau} = k'' + \alpha bk' + \alpha^2 k^3 + \alpha^2 kR_0|_\gamma \tag{1.7}$$

<u>Proof</u>: In general, if a curve γ in a Riemannian manifold evalues in time τ according to $\frac{\partial\gamma}{\partial\tau} = V$, for some vectorfield V along γ , then the curvature of γ evolves according to

$$\frac{\partial}{\partial\tau}k^2 = 2<\nabla_T\nabla_T V,\nabla_T T> - 4k^2<\nabla_T W,T> + 2<R(V,T)T,\nabla_T T>$$

(see [LS1]). Applying this formula to $V = \alpha^2(bT + kN)$ and using the Frenet equations easily yields the result.

□

We recall that for the original flow $\frac{\partial\gamma}{\partial t} = kN$ (where the speed α depends also on σ) the curvature evolves according to

$$\frac{\partial}{\partial t}k = \frac{1}{\alpha}\frac{\partial}{\partial\sigma}\left(\frac{1}{\alpha}\frac{\partial k}{\partial\sigma}\right) + k^3 + kR_0|_\gamma \quad .$$

Thus, the benefit of normalization is that the "Laplace Beltrami" operator (which is troublesome because it "changes" with the curve) is replaced by the ordinary Laplacian. The price one pays, of course, is that the lower order terms become more complicated. (Actually, the equation will assume a much simpler form, below.)

It will also be useful to consider a "rescaled" flow where the curves γ_τ all have unit length. In Euclidean space one can simply enlarge γ_τ by homothety: $\gamma_\tau \longrightarrow \frac{1}{\alpha}\gamma_\tau$. In the general Riemannian case we can view γ_τ as evolving in a manifold (M^n,g) whose metric is itself evolving conformally according to

(1.8) $$g = \alpha^{-2} g_0 \quad .$$

For the Euclidean case this amounts to the same thing, as scaling of the metric is induced by homothety. We will sometimes refer to this "rescaled normalized curve shortening flow" as the rescaled flow.

To describe the evolution of curvature under the rescaled flow we observe first that rescaling affects the curvature of γ_τ and of M^2 according to: $k \longrightarrow \alpha k$, $R_0 \longrightarrow \alpha^2 R_0$. Setting

(1.9) $$\begin{aligned} \kappa &= \alpha k \\ \beta &= \alpha b \\ R &= \alpha^2 R_0 \end{aligned}$$

we have

Theorem 1 If γ is a trajectory of the rescaled curve shortening flow in M^2 then the curvature κ of γ evolves according to

(1.10) $$\frac{\partial \kappa}{\partial \tau} = (\kappa' + \beta\kappa)' + \kappa R|_\gamma$$

Proof: Using equations (1.5), (1.7), and (1.9) one computes

(1.11) $$\frac{\partial \kappa}{\partial \tau} = \kappa'' + \beta\kappa' + \kappa^3 - \kappa\int_0^1 \kappa^2 d\sigma + \kappa R \quad .$$

Also, equations (1.2), (1.9) yield

(1.12) $$\beta' = \kappa^2 - \int_0^1 \kappa^2 \, d\sigma \quad .$$

Combining equations (1.11), (1.12) gives (1.10).

□

We remark that the utility of equation (1.10) is actually independent of its interpretation in terms of the rescaled flow; one may choose to regard it as an equation describing the unrescaled (normalized) flow via formal substitutions (1.9).

For the remainder of this section we restrict our attention to the special case $M = R^2$. Since $R \equiv 0$ in this case, the right hand side of (1.10) is a derivative. This is very suggestive of a fact about the original curve shortening flow:

<u>Corollary 1.3</u>. The total absolute curvature $T(\gamma) = \int_\gamma |k| ds = \int_0^1 |\kappa| d\sigma$ is non-increasing under the curve shortening flow. In fact, $\frac{d}{d\tau} T(\gamma) < 0$ unless k' vanishes whenever k does.

<u>Proof</u>: For any time τ we can divide γ into countably many curves $\gamma_i : [a_i, b_i) \longrightarrow R^2$ for which the curvature vanishes at both endpoints and does not change sign in the interior. (here we are assuming the curvature vanishes <u>somewhere</u> on γ, but if not, then $T \equiv 2\pi$.(rotation number of γ) on a C^2 neighborhood of γ, hence $\frac{dT}{d\tau}(\gamma) = 0$).

Where $\kappa \neq 0$ we clearly have $\frac{\partial}{\partial \tau} \operatorname{sgn}(\kappa) = 0$ and, on the interior of an interval for which κ vanishes identically, equation (1.10) implies $0 = \frac{\partial \kappa}{\partial \tau} = \frac{\partial |\kappa|}{\partial \tau}$. Thus, letting

$\operatorname{sgn}(i) = \operatorname{sgn}(\kappa(\sigma))$, $\sigma \in (a_i, b_i)$, we can wite

$$\frac{d}{d\tau} T(\gamma) = \frac{d}{d\tau} \int_0^1 \operatorname{sgn}(\kappa)\kappa d\sigma = \int_0^1 \operatorname{sgn}(\kappa) \frac{\partial\kappa}{\partial\tau} d\sigma$$

$$= \sum_i \operatorname{sgn}(i) \int_{a_i}^{b_i} \frac{\partial}{\partial\sigma}(\kappa' + \beta\kappa) d\sigma = \sum_i \operatorname{sgn}(i)(\kappa'(b_i) - \kappa'(a_i)) .$$

Clearly no term in the last sum can be positive, and the sum will be negative unless $\kappa'(a_i) = \kappa'(b_i) = 0$ for all i .

□

Since a plane curve γ is convex (in the weak sense) if and only if $T(\gamma) = 2\pi$, we have at once

Corollary 1.4 Convex curves remain convex under the curve shortening flow.

Some less obvious applications of the rescaled flow equations concern estimates on derivatives of κ . In particular, while Corollary 1.3 shows that the L^1 norm of κ remains bounded under the flow, it will be seen below that a slightly stronger bound - e.g., an L^2 bound on κ - would imply that all derivatives of κ are bounded.

In order to prove this and other results we first establish two essential lemmas regarding the time behavior of the Sobolev seminorms of κ

$$(1.13) \qquad x_j = \int (\partial^i \kappa)^2 .$$

Here we use the shorthand notation $\int$ in place of $\int_0^1 ds$. We will also adhere to the convention that the derivative operator $\partial^j = \frac{\partial^j}{\partial s^j}$ is applied only to the term immediately to the right.

<u>Lemma 1.5</u> The time derivatives $\dot{x}_j = \frac{d}{d\tau} x_j$ satisfy the estimates

(1.14) $$\dot{x}_0 \leq - 2x_1 + \sqrt{x_0 x_1}\, x_0 \quad ,$$

and for $j \geq 1$,

(1.15) $$\dot{x}_j \leq - 2x_{j+1} + (3^{j+1}-2) \sqrt{x_0 x_1}\, x_j + 2^{j+1} x_0 x_j \quad .$$

<u>Proof</u>: To begin with we have by interpolation

(1.16) $$x_\mu x_j \leq x_{\mu-\nu}\, x_{j+\nu} \quad \text{for} \quad 0 \leq \nu \leq \mu \leq j \quad .$$

Secondly, since

$$||\partial^\mu (\kappa^2 - x_0)||_\infty \leq \frac{1}{2} \int |\partial^{\mu+1}(\kappa^2 - x_0)|$$

$$\leq \frac{1}{2} \sum_{\nu=0}^{\mu+1} \binom{\mu+1}{\nu} \int |\partial^\nu \kappa|\,|\partial^{\mu+1-\nu}\kappa| \leq \frac{1}{2} \sum_{\nu=0}^{\mu+1} \binom{\mu+1}{\nu} \sqrt{x_\nu}\, \sqrt{x_{\mu+1-\nu}} \quad ,$$

inequality (1.16) gives, for all $\mu \geq 0$,

(1.17) $$||\partial^\mu (\kappa^2 - x_0)||_\infty \leq 2^\mu \sqrt{x_0 x_{\mu+1}} \quad .$$

Similarly, since

$$||\kappa\partial^\nu\kappa||_\infty \le |\int_0^1 \kappa\partial^\nu\kappa ds| + \frac{1}{2}\int|\partial(\kappa\partial^\nu\kappa)|$$

$$\le \sqrt{x_0 x_\nu} + \frac{1}{2}\sqrt{x_1 x_\nu} + \frac{1}{2}\sqrt{x_0 x_{\nu+1}} \quad ,$$

we obtain also

$$(1.18) \qquad ||\kappa\partial^\nu\kappa||_\infty \le \sqrt{x_0 x_\nu} + \sqrt{x_0 x_{\nu+1}}$$

At this point we invoke the rescaled flow equation (1.10) to compute

$$\dot{x}_j = 2\int \partial^j\dot{\kappa}\partial^j\kappa = 2\int \partial^j\kappa \; (\partial^{j+2}\kappa + \partial^{j+1}(\beta\kappa))$$

$$= -2x_{j+1} + 2\sum_{\nu=0}^{j+1}\binom{j+1}{\nu}\int \partial^\nu\beta\,\partial^{j+1-\nu}\kappa\,\partial^j\kappa$$

$$= -2x_{j+1} + \int\beta\,\partial(\partial^j\kappa)^2 ds + 2(j+1)\int\partial\beta(\partial^j\kappa)^2 ds + 2\varphi_j$$

$$= -2x_{j+1} + (2j+1)\int(\kappa^2 - x_0)(\partial^j\kappa)^2 ds + 2\varphi_j \quad ,$$

where we have set

$$\varphi_j = \sum_{\nu=2}^{j+1}\binom{j+1}{\nu}\int\partial^{\nu-1}(\kappa^2 - x_0)\;\partial^{j+1-\nu}\kappa\;\partial^j\kappa \quad .$$

Because of (1.17) the above yields

$$(1.19) \qquad \dot{x}_j \le -2x_{j+1} + (2j+1)\sqrt{x_0 x_1}\;x_j + 2\varphi_j \quad .$$

