Explicit construction of characteristic classes

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Abstract

Let E be a vector bundle over an algebraic manifold X. An explicit local construction of characteristic classes $c_n(E)$ with values in Bigrassmannian cohomology that are defined in § 1 is given. In the special case $n = \dim E$ it reduces to the construction of $c_n(E)$ with values in the Grassmannian cohomology given in [BMS].

Our construction implies immediately an explicit construction of Chern classes with values in $H^n\left(X, \underline{\underline{K}}_n^M\right)$, where $\underline{\underline{K}}_n^M$ is the sheaf of Milnors K-groups.

A construction of classes $c_n(E)$ with values in motivic cohomology is given for $n \leq 3$. For $n = 2$ it could be considered as a motivic analog of the local combinatorial formula of Gabrielov, Gelfand and Losik for the first Pontryagin class ([GGL]). The reason for the restriction $n \leq 3$ is the absence of a good theory of n-logarithms for $n \geq 4$ today. Explicit constructions of the universal Chern classes $c_n \in H^n\left(BGL_{m^\bullet}, \underline{\underline{K}}_n^M\right)$ and for $n \leq 3$ $c_n \in H^{2n}_{\mathcal{M}}(BGL_{m^\bullet}, \mathbf{Z}(n))$ ($H^\bullet_{\mathcal{M}}$: motivic cohomology) are given.

§ 1 Introduction

1. **Chern classes with values in** $H^n\left(X, \underline{\underline{K}}_n^M\right)$. Let L be a line bundle over X. There is the following classical construction of $c_1(L) \in H^1(X, \mathcal{O}^*)$. Choose a Zariski covering $\{U_i\}$ of X such that $L|_{U_i}$ is trivial. Choose non-zero sections $s_i \in \Gamma(U_i, L)$. Then $s_i/s_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ satisfies the cocycle condition and hence define a cohomology class $c_1(L) \in H^1(X, \mathcal{O}^*)$.

Let us define the presheaf of Milnor's K-groups on X as follows: its section over an open set U is the quotient group of $\underbrace{\mathcal{O}^*(U) \otimes \cdots \otimes \mathcal{O}^*(U)}_{n \text{ times}}$ by the subgroup generated by elements

$$g_1 \otimes \cdots \otimes g_k \otimes f \otimes (1-f) \otimes g_{k+3} \otimes \cdots \otimes g_n \, , \quad g_i, f, 1-f \in \mathcal{O}^*(U) \, .$$

Let us denote by $\underline{\underline{K}}_n^M$ the sheaf associated with this presheaf. We will denote by $\{f_1, \cdots, f_n\}$ the image of $f_1 \otimes \cdots \otimes f_n \in \mathcal{O}^*(U)^{\otimes n}$ in $\underline{\underline{K}}_n^M(U)$.

In § 3 for any vector bundle E over X an explicit construction of the Chern classes $c_n(E) \in H^n\left(X, \underline{\underline{K}}_n^M\right)$ will be given.

The construction of $c_n(E^n)$ for an n-dimensional vector bundle E^n follows from [S1] and [BMS], ch. 1. More precisely, let U_i be a Zariski covering such that $E^n|_{U_i}$ is trivial. Choose a section $s_i \in \Gamma(U_i, E^n)$ such that $s_{i_1}(x), \cdots, s_{i_{n+1}}(x)$ are in generic position on $U_{i_1 \cdots i_{n+1}} := U_{i_1} \cap \cdots \cap U_{i_{n+1}}$. Then $s_{i_{n+1}}(x) = \sum_{k=1}^{n} a_{i_k}(x) \cdot s_{i_k}(x)$ and

$$\{a_{i_1}(x), \cdots, a_{i_n}(x)\} \in K_n^M(U_{i_1 \cdots i_{n+1}})$$

is a cocycle in the Čech complex.

I will generalize this construction to vector bundles of arbitrary dimension and show that for $c_1(E)$ it gives exactly the described above cocycle for $c_1(\det E)$.

2. **Applications.** There is a canonical map of sheaves

$$\begin{gathered} \underline{\underline{K}}_n^M \to \Omega^n_{\log} \hookrightarrow \Omega^n_{cl} \hookrightarrow \Omega^n \\ \{f_1, \cdots, f_n\} \mapsto d\log f_1 \wedge \cdots \wedge d\log f_n \end{gathered}$$

Here $\Omega^n_{\log}$ (respectively Ω^n_{cl}) is the sheaf of n-forms with logarithmic singularities at infinity (respectively closed n-forms). Therefore we get a construction of characteristic classes with values in $H^n\left(X, \Omega^n_{\log}\right)$ and $H^n(X, \Omega^n_{cl})$. Note that the Atiyah's construction provides us characteristic classes in $H^n(X, \Omega^n)$ ([A], see also [Har]).

3. **The Grassmannian bicomplex and Bigrassmannian cohomology** (see [G1], [G2], compare with [GGL] and [BMS]). Let Y be a set and $\tilde{C}_n(Y)$ be a free abelian group generated by elements $(y_0, \cdots, y_n)$ of $Y^{n+1} := \underbrace{Y \times \cdots \times Y}_{n+1}$. There is a complex $\left(\tilde{C}_*(Y), d\right)$ where

$$d(y_0, \cdots, y_n) := \sum_{i=0}^{n} (-1)^i (y_0, \cdots, \hat{y}_i, \cdots, y_n) \tag{1.1}$$

This is just the simplicial complex of the simplex whose vertices are labeled by elements of Y. Suppose that a group G acts on Y. Let us call elements of the quotient set $G \setminus Y^{n+1}$ by **configurations** of elements of Y. Denote by $C_n(Y)$ a free abelian group generated by configurations of $(n+1)$ elements of Y. There is a complex $(C_*(Y), d)$, where d is defined by the same formula (1.1) and $C_*(Y) = \tilde{C}_*(Y)_G$. We will also apply this construction to subsets of $G \setminus Y^{n+1}$ of "configurations in generic position".

Now let us denote by $C_n(m)$ a free abelian group generated by configurations of $n+1$ vectors in generic position in an m-dimensional vector space V^m over F (i.e. any m vectors of the configuration are linearly independent). In this case there is another map:

$$d' : C_n(m) \to C_{n-1}(m-1)$$

$$d' : (v_0, \cdots, v_n) \mapsto \sum_{i=0}^{n} (-1)^i (v_0 | v_0, \cdots, \hat{v}_i, \cdots, v_n)$$

Here $(v_i | v_0, \cdots, \hat{v}_i, \cdots, v_n)$ is a configuration of vectors in $V^m / \langle v_i \rangle$ obtained by projection of vectors $v_j \in V^m$, $j \neq i$. Then there is the following bicomplex

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & C_{n+4}(n+2) & \xrightarrow{d} & C_{n+3}(n+2) & \xrightarrow{d} & C_{n+2}(n+2) \\
 & \downarrow d' & & \downarrow d' & & \downarrow d' \\
\cdots \to & C_{n+3}(n+1) & \xrightarrow{d} & C_{n+2}(n+1) & \xrightarrow{d} & C_{n+1}(n+1) \\
 & \downarrow d' & & \downarrow d' & & \downarrow d' \\
\cdots \to & C_{n+2}(n) & \xrightarrow{d} & C_{n+1}(n) & \xrightarrow{d} & C_n(n)
\end{array} \qquad (1.2)$$

We will call it the **Grassmannian bicomplex** (over $X = \mathrm{Spec}F$).

There is a subcomplex $(C_*(n), d)$

$$\to C_{n+2}(n) \xrightarrow{d} C_{n+1}(n) \xrightarrow{d} C_n(n) \qquad (1.3)$$

of the bicomplex (1.2). This is the **Grassmannian complex** introduced in [S2], [BMS], see also [Q2].

Let us denote by $(BC_*(n), \partial)$ the total complex associated with the bicomplex (1.2): $BC_n(n) := C_n(n)$. We will suppose that $BC_n(n)$ placed in degree n and ∂ has degree $+1$.

Now let us give a more geometrical interpretation of the Grassmannian bicomplex that also explains the name.

Let $(e_1, \cdots, e_{p+q+1})$ be a coordinate frame in a vector space V. Let us denote by $\hat{G}^p_q$ the open subset of the Grassmannian of q-dimensional subspaces of P^{p+q} which are in transverse to the coordinate hyperplanes. R. MacPherson constructed in [M] an isomorphism

$$m : \hat{G}^p_q \xrightarrow{\sim} \left\{ \begin{array}{c} \text{configurations of } p+q+1 \text{ vectors in generic} \\ \text{position in a } p\text{-dimensional vector space} \end{array} \right\} \qquad (1.4)$$

Namely, $m(\xi)$ is a configuration formed by images of e_i in V/ξ .

Let

$$\mathbb{Z} : Var \to Ab \tag{1.5}$$

be a functor from the category of algebraic varieties over F to the one of abelian groups that sends a variety X to the free abelian group generated by F-points of X. Applying it to (1.4) we get an isomorphism

$$\mathbb{Z}\left[\hat{G}^p_q\right] \xrightarrow{\sim} C_{p+q}(p) \tag{1.6}$$

For each integer i such that $0 \le i \le p+q$, there are intersection maps a_i and projection maps b_i :

$$\begin{array}{ccc} \hat{G}^p_q & \xrightarrow{a_i} & \hat{G}^p_{q-1} \\ \downarrow b_i & & \\ \hat{G}^{p-1}_q & & \end{array} \tag{1.4}$$

Here the subspace $a_i(\xi)$ is the intersection of ξ with the $i-th$ coordinate hyperplane and the subspace $b_i(\xi)$ is the projection of ξ on the $i-th$ hyperplane by the projection with the center at $i-th$ vertex of the simplex. We get a Bigrassmannian $\hat{G}(n)$:

$$\hat{G}(n): \quad \begin{array}{ccccccc} & & & & & & \downdownarrows \\ & & & & & \rightrightarrows & \hat{G}^{n+2}_0 \\ & & & & \downdownarrows & & b_0 \downdownarrows b_{n+1} \\ & & & \rightrightarrows & \hat{G}^{n+1}_1 & \overset{a_0}{\underset{a_{n+1}}{\rightrightarrows}} & \hat{G}^{n+1}_0 \\ & & \downdownarrows & & b_0 \downdownarrows b_{n+1} & & b_0 \downdownarrows b_n \\ & \rightrightarrows & \hat{G}^n_2 & \overset{a_0}{\underset{a_{n+1}}{\rightrightarrows}} & \hat{G}^n_1 & \overset{a_0.}{\underset{a_n}{\rightrightarrows}} & \hat{G}^n_0 \end{array} \tag{1.7}$$

Applying functor (1.5) to it, considering differentials $d = \Sigma(-1)^i a_i$ and $d' = \Sigma(-1)^i b_i$ and using isomorphism 1.6 we get the Grassmannian bicomplex.

Now let us sheafefy these constructions.

A bicomplex of sheaves on X called the Grassmannian bicomplex $\underline{\underline{\mathbb{Z}}}\left[\hat{G}(n)\right]$ is constructed as follows: For a point $x \in X$, the stalk of $\underline{\underline{\mathbb{Z}}}\left[\hat{G}(n)\right]$ at x is the formal linear combinations of germs at x of maps from X to $\hat{G}^p_q$. The corresponding bicomplex looks as follows

$$\underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}(n)\right]: \quad \begin{array}{ccccc} & & \cdots & & \cdots \\ & & \downarrow & & \downarrow \\ \cdots & \rightarrow & \underline{\underline{\mathbf{Z}}}_1\left[\hat{\mathrm{G}}_1^{n+1}\right] & \xrightarrow{d} & \underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}_0^{n+1}\right] \\ & & \downarrow d' & & \downarrow d' \\ \cdots & \rightarrow & \underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}_1^{n}\right] & \xrightarrow{d} & \underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}_0^{n}\right] \end{array} \tag{1.8}$$

Here $\underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}_0^n\right]$ placed in degree $(n,0)$ and d (respectively d') has degree $(1,0)$ (respectively $(0,1)$). The hypercohomology of the total complex associated with this bicomplex of sheaves is the **Bigrassmannian cohomology** of X. We will denote it as $H^*\left(X, \underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}(n)\right]\right)$. Note that the Grassmannian cohomology of [BMS] maps canonically to the Bigrassmannian one, but there is no inverse map.

In § 2 we will construct explicitely characteristic classes $c_n(E) \in H^{2n}\left(X, \underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}(n)\right]\right)$. There is a homomorphism of complexes of sheaves

$$\underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}(n)\right] \rightarrow \underline{\underline{K}}_n^M[-n] \tag{1.9}$$

(see § 3), that provides a construction of characteristic classes

$$c_n(E) \in H^n\left(X, \underline{\underline{K}}_n^M\right)$$

4. **Polylogarithms** (compare with [GGL], [BMS], [HM]). Now let $F = \mathbb{C}$. Note that $\hat{\mathrm{G}}_0^n$ is almost canonically isomorphic to $(\mathbb{C}^*)^n$. Indeed, according to (1.4) a point $\xi \in \hat{\mathrm{G}}_0^n$ defines an (ordered) configuration of $n+1$ vectors in generic position in $\mathbb{C}^n$: $m(\xi) = (v_0, \cdots, v_n)$. So $v_0 = \sum_{i=1}^{n} z_i v_i$ and the map $\xi \mapsto (z_1, \cdots, z_n)$ provides an isomorphism $\hat{\mathrm{G}}_0^n \xrightarrow{\sim} (\mathbb{C}^*)^n$. Therefore there is a canonical multivalued holomorphic $n-1$ form

$$w_0^n := \frac{1}{n}\sum_{i=1}^{n}(-1)^i \log z_i d\log z_1 \wedge \cdots \wedge d\widehat{\log} z_i \wedge \cdots \wedge z_n \tag{1.10}$$

on $\hat{G}_0^n$.

Consider the mulivalued Deligne complex $\tilde{\mathbf{Q}}(n)$ on a variety Y ($\mathbf{Q}$ placed in degree 0 , d has degree $+1$):

$$\mathbf{Q} \xrightarrow{(2\pi i)^n} \tilde{\Omega}^0(Y) \xrightarrow{d} \tilde{\Omega}^1(Y) \xrightarrow{d} \cdots \xrightarrow{d} \tilde{\Omega}^{p-1}(Y) \rightarrow 0$$

Here $\tilde{\Omega}^i$ represents multivalued holomorphic differential forms, i.e. holomorphic differential forms defined on the universal covering space $\tilde{Y}$ of Y. We wish to consider a triple complex $\mathbf{D}$ which is the mulivalued complex $\tilde{\mathbf{Q}}(n)$ in the vertical direction and is a double complex constructed from the Bigrassmannian $\hat{G}(n)$ in the horizontal directions. All differentials have degree $+1$.

A $2n$-cocycle in the complex $\mathbf{D}$ is just a collection of $(n-1-p-q)$-forms $\{\omega_q^p\}$ such that

$$d\omega_q^p = \Sigma(-1)^i a_i^* \omega_{q-1}^p + \Sigma(-1)^i b_i^* \omega_q^{p-1} \tag{1.11}$$

<u>**Conjecture 1.1**</u> *There exists a $2n$-cocycle $\mathbf{L}_n$ in the triple complex $\mathbf{D}$ such that its ω_0^n-component is given by formula (1.10).*

The collection of forms $\{\omega_q^n\}$ is, of course, the Grassmannian n-logarithm conjectured in [BMS], [HM]. However for an explicit construction of the Chern classes in Deligne cohomology we have to construct the whole Bigrassmannian n-logarithm and it is **not** sufficient to construct only its Grassmannian part. The main construction of this paper (see § 2) gives a construction of

$$c_n(E) \in H_{\mathcal{D}}^{2n}(X, \mathbb{Q}(n))$$

using the Bigrassmannian polylogarithm $\mathbf{L}_n$. The coincidence of this class with the one constructed by A.A. Beilinson [B2] is provided by formula (1.10) (see theorem 5.11). The problem of construction of a collection of forms $\{\omega_q^p\}$ satisfying the condition (1.11) goes back to [GGL], see also [You], where the real-valued forms on the corresponding manifolds over R were considered, (forms $S^{p,q}$).

The most interesting component of $\mathbf{L}_n$ is a multivalued function $P_n := \omega_{n-1}^n$ on $\hat{\mathrm{G}}_{n-1}^n$. The cocycle condition means that it should satisfy two "$2n+1$-term" functional equations

$$\sum_{i=0}^{2n} (-1)^i a_i^* P_n = (2\pi i)^n q_1 \tag{1.12a}$$

$$\sum_{i=0}^{2n} (-1)^i b_i^* P_n = (2\pi i)^n q_2 \tag{1.12b}$$

where $q_1, q_2 \in \mathbb{Q}$. Note that a_i^*, b_i^* have sense after lifting of maps a_i, b_i to the simply connected covering spaces.

Instead of the Deligne complex $\underline{\underline{\mathbb{Q}}}(n)_{\mathcal{D}}$ one could consider the real Deligne complex $\underline{\underline{R}}(n)_{\mathcal{D}}$ that is the total complex of the following bicomplex

$$\underline{\underline{R}}(n)_{\mathcal{D}}: \quad \begin{array}{ccccccccccc} S_X^0 & \xrightarrow{d} & S_X^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & S_X^n & \xrightarrow{d} & S^{n+1} & \xrightarrow{d} & \cdots \\ & & & & & & \uparrow \alpha_n & & \uparrow \alpha_n & & \\ & & & & & & \Omega_X^n & \xrightarrow{\partial} & \Omega_X^{n+1} & \xrightarrow{\partial} & \cdots \end{array} \tag{1.13}$$

where $(S_X^\bullet, d)$ is the de Rham complex of the real-valued forms, $(\Omega^\bullet, \partial)$ is the de Rham complex of holomorphic forms with logarithmic singularities at infinity, $\alpha_n = (-1)^{n-1} \cdot Re$ for odd n and $(-1)^n \mathrm{Im}$ for even and S_X^0 placed in degree 1.

One can consider the triple complex $\mathbf{D}$ which is the complex $\underline{\underline{R}}(n)_{\mathcal{D}}$ in the vertical direction and is a double complex constructed from the Bigrassmannian $\hat{\underline{\underline{\mathrm{G}}}}(n)$ in the horizontal directions. In fact it is more naturally to consider complex for computation of the hypercohomology of the Bigrassmannian $\hat{\mathrm{G}}(n)$ with coefficients in $\underline{\underline{R}}(n)_{\mathcal{D}}$ (for this we should replace the complex $\left(\Omega_X^{\geq n}, \partial\right)$ in (1.13) by its Dolbeaux resolution $\left(\mathcal{D}_X^{\geq n,q}\right)$ for example), but it is not important for our purposes.

<u>**Conjecture 1.1**</u> *′ There exists a $2n$-cocycle $\mathbf{L}_n^{'}$ in the triple complex $\mathbf{D}'$ such that its component over $\hat{\mathrm{G}}_0^n$ is given by the following formulas:*

$$\begin{aligned}
&\omega_0^{n'} = \alpha_n \left(\frac{1}{n} \sum_{i=1}^{n} (-1)^i \log z_i d\log z_1 \wedge \cdots \wedge d\log z_i \wedge \cdots \wedge d\log z_n \right) \in S_{\hat{\mathrm{G}}_0^n}^{n-1} \\
&\omega_0^{n''} = d\log z_1 \wedge \cdots \wedge d\log z_n \in \Omega_X^n \\
&\left(d\omega_0^{n'} + \alpha_n \left(\omega_0^{n''} \right) = 0 \right)
\end{aligned} \qquad (1.14)$$

The corresponding component $P_n^{'}$ of $\mathbf{L}_n^{'}$ on $\hat{\mathrm{G}}_{n-1}^n$ should satisfy the "clean" $(2n+1)$-term equations

$$\sum_{i=0}^{2n} (-1)^i a_i^* P_n^{'} = 0 \qquad (1.14a)$$

$$\sum_{i=0}^{2n} (-1)^i b_i^* P_n^{'} = 0 \qquad (1.14b)$$

From the other hand there are the **classical** polylogarithms $\mathrm{Li}_n(z)$ that are functions of **one** complex variable z. They were defined by Joh. Bernoulli and L. Euler on the unit disc $|z| \leq 1$ by absolutely convergent series

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n},$$

and can be continued analytically to a multivalued function on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ using the inductive formulas

$$\mathrm{Li}_1(z) = -\log(1-z)$$

$$\mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(t) \frac{dt}{t}$$

It turns out that $\mathrm{Li}_n(z)$ has a remarkable single-valued version ($B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6, \cdots$ are Bernoulli numbers) ([Z])

$$\mathcal{L}_n(z) = \begin{matrix} \mathrm{Re}(n : odd) \\ \mathrm{Im}(n : even) \end{matrix} \left(\sum_{k=0}^{n} \frac{B_k \cdot 2^k}{k!} \log^k |z| \cdot \mathrm{Li}_{n-k}(z) \right), \quad n \geq 2$$

$$\mathcal{L}_1(z) = \log|z|$$

For example

$$\mathcal{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z) \cdot \log|z|$$

is the Bloch-Wigner function, and

$$\mathcal{L}_3(z) = \mathrm{Re}\left(\mathrm{Li}_3(z) - \log|z| \cdot \mathrm{Li}_2(z) + \frac{1}{3} \log^2|z| \cdot \mathrm{Li}_1|z| \right)$$

was used in [G1]. The functions $\mathcal{L}_n(z)$ for arbitrary n were written by D. Zagier, [Z1].

Explicit formulas expressing the Bigrassmannian polylogarithms $\mathbf{L}_n$, $\mathbf{L}_n^{'}$ by the classical polylogarithms for $n \leq 3$ were given in [G1] (see also [G2] and § 5 of this paper). For example $\mathbf{L}_3^{'}$, that is a function on the 9-dimensional manifolds $\hat{G}_2^3$, is expressed by $\mathcal{L}_3(z)$. However for $n \geq 4$ the "natural" cocycle $\mathbf{L}_n$ can **not** be expressed by the classical polylogarithms (the reason was explained in S. of § 1 in [G1]).

An interesting geometrical construction of the Grassmannian 2 and 3-logarithms was suggested by M. Hanamura and R. MacPherson [Han-M]. The existence of the Grassmannian n-logarithms for $n \leq 3$ was proved in [HM].

In formulas for $\mathbf{L}_n^{'}$, $(n \leq 3)$, given in s. 9 of [G1]. It is interesting that all forms ω_q^{n+i} are equal to zero for $i > 0$. This means that the Bigrassmannian n-logarithms for $n \leq 3$ reduces essentially to its Grassmannian part $\{\omega_q^n\}$. Thus is a nontrivial fact about the Grassmannian n-logarithms, $n \leq 3$. But this is **not true** for $n \geq 4$. For example, already forms ω_1^{n+1} can not be choosen equal to zero for $n \geq 4$. This is another important difference between cases $n \leq 3$ and $n \geq 4$. It shows why we have to enlarge the Grassmannian polylogarithms to the Bigrassmannian one.

5. **The universal Chern classes** $c_n \in H^n\left(BGL_m, \underline{\underline{K}}_n^M\right)$. Recall that the classifying space for a group G can be represented by the simplicial scheme

$$BG_\bullet : * \leftleftarrows G \leftleftarrows G^2 \cdots$$

In § 4 I will construct explicitely the universal Chern classes $c_n \in H^n\left(BGL_m(F)_\bullet, \underline{\underline{K}}_n^M\right)$, $m \geq n$. This is a refinement of the construction from § 2 and, of course, implies it immediately.

More precisely, a Zariski covering $\{U_i\}_{i \in I}$ defines a simplicial scheme $U_\bullet$:

$$\coprod_{i \in I} U_i \leftleftarrows \coprod_{i_0 < i_1 \in I} U_{i_0 i_1} \leftleftarrows \coprod_{i_0 < i_1 < i_2 \in I} U_{i_0 i_1 i_2} \cdots$$

A G-bundle E over X given by its transition functions $g_{ij} \in \Gamma(U_{ij}, G)$ defines a canonical map of simplicial schemes $u : U_\bullet \to BG_\bullet$. Our G-bundle is the inverse image of the canonical G-bundle $EG_\bullet \xrightarrow{G} BG_\bullet$ over $BG_\bullet$ and $c_n(E) = U^* c_n$.

As a byproduct I get an explicit algebraic construction of cohomology classes generating the ring $H^*(GL_m)$. The existence of such a style description of the usual topological cohomology of GL_m was conjectured by A.A. Beilinson [B3].

6. **The universal motivic Chern classes.** In § 4 an explicit construction of such Chern classes

$$c_n \in H^{2n}_{\mathcal{M}}(BGL_{m^\bullet}, \mathbf{Z}(n)) , \quad n \leq 3$$

will be given. It implies, in particularly, an explicit construction of the Chern classes $c_n(E)$ with values in Deligne cohomology $H^{2n}_{\mathcal{D}}(X, \mathbf{Z}(n))$ by means of the classical n-logarithms $(n \leq 3)$. A cocycle representing the ususal topological characteristic class $c_n(E) \in H^{2n}(X, \mathbf{Z})$ in the Čech comples was constructed by J.L. Brylinsky and D. MacLaughlin [B-M].

A local combinatorial formula for all Pontryagin classes was suggested by I.M. Gelfand and R. MacPherson [GM2].

Let $H^*_{cts}(G,R)$ be the ring of continuous cohomology of a Lie group G. It is known that

$$H^*_{cts}(GL_m(\mathbb{C}),R) = \Lambda^*_R\left(b_1^{(m)}, b_{3,\cdots,}^{(m)} b_{2m-1}^{(m)}\right)$$
$$b_{2k-1}^{(m)} \in H^{2k-1}_{cts}(GL_m(\mathbb{C}),R)$$

As a byproduct of the construction of the universal Chern classes $c_n \in H^{2n}_{\mathcal{D}}(BGL_m(\mathbb{C}),R(n))$, we get an explicite formula for (measurable) cocycles representing classes $b_{2n-1}^{(m)}$ for $n \leq 3$ and arbitrary $m \geq 2n-1$ by means of the classical n-logarithm. The formula for b_1 is well-known: $b_1(g) := \log|\det g|$, $g \in GL_m(\mathbb{C})$, is a 1-cocycle. The formula for $b_3^{(2)}$ was found by D. Wigner in the middle of 70-s, and for $b_5^{(3)}$ by the author ([G1], see also [G2]). A formula for $b_3^{(m)}$ was written also by Kioshi Igusa (unpublished ?).

Note that there is a canonical map

$$H_n\left(X, \underline{\underline{K}}^M_n\right) \to H^n\left(X, \underline{\underline{K}}_n\right)$$

and it was shown by Soule ([Sou]) and by Nesterenko and Suslin [NS] that this map is an isomorphism modulo torsion. This together with characetristic classes $c_n(E) \in H^n\left(X, \underline{\underline{K}}_n\right)$ of Gillet ([Gil]) proves the existence of $c_n(E) \in H^n\left(X, \underline{\underline{K}}^M_n\right)$ but does not give any precise construction.

This work was initiated by A.A. Beilinson who explained to me that there are no explicit construction of the Chern classes with values in $H^n\left(X, \underline{\underline{K}}^M_n\right)$ as well as in $H^n\left(X, \Omega^n_{\log}\right)$ or $H^n\left(X, \Omega^n_{el}\right)$ and emphasized importance of such a construction.

I hope it is clear from the introduction how much I benefited from paper of A.M. Gabrielov, I.M. Gelfand and M.V. Losik [GGL].

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§ 2 Affine flags and Chern classes in Bigrassmannian cohomology

1. **Affine flags.** Let V be a vector space over a field F. By definition a p**-flag** in V is a sequence of subspaces

$$0 \subset L^1 \subset L^2 \subset \cdots \subset L^p , \quad \dim L^i = i ,$$

An **affine** p**-flag** $L^\bullet$ is a p-flag together with choice of vectors $l^i \in L^i/L^{i-1}$, $i = 1, \cdots, p$ $\left(L^0 = 0\right)$. We will denote affine p-flags as $\left(l^1, \cdots, l^p\right)$. Subspaces L^i can be recovered as the ones generated by $l^1, \cdots, l^i$: $L^i = \left\langle l^1, \cdots, l^i \right\rangle$. We will say that an $(n+1)$-tuple of affine flags

$$L_0^\bullet = \left(l_0^1, \cdots, l_0^p\right), \cdots, L_n^\bullet = \left(l_n^1, \cdots, l_n^p\right) \tag{2.1}$$

are in generic position if

$$\dim\left(L_0^{i_0} + \cdots + L_n^{i_n}\right) = i_0 + \cdots + i_n \quad \text{whenever} \quad i_0 + \cdots + i_n \leq \dim V . \tag{2.2}$$

Let $A^p(m)$ be the manifold of all affine p-flags in an m-dimensional vector space V_m . It is a $GL(V_m)$-set, so as usual (see 5.3 of the Introduction) one can consider free abelian groups $C_n(A^p(m))$ of configurations of $(n+1)$-tuples of affine p-flags in generic position in V_m . Further, there is a complex of affine p-flags $C_*(A^p(m))$:

$$\begin{gathered} \cdots \xrightarrow{d} C_{n+1}(A^p(m)) \xrightarrow{d} C_n(A^p(m)) \xrightarrow{d} C_{n-1}(A^p(m)) \xrightarrow{d} \cdots \\ d : (L_0^\bullet, \cdots, L_n^\bullet) \mapsto \sum_{i=0}^{n} (-1)^i \left(L_0^\bullet, \cdots, \widehat{L_i^\bullet}, \cdots, L_n^\bullet\right) \end{gathered} \tag{2.3}$$

In particularly $C_*\left(A^1(m)\right) \equiv C_*(m)$. Let us define a map of complexes

$$T : C_*\left(A^{p+1}(n+p)\right) \to BC_*(n) \tag{2.4}$$

as follows: for

$$a_k^{p+1} = \left(v_0^1, \cdots, v_0^{p+1}; \cdots; v_k^1, \cdots, v_k^{p+1}\right) \in C_k\left(A^{p+1}(n+p)\right) \quad (k \geq n)$$

set

$$\begin{gathered} T\left(a_k^{p+1}\right) := \bigoplus_{q=0}^{k-n} \sum_{\substack{i_0+\cdots+i_k=p-q \\ i_k \geq 0}} \left(L_0^{i_0} \oplus \cdots \oplus L_m^{i_m} | v_0^{i_0+1}, \cdots, v_k^{i_k+1}\right) \in \\ \in \bigoplus_{q=0}^{k-n} C_k(n+q) =: BC_k(n) \end{gathered} \tag{2.5}$$