In particular, this gives inequality (1.14) of the lemma. To obtain the second inequality of the lemma it remains to bound φ_j for $j \ge 1$:

$$\varphi_j = \sum_{\mu=1}^{j} \binom{j+1}{\mu+1} \int \partial^\mu(\kappa^2 - x_0)\partial^{j-\mu}\kappa\partial^j\kappa$$

$$= \sum_{\mu=1}^{j-1} \binom{j+1}{\mu+1} \int \partial^\mu(\kappa^2 - x_0)\partial^{j-\mu}\kappa\partial^j\kappa \; +$$

$$\sum_{\nu=0}^{j-1} \binom{j}{\nu} \int \kappa\partial^\nu\kappa\partial^{j-\nu}\kappa\partial^j\kappa \;+\; \int \kappa^2 (\partial^j\kappa)^2$$

$$\le \sum_{\mu=1}^{j-1} \binom{j+1}{\mu+1} ||\partial^\mu(\kappa^2 - x_0)||_\infty \sqrt{x_{j-\mu}}\,\sqrt{x_j}$$

$$+ \sum_{\nu=0}^{j-1} \binom{j}{\nu} ||\kappa\partial^\nu\kappa||_\infty \sqrt{x_{j-\nu}}\,\sqrt{x_j}$$

$$+ \dot{x}_0 x_j + ||\kappa^2 - x_0||_\infty x_j \quad .$$

Using the estimates (1.16) and (1.17), (1.16) and (1.18), respectively, we bound each term separately:

$$2\varphi_j \le 2 \sum_{\mu=1}^{j-1} \binom{j+1}{\mu+1} 2^\mu \sqrt{x_0 x_1}\; x_j$$

$$+ 2 \sum_{\nu=0}^{j-1} \binom{j}{\nu} (x_0 + \sqrt{x_0 x_1})\, x_j$$

$$+ 2 (x_0 + \sqrt{x_0 x_1})\, x_j$$

$$= (3^{j+1} - (2j+3)) \sqrt{x_0 x_1}\; x_j + 2^{j+1} x_0 x_j \quad .$$

Inequality (1.15) now follows from (1.19).

□

<u>Lemma 1.6</u> The geometric mean $m = \sqrt{x_0 x_1}$ obeys

(1.20) $$\dot{m} \le 2m \left(- \frac{m^2}{x_0^2} + 2m + x_0 \right) ,$$

and for $j \geq 1$ one has

$$\dot{x}_j \leq x_j \left(-2\frac{x_j}{x_{j-1}} + (3^{j+1}-2)m + 2^{j+1}x_0\right) \tag{1.21}$$

<u>Proof</u>: Using inequalities (1.14) and (1.15) and then (1.16) one computes in a straightforward manner

$$2m\,\dot{m} = x_0\dot{x}_1 + \dot{x}_0 x_1 \leq -2x_0x_2 + 7\sqrt{x_0x_1}\,x_0x_1 + 4x_0^2x_1$$
$$-2\,x_1^2 + \sqrt{x_0x_1}\,x_0x_1 \leq 4m^2\left(-\frac{m^2}{x_0^2} + 2m + x_0\right) .$$

Inequality (1.21) follows from (1.15) and (1.16).

□

We are now in a position to bound the higher Sobolev norms in terms of $x_0(\tau)$; more precisely, our bounds will be given in terms of the quantities

$$C(\tau) = \max\{x_0(\tau') : 0 \leq \tau' \leq \tau\} .$$

Note that $C(\tau) \geq 4\pi^2 > 32$.

<u>Proposition 1.7</u> The functions $m(\tau)$ and $x_j(\tau)$, $j \geq 1$, are bounded on the whole existence interval in terms of $C(\tau)$ and the initial data $m(0)$ and $x_1(0),\ldots,x_j(0)$, respectively. Asymptotically,

i.e., for $\tau \cdot C(\tau)$ sufficiently large, one has the following bounds depending only on $C(\tau)$:

(1.22) $$m(\tau) \leq 3\,C(\tau)^2$$

(1.23) $$x_j(\tau) \leq 3^{\frac{1}{2}j(j+5)} \cdot C(\tau)^{2j+1}$$

Proof: Observe first that the right hand sides of (1.22), (1.23) are both non-decreasing. Thus it will suffice to prove that exponential decay holds for $m(\tau)$ as long as (1.21) is violated, and to argue similarly for x_j .

a) A direct calculation shows that

$$-m^2+2x_0^2m+x_0^3 \leq -\tfrac{1}{4}x_0^2 \quad \text{as long as} \quad m^2 \geq 2x_0^2+\tfrac{1}{2}x_0 \quad .$$

Therefore, it follows from inequality (1.20) that one has $\dot{m} \leq -\frac{1}{2}m$ as long as $m(\tau) \geq 3C(\tau)^2 (>2x_0(\tau)^2 + \frac{1}{2}x_0(\tau))$.

b) By inequality (1.21) one has $\dot{x}_j \leq -2^{j+10}x_j$ as long as $x_j \leq x_{j-1}(\frac{1}{2}\cdot 3^{j+1}m + 2^{j+1}C(\tau)^2)$ holds. By induction, the boundedness of x_j, $j \geq 1$, now follow from the bounds on m and x_{j-1} .

In order to obtain the estimate (1.23) we may assume $\tau \cdot C(\tau)$ large enough so that $m(\tau) \leq 3C(\tau)^2$. One then argues inductively using $\frac{1}{2}\cdot 3^{j+1}m + 2^{j+1}C(\tau)^2 \leq 3^{j+2}C(\tau)^2$.

□

<u>Remark</u> The asymptotic bounds of Proposition 1.7 reflect the partially smoothing nature of the flow. This phenomenon is due to the Laplacian on the right side of equation (1.10) and to this extent comparison with the behavior of the heat equation is appropriate. However, the non-linear lower order term generally disrupts the familiar total smoothing phenomenon. In fact, the existence of the homothetic solutions, obtained below, shows that one cannot expect the x_j to decay to zero for $j \leq 1$ even if x_0 remains bounded. In this sense, Proposition 1.7 provides the best possible general set of estimates.

Moreover, even when x_0 is not bounded - so the flow is approaching a singularity - the above estimates are adequately describing the behavior of the x_j. This is the essential point of the following complementary proposition, which gives large <u>lower</u> bounds on the x_j when κ is L^2-far from constant.

<u>Proposition 1.8</u> Letting $|| \ ||_1$ denote the L'-norm, one has

$$\int |\kappa|^p \geq ||\kappa||_1^{2-p} \, x_0^{p-1} \qquad \text{for} \quad p \geq 2 \quad , \tag{1.24}$$

and for all $j \geq 1$,

$$x_j \geq x_0 \left(\frac{x_0}{||\kappa||_1^2} - 1 \right)^{2j} \quad . \tag{1.25}$$

<u>Proof</u>: (1.24) follows directly from Hölder's inequality. Using this estimate with $p=4$ and (1.17) one obtains

$$x_0^2 \left(\frac{x_0}{||\kappa||_1^2} - 1 \right) \leq \int (\kappa^2 - x_0)\kappa^2 \leq ||\kappa^2 - x_0||_\infty x_0 \leq x_0 \sqrt{x_0 x_1} \quad .$$

This proves (1.25) in case $j = 1$. Inequality (1.16) now yields the general result by induction.

□

Finally, we have the following general estimate on the existence interval for trajectories of the flow:

Proposition 1.9 x_0 satisfies the differential inequality

(1.26) $$\dot{x}_0 \le \frac{1}{8} x_0^3 \quad .$$

Therefore the solution exists at least as long as $\tau < \tau_E = 4\, x_0(0)^{-2}$.

Proof: The differential inequality follows from (1.14) because of $\sqrt{x_0 x_1}\; x_0 \le 2 \cdot x_1 + \frac{1}{8} x_0^3$ (which is just a special case of $\sqrt{ab} \le \frac{1}{2}(\frac{a}{\varepsilon} + \varepsilon b)$) . Note that Proposition 1.7 implies κ is C^∞ as long as $x_0(\tau)$ is bounded. Hence, integrating the comparison equation yields the estimate on the existence interval.

□

2. Closed Homothetic Solutions

As the nonlinear evolution equations of curve shortening apparently do not lend themselves to a general partial differential equations approach, the possibility of writing down some special solutions analytically is of particular interest. It is natural to look first for homothetic solutions in $\mathbb{R}^2$, i.e., curves which do not change shape at all under the curve shortening flow, only size.

To begin our study of closed homothetic solutions we observe first that any such constant speed curve $\gamma : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^2$ represents a "steady state", i.e., time independent solution of the rescaled evolution equation (1.10). Thus, γ would have to satisfy a system of the form

$$(2.1) \qquad \begin{aligned} \kappa' &= -\beta\kappa + \mu_1 \\ \beta' &= \kappa^2 + \mu_2 \end{aligned}$$

for appropriate constants μ_1, μ_2 and some function β having average value zero. Since by definition κ depends only on the similarity class of γ it will be sufficient to consider only unit speed curves $\gamma : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^2$. In this case κ is actually the curvature of γ (we prefer not to write k since k should be thought of as evolving in time, even for homothetic solutions).

From (2.1) we will shortly derive much more useful equations, but we first note that some significant information can be read off directly:

<u>Proposition 2.1</u> Let γ be a C^4 (hence C^∞) closed curve representing a homothetic solution. Then the curvature of γ is nowhere vanishing. Thus, we might as well assume from now on that $\kappa > 0$.

<u>Proof</u>: Suppose κ vanishes somewhere. Since the total absolute curvature of γ obviously does not change under the curve shortening flow it follows from Corollary 1. that κ, κ' must vanish simultaneously. Uniqueness of solutions to the initial value problem for eq. (2.1) now implies $\kappa \equiv 0$, which is absurd.