<u>**Key lemma 2.1**</u> *T is a homomorphism of complexes.*

Proof: Let $T_k(n+q) : C_k\left(A^{p+1}(n+p)\right) \to C_k(n+q)$ be the $C_k(n+q)$-component of the map P. We have to prove that (see 2.6)

$$d \circ T_k(n+q) = T_{k-1}(n+q) - d' \circ T_k(n+q+1)$$

$$
\begin{array}{ccc}
a_k^{p+1} \in C_k\big(A^{p+1}(n+q)\big) & \to & C_k(n+q+1) \\
\downarrow & \searrow & \downarrow \\
C_k(n+q) & \to & C_{k-1}(n+q)
\end{array}
\tag{2.6}
$$

For a given partition $i_0 + \cdots + i_k = p - q$ let us consider the expression

$$
\begin{aligned}
&d\Big(L_0^{i_0} \oplus \cdots \oplus L_k^{i_k} | v_0^{i_0+1}, \cdots, v_k^{i_k+1}\Big) = \\
&= \sum_{j=0}^{k} (-1)^j \Big(L_0^{i_0} \oplus \cdots \oplus L_k^{i_k} | v_0^{i_0+1}, \cdots, \widehat{v_j^{i_j+1}}, \cdots, v_k^{i_k+1}\Big)
\end{aligned}
\tag{2.7}
$$

If $i_j = 1$ then the corresponding term in 2.6 will appear in formula for $T_{k-1}(n+q)\Big(a_k^{p+1}\Big)$. In the case $i_j > 1$ such term will be in formula for

$$
d'\Big(L_0^{i_0} \oplus \cdots \oplus L_j^{i_j - 1} \oplus \cdots \oplus L_k^{i_k} | v_0^{i_0+1}, \cdots, v_j^{i_j}, \cdots, v_k^{i_k+1}\Big) .
$$

□

2. **A construction of Chern classes in Bigrassmannian cohomology.** Let us denote by $\mathcal{A}_E^p(X)$ the bundle of affine p-flags in fibers of a vector bundle E over X. Choose a Zariski covering $\{U_i\}$ of X such that E/U_i is trivial. Choose sections

$$
L_i^\bullet(x) \in \Gamma\big(U_i, \mathcal{A}_E^p(x)\big)
$$

such that for any $i_0 < \cdots < i_n$ affine p-flags $L_{i_0}^\bullet(x), \cdots, L_{i_n}^\bullet(x)$ are in generic position for every $x \in U_{i_0, \cdots, i_n}$.

Theorem 2.2 $T\big(L_{i_0}^\bullet(x), \cdots, L_{i_n}^\bullet(x)\big) \in \underline{\underline{\mathbb{Z}}}\Big[\hat{G}(n)\Big](U_{i_0 \cdots i_n})$ *is a cocycle in the Čech complex for the covering* $\{U_i\}$ *with values in the Bigrassmannian complex.*

Proof: Follows immediately from the Key lemma 2.1. □

A different choice of sections $L_i^\bullet(x)$ gives a cocycle that is canonically cohomologous to the previous one. So the cohomology class $c_n(E)$ of this cocycle is well-defined.

§ 3 Chern classes with values in $H^n\left(X, \underline{\underline{K}}^M_n\right)$

1. In § 2 we have constructed Chern classes with values in $H^{2n}\left(\underline{\underline{BC}}_*(n)\right)$. To obtain Chern classes with values in $H^n\left(X, \underline{\underline{K}}^M_n\right)$ it is sufficient to define a homomorphism

$$BC_*(n) \to K_n^M(F)[-n] \tag{3.1}$$

i.e. a homomorphism $\overline{f}_n(n) : C_n(n) \to K_n^M(F)$ such that $f_n(n) \circ d = f_n(n) \circ d' = 0$, :

$$\begin{array}{ccccc} & & & & \downarrow d' \\ & & C_{n+2}(n+1) & \xrightarrow{d} & C_{n+1}(n+1) \\ & & \downarrow d' & & \downarrow d' \\ \xrightarrow{d} & & C_{n+1}(n) & \xrightarrow{d} & C_n(n) \\ & & & & \downarrow \\ & & & & K_n^M(F) \end{array}$$

Now let us define a homomorphism

$$f_n(n) : C_n(n) \to \Lambda^n F^*$$

as follows (compare with s. 2 of § 3 in [G2]). Choose a volume form $\omega \in \det(V^n)^* \equiv \Lambda^n(V^n)^*$ (where $\dim V^n = n$). Set

$$\begin{aligned} \Delta(v_1, \cdots, v_n) &:= \langle w, v_1 \wedge \cdots \wedge v_n \rangle \in F^* , \quad v_i \in V^n \\ f_n(n)(v_0, \cdots, v_n) &:= \text{Alt} \bigwedge_{1 \le i \le n} \Delta(v_0, \cdots, \widehat{v_i}, \cdots, v_n) \in \Lambda^n F^* \end{aligned} \tag{3.2}$$

Here Alt $g(v_0, \cdots, v_n) := \sum_{\sigma \in S_{n+1}} (-1)^{|\sigma|} g\left(v_{\sigma(0)}, \cdots, v_{\sigma(n)}\right)$. For example, up to a 2-torsion

$$\begin{aligned} &f_2(2)(v_0, v_1, v_2) := \\ &2(\Delta(v_0, v_2) \wedge \Delta(v_0, v_1) - \Delta(v_1, v_2) \wedge \Delta(v_0, v_1) + \Delta(v_0, v_2) \wedge \Delta(v_1, v_2)) \end{aligned}$$

Lemma 3.1 *$f_n(n)(v_0, \cdots, v_n)$ does not depend on w.*

Proof: Let $f_n'(n)$ be a homomorphism defined using another volume form $w' = \lambda w$. Then

$$\left(f_n(n) - f_n'(n)\right)(v_0, \cdots, v_n) = \lambda \wedge \Sigma \Lambda_{i,j}$$

where $\Lambda_{i,j} \in \Lambda^{n-1} F^*$ and depends on $v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_n$. So $\Lambda_{i,j}$ is symmetric on v_i, v_j . But the left-hand side is skew-symmetric by definition. So $\Lambda_{i,j} = 0$.

□

Lemma 3.2 *The composition*

$$C_{n+1}(n+1) \xrightarrow{d'} C_n(n) \xrightarrow{f_n(n)} \Lambda^n F^* \tag{3.2}$$

is equal to zero modulo 2-torsion.

Proof: (Compare with proof of lemma 3.4 in [G1])

$$f_n(n)\circ d'(v_0,\cdots,v_{n+1}) = \text{Alt}\bigwedge_{j=2}^{n+1}\Delta(v_0,v_1,\cdots,\widehat{v_j},\cdots,v_{n+1}) = 0$$

because $\Delta(v_0,v_1,\cdots,\widehat{v_j},\cdots,v_{n+1})$ is invariant under the switch of v_0 and v_1 modulo 2-torsion.

□

Proposition 3.3 *The composition*

$$C_{n+1}(n)\xrightarrow{d} C_n(n)\xrightarrow{\overline{f}_n(n)} K_n^M(F)$$

is equal to zero.

Proof: (Compare with proof of proposition 2.4 in [S1]). There is a duality $*: C_{m+n-1}(m)\to C_{m+n-1}(n)$, $*^2 = id$ that satisfies the following properties (see s.8 of § 3 in [G2]).

1. $*$ commutes with the action of the permutation group S_{m+n}.
2. If $*(l_1,\cdots,l_{m+n}) = \left(l'_1,\cdots,l'_{m+n}\right)$ then

$$*\left(l_1,\cdots,\widehat{l_i},\cdots,l_{m+n}\right) = \left(l'_i|l'_1,\cdots,\widehat{l_i},\cdots,\widehat{l}_{m+n}\right)$$

3. Choose volume forms in V_m and V_n ; consider partition

$$\{1,\cdots,m+n\} = \{i_1<\cdots<i_m\}\cup\{j_1<\cdots<j_n\}$$

Then $\frac{\Delta(l_{i_1},\cdots,l_{i_m})}{\Delta(l_{j_1},\cdots,l_{j_n})}$ does not depend on a partition.

This duality can be defined as follows. A configuration of $(m+n)$ vectors in an m-dimensional coordinate vector space can be represented as columns of the $m\times(m+n)$ matrix (I_m, A). The dual configuration is represented by $n\times(m+n)$ matrix $\left(-A^t, I_n\right)$. Using the duality we can reformulate proposition 3.3 as follows: the composition

$$C_{n+1}(2)\xrightarrow{d'} C_n(1)\xrightarrow{\tilde{f}_n(n)} K_n^M(F)$$

is equal to 0. Here

$$\tilde{f}_n(n)(v_0,\cdots,v_n) := \text{Alt}\Delta(v_0)\wedge\Delta(v_1)\wedge\cdots\wedge\Delta(v_{n-1})\in\Lambda^n F^*$$

Consider the following diagram

$$\begin{array}{ccc} C_{n+1}(2) & \xrightarrow{d'} & C_n(1) \\ \downarrow \tilde{f}_{n+1}(n) & & \downarrow \tilde{f}_n(n) \\ \mathbf{Z}[P_F^1\setminus\{0,1,\infty\}]\otimes\Lambda^{n-2}F^* & \xrightarrow{\delta} & \Lambda^n F^* \end{array}$$

Here $\mathbf{Z}[P_F^1\setminus\{0,1,\infty\}]$ is a free abelian group generated by symbols $\{x\}$ where $x\in P_F^1\setminus\{0,1,\infty\}$, $\delta:\{x\}\otimes y_1\wedge\cdots\wedge y_{n-2}\mapsto(1-x)\wedge x\wedge y_1\wedge\cdots\wedge y_{n-2}$. Note that by definition $\text{Coker}\delta = K_n^M(F)$. The homomorphism $\tilde{f}_{n+1}(n)$ is defined as follows:

$$\tilde{f}_{n+1}(n)(v_0,\cdots,v_{n+1}) := n![v_0,\cdots,v_{n+1}]$$

where $[v_0,\cdots,v_{n+1}]$ is defined by induction:

$$[v_0,v_1,v_2,v_3] := \{r(v_0,v_1,v_2,v_3)\} \in \mathbf{Z}[P^1_F \setminus \{0,1,\infty\}]$$
$$[v_0,\cdots,v_{n+1}] := \gamma_n^{-1}\cdot \mathrm{Alt}(\varepsilon_1 \cdot C^1_{n+1}[v_1,\cdots,v_{n+1}] \otimes \Delta(v_0,v_1)$$
$$+\sum_{k=2}^{n-2} \varepsilon_k C^k_{n+1}[v_0,v_{k+1},\cdots,v_{n+1}] \otimes \Delta(v_0,v_1)\wedge\cdots\wedge\Delta(v_0,v_k)$$

Here $\varepsilon_i = \pm 1$. More precisely, $\gamma_n = 2^{n+1} - \left(2 + C^{n+1}_{n+1} + C^n_{n+1} + C^{n-1}_{n+1}\right)$, $\varepsilon_1 = +1$ and $\varepsilon_i = (-1)^i$, is for even n and $\gamma_n = 2^{n+1} - \left(C^{n+1}_{n+1} + C^n_{n+1} + C^{n-1}_{n+1}\right)$, $\varepsilon_1 = -1$, $\varepsilon_i = +1$, $i > 1$ for odd n . To prove the last formula one can wright

$$[v_0,\cdots,v_{n+1}] = \mathrm{Alt}(\alpha_1\cdot[v_1,\cdots,v_{n+1}]\otimes\Delta(v_0,v_1)+$$
$$+\sum_{k=2}^{n-3}\alpha_k[v_0,v_k,\cdots,v_{n+1}]\otimes\Delta(v_0,v_1)\wedge\cdots\wedge\Delta(v_0,v_k))$$

with some unknown coefficients α_i . Then the condition $\delta[v_0,\cdots,v_{n+1}] = \frac{1}{n!}\mathrm{Alt}\Delta(v_0,v_1)\wedge\cdots\wedge\Delta(v_0,v_n)$ gives exactly $n-3$ simple linear equations on α_i .

2. We get the following construction of the Chern classes $c_n(E) \in H^n\left(X, \underline{\underline{K}}^M_n\right)$. Choose a Zariski covering $\{U_i\}$ of X such that $E|_{U_i}$ is trivial. Choose sections $L^\bullet_i(x) \in \Gamma\left(U_i, \mathcal{A}^p_E(x)\right)$ such that for any $i_0 < \cdots < i_n$ affine flags $L^\bullet_{i_0}(x),\cdots,L^\bullet_{i_n}(x)$ are in generic position for every $x \in U_{i_0,\cdots i_n}$.

<u>**Theorem 3.4**</u>

$$\overline{f}_n(n)\left(P\left(L^\bullet_{i_0}(x),\cdots,L^\bullet_{i_n}(x)\right)\right) \in \underline{\underline{K}}^M_n(\mathcal{O}^*(U_{i_0,\cdots,i_n})) \qquad (3.5)$$

is a cocycle in the Čech complex for the covering $\{U_i\}$.

Proof: Follows immediately from lemmas 3.2, 3.3 and theorem 2.2.

□

By definition $c_n(E)$ is the cohomology class of the cocycle from theorem 3.4. It does not depend from the choice of sections $L^\bullet_i(x)$.

<u>**Example 3.5**</u> Recall that $c_1(E) = c_1(\det E)$. So $c_1(E)$ can be computed as follows: choose $m = \dim E$ linearly independent sections $l^\alpha_i(x)$ $(1 \le \alpha \le m)$ of $E|_{U_i}$. Then $(l^\alpha_i(x)) = g_{ij}(x)\cdot\left(l^\beta_i(x)\right)$ where $g_{ij}(x) \in \mathrm{GL}_n(F)$ is the transition matrix and $\det g_{ij}(x)$ is a 1-cocycle representing $c_1(E)$.

Now let $\left(l^1_1,\cdots,l^m_i\right)$ is the affine flag corresponding to the m-tuple of vectors $\left(l^1_i;\cdots;l^m_i\right)$. Let us prove that the cocycle 3.5 we get for these flags is exactly $\det g_{ij}$.

<u>**Proposition 3.6**</u> $f_1(1)\left(c\left(\left(l^1_i,\cdots,l^m_i\right),\left(l^1_j,\cdots,l^m_j\right)\right)\right) = \det g_{ij}$.

Proof: Let us say that a frame $\left(f^1;\cdots;f^m\right)$ is associated with an affine m-flag $\left(l^1,\cdots,l^m\right)$ if

$$\left\langle f^1,\cdots,f^k\right\rangle = \left\langle l^1,\cdots l^k\right\rangle \equiv L^k$$

and the images of f^{k+1} and l^{k+1} in L^{k+1}/L^k are coincide.

The set of all frames associated with a given affine m-flag is a principal homogeneous space over the group of upper triangular matrices.

Lemma-construction 3.7 *For 2 affine m-flags in generic position in V^m :*

$$L_1^\bullet = (v_1, \cdots, v_m) \quad \text{and} \quad L_2^\bullet = (w_1, \cdots, w_m)$$

there are just 2 frames associated with both of them.

Proof: We have the following isomorphisms of 1-dimensional vector spaces:

$$s_1 : L_1^k / L_1^{k-1} \xrightarrow{\sim} L_1^k \cap L_2^{m-k+1}$$
$$s_2 : L_2^{m-k+1} / L_2^{m-k} \xrightarrow{\sim} L_1^k \cap L_2^{m-k+1}$$

Put $f_1^k := s_1(v_k)$, $f_2^{m-k+1} := s_2(w_{m-k+1})$. Then the frames $(f_1^1; \cdots; f_1^m)$ and $(f_2^1; \cdots; f_2^m)$ associated with both $L_1^\bullet$ and $L_2^\bullet$.

□

Let $f_1^k = \lambda_k \cdot f_2^k$, $\lambda_k \in F^*$, and

$$(v_1; \cdots; v_m) = g \cdot (w_1, \cdots, w_m) , \quad q \in \mathrm{GL}_m(F) .$$

Then $\det g = \prod_{k=1}^{m} \lambda_k$ because $g = n_+ \cdot \lambda \cdot n_-$:

$$(\omega_i) \xrightarrow{n_-} \left(f_2^k\right) \xrightarrow{\lambda=(\lambda_k)} \left(f_1^k\right) \xrightarrow{n_+} (v_j)$$

where $n_-(n_+)$ is a lower (upper) triangular matrix and λ is a diagonal one with entries λ_k (the Gauss decomposition).

From the other hand the left-hand side in proposition 2.4 is equal to

$$f_1(1)\left(\sum_{k=1}^{m} \left(L_1^k \oplus L_2^{m-k} | l_1^k, l_2^{m-k+1}\right)\right) = f_1(1)\left(f_1^k, f_2^k\right) = \prod_{k=1}^{m} \lambda_k .$$

□

§ 4 The universal Chern class $c_n \in H^n\left(\mathrm{BGL}(m)_{\cdot}, \underline{\underline{K}}{}_n^M\right)$

1. **The Gersten resolution to Milnor's K-theory ([Ka]).** Let F be a field with a discrete valuation v and the residue class $\overline{F}_v (= \overline{F})$. The group of units U has a natural homomorphism $U \to \overline{F}^*$, $u \mapsto \overline{u}$. An element $\pi \in F^*$ is prime if $\mathrm{ord}_v(\pi) = 1$. There is a canonical homomorphism (see [M1]):

$$\partial : K_{n+1}^M(F) \to K_n^M\left(\overline{F}_v\right) \qquad (n \geq 0)$$

uniquely defined by properties $(u_i \in U)$

$$1. \quad \partial(\{\pi, u_1, \cdots, u_n\}) = \{\overline{u}_1, \cdots, \overline{u}_n\}$$
$$2. \quad \partial(\{u_1, \cdots, u_{n+1}\}) = 0$$

Let X be an excellent scheme (EGA [3] IV § 7), $X_{(i)}$ the set of all codimension i points x, $F(x)$ the field of functions corresponding to a point $x \in X_{(i)}$.
There is a sequence of group $\mathcal{K}(n)$. (Here $K_n^M(x) := K_n^M(F(x))$):

$$K_n^M(F(X)) \xrightarrow{\partial} \bigoplus_{x \in X_{(1)}} K_{n-1}^M(x) \xrightarrow{\partial} \bigoplus_{x \in X_{(2)}} K_{n-2}^M(x) \to \cdots \to \bigoplus_{x \in X_{(r)}} \mathbf{Z} \qquad (4.1)$$

We will follow [Ka] in the definition of ∂. Let us define for $y \in X_{(i)}$ and $x \in X_{i+1}$ a homomorphism

$$\partial_x^y : K_{*+1}^M(y) \to K_*^M(x)$$

as follows. Let Y be the normalisation of the reduced scheme $\{\overline{y}\}$. Set

$$\partial_x^y : \sum_{x'} N_{F(x')/F(x)} \circ \partial_{x'}$$

where x' ranges over all points of Y lying over x, $\partial_{x'} : K_{*+1}^M(y) \to K_*(x)$ is the tame symbol associated with the discrete valuation ring $\mathcal{O}_{Y,x'}$ and $N_{F(x')/F(x)}$ is the norm map $K_*^M(x') \to K_*^M(x)$ (see [BT], ch. I § 5 and [Ka], § 1.7). The coboundary ∂ is by definition the sum of these homomorphism ∂_x^y.

<u>**Proposition 4.1**</u> $\partial^2 = 0$.

Proof: See proof of proposition 1 in [Ka].

<u>**Theorem 4.2**</u> *The complex $\mathcal{K}(n)_\bullet$ is exact.*

□

2. **Explicit formula for a class $c \in H^n\left(\mathrm{BGL}(m)_\bullet, \underline{\underline{K}}{}_n^M\right)$**. Set $G := \underbrace{G \times \cdots \times G}_{n\ times}$.

Recall that

$$BG_\bullet := pt \overset{s_0}{\underset{s_1}{\leftleftarrows}} G \overset{s_0}{\underset{s_2}{\leftleftarrows}} G^2 \cdots$$

is the symplicial scheme representing the classifying space for a group G. We will compute $H^n\left(BG_\bullet, \underline{\underline{K}}{}_n^M\right)$ using the Gersten resolution (4.1). So cochain we have to construct lives in the following bicomplex $(G := \mathrm{GL}(m))$

$$\begin{array}{ccccc}
\cdots & & & & \\
\uparrow \partial & & & & \\
\bigoplus_{x \in G^{n-2}_{(2)}} K^M_{n-2}(F(x)) & \xrightarrow{s^*} & \bigoplus_{x \in G^{n-1}_{(2)}} K^M_{n-2}(F(x)) & & \\
& & \downarrow \partial & & \\
& & \bigoplus_{x \in G^{n-1}_{(1)}} K^M_{n-1}(F(x)) & \xrightarrow{s^*} & \bigoplus_{x \in G^{n}_{(1)}} K^M_{n-1}(F(x)) \\
& & & & \uparrow \partial \\
& & & & K^M_n(F(G^n))
\end{array} \tag{4.2}$$

For each partition $j_0 + \cdots + j_r = m - n$ a codimension $(n-r)$ irreducible subvariety $D(j_0, \cdots, j_r) \in G^r_{(n-r)}$ and an element $\omega(j_0, \cdots, j_r) \in K^M_r(D(j_0, \cdots, j_r))$ will be defined such that a collection of all these elements forms a cocycle in (4.2).

Recall that $A^{m-n+1}(m)$ be the manifold of affine $m-n+1$ flags in V^m . Let us define for a partition $j_0 + \cdots + j_r = m - r$ a codimension $n-r$ manifold

$$\tilde{D}_{j_0,\cdots,j_r} \subset \underbrace{A^{m-n+1}(m) \times \cdots \times A^{m-n+1}(m)}_{r+1 \ times}$$

as follows: $(L^\bullet_0, \cdots, L^\bullet_r) \in \tilde{D}_{j_0,\cdots,j_r}$ if and only if

$$\dim \left(\bigoplus_{p=0}^{r} L_p^{j_p+1} \right) = r + \sum_{p=0} j_p = \dim \left(\bigoplus_{p=0}^{r} L_p^{j_p+1} \right) - 1 \, .$$

Note that for generic $(L^\bullet_0, \cdots, L^\bullet_r) \in \tilde{D}_{j_0,\cdots,j_r}$ the sum $\bigoplus_{p=0}^{r} L_p^{j_p}$ is direct and the configurations of $r+1$ vectors

$$\left(\bigoplus_{p=0}^{r} L_p^{j_p} | l_0^{j_0+1}, \cdots, l_r^{j_r+1} \right) \tag{4.3}$$

in $V^m / \bigoplus_{p=0}^{r} L_p^{j_p}$ generates a subspace of dimension r . Recall that there is a homomorphism (see 3.2)

$$\overline{f}_r(r) : C_r(r) \to \Lambda^r F^* \to K^M_r(F) \, .$$

Applying it to the configuration of $r+1$ vectors (4.3) we get an element

$$\tilde{\omega}_{j_0,\cdots,j_r} \in K^M_r \left(F \left(\tilde{D}_{j_0,\cdots,j_r} \right) \right) \tag{4.4}$$

Now choose $a \in A^{m-n+1}(m)$. Set

$$D_{j_0 \cdots j_r; a} := \left\{ (g_1, \cdots, g_r) \in G^r | (a, g_1 a, \cdots, g_r a) \in \tilde{D}_{j_0,\cdots,j_r} \right\}$$

Then $D_{j_0,\cdots,j_r;a} \in G^r_{(n-r)}$ and $\tilde{w}_{j_0,\cdots,j_r}$ induces an element

$$\omega_{j_0,\cdots,j_r;a} \in K^M_r(F(D_{j_0,\cdots,j_r;a})) \tag{4.5}$$

Set

$$\tilde{\omega}_r := \sum_{j_0+\cdots+j_r=m-n} \tilde{\omega}_{j_0,\cdots,j_r} \in \bigoplus_{j_0+\cdots+j_r=m-n} K_r^M\left(F\left(\tilde{D}_{j_0,\cdots,j_r}\right)\right)$$

$$\omega_r := \sum_{j_0+\cdots+j_r=m-n} \omega_{j_0,\cdots,j_r;a} \in \bigoplus_{j_0+\cdots+j_r=m-n} K_r^M(F(D_{j_0,\cdots,j_r;a}))$$

Theorem 4.3 *Collection of elements* ω_r *defines a cocycle in the bicomplex (4.2).*

Proof: Choose a partition $i_0 + \cdots + i_r = m - r$. Let $\tilde{\mathcal{E}}$ be a subvariety in the manifold of $(r+1)$-tuples of affine $(m-r+1)$-flags in V^m defined as follows:

$$\tilde{\mathcal{E}}_{i_0,\cdots i_r} := \left\{ (L_0^\bullet, \cdots, L_r^\bullet) | \dim \left(\bigoplus_{p=0}^{r} L_p^{i_p} \right) = \left(\sum_{p=0}^{r} i_p \right) - 1 \right\}$$

This is a codimension $n - r + 1$ irreducible subvariety.

Proposition 4.4 *The component of* $\partial\tilde{\omega}_r$ *on* $\tilde{\mathcal{E}}_{i_0,\cdots,i_r}$ *is non zero if* $i_k = 0$ *for some* k *but* $i_p > 0$ *for* $p \neq k$. *In this case it is equal to*

$$\overline{f}_r(r)\left(\bigoplus_{p\neq k} L_p^{i_p-1} | l_0^{i_0}, \cdots, \widehat{l_k^{i_k}}, \cdots, l_r^{i_r} \right) \tag{4.6}$$

Proof: Let $j_0 + \cdots + j_r = m - n$ and

$$\left(l_0^1, \cdots, l_0^{m-n+1}; \cdots; l_r^1, \cdots, l_r^{m-n+1}\right) \equiv (L_0^\bullet, \cdots, L_r^\bullet) \in \tilde{D}_{j_0,\cdots,j_r}$$

Choose a volume form in the codimension n-subspace $\left\langle l_0^1, \cdots, l_0^{j_0+1}, \cdots, l_r^1, \cdots, l_r^{j_r+1} \right\rangle$. Then we can compute the determinant $\Delta(v_1, \cdots, v_{m-n+r})$ for any $m - n + r$ vectors in this subspace. Set

$$\Delta(j_{k+1}) := \Delta\left(l_0^1, \cdots, l_0^{j_0+1}, \cdots, \widehat{l_k^{j_{k+1}}}, \cdots, l_r^1, \cdots, l_r^{j_r+1} \right)$$

Then by definition

$$\tilde{\omega}_{j_0,\cdots,j_r} = \sum_{k=0}^{r} (-1)^k \left\{ \Delta(j_0+1), \cdots, \Delta(\widehat{j_k+1}), \cdots, \Delta(j_r+1) \right\} \tag{4.7}$$

The coboundary $\partial\tilde{\omega}_{j_0,\cdots,j_r}$ can be nonzero on divisors $\Delta(j_{k+1}) = 0$ in $\tilde{D}_{j_0,\cdots,j_r}$ only. The component of $\partial\tilde{\omega}_{j_0,\cdots,j_r}$ on the divisor $\Delta(j_{k+1}) = 0$ is equal to

$$s\left(\bigoplus_{p=0}^{r} L_p^{j_p} \oplus l_k^{j_{k+1}} | l_0^{j_0+1}, \cdots, \widehat{l_k^{j_{k+1}}}, \cdots, l_r^{j_r+1} \right) \tag{4.8}$$

This formula implies immediately that the component of $\partial\tilde{\omega}_r$ on $\tilde{\mathcal{E}}_{i_0,\cdots i_r}$ is zero if $i_{k_1} = i_{k_2} = 0$ for some $k_1 \neq k_2$.

It follows from (4.8) that in the case $i_p > 0$ for all p the component of $\partial\tilde{\omega}_r$ on $\tilde{\mathcal{E}}_{i_0,\cdots,i_r}$ is

$$f_r(r)\left(\sum_{k=0}^{r} (-1)^k \left(\bigoplus_{p=0}^{r} L_p^{i_p-1} + l_k^{i_k} | l_0^{i_0}, \cdots, \widehat{l_k^{i_k}}, \cdots, l_r^{i_r} \right) \right) \tag{4.9}$$

Note that $\left(\bigoplus_{p=0}^{r} L^{i_p-1}|l_0^{i_0},\cdots,l_r^{i_r}\right)$ is a configuration of $m+1$ vectors in an m-dimensional space (4.9) is equal to

$$f_r(r)\circ d'\left(\bigoplus_{p=0}^{r} L_p^{i_p-1}|l_0^{i_0},\cdots,l_r^{i_r}\right)$$

But this is equal to zero according to lemma 3.2.

Now suppose that $i_k=0$, $i_p\neq 0$ for $p\neq k$. Then (4.8) implies that the component of $\partial(\tilde{\omega}_r)$ on $\tilde{\mathcal{E}}_{i_0,\cdots,i_r}$ is exactly (4.6).

□

3. **Relation to the classical construction of Chern cycles.** Suppose that a vector bundle E ver X s sufficiently many sections. Consider first of all the case when $\dim E=n$ and we are interested in $c_n(E)\in CH^n(X)$. Choose a section $s_0(x)\in\Gamma(X,E)$ that is transversal to the zero section of E. Then the subvariety

$$D_0:=\{x\in X|s_0(x)=0\}$$

has codimension n and represents the class $c_n(E)\in CH^n(X)$. Now let $s_1(x)$ be another generic section of E (i.e. it is transversal to the zero section of E too). Then

$$D_1:=\{x\in X|s_1(x)=0\}$$

should represent the same class in $CH^n(X)$. To see this let us consider a codimension $(n-1)$ subvariety

$$D_{01}:=\{x\in X|\exists\lambda_0,\lambda_1\in\mathbb{C}\quad\text{such that}\quad\lambda_0 s_0(x)+\lambda_1 s_1(x)=0\}$$

There is a canonical rational function

$$\lambda_{01}:=\frac{\lambda_0}{\lambda_1}\in F(D_{01})\quad\text{and}\quad\mathrm{Div}(\lambda_{01})=D_0-D_1$$

So D_0 and D_1 are canonically rationally equivalent cycles. Now let $s_2(x)$ be the third generic section of E. Put

$$D_{012}=\{x\in X|\dim\langle s_0(x),s_1(x),s_2(x)\rangle=2\}$$

Then $\operatorname{codim} D_{012}=n-2$ and there is a canonical element

$$\lambda_{012}:=f_2(2)(s_0,s_1,s_2)\in K_2(F(D_{012}))$$
$$\partial(\lambda_{012})=\lambda_{01}-\lambda_{02}+\lambda_{12}$$

where $\partial: K_2(F(Y))\to\coprod_{y\in Y_{(1)}}F(y)^*$ is the tame symbol. Continuing this process we get for $r+1$ generic sections $s_0(x),\cdots,s_r(x)$ of E a codimension $(n-r)$ subvariety

$$D_{01\cdots r}:=\{x\in X|\dim\langle s_0(x),\cdots,s_r(x)\rangle=r\}$$

and a canonical element

$$\lambda_{01\cdots r} := f_r(z)(s_0, \cdots, s_r) \in K_r^M(F(D_{01\cdots r}))$$

satisfying the relation

$$\partial(\lambda_{01\cdots r}) = \sum_{i=0}^{r} (-1)^i \lambda_{01\cdots\hat{i}\cdots r}$$

(∂ is the differential in complex (4.1).