□

The periodicity of κ and β has important consequences. We claim that μ_1 is actually zero and we observe that μ_2 is in fact negative, as it is the average value of $-\kappa^2$. Thus, we rewrite (2.1) as

$$(2.1') \qquad \begin{aligned} \kappa' &= -\beta\kappa \\ \beta' &= \kappa^2 - \lambda^2 \quad . \end{aligned}$$

To prove the claim we note first that taking an antiderivative $\int\beta$ preserves periodicity since β has average value zero. Therefore, the functions $f = \kappa e^{\int\beta}$ and $f' = (-\beta\kappa+\mu_1)\, e^{\int\beta} + \beta f = \mu_1 e^{\int\beta}$ are also periodic. But $\operatorname{sign}(f') = \operatorname{sign}(\mu_1) =$ constant, so evidently $\mu_1 = 0$.

From (2.1') one readily verifies the following

Proposition 2.2 Let $\gamma : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^2$ be a smooth closed curve of unit speed representing a homothetic solution. Then the function

(2.2) $$B = 2\ln\frac{\kappa}{\lambda}$$

satisfies the steady state equation:

(2.3) $$B'' + 2\lambda^2(e^B - 1) = 0$$

Such an equation is particularly welcome as it yields a first integral:

(2.4) $$\frac{1}{2}(B')^2 + 2\lambda^2 V(B) = 2\lambda^2\eta \quad ,$$
$$V(B) = e^B - B - 1$$

with η a non-negative constant. Indeed, the solution B to the steady state equation is obtained by inverting the integral

$$\sigma(B) = \frac{1}{2\lambda}\int_0^B \frac{dy}{\sqrt{\eta - V(y)}}$$

Remark: Any such normalized homothetic solution γ (unit speed, length one) determines a whole conformal class of homothetic solutions (unit speed, length ℓ) $\gamma_\ell : \mathbb{R}/\ell\mathbb{Z} \longrightarrow \mathbb{R}^2$ defined by $\lambda_\ell(s) = \ell\lambda(\frac{s}{\ell})$. This evident scaling property of homothetic solutions corresponds to the fact that the differential equations (2.1') and (2.4) continue to hold when $\kappa, \beta, \lambda, B$ are replaced by $\kappa_\ell(s) = \frac{1}{\ell}\kappa(\frac{s}{\ell})$, $\beta_\ell(s) = \frac{1}{\ell}\beta(\frac{s}{\ell})$, $\lambda_\ell = \frac{1}{\ell}\lambda$, $B_\ell(s) = B(\frac{s}{\ell})$, respectively. The important point here is that the constant η appearing in equation (2.4) is unaffected by this transformation. Hence, η should be thought of as determining the <u>shape</u> of γ and λ (which is inversely proportional to ℓ) as measuring the <u>scale</u> of γ. The initial condition $B(0) = B_0$ for equation (2.4) is geometrically insignificant as it affects only the location of the initial point $\gamma(0)$ on the curve.

Our aim now is to use the above equations to find all smooth closed curves γ representing (normalized) homothetic solutions. We begin by noting that solutions to (2.3) (hence also (2.1')) are globally defined as functions on $\mathbb{R}$ since the first integral implies a uniform bound on B. We remark that even system (2.1) with <u>arbitrary</u> constants μ_1, μ_2 has only global solutions since the differential inequality $\frac{1}{2}|(\kappa^2+\beta^2)'| = |\mu_1\kappa+\mu_2\beta| \le \sqrt{\mu_1^2+\mu_2^2}\,\sqrt{\kappa^2+\beta^2}$ also gives an a priori bound.

Thus, given a solution B of equation (2.3), equation (2.2) and the Frenet equations determine a curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$. Of course, the resulting γ may or may not be closed. However, we at least have

<u>Proposition 2.3</u> Any solution B of equation (2.3) is periodic and oscillates between the roots B_+, B_- of the equation

(2.5) $$\eta = V(B) \quad .$$

As the constant η tends to its lower limit, zero, the period of B tends to $\sqrt{2}\,\pi/\lambda$. In general, we have

$$\text{period}(\eta,\lambda) = \frac{1}{\lambda}\,\text{period}(\eta,1) \quad .$$

Proof: Given that B is periodic, the last statement is apparent from the above Remark. That B is bounded by B_+, B_- follows at once from equation (2.4). The fact that B actually achieves these bounds - i.e., that B has finite period - can be seen as a consequence of the Sturm Comparison Theorem.

Specifically, we rewrite equation (2.3) as $B'' + 2\lambda^2 T(B)B = 0$, where $T(B) = \frac{e^B - 1}{B}$. We note that $T(B)$ is a positive, monotone increasing function of B , hence $0 < T(B_-) \le T(B) \le T(B_+)$. It follows that $\frac{2\pi}{\sqrt{2\lambda^2 T(B_+)}} \le \text{period}(B) \le \frac{2\pi}{\sqrt{2\lambda^2 T(B_-)}}$. As η tends to zero, B_+ and B_- also tend to zero, so $T(B)$ is forced to approach $T(0) = 1$.

□

To settle the closedness question we must still investigate the global behavior of curves γ arising from equations (2.2), (2.3) and the Frenet equations. The global situation is easily clarified via

Proposition 2.4 Let κ, B satisfy equations (2.2), (2.3) and let γ be a unit speed curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ having curvature κ. Then the vectorfield $J = \kappa T + \frac{\kappa'}{\kappa} N$ extends to a non-constant Killing field on $\mathbb{R}^2$, thus giving rise to a natural polar coordinate system (r,θ) on $\mathbb{R}^2$ - the origin being the zero of J. In fact, $J = \lambda^2 \frac{\partial}{\partial\theta}\Big|_\gamma$.

Proof: In general, if γ is a curve in a simply connected space form M^2, a vectorfield J along γ extends to a Killing field on M^2 if and only if $\langle\nabla_T J, T\rangle = 0$ and $\langle\nabla_T\nabla_T J, N\rangle = 0$ (see [LS1]).

For the above J equations (2.1') yield $\nabla_T J = \lambda^2 N$, so J obviously satisfies the general criterion. The last statement now follows from the observation that at a critical point of κ we have $T = \frac{J}{|J|}$; so on the one hand, $\frac{1}{|J|}\nabla_J J = \lambda^2 N$, while on the other hand, we know that $\frac{1}{|\frac{\partial}{\partial\theta}|}\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = N$.

□

Corollary 2.5 If we set $\gamma(s) = (r(s), \theta(s))$ then the extrema of $r(s)$ satisfy

$$\frac{r_{max}}{\kappa_{max}} = \frac{r_{min}}{\kappa_{min}} = \lambda^{-2} .$$

In fact, $r(s)$ is explicitly related to $\kappa(s)$ by

$$\kappa = C\cdot e^{\frac{1}{2}\lambda^2 r^2}$$

where C is a constant determined by λ and η .

<u>Proof</u>: Note that at extrema of r , J and T are parallel. Therefore the proposition shows that extrema of r and κ coincide. The first statement now follows upon comparing J and $\frac{\partial}{\partial\theta}$ at an extremum of $\kappa(s)$.

To obtain the formula for κ we note first that equations (2.2), (2.4) yield $(\frac{\kappa'}{\kappa})^2 + \kappa^2 = \lambda^2(\eta+1 + 2\ln\frac{\kappa}{\lambda})$. Hence, choosing our initial point so that $r_{min} = r(0)$ and noting that $\frac{\partial}{\partial r}$ is a unit vector perpendicular to J we have

$$r - r_{min} = \int_0^s < T, \frac{\partial}{\partial r} > ds = \int_0^s \frac{\kappa' ds}{\kappa\sqrt{(\frac{\kappa'}{\kappa})^2 + \kappa^2}} =$$

$$= \int_{\kappa_{min}}^{\kappa} \frac{d\kappa}{\lambda\kappa\sqrt{\eta+1+2\ln\frac{\kappa}{\lambda}}} = \frac{1}{\lambda}\sqrt{\eta+1+2\ln\frac{\kappa}{\lambda}} - \frac{1}{\lambda^2}\kappa_{min} \quad .$$

Using the first part of the corollary and solving for κ gives the desired formula.

□

<u>Proposition 2.6</u> All non-circular homothetic solutions are transcendental curves.

<u>Proof</u>: Observe first that the curvature function of an algebraic curve with respect to any algebraic parameter is algebraic. If the

non-circular homothetic solutions were algebraic then r^2 would be such an algebraic parameter. However Corollary 2.5 states that κ depends exponentially on r^2.

□

Thus, in each period of κ, γ behaves as follows (we might as well assume $\kappa_{min} = \kappa(0)$) : γ begins at a point of tangency to an inner circle $r = r_{min} = \frac{1}{\lambda^2}\kappa_{min}$, proceeds to a point of tangency to an outer circle $r = r_{max} = \frac{1}{\lambda^2}\kappa_{max}$, and then returns. The angular progress of γ per period coincides with the angular change of the unit tangent vector T per period, and γ closes up smoothly if and only if this change $\Delta\theta$ is rationally related to π. [Note that these curves behave qualitatively like cyclids, but that Proposition 2.6 rules out the possibility that any non-circular homothetic solution actually is a cyclid.]

Consequently, in order to describe all closed homothetic solutions in $\mathbb{R}^2$ it remains only to investigate the behavior of $\Delta\theta$ over the set of solutions to (2.3). Using equations (2.2), (2.4), and setting $L = \text{per}\,\kappa$, we obtain

$$(2.6) \quad \Delta\theta = \Theta(\eta) = \int_0^L \kappa\, ds = 2\int_0^{L/2} \frac{\kappa\, dB}{\sqrt{B'^2}} = \int_{B_-}^{B_+} \frac{dB}{\sqrt{e^{-B}(\eta - V(B))}}$$

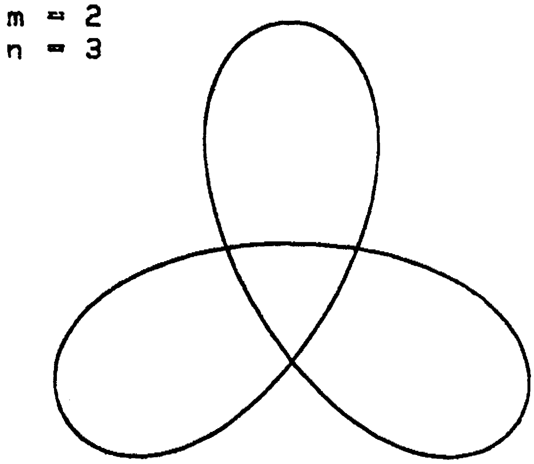
<u>Proposition 2.7</u> As η increases from zero to infinity $\Theta(\eta)$ decreases monotonically from $\sqrt{2}\,\pi$ to π.