Now let E be a vector bundle of dimension $m > n$ and $p = m - n + 1$. Let

$$L_0(x) = \left(l_0^1(x), \cdots, l_0^p(x)\right)$$

is a **generic** section of the bundle of affine p-flags on X. Put

$$D_0 := \left\{x \in X | l_0^1(x) \wedge \cdots \wedge l_0^p(x) = 0 \ , \quad \text{but} \quad l_0^1(x) \wedge \cdots \wedge l_0^{p-1}(x) \neq 0\right\}$$

It is well-known (see, for example, s. 3 of ch. III in [GH] that the image of the cycle D_0 in the Chow group $CH^n(X)$ is just $c_n(E)$. Let $L_0^\bullet(x), \cdots, L_r^\bullet(x)$ be $r+1$ generic sections of the bundle of affine p-flags. For any partition $j_0 + \cdots + j_r = p - 1$, $j_k \geq 0$, put

$$\begin{gathered} D(j_0, \cdots, j_r) := \{x \in X | (r+1) - \text{tuple of vectors} \\ \left(\mathrm{L}_0^{j_0} + \cdots + \mathrm{L}_r^{j_r} | \ell_0^{j_0+1}, \cdots, \ell_r^{j_2+1}\right) \quad \text{generates r} - \text{dimensional} \\ \text{vector space and} \quad \dim \bigoplus_{k=0}^{r} L_0^{j_k} = \sum_{k=0}^{r} j_k\} \end{gathered} \tag{4.10}$$

Then $D(j_0, \cdots j_r)$ is a codimension $n - r$ cycle in X. There is a canonical element

$$\overline{f}_r(r)\Big(\Big(L_0^{j_0} \oplus \cdots \oplus L_r^{j_r} | l_0^{j_0+1}, \cdots, l_r^{j_2+1}\Big)\Big) \in K_r^M(F(D(j_0, \cdots, j_r))) \tag{4.11}$$

Let us define an element

$$\lambda_{01\cdots r} \in \coprod_{j_0+\cdots+j_2=p-1} K_r^M(F(D(j_0, \cdots, j_r))) \subset \coprod_{x \in X^{n-r}} K_r^M(F(x))$$

as the sum of elements (4.11):

$$\lambda_{01\cdots r} := \sum_{j_0+\cdots+j_2=p-1} \overline{f}_r(r)\left(\bigoplus_{k=0}^{r} L_k^{j_k} | l_0^{j_0+1}, \cdots, l^{j_r+1}\right)$$

Theorem 4.5 $\partial(\lambda_{01\cdots r}) = \sum\limits_{i=0}^{r} (-1)^i \lambda_{01\cdots\hat{i}\cdots r}$

Proof: Follows immediately from proof of theorem 4.4.

□

4. **An algebraic construction of ring generators of** $H^*(GL_m(\mathbb{C}))$. I will construct a non-zero class in $W_0 H^{2n-1}(GL_m(\mathbb{C}), \mathbb{Q}(n))$. This vector space is one-dimensional for $m \geq n$.

Let us define for any $0 \leq j \leq m-n$ a subvariety $\tilde{D}_j \subset A^{m-n+1}(m) \times A^{m-n+1}(m)$ as follows:

$$\tilde{D}_j := \{(L_1^\bullet, L_2^\bullet) \text{ such that } \left(L_1^j + L_2^{m-n-j} | l_1^{j+1}, l_2^{m-n-j+1}\right) \\ \text{is a pair of collinear nonzero vectors}\} \tag{4.12}$$

There is canonical invertible function $\tilde{f}_j$ on $\tilde{D}_j$: the ratio $\frac{\text{first vector}}{\text{second vector}}$ (see 4.12). Now choose an affine $(m-n+1)$-flag $L^\bullet$ in V_m . Set

$$GL(V_m) \supset D_j := \left\{g \in GL(V_m) | (gL^\bullet, L^\bullet) \in \tilde{D}_j\right\}$$

There is canonical function $f_j \in \mathcal{O}(D_j)^*$.

Theorem 4.6. *The current* $\sum_j d\log f_j$ *represents a nonzero class in* $W_0 H^{2n-1}(GL_m(\mathbb{C}), \mathbb{Q}(n))$.

Proof: Let us prove that $\sum_j \mathrm{div} f_j = 0$ where $\mathrm{div} f_j$ is the divisor of f_j on $\overline{D_j}$ considered as a codimension n cycle on $GL(V_m)$. Note that $\mathrm{div}\tilde{f}_j = Z_j^+ - Z_j^-$ where

$$Z_j^+ = \{(L_1^\bullet, L_2^\bullet) | < L_1^{j+1}, L_2^{m-n-j} > = < L_1^j, L_2^{m-n-j} > \\ \text{and} \quad L_1^j \cap L_2^{m-n-j} = 0\}$$

$$Z_j^- = \{(L_1^\bullet, L_2^\bullet) | < L_1^j, L_2^{m-n-j+1} > = < L_1^j, L_2^{m-n-j} > \\ \text{and} \quad L_1^j \cap L_2^{m-n-j} = 0\}$$

Therefore it is easy to see that $\sum_j \mathrm{div}\tilde{f}_j = 0$ and hence $\sum_j \mathrm{div} f_j = 0$. So the current $\sum_j d\log f_j$ represents a class in $W_0 H^{2n-1}(GL_m(\mathbb{C}), \mathbb{Q}(n))$. It remains to prove that it is nontrivial.

Let $Gr(N-m, N)$ be the Grassmannian of codimension m subspaces in V_N . There is canonical m-dimensional bundle E over it: the fiber over plane h is V_N/h . Let us choose an affine $m-n+1$ flag $L^1 \subset \cdots \subset L^{m-n+1}$ in V_N . It defines a Chern cycle $c_m(E; L^\bullet) \subset Gr(N-m, N)$. Let $\pi : \tilde{E} \to Gr(N-m, N)$ be the bundle of frames $(e_1, \cdots, e_n)$ in fibers of E. This is a principle GL_m-bundle. Let us construct a cycle $B_m \subset \tilde{E}$ together with a rational function $g_m \in k(B_m)$ such that

$$\mathrm{div} g_m = \pi^{-1}(c_m(E; L^\bullet)) \tag{4.13}$$

and for generic $h \in Gr(N-m, N)$ the intersection

$$(B_m, g_m) \cap \pi^{-1}(h) \text{ coincides with } \sum_j (D_j, f_j) \tag{4.14}$$

constructed using the projection of the flag $L^\bullet$ onto V_N/h . (More precisely, a reper $(e_1, \cdots, e_m)$ defines an affine $(m-n+1)$-flag $(e_1; \cdots; e_{m-n+1})$ and this flag together with the projection of $L^\bullet$ should satisfy 4.12). Conditions 4.13 and 4.14 just means that the cohomology class of the current $\sum_j d\log f_j$ is the transgression of the $m-th$ Chern class of the universal bundle. Moreover, they give a precise description of the cycle B_m : it is closure of union of cycles $\sum D_j \subset \pi^{-1}(h)$ constructed using the projection of $L^\bullet$; here h runs through an open part in $\mathrm{Gr}(N-m, N)$. It is easy to see that for the natural invertible function g_m on B_m (4.13) holds.

■

§ 5 Explicit formulas for the universal motivic Chern classes $c_n \in H^{2n}(BGL_m, \mathbb{Q}(n))$ for $n \leq 3$

First of all I have to recall what are the motivic complexes. So for convenience of the reader I will reproduce in S. 1-3 basic definition and results from [G1], [G2].

1. **Motivic complexes.** Let F be an arbitrary field. Denote by $\mathbf{Z}[P^1_F]$ a free abelian group generated by symbols $\{x\}$ where x runs all F-points of P^1. Let us define subgroups $R_n(F) \subset \mathbf{Z}[P^1_F]$ $(n \leq 3)$ as follows:

$R_1(F) :=$ a subgroup generated by $\{xy\} - \{x\} - \{y\}$ where x, y run through all elements of F^*

$R_2(F)$:= a subgroup generated by $\sum_{i=0}^{4}(-1)^i\{r(x_0, \cdots, \widehat{x_i}, \cdots x_4)\}$ where $(x_0, \cdots x_4)$ runs through all configuration of 5 distinct points of P^1_F and $r(x_1, \cdots, x_4) := \frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$ is the cross-ratio

$R_3(F)$:= a subgroup generated by $\sum_{i=0}^{6}(-1)^i\left\{r_3\left(l_0, \cdots, \widehat{l_i}, \cdots, l_0\right)\right\}$ where $(l_0, \cdots, l_6)$ runs through all configuration of 7 points in P^2_F in generic position and $r_3(l_1, \cdots, l_6) \in \mathbf{Z}[P^1_F]$ is the generalized cross-ratio:

$$r_3(l_1, \cdots, l_6) := \mathrm{Alt}\left\{\frac{\Delta\left(\tilde{l}_1\tilde{l}_2\tilde{l}_4\right) \cdot \Delta\left(\tilde{l}_2\tilde{l}_3\tilde{l}_5\right) \cdot \Delta\left(\tilde{l}_3\tilde{l}_1\tilde{l}_6\right)}{\Delta\left(\tilde{l}_1\tilde{l}_2\tilde{l}_5\right) \cdot \Delta\left(\tilde{l}_2\tilde{l}_3\tilde{l}_6\right) \cdot \Delta\left(\tilde{l}_3\tilde{l}_1\tilde{l}_4\right)}\right\} \tag{5.1}$$

where $\mathrm{Alt}\ f(l_1, \cdots, l_6) := \sum_{\sigma \in S_6}(-1)^{|\sigma|} f\left(l_{\sigma(1)}, \cdots, l_{\sigma(6)}\right)$.

Here $\tilde{l}_i$ are vectors in $V^3 \setminus 0$ that projects to the points $l_i \in P(V^3)$. The right-hand side of (4.1) does not depend neither from the volume form in V^3, nor from the length of vectors l_i. So the cross-ration of 6 points in P^2_F is well-defined. Put

$$B_n(F) := \frac{\mathbf{Z}[P^1_F]}{R_n(F), \{0\}, \{\infty\}}$$

There is a canonical isomorphism $B_1(F) \xrightarrow{\sim} F^*$ provided by the map $\{x\} \mapsto x$; $\{0\}$, $\{\infty\} \mapsto 1$. Let us consider the following complexes $B_F(n)$:

$$\begin{aligned} &B_F(1) : F^* \\ &B_F(2) : B_2(F) \xrightarrow{\delta_2} \Lambda^2 F^* \\ &B_F(3) : B_3(F) \xrightarrow{\delta_3} B_2(F) \otimes F^* \xrightarrow{\delta_3} \Lambda^3 F^* \end{aligned} \tag{5.2}$$

Here

$$\delta_2\{x\} := (1-x) \wedge x$$
$$\delta_3\{x\} := \{x\} \otimes x; \quad \delta_3\{x\} \otimes y := (1-x) \wedge x \wedge y$$

and by definition $\delta_n\{0\} = \delta_n\{\infty\} = 0$, $(n = 2, 3)$. Note that $\delta_3 \circ \delta_3(\{x\}) = (1-x) \wedge x \wedge x = 0$, so $B_F(3)$ is a complex.

<u>**Theorem 5.1**</u> $\delta_n(R_n(F)) = 0$

Proof: See § 3 of [G2] or theorem 5. below.

In complexes (5.2) groups $B_n(F)$ placed in degree 1 and δ_n has degree $+1$.

The complex $B_F(2)$ is the well-known Bloch-Suslin complex.

2. **The motivic complexes $\Gamma(X;n)$ for a regular scheme** X $(n \le 3)$. Let F be a field with a discrete valuation v and the residue class $\overline{F}_v$. Let us construct a canonical homomorphism of complexes

$$\partial_v : B_F(n) \to B_{\overline{F}_v}(n-1)[-1]$$

There is a homomorphism $\theta : \Lambda^n F^* \to \Lambda^{n-1}\overline{F}_v^*$ uniquely defined by the following properties ($u_i \in U$, $u \mapsto \overline{u}$ is the natural homomorphism $U \to \overline{F}_v^*$ and π is a prime: $\mathrm{ord}_v \pi = 1$) :

$$\begin{array}{ll} 1. & \theta(\pi \wedge u_1 \wedge \cdots \wedge u_{n-1}) = \overline{u}_1 \wedge \cdots \wedge \overline{u}_{n-1} \\ 2. & \theta(u_1 \wedge \cdots \wedge u_n) = 0 \end{array}$$

It clearly does not depend on the choice of π .

Let us define a homomorphism $s_v : \mathbf{Z}[P^1_F] \to \mathbf{Z}\left[P^1_{\overline{F}_v}\right]$ as follows .

$$s_v\{x\} = \begin{cases} \{\overline{x}\} & \text{if } x \text{ is a unit} \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

Proposition 5.2 *Homomorphism (5.4) induces a homomorphism*

$$s_v : B_n(F) \to B_n\left(\overline{F}_v\right) , \quad n = 2, 3 .$$

Proof: Straightforward but tedious computations using explicit formula (3.17) from [G3] for generators of the subgroup $R_3(F)$.

To avoid such computations one can consider subgroups $\mathcal{R}_n(F) \subset \mathbf{Z}[P^1_F]$ defined in s. 4 of § 1 in [G3]. Then more or less by definition $s_v(\mathcal{R}_n(F)) = \mathcal{R}_n\left(\overline{F}_v\right)$ and $\delta(R_n(F)) = 0$. So there are corresponding groups $\mathcal{B}_n(F) := \frac{\mathbf{Z}[P^1_F]}{\mathcal{R}_n(F)}$ together with homomorphisms s_v : $\mathcal{B}_n(F) \to \mathcal{B}_n\left(\overline{F}_v\right)$.

$\square$

Set

$$\partial_v := s_v \otimes \theta : B_k(F) \otimes \Lambda^{n-k} F^* \to B_k\left(\overline{F}_v\right) \otimes \Lambda^{n-k-1}\overline{F}_v^* \tag{5.5}$$

Lemma 5.3 *The homomorphism ∂_v commutes with the coboundary δ and hence defines a homomorphism of complexes (5.3).*

Proof: Straightforward computation. See also s. 14 of § 1 in [G2] where the corresponding fact proved for groups $\mathcal{B}_n(F)$.

$\square$

Now let X be an arbitrary regular scheme, $X_{(i)}$ the set of all codimension i points of X, $F(x)$ the field of functions corresponding to a point $x \in X_{(i)}$. We define the motivic complexes $\Gamma(X, n)$ as the total complexes associated with the following bicomplexes:

$$\Gamma(X,1): \quad F(X)^* \xrightarrow{\partial_1} \coprod_{x \in X_{(1)}} \mathbf{Z}$$

$$\Gamma(X,2): \quad \begin{array}{ccccc} \Lambda^2 F(X)^* & \xrightarrow{\partial_1} & \coprod_{x\in X_{(1)}} F(x)^* & \xrightarrow{\partial_2} & \coprod_{x\in X_{(2)}} \mathbf{Z} \\ \uparrow \delta & & & & \\ B_2(F(X)) & & & & \end{array}$$

$$\Gamma(X;3): \quad \begin{array}{ccccccc} \Lambda^3 F(X)^* & \xrightarrow{\partial_1} & \coprod_{x\in X_{(1)}} \Lambda^2 F(x)^* & \xrightarrow{\partial_2} & \coprod_{x\in X_{(2)}} F(x)^* & \xrightarrow{\partial_3} & \coprod_{x\in X_{(3)}} \mathbf{Z} \\ \uparrow \delta & & \uparrow \delta & & & & \\ B_2(F(X)) \otimes F(X)^* & \xrightarrow{\partial_1} & B_2(F(X)) & & & & \\ \uparrow \delta & & & & & & \\ B_3(F(X)) & & & & & & \end{array}$$

where $B_n(F(X))$ is placed in degree 1 and coboundaries have degree $+1$.

The coboundaries ∂_i are defined as follows. $\partial_1 := \coprod_{x\in X_{(1)}} \partial_{v_x}$. The others are a little bit more complicated. Let $x \in X_{(k)}$ and $v_1(y), \cdots, v_m(y)$ be all discrete valuations of the field $F(x)$ over a point $y \in X_{(k+1)}$, $y \in \overline{x}$. Then $\overline{F(x)}_i := \overline{F(x)}_{v_i(y)} \supset F(y)$. (If $\overline{x}$ is nonsingular at the point y, then $\overline{F(x)}_i = F(y)$ and $m = 1$). Let us define a homomorphism $\partial_2 : \Lambda^2 F(x) \to F(y)^*$ as the composition

$$\Lambda^2 F(x)^* \xrightarrow{\oplus \partial_{v_i(y)}} \bigoplus_{i=1}^{m} \overline{F(x)}_i^* \xrightarrow{\oplus N_{\overline{F(x)}_i/F(y)}} F(y)^*$$

and $F(x)^* \xrightarrow{\oplus \partial_{v_i}} \bigoplus_{i=1}^{m} \mathbf{Z} \xrightarrow{\Sigma} \mathbf{Z}$.

3. **Motivic Chern classes** $c_n \in H^{2n}_{\mathcal{M}}(BGL_m(F)_\bullet, \mathbf{Z}(n))$, $n \leq 3$. Recall that

$$BG_. := pt \overset{s_0}{\underset{s_1}{\leftleftarrows}} G \overset{s_0}{\underset{s_2}{\leftleftarrows}} G^2 \cdots$$

We have to construct a 2n-cocycle c_n in the bicomplex

$$\Gamma(G;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^n;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma\left(G^{2n-1};n\right) \tag{4.7}$$

where $s^* = \Sigma(-1)^i s_i$. Its components in

$$\Gamma(G;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^n;n) \tag{128}$$

should be in the following part of the bicomplex:

$$
\begin{array}{ccccccc}
\bigoplus_{x\in G_{(n)}} \mathbb{Z} & & & & & & \\
\uparrow \partial & & & & & & \\
\bigoplus_{x\in G_{(n-1)}} F(x)^* & \xrightarrow{s^*} & \bigoplus_{x\in G^2_{(n-1)}} F(x)^* & & & & \\
 & & \uparrow \partial & & & & \\
 & & \bigoplus_{x\in G^2_{(n-2)}} \Lambda^2 F(x)^* & \xrightarrow{s^*} & \cdots & & \\
 & & & & \cdots & \xrightarrow{s^*} & \bigoplus_{x\in G^n_{(1)}} \Lambda^n F(x)^* \\
 & & & & & & \uparrow \partial \\
 & & & & & & \Lambda^n F(G^n)^*
\end{array}
\tag{5.8}
$$

In fact the components of c_n in 5.8 were already constructed in § 4. Recall this construction. Let a be an affine $(m-n+1)$-flag in an m-dimensional vector space V^m . For each partition $j_0+\cdots+j_r = m-n$ irreducible subvarieties

$$D_{j_0,\cdots,j_r;a} \in G^r_{(n-r)}$$

together with elements

$$\tilde{\omega}_{j_0,\cdots,j_r;a} \in \Lambda^r F(D_{j_0,\cdots,j_r;a})^* \tag{5.9}$$

were constructed. More precisely, if

$$(L_0^\bullet,\cdots,L_r^\bullet) := (a, g_1 a,\cdots, g_r a)$$

where $(g_1,\cdots,g_r) \in D_{j_0,\cdots,j_r;a} \subset G^r$ then

$$\left(\bigoplus_{p=0}^{r} L_p^{j_p} | l_0^{j_0+1},\cdots, l_r^{j_2+1}\right)$$

is a configuration of $r{+}1$ vectors in an r-dimensional vector space. Applying homomorphism $f_r(r): C_r(r) \to \Lambda^r F^*$ to it we get the element (5.9). The collection of elements

$$
\begin{aligned}
\tilde{\omega}_r := \sum_{j_0+\cdots+j_r=m-n} \tilde{\omega}_{j_0,\cdots,j_r;a} \in & \bigoplus_{j_0+\cdots+j_2=m-n} \Lambda^r F(D_{j_0,\cdots,j_r;a})^* \\
\in & \bigoplus_{x\in G^r_{(n-r)}} \Lambda^r F(x)^*
\end{aligned}
\tag{5.10}
$$

forms a cocycle in the bicomplex (5.8). (The proof of this fact is absolutely the same as the one of theorem 4.3). The components of c_n in the bicomplex

$$\Gamma(G^n;n) \xrightarrow{s^*} \Gamma(G^{n+1};n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^{2n-1};n) \tag{5.11}$$

are constructed as follows. There is a homomorphism of complexes (see (2.4), (2.5))

$$T: C_*(A^{m-n+1}(m)) \to BC_*(n)$$

where $BC_*(n)$ is the total complex for the Grassmannian bicomplex (1.2).

We will construct homomorphisms of complexes

$$f(n) : BC_*(n) \to B_F(n) \qquad (n \leq 3) \tag{5.12}$$

such that for $r \geq n+1$ the ∂-coboundaries of elements

$$f(n) \circ P(a, g_1 a, \cdots, g_r a) \tag{5.13}$$

are equal to zero. The collection of elements (5.10) and (5.13) form a cocycle c_n in the bicomplex (5.7).

Let us describe the construction of the homomoprhism (5.12).

a) $n = 1$. $f_1(1) : C_1(1) \to F^*$ is the only homomorphism we need. It is very easy to check that $f_1(1) \circ d' : C_2(2) \to F^*$ and $f_1(1) \circ d : C_2(1) \to F^*$ are equal to zero, so we get a homomorphism $f(1) : BC_*(1) \to F^*[-1]$.

b) $n = 2$. We have to construct a homomorphism from the total complex associated with the bicomplex

$$\begin{array}{ccccc} & \downarrow & & \downarrow \\ \to & C_4(3) & \xrightarrow{d} & C_3(3) \\ & \downarrow d' & & \downarrow d' \\ \xrightarrow{d} & C_3(2) & \xrightarrow{d} & C_2(2) \end{array}$$

to the complex

$$0 \to B_2(F) \to \Lambda^2 F^*$$

A homomorphism $f_2(2) : C_2(2) \to \Lambda^2 F^*$ was defined by formula (3.2). Lemma 3.2 shows that one can put a map from $C_3(3)$ to $B_2(F)$ equals to zero. Let us define a homomorphism

$$f_3(2) : C_3(2) \to B_2(F)$$

setting

$$(l_0, \cdots, l_3) \mapsto \{r(\bar{l}_0, \cdots, \bar{l}_3)\}_2$$

where $(\bar{l}_0, \cdots, \bar{l}_3)$ is a configuration of 4 points in P^1_F corresponding to the one $(l_0, \cdots, l_3)$ of 4 vectors in V^2. Then $f_3(2) \circ d : C_4(2) \to B_2(F)$ is zero by definition of the group $B_2(F)$.

Lemma 5.4 $f_3(2) \circ d' = 0$

Proof: We have to prove that for $(l_0, \cdots, l_4) \in C_4(3)$

$$\sum_{i=0}^{4} (-1)^i \left\{ r\left(\bar{l}_i | \bar{l}_0, \cdots, \hat{l}_i, \cdots, \bar{l}_4\right) \right\}_2 = 0 \quad \text{in} \quad B_2(F). \tag{5.14}$$

There is a conic (a curve of order 2) passing through 5 points $\bar{l}_0, \cdots, \bar{l}_u$ in P^2_F. Let us consider it as a projective line. Then (5.14) is just the 5-term relation for 5 points $\bar{l}_i$ on this projective line.

□

So we have defined a homomorphism $f(r) : BC_*(r) \to B_F(2)$. It is non-zero only on the Grassmannian subcomplex $C_*(2) \subset BC_*(2)$.

c) $n = 3$. We have to define a homomorphism from the total complex associated with the bicomplex

$$\begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow \\ \to & C_6(4) & \to & C_5(4) & \to & C_4(4) \\ & \downarrow & & \downarrow & & \downarrow \\ \to & C_5(3) & \to & C_4(3) & \to & C_3(3) \end{array}$$

to the complex

$$B_3(F) \to B_2(F) \otimes F^* \to \Lambda^3 F^*$$

A homomorphism $f_3(3) : C_3(3) \to \Lambda^3 F^*$ was defined by formula (3.2). Set

$$\begin{aligned} &f_4(3) : C_4(3) \to B_2(F) \otimes F^* \\ &f_4(3) : (l_0, \cdots, l_4) \mapsto \frac{1}{2}\mathrm{Alt}\{r(\bar{l}_0|\bar{l}_1, \cdots, \bar{l}_u)\}_2 \otimes \Delta(l_0, l_1, l_2) \end{aligned} \tag{5.15}$$

Proposition 5.5 $f_4(3)$ *does not depend on the choice of the volume form* $\omega_3 \in \Lambda^3(V^3)^*$ *that we need for the definition of* $\Delta(l_0, l_1, l_2)$.

Proof: The difference between the right-hand sides of (5.15) computed using $\lambda \cdot \omega_3$ and ω_3 is proportional to (right-hand side of (5.14) $\otimes\lambda$. So it is zero by lemma 5.4.

□

Proposition 5.6 $f_3(3) \circ d = \delta \circ f_4(3)$

Proof: Direct calculation using the formula

$$r(\bar{l}_1, \cdots, \bar{l}_u) = \frac{\Delta(l_1, l_3) \cdot \Delta(l_2, l_4)}{\Delta(l_1, l_4) \cdot \Delta(l_2, l_3)}$$

□

Now set

$$\begin{aligned} &f_5(3) : C_5(3) \to B_3(F) \\ &f_5(3) : (l_0, \cdots, l_5) \mapsto \mathrm{Alt}\left\{\frac{\Delta(l_0, l_1, l_3) \cdot \Delta(l_1, l_2, l_4) \cdot \Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4) \cdot \Delta(l_1, l_2, l_5) \cdot \Delta(l_2, l_0, l_3)}\right\}_3 \end{aligned} \tag{5.16}$$

Theorem 5.7 $f_4(3) \circ d = \delta \circ f_5(3)$

Proof: See proof of theorem 3.10 in [G3].

□

Proposition 5.8 $f_k(3) \circ d' = 0$ *for* $k = 3, 4, 5$.

Proof: For $k = 3$ this is lemma 3.2. For $k = 4, 5$ see theorem 3.12 in [G3].

□

Proposition 5.9 $f_5(3) \circ d = 0$ *in* $B_3(F)$.

Proof: Follows immediately from the definition of the group $B_3(F)$.

□

So one can define a homomorphism $f(3) : BC_*(3) \to B_F(3)$ using homomorphisms $f_k(3)$ on the subcomplex $C_*(3) \subset BC_*(3)$ and zeros otherwise.

Now consider an element

$$f_4(3) \circ P(a, g_1 a, \cdots, g_4 a) \in B_2\big(F(G^4)\big) \otimes F(G^4)^*$$

Then

$$\partial_1 \circ f_4(3) \circ P(a, g_1 a, \cdots, g_4 a) \in \bigoplus_{x \in G^4_{(1)}} B_2(F(x)) \tag{5.17}$$

<u>**Lemma 5.10**</u> *The left-hand side of (5.17) is euqal to zero.*

Proof: It follows from the definition (5.5) of ∂_v and the following remark: $\Delta(l_0, l_1, l_2)$ appears in formula (5.15) with factor $\{r(\bar{l}_3|\bar{l}_0, \bar{l}_1, \bar{l}_2, \bar{l}_4)\}_2 - \{r(\bar{l}_4|\bar{l}_0, \bar{l}_1, \bar{l}_2, \bar{l}_3)\}_2$ that is obviously zero if $\Delta(l_0, l_1, l_2) = 0$.

□

So we have proved that the collection of elements (5.10) and (5.13) form a cocycle in the bicomplex (5.7). The cohomology class of this cocycle does not depend from the choice of an affine $(m-n+1)$-flag a. (Different flags give cocycles that are canonically cohomologous).

4. **Chern classes in Deligne cohomology.** Let us suppose that there exists a $2n$-cocycle $\mathbf{L}'_n$ from conjecture 1.1' (A precise construction of this cocycle for $n \leq 3$ can be found in § 9 of [G1], see also [G2]). The main construction of § 2 gives an explicit construction of Chern classes in Bigrassmannian cohomology and hence, applying $\mathbf{L}'_n$, in real Deligne cohomology. We will see in the next section that these Chern classes coincides with the classical ones (see theorem 5.11)

5. **The universal Chern classes in Deligne cohomology.** Assuming existence of $\mathbf{L}'_n$ we will construct

$$c_n \in H^{2n}_{\mathcal{D}}(BGL_m(\mathbb{C}), R(n))$$

The Dolbeaux resolution of the complex associated with the bicomplex 1.13 provides us a complex computing real Deligne cohomology of an algebraic manifold over $\mathbb{C}$. We will denote this complex as $R(X, n)$. We have to construct a $2n$-cocycle in the bicomplex

$$R(G, n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} R(G^n, n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} R\big(G^{2n-1}, n\big) \tag{5.18}$$

(compare with 4.7). First of all let us construct its components in

$$R(G, n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} R(G^n, n) \tag{5.19}$$

If $Y \hookrightarrow X$ is a subvariety of codimension d then there is a canonical homomorphism of complexes $i_* : R(Y, n) \to R(X, n+d)[2d]$. In s. 3 we have constructed a chain (5.10) in the bicomplex 5.8 corresponding to an affine $(m-n+1)$-flag a in V_m . Each component of this chain lies in $\Lambda^r F(x)^*$ where x is a codimension $n-r$ point in G^r. There is canonical map

$$\Lambda^r \mathbb{C}(x)^* \to R(\mathrm{Spec}\ \mathbb{C}(x), r)$$

$$f_1 \wedge \cdots \wedge f_r \mapsto$$
$$\left\{ \alpha_r \left(\frac{1}{r} \sum_{i=1}^{r} (-1)^i \log f_i d\log f_1 \wedge \cdots \wedge \widehat{d\log f_i} \wedge \cdots \wedge d\log f_r \right), d\log f_1 \wedge \cdots \wedge \log f_r \right\}$$

commuting with residue homomorphisms. Here $\alpha_r = (-1)^{r-1} \cdot Re$ for odd r and $(-1)^{r-1}\mathrm{Im}$ for even. So we get a chain in (5.19).