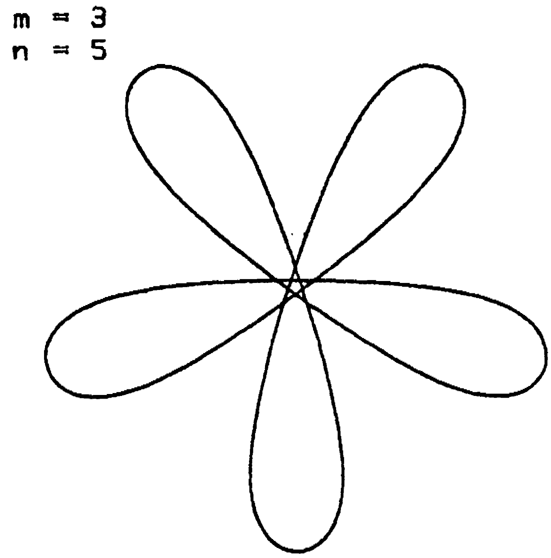
The verification of the monotone behavior of $\Theta(\eta)$ presents considerable technical difficulties and we therefore treat the integral $\Theta(\eta)$ in the appendix.

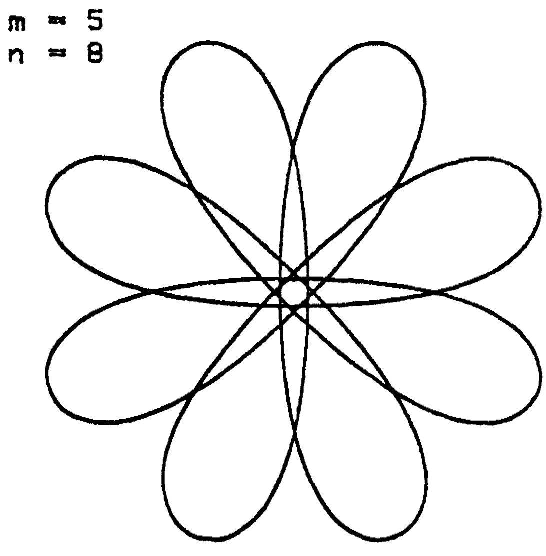
To summarize the discussion of this section we have

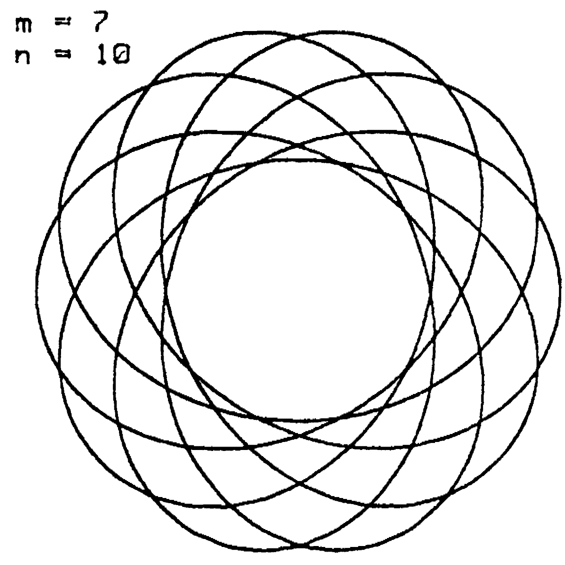
<u>Theorem 2</u> Let $\gamma : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^2$ be a regular closed curve representing a homothetic solution of the curve shortening flow. Then γ is an m-covered circle γ_m, $m = 1,2,3,\ldots,$ or γ is a member of the family of closed homothetic solutions $\{\gamma_{m,n}\}$. having the following description: if $m > 1$ and n are integers satisfying $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}$ there is (up to congruence) a unique curve $\gamma_{m,n}$ which closes up in n periods of its curvature function $\kappa > 0$ - a solution to equations (2.2), (2.3) - while making m orbits about the fixed point of the associated Killing field J .

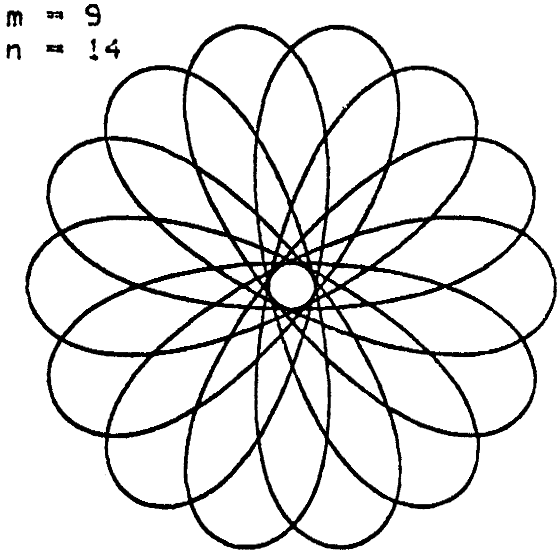
Six of the solutions $\gamma_{m,n}$ are pictured in Figure 1 below (the scale is not consistent, but only the similarity class matters anyway). The curves were generated by computer by investing $\Theta(\eta)$ and then solving the system (2.1') together with the Frenet equations, using the initial conditions $\beta(0) = 0$, $\kappa(0) = \kappa_{min}(\eta)$. This system is particularly well behaved because the right hand side is quadratic in terms which can be estimated linearly, given a uniform bound on κ .

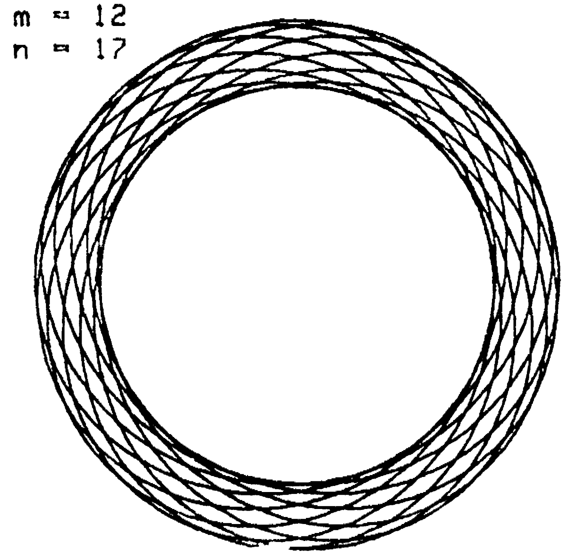

m = 2
n = 3


m = 3
n = 5


m = 5
n = 8


m = 7
n = 10


m = 9
n = 14


m = 12
n = 17

3. Asymptotic Behavior of Curves on 2-Manifolds

Suppose $\gamma = \gamma_0 : \mathbb{R}/\mathbb{Z} \longrightarrow M^2$ is a smooth immersion for which the rescaled curve-shortening trajectory γ_τ is defined for all time $\tau > 0$. In Section 1 it was proved that the L^1 norm of κ remains bounded (at least in the planar case). If we assume that in fact κ_τ converges in L^1 then, owing to our knowledge of homothetic solutions in the plane, a great deal can be said about the possible asymptotic behavior of γ_τ. To begin with, we have

Proposition 3. If $\kappa_\tau(\sigma) = \kappa(\tau,\sigma)$ is an all time solution to the system (1.10), (1.12) which converges in $L^1(\mathbb{R}/\mathbb{Z})$ as $\tau \to \infty$ then the limit $\kappa = \kappa_\infty$ is in $C^\infty(\mathbb{R}/\mathbb{Z})$ and satisfies equation (2.3) (with B defined by equation (2.2)).

Proof: We begin by integrating equation (1.10) twice:

$$\frac{\partial}{\partial\tau} \int_0^\sigma\int_0^u \kappa \, dvdu = \kappa + \int_0^\sigma \beta\kappa du + \int_0^\sigma\int_0^u \kappa R \, dvdu + C\sigma + D .$$

Since κ converges in L^1, the right hand side of the above equation clearly converges in L^1 to some function $H(\sigma)$. We claim that in fact $H(\sigma) \equiv 0$. For suppose $H(\sigma_0) = H_0 > 0$ for some σ_0; then for all sufficiently large τ, $2\frac{\partial}{\partial\tau}\int_0^{\sigma_0}\int_0^u \kappa R dvdu > H_0$, hence $\int_0^{\sigma_0}\int_0^u \kappa dvdu$ must tend to infinity, contradicting convergence of κ.

Setting the right hand side of the above equation equal to zero, it follows by induction that κ is C^∞. Differentiating twice and observing that κR tends to zero in L^1 as $\tau \to \infty$, we recover equations (2.1').

□

We would like to interpret the above result in terms of convergence of curves. For plane curves this is particularly easy to do since the homothetic curvature κ is just the ordinary curvature k, once we rescale curves by homothety to have unit speed. Since we are interested in asymptotic behavior as curves shrink to a point it is appropriate to generalize this to manifolds via the exponential map.

To be specific, suppose γ_τ converges to a point $p \in M^2$. Then for sufficiently large τ we can define a curve $\bar{\gamma}_\tau$ in T_pM by $\bar{\gamma}_\tau = \exp_p^{-1}(\gamma_\tau)/L(\gamma_\tau)$. Of course, if $M^2 = \mathbb{R}^2$ then $\bar{\gamma}_\tau$ is just γ_τ rescaled as above.

<u>Theorem 3</u>. Suppose $\gamma_\tau \subset M^2$ is an all time solution to the normalized curve shortening flow and suppose κ_τ converges in L^1. Then either γ_τ converges C^1 to a geodesic or γ_τ converges to a point $p \in M^2$, and $\bar{\gamma}_\tau$ converges C^1 to one of the (similarity classes of) homothetic solutions γ_m or $\gamma_{m,n}$ of Theorem 2.

<u>Proof</u>: Suppose γ_τ does not converge to a geodesic. Then there exist constants τ_0, $C > 0$ such that $\tau > \tau_0$ implies $\int_0^1 k_\tau^2 d\sigma > C$.

Hence, equation (3) implies $\frac{d\alpha}{d\tau} < C\,\alpha^3$. If follows that $L(\gamma_\tau)$ tends to zero.

We claim that in fact γ_τ converges to a point $p \in M^2$. To see this, we consider $\tau_i \to \infty$ and, for each i , a "strongly convex set" $\Gamma_i \subset M^2$ (i.e. $\partial\Gamma_i$ has strictly positive inward curvature) which contains γ_{τ_i} and has diameter a bounded multiple of $\mathrm{diam}(\gamma_{\tau_i})$. For $\tau > \tau_i$, the curve γ_τ must remain inside the fixed set Γ_i ; if γ_τ ever touches $\partial\Gamma_i$, the curvature normal kN is actually pointing inward in a small neighborhood of any point of first order contact. So γ_τ can never cross $\partial\Gamma_i$. We thus have a nested family of compact sets whose diameters tend to zero.

It now makes sense to define $\bar{\gamma}_\tau$ for sufficiently large τ . Since the Christoffel symbols for normal coordinates tend to zero at the origin, the curvature of $\bar{\gamma}_\tau$ approaches κ_τ as $\tau \to \infty$. The theorem now follows from Proposition 3.1 and Theorem 2.

□

[We remark that the above theorem would still hold if $\bar{\gamma}_\tau$ were defined simply as the rescaled trajectory, $\bar{\gamma}_\tau \subset (M^2, g = \alpha^{-2} g_0)$, and convergence of $\bar{\gamma}_\tau$ were appropriately defined. Since the curvature of (M^2, g) converges to zero at any point p , it is clear how to define the limit of $\bar{\gamma}_\tau$ as a plane curve.]

Since the trajectories of the curve shortening flow are necessarily regular homotopies one can be still more specific about asymptotic behavior if one specifies topological information about the initial curve γ_0 :

<u>Corollary 3.2</u> Assume, e.g., M^2 is topologically $\mathbb{R}^2$ or the 2-torus, and γ_τ converges as in Theorem 3. Then if γ_0 is a simple curve (or regularly homotopic to a simple curve) either γ_τ converges to a geodesic or $\bar{\gamma}_\tau$ converges to a circle.

<u>Proof</u>: Suppose γ_τ does not converge to a geodesic. Then $\bar{\gamma}_\tau$ must converge C^1 to one of the γ_m or $\gamma_{m,n}$, and the same statement must hold when the regular homotopy γ_τ is lifted to the universal cover $\mathbb{R}^2$. But now the Whitney-Graustein Theorem implies that the limit must have rotation index ± 1 . Hence, $m = 1$, so the curves $\gamma_{m,n}$ are ruled out.

□

For $M = S^2$ one would need an additional geometric hypothesis; for there are only two regular homotopy classes of immersed circles in S^2 (see [S]) , and simple curves are regularly homotopic to curves resembling $\gamma_{m,n}$ for any odd m . The proof of Theorem 3 provides one such hypothesis:

<u>Corollary 3.3</u> The previous Corollary holds for arbitrary M^2 if γ_0 is assumed also to lie in a strongly convex domain $\Gamma \subset M^2$.

Finally, we observe that application of Theorem 3 yields a particularly simple proof of some non-convergence results for the curve shortening flow:

Corollary 3.4 Suppose $\gamma_0 : \mathbb{R}/\mathbb{Z} \longrightarrow M^2$ is not null-homotopic and not regularly homotopic to a geodesic; alternatively, suppose $\gamma_0 : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^2$ has rotation index zero (e.g., γ_0 looks like a figure eight). Then the (normalized) curvature κ_τ of the resulting curve shortening trajectory γ_τ cannot converge in L^1 .

Before stating the second non-convergence result we recall that for closed curves $c: \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ the algebraic area A(c) is defined in terms of an integral of the winding number N(x,c) over almost all points x in $\mathbb{R}^2$:

$$A(c) := \int_{\mathbb{R}^2} N(x,c)\, dx = \frac{1}{2} \int_c \det(c,c')\, d\sigma.$$

The criterion will also involve the rotation index

$$m = \operatorname{Ind}(c) = \frac{1}{2\pi} \int_c k\, ds.$$

Proposition 3.5 Suppose γ is a regular closed curve in $\mathbb{R}^2$ satisfying one of the following two conditions:

i) $A(\gamma) \neq 0$ and $A(\gamma) \cdot \operatorname{Ind}(\gamma) \leq 0$

ii) $k \geq 0$ along γ and $N(p,\gamma) < 0$ for some point $p \in \mathbb{R}^2$.

Let γ_t be the trajectory of the curve shortening flow with initial curve γ.

Then the L^2- norm of the rescaled curvature κ diverges before the length of γ_t approaches zero, hence, within finite time $\tau(t)$.

Proof: i) For any differentiable family of curves $c_t:\mathbb{R}/\mathbb{Z}\to\mathbb{R}^2$ the differential of the algebraic area can be calculated in a straightforward manner:

$$\frac{d}{dt}A(c_t) = -\int_0^1 \det(c_t', \frac{\partial}{\partial t}c_t)\,d\sigma.$$

Using the flow equations (1.1) we obtain

$$\frac{d}{dt}A(\gamma_t) = -\int_{\gamma_t} k ds = -2\pi m,$$

and by assumption i),

$$|A(\gamma_t)| \geq |A(\gamma)| > 0.$$

We make use of Corollary 1.3 to bound the winding numbers:

$$|N(x,\gamma_t)| \leq \frac{1}{2\pi}\int_0^1 |k_t| ds \leq \frac{1}{2\pi}\int_0^1 |k| ds, \qquad \forall x\in\mathbb{R}^2,$$

and we deduce a uniform positive lower bound on the enclosed area, i.e., on area$\{x: N(x,\gamma_t)\neq 0\}$. Thus, the isoperimetric inequality yields a uniform positive lower bound on the length of γ_t.

However, according to the proof of Theorem 3, the length would have to approach zero if the flow existed for all $\tau>0$. In view of Proposition 1.7 the L^2-norm of κ must therefore diverge.

ii) In this case we look at the set $F_t:=\{x\in\mathbb{R}^2 : N(x,\gamma_t)\leq -1\}$, and its area, $a_t:=$ area(F_t). By hypothesis $a_0>0$. Our goal is to show that a_t is non-decreasing, then again use the isoperimetric inequality and finish the proof as above.

Notice that for an arbitrary differentiable family of curves $c_t: \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ the function $t \to a_t$ is only Lipschitz rather than C^1 (differentiability may be lost at non-transversal intersections). The upper and lower Dini derivatives are given as follows:

$$D_t^+ a_t = \int_{\partial\overline{F}_t} \max\{\det(c_t'/|c_t'|, \tfrac{\partial}{\partial t}c_t)(\sigma) : \sigma \in c_t^{-1}\{p\}\} ds(p)$$

$$D_t^- a_t = \int_{\partial\overline{F}_t} \min\{\det(c_t'/|c_t'|, \tfrac{\partial}{\partial t}c_t)(\sigma) : \sigma \in c_t^{-1}\{p\}\} ds(p) \; .$$

In our case, since $k_t \geq 0$ (by Corollary 1.3), we obtain the desired estimate:

$$D_t^- a_t = \int_{\partial\overline{F}_t} \min\{k(\sigma) : \sigma \in c_t^{-1}\{p\}\} \; ds(p) \geq 0.$$

Actually, for trajectories of the curve shortening flow, a_t is differentiable, since k_t is constant in almost all fibers $c_t^{-1}\{p\}$.

□

Figure 2, below, shows some initial curves which must develop singularities by the preceeding criterion.

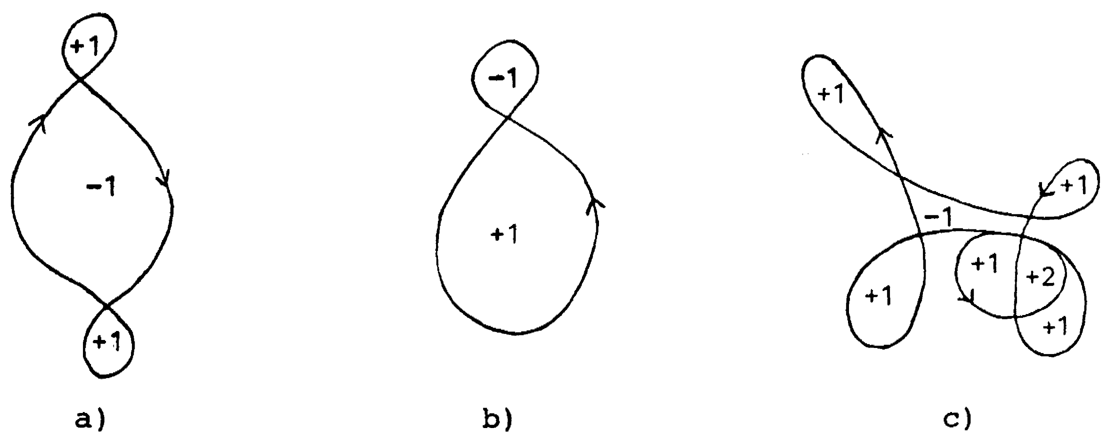


Figure 2

Remark: In view of such examples as the figure eight, a full account of asymptotics of the curve shortening flow would require that convergence be discussed in the sense of distributions. For example, for a standard symmetrical figure eight it is an open question whether the limiting curvature will be a sum of two Dirac measures $\delta=\pi(\delta_0 - \delta_{\frac{1}{2}})$ or some other type of singularity.