The components of c_n in the bicomplex

$$R(G^n, n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} R\left(Gr^{2n-1}, n\right)$$

are constructed as a composition of the homomorphism of complexes

$$T : C_*\left(A^{m-n+1}(m)\right) \to BC_*(n)$$

with the $2n$-cocyle $\mathbf{L}_n'$ that lives on $BC_*(n)$. More precisely to construct a $R\left(G^k, n\right)$-component of c_n we have to restrict homomorphism T to elements $(a, g_1 a, \cdots, g_k a)$ where a is a given affine $(m-n+1)$-flag in V_m.

Theorem 5.11 *a) The constructed chain c_n is a cocyle in 5.18*

b) The cohomology class of c_n coincides with the usual Chern class in $H_D^{2n}(BGL_m(\mathbb{C}), R(n))$.

Proof: a) follows from the definition and previous results.

The proof of b) is in complete analogy to the one of the theorem 5.10 in [G2]. Let $\pi : EG_\bullet \to BG_\bullet$ be the universal G-bundle then $EP_{(p)} = BG_{(p+1)}$ and so any i-cochain $c_{(\bullet)}$ for $BG_\bullet$ defines an $(i-1)$-cochain $\tilde{c}_{(\bullet)}$ for $EG_\bullet$: $\tilde{c}_{(p)} := c_{(p+1)}$. Moreover, if $c_{(0)} = 0$ and $c_{(\bullet)}$ is a cocycle then $d\tilde{c}_{(\bullet)} = c_{(\bullet)}$. Therefore $c_{(1)} = \tilde{c}|_G$ is the transgression of the cocycle $c_{(\bullet)}$.

Applying this to the constructed cocycle c_n we get a cocycle c_n' in $H_D^{2n-1}(GL_m(\mathbb{C}), R(n))$. The usual exact sequence for Deligne cohomology gives us

$$\cdots \to H_D^{2n-1}(GL_m(\mathbb{C}), R(n)) \xrightarrow{\alpha} H^{2n-1}(GL_m(\mathbb{C}), R(n)) \cap$$
$$\cap H^{2n-1}\left(GL_m(\mathbb{C}), \Omega^{\geq n}\right)$$

It follows from definitions that $\alpha\left(c_n'\right)$ coincides with the class constructed in s.4 of § 4. It is nontrivial according to theorem 4.6. Theorem 5.11 is proved.

6. **Explicit formulas for measurable cocycles of** $GL(\mathbb{C})$. We will suppose that there exist a function P_n' on $\hat{G}_{n-1}^n$ satisfying $(2n+1)$-term relations (1.14). Recall that such a function can be considered as a function on configurations of $2n$ vectors in generic position in $\mathbb{C}^n$ satisfying the equation

$$\sum_{i=0}^{2n} (-1)^i P_n'\left(l_0, \cdots, \hat{l}_i, \cdots, l_{2n}\right) = 0 \qquad (5.20a)$$

$$\sum_{i=0}^{2n}(-1)^i P_n'\Big(l_i|l_0,\cdots,\hat{l}_i,\cdots,l_{2n}\Big)=0 \qquad (5.20b)$$

We will assume also that P_n' is a component of a $2n$-cocycle $\mathbf{L}_n'$ from conjecture 1.1'.

Theorem 5.12 *Let a be an affine $(m-n+1)$-flag in V_m . Then $P_n(T(g_0a,\cdots,g_{2n-1}a))$ is a $2n$-cocycle of $GL_m(\mathbb{C})$. Its cohomology class coincides with the Borel class in $H^{2n-1}_{(m)}(GL_m(\mathbb{C}),R)$ $(m\geq n)$.*

Recall that here $T: C_*\big(A^{m-n+1}(n)\big)\to BC_*(n)$ is a homomorphism of complexes. The cocycle condition follows just from this fact and $(2n+1)$-terms equations (5.20).

Let G^δ be the Lie group made discrete. The morphism of groups $GL_m(\mathbb{C})^\delta\to GL_m(\mathbb{C})$ provides a morphism

$$e: BGL_m(\mathbb{C})^\delta_\bullet\to BGL_m(\mathbb{C})_\bullet$$

Therefore

$$\begin{aligned}e^*: H^{2n}_{\mathcal{D}}(BGL_m(\mathbb{C})_\bullet,R(n))&\to H^{2n}_{\mathcal{D}}\Big(BGL_m(\mathbb{C})^\delta_\bullet,R(n)\Big)\\ = H^{2n-1}\big(BGL_m(\mathbb{C})_\bullet,S^0\big)&\equiv H^{2n-1}_{(m)}(GL_m(\mathbb{C}),R(n-1))\end{aligned}$$

Here S^0 is a sheaf of smooth functions. It is known that e^* maps the indecomposable class in $H^{2n}_{\mathcal{D}}(BGL_m(\mathbb{C}),\mathbf{Z}(n))$ just to the Borel class in $H^{2n-1}_{(m)}(GL_m(\mathbb{C}),R(n-1))$ (see [B2], [DMZ]. The arguments in proof of theorem 5.11 show that the constructed class $c_n\in H^{2n}_{\mathcal{D}}(BGL_m(\mathbb{C})_\bullet,R(n))$ lies in

$$\mathrm{Im}\ H^{2n}_{\mathcal{D}}(BGL_m(\mathbb{C})_\bullet,\mathbf{Z}(n))\to H^{2n}_{\mathcal{D}}(BGL_m(\mathbb{C})_\bullet,R(n))$$

and in fact coincides with the image of the standard class in $H^{2n}_{\mathcal{D}}(BGL_m(\mathbb{C})_\bullet,\mathbf{Z}(n))$. In our case $e^*(c_n)$ coincides with $P_n(T(g_0a,\cdots,g_{2n-1},a))$ just by definition. Theorem 5.12 is proved.

Remark 5.13 Explicit formulas for functions P_n are known for $n\leq 3$:

$$\begin{aligned}P_2(l_1,\cdots,l_4)&:=\mathcal{L}_2(r(l_1,\cdots,l_4))\\ P_3(l_1,\cdots,l_6)&:=\mathcal{L}_3(r_3(l_1,\cdots,l_6))\ .\end{aligned}$$

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Abstract

Let E be a vector bundle over an algebraic manifold X. An explicit local construction of characteristic classes $c_n(E)$ with values in Bigrassmannian cohomology that are defined in § 1 is given. In the special case $n = \dim E$ it reduces to the construction of $c_n(E)$ with values in the Grassmannian cohomology given in [BMS].

Our construction implies immediately an explicit construction of Chern classes with values in $H^n\left(X, \underline{\underline{K}}_n^M\right)$, where $\underline{\underline{K}}_n^M$ is the sheaf of Milnors K-groups.

A construction of classes $c_n(E)$ with values in motivic cohomology is given for $n \leq 3$. For $n = 2$ it could be considered as a motivic analog of the local combinatorial formula of Gabrielov, Gelfand and Losik for the first Pontryagin class ([GGL]). The reason for the restriction $n \leq 3$ is the absence of a good theory of n-logarithms for $n \geq 4$ today. Explicit constructions of the universal Chern classes $c_n \in H^n\left(BGL_{m^\bullet}, \underline{\underline{K}}_n^M\right)$ and for $n \leq 3$ $c_n \in H^{2n}_{\mathcal{M}}(BGL_{m^\bullet}, \mathbf{Z}(n))$ ($H^\bullet_{\mathcal{M}}$: motivic cohomology) are given.

§ 1 Introduction

1. **Chern classes with values in** $H^n\left(X,\underline{\underline{K}}_n^M\right)$. Let L be a line bundle over X. There is the following classical construction of $c_1(L) \in H^1(X,\mathcal{O}^*)$. Choose a Zariski covering $\{U_i\}$ of X such that $L|_{U_i}$ is trivial. Choose non-zero sections $s_i \in \Gamma(U_i,L)$. Then $s_i/s_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ satisfies the cocycle condition and hence define a cohomology class $c_1(L) \in H^1(X,\mathcal{O}^*)$.

Let us define the presheaf of Milnor's K-groups on X as follows: its section over an open set U is the quotient group of $\underbrace{\mathcal{O}^*(U)\otimes\cdots\otimes\mathcal{O}^*(U)}_{n \text{ times}}$ by the subgroup generated by elements

$$g_1\otimes\cdots\otimes g_k\otimes f\otimes(1-f)\otimes g_{k+3}\otimes\cdots\otimes g_n\,,\quad g_i,f,1-f\in\mathcal{O}^*(U)\,.$$

Let us denote by $\underline{\underline{K}}_n^M$ the sheaf associated with this presheaf. We will denote by $\{f_1,\cdots,f_n\}$ the image of $f_1\otimes\cdots\otimes f_n \in \mathcal{O}^*(U)^{\otimes n}$ in $\underline{\underline{K}}_n^M(U)$.

In § 3 for any vector bundle E over X an explicit construction of the Chern classes $c_n(E) \in H^n\left(X,\underline{\underline{K}}_n^M\right)$ will be given.

The construction of $c_n(E^n)$ for an n-dimensional vector bundle E^n follows from [S1] and [BMS], ch. 1. More precisely, let U_i be a Zariski covering such that $E^n|_{U_i}$ is trivial. Choose a section $s_i \in \Gamma(U_i,E^n)$ such that $s_{i_1}(x),\cdots,s_{i_{n+1}}(x)$ are in generic position on $U_{i_1\cdots i_{n+1}} := U_{i_1}\cap\cdots\cap U_{i_{n+1}}$. Then $s_{i_{n+1}}(x) = \sum_{k=1}^{n} a_{i_k}(x)\cdot s_{i_k}(x)$ and

$$\{a_{i_1}(x),\cdots,a_{i_n}(x)\}\in K_n^M(U_{i_1\cdots i_{n+1}})$$

is a cocycle in the Čech complex.

I will generalize this construction to vector bundles of arbitrary dimension and show that for $c_1(E)$ it gives exactly the described above cocycle for $c_1(\det E)$.

2. **Applications.** There is a canonical map of sheaves

$$\begin{aligned}&\underline{\underline{K}}_n^M \to \Omega^n_{\log} \hookrightarrow \Omega^n_{cl} \hookrightarrow \Omega^n\\ &\{f_1,\cdots,f_n\}\mapsto d\log f_1\wedge\cdots\wedge d\log f_n\end{aligned}$$

Here $\Omega^n_{\log}$ (respectively Ω^n_{cl}) is the sheaf of n-forms with logarithmic singularities at infinity (respectively closed n-forms). Therefore we get a construction of characteristic classes with values in $H^n\left(X,\Omega^n_{\log}\right)$ and $H^n\left(X,\Omega^n_{cl}\right)$. Note that the Atiyah's construction provides us characteristic classes in $H^n(X,\Omega^n)$ ([A], see also [Har]).

3. **The Grassmannian bicomplex and Bigrassmannian cohomology** (see [G1], [G2], compare with [GGL] and [BMS]). Let Y be a set and $\tilde{C}_n(Y)$ be a free abelian group generated by elements $(y_0,\cdots,y_n)$ of $Y^{n+1} := \underbrace{Y\times\cdots\times Y}_{n+1}$. There is a complex $\left(\tilde{C}_*(Y),d\right)$ where

$$d(y_0,\cdots,y_n) := \sum_{i=0}^{n}(-1)^i(y_0,\cdots,\hat{y}_i,\cdots,y_n) \tag{1.1}$$

This is just the simplicial complex of the simplex whose vertices are labeled by elements of Y. Suppose that a group G acts on Y. Let us call elements of the quotient set $G \setminus Y^{n+1}$ by **configurations** of elements of Y. Denote by $C_n(Y)$ a free abelian group generated by configurations of $(n+1)$ elements of Y. There is a complex $(C_*(Y), d)$, where d is defined by the same formula (1.1) and $C_*(Y) = \tilde{C}_*(Y)_G$. We will also apply this construction to subsets of $G \setminus Y^{n+1}$ of "configurations in generic position".

Now let us denote by $C_n(m)$ a free abelian group generated by configurations of $n+1$ vectors in generic position in an m-dimensional vector space V^m over F (i.e. any m vectors of the configuration are linearly independent). In this case there is another map:

$$d' : C_n(m) \to C_{n-1}(m-1)$$

$$d' : (v_0, \cdots, v_n) \mapsto \sum_{i=0}^{n} (-1)^i (v_0 | v_0, \cdots, \hat{v}_i, \cdots, v_n)$$

Here $(v_i | v_0, \cdots, \hat{v}_i, \cdots, v_n)$ is a configuration of vectors in $V^m/\langle v_i \rangle$ obtained by projection of vectors $v_j \in V^m$, $j \neq i$. Then there is the following bicomplex

$$\begin{array}{ccccccc} & \cdots & & \cdots & & \cdots \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & C_{n+4}(n+2) & \xrightarrow{d} & C_{n+3}(n+2) & \xrightarrow{d} & C_{n+2}(n+2) \\ & \downarrow d' & & \downarrow d' & & \downarrow d' \\ \cdots \to & C_{n+3}(n+1) & \xrightarrow{d} & C_{n+2}(n+1) & \xrightarrow{d} & C_{n+1}(n+1) \\ & \downarrow d' & & \downarrow d' & & \downarrow d' \\ \cdots \to & C_{n+2}(n) & \xrightarrow{d} & C_{n+1}(n) & \xrightarrow{d} & C_n(n) \end{array} \quad (1.2)$$

We will call it the **Grassmannian bicomplex** (over $X = \mathrm{Spec}F$).

There is a subcomplex $(C_*(n), d)$

$$\to C_{n+2}(n) \xrightarrow{d} C_{n+1}(n) \xrightarrow{d} C_n(n) \quad (1.3)$$

of the bicomplex (1.2). This is the **Grassmannian complex** introduced in [S2], [BMS], see also [Q2].

Let us denote by $(BC_*(n), \partial)$ the total complex associated with the bicomplex (1.2): $BC_n(n) := C_n(n)$. We will suppose that $BC_n(n)$ placed in degree n and ∂ has degree $+1$.

Now let us give a more geometrical interpretation of the Grassmannian bicomplex that also explains the name.

Let $(e_1, \cdots, e_{p+q+1})$ be a coordinate frame in a vector space V. Let us denote by $\hat{\mathrm{G}}^p_q$ the open subset of the Grassmannian of q-dimensional subspaces of $\mathbf{P}^{p+q}$ which are in transverse to the coordinate hyperplanes. R. MacPherson constructed in [M] an isomorphism

$$m : \hat{\mathrm{G}}^p_q \xrightarrow{\sim} \left\{ \begin{array}{c} \text{configurations of } p+q+1 \text{ vectors in generic} \\ \text{position in a } p\text{-dimensional vector space} \end{array} \right\} \quad (1.4)$$

Namely, $m(\xi)$ is a configuration formed by images of e_i in V/ξ .