4. Homothetic Solutions as Saddle Points

In the discussion of the homothetic solutions $\gamma_{m,n}$ in Section 2 it was natural to regard the closed curves $\gamma_{m,n}$ as belonging to a larger family of homothetic solutions which are not necessarily closed. Viewed in this context the curves $\gamma_{m,n}$ occur almost incidentally; by continuity, the real number $\Delta\theta$ (=angular progress per period) must sometimes be rational. Moreover, the proof that $\pi < \Delta\theta < \sqrt{2}\pi$ (in the appendix) leaves the condition $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}$ looking rather arbitrary.

One goal of the present section is to impart the opposite impression - namely, that there is a good reason for the existence of each $\gamma_{m,n}$ - and, at the same time, to offer a partial explanation for the following rather curious coincidence: the classification of the (non-circular) closed free elasticae $\sigma_{m,n}$ in the hyperbolic plane (critical points of $\int_\sigma k^2 ds : \mathrm{Imm}(S^1, H) \to \mathbb{R}$) follows the very same arithmetic condition, $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}$, and qualitative description (see [LS1]).

The point is that the curves $\gamma_{m,n}$ ($\sigma_{m,n}$) are precisely the saddle type critical points one is led to expect if one combines stability computations for the multiple circle solutions γ^m (σ^m) with knowledge of trajectories tending to singular curves.

Before proceeding to concrete results we first describe the overall picture. Observe that $\gamma_{m,n}$ is "fixed" by the group $G=G(m,n)=\langle g\rangle \cong \mathbb{Z}_n$, where g corresponds to rotation by $\theta=\frac{2\pi m}{n}$. Figure 3 describes, for the case m=2, n=3, a G-equivariant regular homotopy beginning at γ^m, passing through $\gamma_{m,n}$, and tending to a singular curve $\Gamma_{m,n}$.

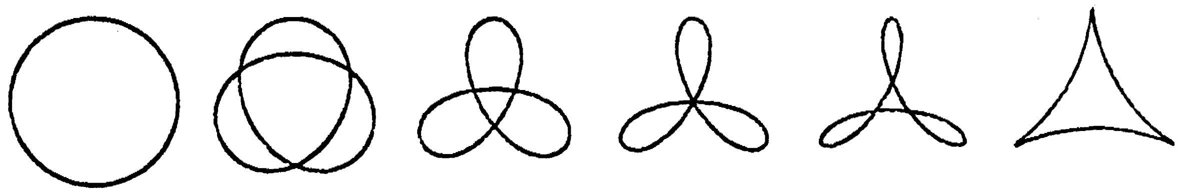


Figure 3

A reasonable conjecture associated with this picture is that if $\varepsilon>0$ is a small number and N is the outward pointing normal along $\gamma_{m,n}$, then the rescaled curve shortening flow carries $\gamma_+ = \gamma_{m,n} + \varepsilon N$ to γ^m and $\gamma_- = \gamma_{m,n} - \varepsilon N$ to a singular curve resembling $\Gamma_{m,n}$ of Figure 3(f).

The evidence for such a conjecture comes in two parts. By Proposition 3.5, any curve resembling (d) or (e) of Figure 3 must tend to such a singular curve; though Proposition 3.5 does not directly apply to (d), the direction of the curvature normal vector kN implies that after a short time t>0 the hypothesis will be satisfied. [In this connection it is interesting to note that by Corollary 2.5 there exist homothetic solutions of unit diameter with arbitrarily small "positive centers". Hence, Figure 3(d) represents an extreme case for such a general divergence argument.]

On the other hand, Proposition 4.2, below, suggests that γ^m ought to attract curves resembling (b) of Figure 3. We note that this requires the condition $\frac{m}{n}<\frac{\sqrt{2}}{2}$, while the condition $\frac{1}{2}<\frac{m}{n}$ is implicit in Figure 3(d); for each of the n petals must contribute at least π to the total rotation $2\pi m$. Thus, the arithmetic condition of the classification will have been explained.

We turn now to the linear stability analysis for circles. The The multiple circles γ^m have curvature functions $\kappa_m = 2\pi m$, which represent critical points of the rescaled flow on the manifold $\Omega=\{\kappa\in C^\infty(\mathbb{R}/\mathbb{Z})$: if $\gamma(s)$ has unit length and curvature $\kappa(s)$ then γ is a regular closed curve $\}$.

Setting $\alpha(u)=\int_0^u \kappa(s)ds$, we can write
$\Omega=\{\kappa\in C^\infty(\mathbb{R}/\mathbb{Z}) : \int_0^1 \kappa ds=2\pi m$, m an integer, and $\int_0^1 e^{i\alpha(s)}ds = 0\}$.

It follows that the tangent space to Ω at $\kappa\in\Omega$ can be identified with $T_\kappa\Omega=\{h\in C^\infty(\mathbb{R}/\mathbb{Z}): 0=\int_0^1 h(s)ds = \int_0^1\int_0^s h(u)du\, e^{i\alpha(s)}ds\} = \{h\in C^\infty(\mathbb{R}/\mathbb{Z}) : 0=\int_0^1 hds=\int_0^1 h(s)\int_0^s e^{i\alpha(s)}duds\}$. In particular, the tangent space at the m-fold circle is given by

(4.1) $$T_{\kappa_m}\Omega =\{h\in C^\infty(\mathbb{R}/\mathbb{Z}): 0=\int_0^1 hds=\int_0^1 h(s)e^{i\kappa_m s}ds\}.$$

We wish to consider now the linearization of the flow $\dot\kappa=\kappa''+(\beta\kappa)' \equiv P(\kappa)$ at some fixed $\kappa\in\Omega$:

(4.2) $$\dot h=DP(\kappa)h = h''+(\beta h)' +(D\beta(\kappa)h\cdot \kappa)' \equiv Lh ,$$

where

(4.3) $$\tfrac{1}{2}D\beta(\kappa)h = \int_0^s h\kappa ds - s\int_0^1 h\kappa ds .$$

For the special case $\kappa=\kappa_m \equiv 2\pi m$, equations (4.2), (4.3), and the facts $\beta(\kappa_m)=0$, $\int_0^1 hds=0$, imply that the linear map $L:T_{\kappa_m}\Omega \to T_{\kappa_m}\Omega$ is given by:

(4.4) $$Lh = h'' + 2\kappa_m^2 h.$$

Proposition 4.1 The multiple circles γ^m, $|m|>1$, are linearly unstable critical points of the rescaled flow.

Proof: Set $h(s)=\cos 2\pi s$. Then for $m\neq\pm1$, $h\in T_{\kappa_m}$. Thus we have found a positive eigenvalue: $Lh=4\pi^2 h$.

□

Remark: We note that the above proposition has a simple geometric interpretation. Consider, e.g., the case m=2. Then varying $\kappa_2\equiv 4\pi$ in the direction of $h=\cos 2\pi s$ corresponds to shrinking one circle of γ^2 while enlarging the other (one should picture a pair of tangent circles of slightly different radii, one inside the other). The proposition may be interpreted as saying that the flow does not tend to restore such perturbations to circularity, but rather, it amplifies the inequality in size. This shows once again the striking difference between the embedded and non-embedded cases of the curve shortening problem.

On the other hand, we will show now that if one restricts L to the appropriate subspace of "symmetric" vectors in $T_{\kappa_m}\Omega$ then positive eigenvalues no longer appear. Specifically, we consider the group $G=G(m,n)=\langle g\rangle\cong\mathbb{Z}_n$ introduced above, which acts on Ω by

(4.5) $$g\kappa(s) = \kappa(s-2\pi m/n).$$

Let Ω^G be the fixed point set of G, $\Omega^G=\{\kappa\in\Omega:\ g\kappa=\kappa\}$. Note that the flow on Ω restricts to a flow on the submanifold Ω^G.

Now the tangent space to Ω^G is given by $T_\kappa\Omega^G=\{h\in T_\kappa\Omega\colon gh=h\}$. In particular, it follows from equations (4.1) and (4.5) that any $h\in T_{\kappa_m}\Omega^G$ has Fourier series representation of the form

$$h(s) = \sum_{j=1}^{\infty} a_j\cos(2\pi jns) + b_j\sin(2\pi jns) \tag{4.6}$$

(in fact $a_j=b_j=0$ in case $jn=m$).

Substitution of the series (4.6) into formula (4.4) yields

$$Lh=(2\pi)^2 \sum_{j=1}^{\infty} (2m^2-j^2n^2)(a_j\cos(2\pi jns) + b_j\sin(2\pi jns)). \tag{4.7}$$

From formula (4.7) follows at once

<u>Proposition 4.2</u> The linear map from $T_{\kappa_m}\Omega^{G(m,n)}$ to itself has strictly negative spectrum precisely when $|\frac{m}{n}|<\frac{\sqrt{2}}{2}$; In this case the eigenvalues λ_i of L are bounded from above by $\lambda_i<(\frac{m}{n}-\frac{\sqrt{2}}{2})<0$. In other words, the flow on $\Omega^{G(m,n)}$ is linearly stable at κ_m exactly when m and n satisfy the above inequality.

5. Appendix

Here we discuss the function $\Theta : (0,\infty) \longrightarrow \mathbb{R}$ which arose in Section 2:

$$\Theta(\eta) = \int_{B_-}^{B_+} \frac{dB}{\sqrt{e^{-B}(\eta - V(B))}} \quad .$$

The convex potential function $V(B) = e^B - B - 1$ assumes any value $\eta \geq 0$ at precisely two points $B_-(\eta) \leq 0 \leq B_+(\eta)$. Thus it defines a bijection $\frac{1}{2}(B_+ - B_-) : [0,\infty) \longrightarrow [0,\infty)$. The inverse of this bijection will be denoted by ρ. The parameter $w = \rho^{-1}(\eta)$ will be useful since it provides explicit analytic expressions:

<u>Lemma 5.1</u> Setting $\sigma = (\sinh w)/w$, one has

$$\rho(w) = w \coth w - 1 + \ln \sigma$$

$$B_+ = B_+ \circ \rho(w) = w - \ln \sigma$$

$$B_- = B_- \circ \rho(w) = -w - \ln \sigma$$

<u>Proof</u>: In addition to $w = \frac{1}{2}(B_+ - B_-)$ we shall consider $u = \frac{1}{2}(B_+ + B_-)$. We have

$$e^{u+w} - u - w = 1 + \eta = e^{u-w} - (u-w) \quad ,$$

from which we conclude

$$w = e^u \sinh w$$

$$1 + \eta = e^u \cosh w - u \quad .$$

Eliminating u by means of the first of these two equations yields the desired expression for ρ . The other claims are also immediate from

$$u = -\ln \sigma.$$

□

When discussing monotonicity and range of Θ we can replace Θ by $\Theta \circ \rho$. In the following computation, and thereafter, we will simplify notation by suppressing the arguments $\rho(w)$ and η in the functions B_+ and B_- :

$$\Theta \circ \rho(w) = \left(\int_{B_-}^{0} + \int_{0}^{B_+} \right) e^{\frac{1}{2}B} \frac{dB}{\sqrt{\rho(w) - V(B)}} = \int_{0}^{\rho(w)} h(\eta) \frac{d\eta}{\sqrt{\eta(\rho(w) - \eta)}} ,$$

where

$$(5.1) \qquad h(\eta) = \sqrt{\eta} \left(\frac{e^{\frac{1}{2}B_+}}{V'(B_+)} - \frac{e^{\frac{1}{2}B_-}}{V'(B_-)} \right) = \frac{\sqrt{\eta}}{2} \left(|\sinh \tfrac{1}{2}B_+|^{-1} + |\sinh \tfrac{1}{2}B_-|^{-1} \right) .$$

Thus we have

$$(5.2) \qquad \Theta \circ \rho(w) = \int_0^1 h(\rho(w) \cdot x) \frac{dx}{\sqrt{x(1-x)}} .$$

Proposition 5.2

$$\lim_{\eta \to 0} \Theta(\eta) = \lim_{w \to 0} \Theta \circ \rho(w) = \pi\sqrt{2}$$

$$\lim_{\eta \to \infty} \Theta(\eta) = \lim_{w \to \infty} \Theta \circ \rho(w) = \pi$$

Proof: Clearly

$$\frac{\sqrt{V(B)}}{2\cdot\sinh\frac{B}{2}} \longrightarrow \begin{cases} 0 & \\ \frac{\sqrt{2}}{2} & \text{as} \\ 1 & \end{cases} \quad \begin{matrix} B \longrightarrow -\infty \\ B \longrightarrow 0 \\ B \longrightarrow \infty \end{matrix} .$$

It follows that

$$n(\eta) \longrightarrow \begin{cases} \sqrt{2} & \text{as} \\ 1 & \end{cases} \quad \begin{matrix} \eta \longrightarrow 0 \\ \eta \longrightarrow \infty \end{matrix}$$

This yields the claim, since $\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi$.

□

Computer plots of the functions $\frac{1}{\pi}\Theta\circ\rho$ and $h\circ\rho$ will provide some intuition for further analysis (see Figure 4) .

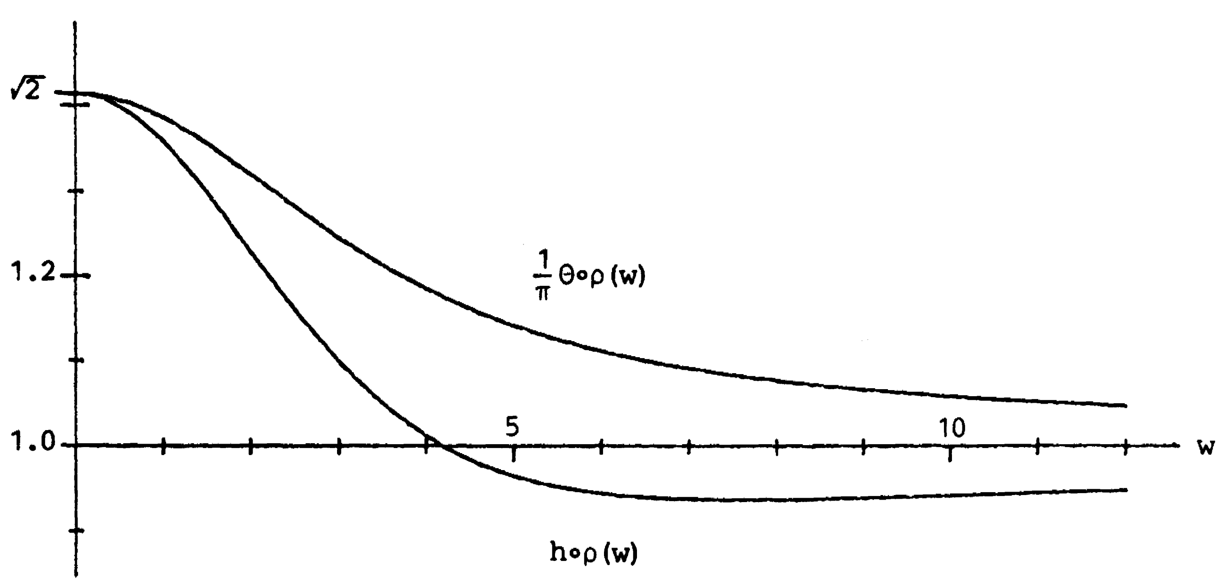


Figure 4

We observe - and later in this section will actually prove - that $h\circ\rho$ is monotone decaying for small w. In view of formula (5.2) this fact implies that $\Theta\circ\rho$ is actually decaying in the same neighborhood of 0. However the graph also indicates that $h\circ\rho$ is not globally decaying; thus for large w the monotonicity of $\Theta\circ\rho$ has to be derived from other ideas. We carry out these arguments next.

The concave function $B \longrightarrow e^{-B}(\eta-V(B))$ takes its maximum at $B = -\eta$; for large η this point gets arbitrarily close to $B_- + 1$ (c.f. Lemma 5.1). This motivates the following splitting (for $w \geq 1$):

(5.3)
$$\Theta_1(\eta) = \int_{B_-}^{B_-+1} \frac{dB}{\sqrt{e^{-B}(\eta-V(B))}}$$

$$\Theta_2(\eta) = \int_{B_-+1}^{B_+} \frac{dB}{\sqrt{e^{-B}(\eta-V(B))}} .$$

<u>Lemma 5.3</u> Θ_1 is monotone decaying and $\lim_{\eta\to\infty} \Theta_1(\eta) = 0$.

<u>Proof</u>: A direct calculation yields $(x := B-B_-)$:

$$\Theta_1\circ\rho(w) = \int_0^1 \frac{dx}{\sqrt{(1+xe^{-B_-})e^{-x}-1}} = \int_0^1 \frac{dx}{\sqrt{(1+xT(2w))e^{-x}-1}} .$$

This already proves the lemma, since the function $T(2w) = (e^{2w}-1)/2w$ is monotonic and tends to infinity with w.

□

<u>Lemma 5.4</u> $(\Theta_2\circ\rho)'(w) \le -\frac{T'(-2w)}{1-T(-2w)}\,\Theta_2\circ\rho(w) + \left(\frac{e}{T(2w)+1-e}\right)^{\frac{1}{2}}$

<u>Proof</u>: Setting $D(x) = (1-xT(-2w))e^x-1$, the substitution $x = B_+-B_-$ yields

$$\Theta_2\circ\rho(w) = \int_0^{2w-1} \frac{dx}{\sqrt{D(x)}} ,$$

hence,

$$(\Theta_2\circ\rho)'(w) = -\frac{1}{2}\int_0^{2w-1} \frac{2T'(-2w)xe^x}{D(x)} \cdot \frac{dx}{\sqrt{D(x)}} + \frac{1}{\sqrt{D(2w-1)}}$$

$$= -\int_0^{2w-1} \frac{T'(-2w)}{T(-x)-T(-2w)} \cdot \frac{dx}{\sqrt{D(x)}} + \left(\frac{e}{1-e+T(2w)}\right)^{\frac{1}{2}} .$$

The claim follows, since by the monotonicity and convexity properties of T we have

$$\frac{T'(-2w)}{T(-x)-T(-2w)} \ge \frac{T'(-2w)}{1-T(-2w)} .$$

□

Observe that for $w \ge 2$ one has

$$\frac{T'(-2w)}{1-T(-2w)} = \frac{1}{4w^2} \cdot \frac{1-(1+2w)e^{-2w}}{1-(1-e^{-2w})/2w} \ge \frac{1}{4w^2} .$$

Hence we conclude:

$$(5.4) \qquad (\Theta_2 \circ \rho)'(w) \le -\frac{1}{4w^2}\,\Theta_2 \circ \rho(w) + \left(\frac{e^{2w-1}}{2w} - 1\right)^{-\frac{1}{2}} .$$

<u>Proposition 5.5</u> The function $\Theta \circ \rho$ is monotone decreasing and hence $> \pi$ on the interval $[5.22, \infty)$.

<u>Proof</u>: Since $\Theta = \Theta_1 + \Theta_2$, it is sufficient in view of Lemma 5.3 to prove the claim for $\Theta_2 \circ \rho$. We already know that

$$(5.5) \qquad \lim_{w \to \infty} \Theta_2 \circ \rho(w) = \lim_{w} \Theta \circ \rho(w) - \lim_{w \to \infty} \Theta_1 \circ \rho(w) = \pi .$$

A computation yields

$$4w^2 \left(\frac{e^{2w-1}}{2w} - 1\right)^{-\frac{1}{2}} \le \pi - 6 \cdot 10^{-4} \quad \text{for} \quad w \in [5.22, \infty) .$$

From equation (5.4) we thus get on this interval the differential inequality

$$(\Theta_2 \circ \rho)'(w) \le -\frac{1}{4w^2}\left(\Theta_2 \circ \rho(w) - \pi + 6 \cdot 10^{-4}\right) ,$$

which gives decay wherever $\Theta_2 \circ \rho > \pi - 6 \cdot 10^{-4}$. The result now follows from equation (5.5).

□

As mentioned earlier, for $0 \leq w \leq 5.22$ monotonicity is going to be deduced from monotonicity of $h \circ \rho$. Setting $\sigma = (\sinh w)/w$ as before, one computes

$$h \circ \rho(w) = \sqrt{\rho(w)} \left(\frac{\sqrt{T(2w)}}{T(2w)-1} + \frac{\sqrt{T(-2w)}}{1-T(-2w)} \right)$$

$$(5.6) \qquad = \sqrt{\rho(w)} \sqrt{\sigma} \left(\frac{e^{w/2}}{\sigma e^{w}-1} + \frac{e^{-w/2}}{1-\sigma e^{-w}} \right)$$

$$= [\cosh w + \sigma(-1+\ln\sigma)]^{\frac{1}{2}} \cdot 2 \sinh\frac{w}{2} \cdot \frac{1+\sigma}{(\sinh 2w)/w - \sigma^2 - 1}$$

For the purpose of analyzing $h \circ \rho$ in detail it is more convenient to pass to the variable $x = w^2$ and consider $H(x) := h \circ \rho(\sqrt{x})$. We then use the function

$$(5.7) \qquad \sigma(x) = \frac{\sinh\sqrt{x}}{\sqrt{x}} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^k$$

For later use we state some elementary facts regarding σ :

<u>Lemma 5.6</u>

i) $\sigma'(x) = \sum_{k=0}^{\infty} \frac{k+1}{(2k+3)!} x^k$

ii) $2w^2\sigma'(w^2) = \cosh w - \sigma(w^2)$

iii) $2\sigma(4x)-1-\sigma(x)^2 = 2\sigma(4x)-1-\frac{1}{2}(\cosh(2\sqrt{x})-1)$

$$= 8x \sum_{k=0}^{\infty} \frac{2k+3}{(2k+4)!} (4x)^k$$

We now rewrite equation (5.6) in terms of σ and σ' :

$$(5.8) \qquad H(x) = x\sqrt{2\sigma' + \sigma\frac{\ln\sigma}{x}} \cdot \sigma(\tfrac{x}{4}) \cdot \frac{1 + \sigma(x)}{2\sigma(4x) - \sigma(x)^2 - 1} .$$

In order to show that H is monotone decaying we compute its logarithmic derivative:

$$(5.9) \qquad \frac{H'}{H} = f_1 + f_2 + (f_{3a} - f_{3b}) - f_4 ,$$

where

$$f_1 = \frac{1}{2}\,[\ln(2\sigma' + \sigma\frac{\ln\sigma}{x})]'$$

$$f_2 = \frac{1}{4}\,\frac{\sigma'}{\sigma}\;(\tfrac{x}{4})$$

$$f_{3a} = \frac{1}{5} \cdot \frac{1+5\sigma'}{1+\sigma}$$

$$f_{3b} = \frac{1}{5}\;\frac{1}{1+\sigma}$$

$$f_4 = \left(\ln\left(\sum_{k=0}^{\infty} \frac{2k+3}{(2k+4)!}\,x^k\right)\right)' .$$

We will need to make use of the following elementary

<u>Lemma 5.7</u> Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $b(x) = \sum_{k=0}^{\infty} b_k x^k$ be the power series with positive coefficients a_k, b_k, converging on $D \subset \mathbb{R}$. If a_k/b_k is a non-increasing sequence then the function $A(x)/B(x)$ is non-increasing on $D \cap [0,\infty)$.

Combining this lemma with the previous formulas and the power series representations for σ, σ', and σ'', we obtain

<u>Lemma 5.8</u> The functions f_2, f_{3a}, f_{3b}, f_4, $\frac{\sigma'}{\sigma}$ and $\frac{\sigma''}{\sigma'}$ are positive and non-increasing.

The next step will be to write f_1 in terms of monotonic functions, too. For this purpose it will be useful to consider the auxiliary functions

$$\text{(5.10)} \qquad r(x) := \left(\frac{\sigma'}{\sigma}\right)^{-1} \cdot \frac{\ln\sigma}{x} - 1 = \frac{1}{30}x + O(x^2)$$

$$\varphi(x) := 1+x\left(\frac{\sigma''}{\sigma'} - \frac{\sigma'}{\sigma}\right) = 1+x\left(\ln\frac{\sigma'}{\sigma}\right)' = 1-\frac{1}{15}x + O(x^2)$$

<u>Lemma 5.9</u> $0 < \varphi(x) \leq \varphi(0) = 1$, and φ is monotonic.

<u>Proof</u>: Using i) of Lemma 5.6 and formula (5.7) we calculate

$$\varphi(w^2) = 1 + \frac{w}{2} \frac{d}{dw} \left(\ln\frac{\sigma'}{\sigma}(w^2)\right) = 1 - \frac{w}{2} \frac{d}{dw} \left(\ln\frac{1}{w}\left(\coth w - \frac{1}{w}\right)\right)$$

$$= \frac{1}{2} \frac{w\sinh(2w) - 2w^2}{w\sinh(2w) - \cosh(2w) + 1}$$

Using $x = w^2$, the numerator and denominator can be represented by power series as

$$x \sum_{k=0}^{\infty} \frac{1}{(2k+3)!} (4x)^k$$

and

$$2x \sum_{k=0}^{\infty} \frac{2k+2}{(2k+4)!} (4x)^k ,$$

respectively. Monotonicity now follows from Lemma 5.7.

□

<u>Lemma 5.10</u> i) $r \geq 0$

ii) $1 + r \leq \frac{1}{\varphi}$

iii) r and $\frac{r}{3+r}$ are non-decreasing.

<u>Proof</u>: i) The function $\psi(x) := x\frac{\sigma'}{\sigma} - \ln\sigma$ vanishes at $x = 0$ and has derivative $\psi' = x(\frac{\sigma'}{\sigma})' \leq 0$ (recall that $\frac{\sigma'}{\sigma}$ is non-increasing).

ii), iii) We compute:

$$\frac{r'}{1+r} = -\left(\frac{\sigma''}{\sigma'} - \frac{\sigma'}{\sigma}\right) + \frac{\sigma'}{\sigma\ln\sigma} - \frac{1}{x}$$

$$\Longrightarrow r' = \frac{1}{x}\,(1 - \varphi(x)\,(1+r)) \quad .$$

Observe that $r'(0) = 1/30$. Hence the right hand side will stay positive since φ is monotone decaying. This proves all the remaining claims.

□

This lemma enables us to write f_1 in terms of monotonic functions. Setting $g = 2\sigma' + \sigma\frac{\ln\sigma}{x}$, we have

$$g' = 2\sigma'' + \sigma'\frac{\ln\sigma}{x} + \frac{\sigma'}{x} - \frac{\sigma\ln\sigma}{x^2} = 2\sigma'' + \frac{\sigma'^2}{\sigma}(1+r) - r\frac{\sigma'}{x} \quad ,$$

hence

$$2f_1 = \frac{2}{3+r}\frac{\sigma''}{\sigma'} + \frac{1+r}{3+r}\frac{\sigma'}{\sigma} - \frac{r}{3+r}\frac{1}{x} \quad . \tag{5.11}$$

We consider intervals $[u,v]\subset[0,\infty)$ and put:

$$(5.12)\qquad \begin{aligned} F_+(u,v) &= \frac{1}{3+r}\left(\frac{\sigma''}{\sigma'}+\frac{1}{2}\,\frac{\sigma'}{\sigma}\right)(u) + (f_2+f_{3a})(v) + \frac{r}{3+r}(v)\cdot\frac{\sigma'}{2\sigma}(u) \\ F_-(u,v) &= (f_{3b}+f_4)(v) + \frac{r}{3+r}(u)\cdot\frac{1}{2v} \quad . \end{aligned}$$

Lemma 5.8, formula (5.11) and Lemma 5.10 iii) now yield

$$(5.13)\qquad \frac{H'}{H}(x) \le F(u,v) := F_+(u,v) - F_-(u,v) \quad , \forall x \in [u,v] \quad .$$

This estimate provides a numerical criterion for proving monotonicity of the positive function H. In fact, the following table establishes the proposition below:

w_i	$F_-(w_i^2, w_{i+1}^2)$	$F(w_i^2, w_{i+1}^2)\cdot 10^4$
0	.2883	-.349
1.6502	.2618	-.373
2.3296	.2397	-.439
2.82	.2222	-.328
3.2076	.2083	-.314
3.5278	.1972	-.309
3.7996	.1881	-.305
4.0352	.1806	-.364
4.2426	.1743	-.334
4.4268	.169	-.304
4.5924	.1644	-.358
4.7418	.1604	-.344
4.8778	.1569	-.348
5.0024	.1538	-.331
5.117	.1511	-.331
5.2226		

Proposition 5.11 The functions $h \circ \rho$ and $\Theta \circ \rho$ are decaying on $[0, 5.22]$.

In view of Proposition 5.5 we have monotonicity of the functions Θ and $\Theta \circ \rho$ on $[0, \infty)$, and the range of Θ can be read off from Proposition 5.2.

REFERENCES

[G1] Gage, M., An isoperimetric inequality with applications to curve shortening, Duke Math. J., Vol 50, No 10, (1983) 1225-1229.

[G2] ________, Curve shortening makes convex curves circular, Invent. Math., 76 (1984) 357-364.

[G-H] Gage, M. and Hamilton, R.S., The heat equation shrinking convex plane curves, preprint (1984).

[L-S] Langer, J. and Singer, D., On the total squared curvature of closed curves, J. Diff. Geo., 20 (1984).

[S] Smale, S., Regular curves on Riemannian manifolds, Trans. Am. Math. Soc., 87 (1958), 492-512.