Let

$$\mathbf{Z} : Var \rightarrow Ab \tag{1.5}$$

be a functor from the category of algebraic varieties over F to the one of abelian groups that sends a variety X to the free abelian group generated by F-points of X. Applying it to (1.4) we get an isomorphism

$$\mathbf{Z}\left[\hat{\mathrm{G}}^p_q\right] \xrightarrow{\sim} C_{p+q}(p) \tag{1.6}$$

For each integer i such that $0 \leq i \leq p+q$, there are intersection maps a_i and projection maps b_i :

$$\begin{array}{ccc} \hat{\mathrm{G}}^p_q & \overset{a_i}{\rightarrow} & \hat{\mathrm{G}}^p_{q-1} \\ \downarrow b_i & & \\ \hat{\mathrm{G}}^{p-1}_q & & \end{array} \tag{1.4}$$

Here the subspace $a_i(\xi)$ is the intersection of ξ with the $i-th$ coordinate hyperplane and the subspace $b_i(\xi)$ is the projection of ξ on the $i-th$ hyperplane by the projection with the center at $i-th$ vertex of the simplex. We get a Bigrassmannian $\hat{\mathrm{G}}(n)$:

$$\hat{\mathrm{G}}(n): \quad \begin{array}{cccccc} & & & & & \downdownarrows \\ & & & & \rightrightarrows & \hat{\mathrm{G}}^{n+2}_0 \\ & & & \downdownarrows & & b_0 \downdownarrows b_{n+1} \\ & & \rightrightarrows & \hat{\mathrm{G}}^{n+1}_1 & \underset{a_{n+1}}{\overset{a_0}{\rightrightarrows}} & \hat{\mathrm{G}}^{n+1}_0 \\ & \downdownarrows & & b_0 \downdownarrows b_{n+1} & & b_0 \downdownarrows b_n \\ \rightrightarrows & \hat{\mathrm{G}}^n_2 & \underset{a_{n+1}}{\overset{a_0}{\rightrightarrows}} & \hat{\mathrm{G}}^n_1 & \underset{a_n}{\overset{a_0}{\rightrightarrows}} & \hat{\mathrm{G}}^n_0 \end{array} \tag{1.7}$$

Applying functor (1.5) to it, considering differentials $d = \Sigma(-1)^i a_i$ and $d' = \Sigma(-1)^i b_i$ and using isomorphism 1.6 we get the Grassmannian bicomplex.

Now let us sheafefy these constructions.

A bicomplex of sheaves on X called the Grassmannian bicomplex $\underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}(n)\right]$ is constructed as follows: For a point $x \in X$, the stalk of $\underline{\underline{\mathbf{Z}}}\left[\hat{\mathrm{G}}(n)\right]$ at x is the formal linear combinations of germs at x of maps from X to $\hat{\mathrm{G}}^p_q$. The corresponding bicomplex looks as follows should replace the complex $\left(\Omega_X^{\geq n}, \partial\right)$ in (1.13) by its Dolbeaux resolution $\left(\mathcal{D}_X^{\geq n,q}\right)$ for example), but it is not important for our purposes.

Conjecture 1.1 ' *There exists a* $2n$*-cocycle* $\mathbf{L}_n^{'}$ *in the triple complex* $\mathbf{D}'$ *such that its component over* $\hat{\mathrm{G}}_0^n$ *is given by the following formulas:*

$$\begin{aligned} \omega_0^{n'} &= \alpha_n\left(\frac{1}{n}\sum_{i=1}^{n}(-1)^i \log z_i d\log z_1 \wedge \cdots \wedge d\log z_i \wedge \cdots \wedge d\log z_n\right) \in S_{\hat{\mathrm{G}}_0^n}^{n-1} \\ \omega_0^{n''} &= d\log z_1 \wedge \cdots \wedge d\log z_n \in \Omega_X^n \\ &\left(d\omega_0^{n'} + \alpha_n\left(\omega_0^{n''}\right) = 0\right) \end{aligned} \tag{1.14}$$

The corresponding component $P_n^{'}$ *of* $\mathbf{L}_n^{'}$ *on* $\hat{\mathrm{G}}_{n-1}^n$ *should satisfy the "clean"* $(2n+1)$*-term equations*

$$\sum_{i=0}^{2n}(-1)^i a_i^* P_n^{'} = 0 \tag{1.14a}$$

$$\sum_{i=0}^{2n}(-1)^i b_i^* P_n^{'} = 0 \tag{1.14b}$$

From the other hand there are the **classical** polylogarithms $\mathrm{Li}_n(z)$ that are functions of **one** complex variable z. They were defined by Joh. Bernoulli and L. Euler on the unit disc $|z| \leq 1$ by absolutely convergent series

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n},$$

and can be continued analytically to a multivalued function on $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ using the inductive formulas

$$\mathrm{Li}_1(z) = -\log(1-z)$$

$$\mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(t)\frac{dt}{t}$$

It turns out that $\mathrm{Li}_n(z)$ has a remarkable single-valued version ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6, \cdots$ are Bernoulli numbers) ([Z])

$$\mathcal{L}_n(z) = \begin{matrix}\mathrm{Re}(n: odd) \\ \mathrm{Im}(n: even)\end{matrix}\left(\sum_{k=0}^{n}\frac{B_k \cdot 2^k}{k!}\log^k|z| \cdot \mathrm{Li}_{n-k}(z)\right), \quad n \geq 2$$

$$\mathcal{L}_1(z) = \log|z|$$

For example

$$\mathcal{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\cdot\log|z|$$

is the Bloch-Wigner function, and

$$\mathcal{L}_3(z) = \mathrm{Re}\left(\mathrm{Li}_3(z) - \log|z|\cdot\mathrm{Li}_2(z) + \frac{1}{3}\log^2|z|\cdot\mathrm{Li}_1|z|\right)$$

§ 2 Affine flags and Chern classes in Bigrassmannian cohomology

1. **Affine flags.** Let V be a vector space over a field F. By definition a p**-flag** in V is a sequence of subspaces

$$0 \subset L^1 \subset L^2 \subset \cdots \subset L^p , \quad \dim L^i = i ,$$

An **affine** p**-flag** $L^\bullet$ is a p-flag together with choice of vectors $l^i \in L^i/L^{i-1}$, $i = 1, \cdots, p$ $\left(L^0 = 0\right)$. We will denote affine p-flags as $\left(l^1, \cdots, l^p\right)$. Subspaces L^i can be recovered as the ones generated by $l^1, \cdots, l^i$: $L^i = \left\langle l^1, \cdots, l^i \right\rangle$. We will say that an $(n+1)$-tuple of affine flags

$$L_0^\bullet = \left(l_0^1, \cdots, l_0^p\right), \cdots, L_n^\bullet = \left(l_n^1, \cdots, l_n^p\right) \tag{2.1}$$

are in generic position if

$$\dim \left(L_0^{i_0} + \cdots + L_n^{i_n}\right) = i_0 + \cdots + i_n \quad \text{whenever} \quad i_0 + \cdots + i_n \leq \dim V . \tag{2.2}$$

Let $A^p(m)$ be the manifold of all affine p-flags in an m-dimensional vector space V_m . It is a $GL(V_m)$-set, so as usual (see 5.3 of the Introduction) one can consider free abelian groups $C_n(A^p(m))$ of configurations of $(n+1)$-tuples of affine p-flags in generic position in V_m . Further, there is a complex of affine p-flags $C_*(A^p(m))$:

$$\begin{gathered} \cdots \xrightarrow{d} C_{n+1}(A^p(m)) \xrightarrow{d} C_n(A^p(m)) \xrightarrow{d} C_{n-1}(A^p(m)) \xrightarrow{d} \cdots \\ d : (L_0^\bullet, \cdots, L_n^\bullet) \mapsto \sum_{i=0}^{n} (-1)^i \left(L_0^\bullet, \cdots, \widehat{L_i^\bullet}, \cdots, L_n^\bullet\right) \end{gathered} \tag{2.3}$$

In particularly $C_*\left(A^1(m)\right) \equiv C_*(m)$. Let us define a map of complexes

$$T : C_*\left(A^{p+1}(n+p)\right) \to BC_*(n) \tag{2.4}$$

as follows: for

$$a_k^{p+1} = \left(v_0^1, \cdots, v_0^{p+1}; \cdots; v_k^1, \cdots, v_k^{p+1}\right) \in C_k\left(A^{p+1}(n+p)\right) \quad (k \geq n)$$

set

$$\begin{aligned} T\left(a_k^{p+1}\right) &:= \bigoplus_{q=0}^{k-n} \sum_{\substack{i_0+\cdots+i_k=p-q \\ i_k \geq 0}} \left(L_0^{i_0} \oplus \cdots \oplus L_m^{i_m} | v_0^{i_0+1}, \cdots, v_k^{i_k+1}\right) \in \\ &\in \bigoplus_{q=0}^{k-n} C_k(n+q) =: BC_k(n) \end{aligned} \tag{2.5}$$

<u>**Key lemma 2.1**</u> *T is a homomorphism of complexes.*

Proof: Let $T_k(n+q) : C_k\left(A^{p+1}(n+p)\right) \to C_k(n+q)$ be the $C_k(n+q)$-component of the map P. We have to prove that (see 2.6)

$$d \circ T_k(n+q) = T_{k-1}(n+q) - d' \circ T_k(n+q+1)$$

$$
\begin{array}{ccc}
a_k^{p+1} \in C_k\big(A^{p+1}(n+q)\big) & \rightarrow & C_k(n+q+1) \\
\downarrow & \searrow & \downarrow \\
C_k(n+q) & \rightarrow & C_{k-1}(n+q)
\end{array}
\tag{2.6}
$$

For a given partition $i_0 + \cdots + i_k = p - q$ let us consider the expression

$$
\begin{aligned}
& d\Big(L_0^{i_0} \oplus \cdots \oplus L_k^{i_k} | v_0^{i_0+1}, \cdots, v_k^{i_k+1}\Big) = \\
& = \sum_{j=0}^{k} (-1)^j \left(L_0^{i_0} \oplus \cdots \oplus L_k^{i_k} | v_0^{i_0+1}, \cdots, \widehat{v_j^{i_j+1}}, \cdots, v_k^{i_k+1} \right)
\end{aligned}
\tag{2.7}
$$

If $i_j = 1$ then the corresponding term in 2.6 will appear in formula for $T_{k-1}(n+q)\Big(a_k^{p+1}\Big)$. In the case $i_j > 1$ such term will be in formula for

$$
d'\left(L_0^{i_0} \oplus \cdots \oplus L_j^{i_j - 1}{}' \oplus \cdots \oplus L_k^{i_k} | v_0^{i_0+1}, \cdots, v_j^{i_j}, \cdots, v_k^{i_k+1}\right) .
$$

□

2. **A construction of Chern classes in Bigrassmannian cohomology.** Let us denote by $\mathcal{A}_E^p(X)$ the bundle of affine p-flags in fibers of a vector bundle E over X. Choose a Zariski covering $\{U_i\}$ of X such that E/U_i is trivial. Choose sections

$$
L_i^\bullet(x) \in \Gamma\big(U_i, \mathcal{A}_E^p(x)\big)
$$

such that for any $i_0 < \cdots < i_n$ affine p-flags $L_{i_0}^\bullet(x), \cdots, L_{i_n}^\bullet(x)$ are in generic position for every $x \in U_{i_0, \cdots, i_n}$.

Theorem 2.2 $T\big(L_{i_0}^\bullet(x), \cdots, L_{i_n}^\bullet(x)\big) \in \underline{\underline{\mathbf{Z}}}\Big[\hat{G}(n)\Big](U_{i_0 \cdots i_n})$ *is a cocycle in the Čech complex for the covering* $\{U_i\}$ *with values in the Bigrassmannian complex.*

Proof: Follows immediately from the Key lemma 2.1. □

A different choice of sections $L_i^\bullet(x)$ gives a cocycle that is canonically cohomologous to the previous one. So the cohomology class $c_n(E)$ of this cocycle is well-defined.

Proof: (Compare with proof of lemma 3.4 in [G1])

$$f_n(n) \circ d'(v_0, \cdots, v_{n+1}) = \mathrm{Alt} \bigwedge_{j=2}^{n+1} \Delta(v_0, v_1, \cdots, \widehat{v_j}, \cdots, v_{n+1}) = 0$$

because $\Delta(v_0, v_1, \cdots, \widehat{v_j}, \cdots, v_{n+1})$ is invariant under the switch of v_0 and v_1 modulo 2-torsion.

□

<u>**Proposition 3.3**</u> *The composition*

$$C_{n+1}(n) \xrightarrow{d} C_n(n) \xrightarrow{\overline{f}_n(n)} K_n^M(F)$$

is equal to zero.

Proof: (Compare with proof of proposition 2.4 in [S1]). There is a duality $* : C_{m+n-1}(m) \to C_{m+n-1}(n)$, $*^2 = id$ that satisfies the following properties (see s.8 of § 3 in [G2]).

1. $*$ commutes with the action of the permutation group S_{m+n} .
2. If $*(l_1, \cdots, l_{m+n}) = \left(l_1', \cdots, l_{m+n}'\right)$ then

$$*\left(l_1, \cdots, \widehat{l_i}, \cdots, l_{m+n}\right) = \left(l_i' | l_1', \cdots, \widehat{l_i}, \cdots, \widehat{l}_{m+n}\right)$$

3. Choose volume forms in V_m and V_n ; consider partition

$$\{1, \cdots, m+n\} = \{i_1 < \cdots < i_m\} \cup \{j_1 < \cdots < j_n\}$$

Then $\frac{\Delta\left(l_{i_1}, \cdots, l_{i_m}\right)}{\Delta\left(l_{j_1}, \cdots, l_{j_n}\right)}$ does not depend on a partition.

This duality can be defined as follows. A configuration of $(m+n)$ vectors in an m-dimensional coordinate vector space can be represented as columns of the $m \times (m+n)$ matrix (I_m, A) . The dual configuration is represented by $n \times (m+n)$ matrix $\left(-A^t, I_n\right)$. Using the duality we can reformulate proposition 3.3 as follows: the composition

$$C_{n+1}(2) \xrightarrow{d'} C_n(1) \xrightarrow{\tilde{f}_n(n)} K_n^M(F)$$

is equal to 0. Here

$$\tilde{f}_n(n)(v_0, \cdots, v_n) := \mathrm{Alt}\Delta(v_0) \wedge \Delta(v_1) \wedge \cdots \wedge \Delta(v_{n-1}) \in \Lambda^n F^*$$

Consider the following diagram

$$\begin{array}{ccc} C_{n+1}(2) & \xrightarrow{d'} & C_n(1) \\ \downarrow \tilde{f}_{n+1}(n) & & \downarrow \tilde{f}_n(n) \\ \mathbf{Z}[P_F^1 \setminus \{0,1,\infty\}] \otimes \Lambda^{n-2} F^* & \xrightarrow{\delta} & \Lambda^n F^* \end{array}$$

Here $\mathbf{Z}[P_F^1 \setminus \{0,1,\infty\}]$ is a free abelian group generated by symbols $\{x\}$ where $x \in P_F^1 \setminus \{0,1,\infty\}$, $\delta : \{x\} \otimes y_1 \wedge \cdots \wedge y_{n-2} \mapsto (1-x) \wedge x \wedge y_1 \wedge \cdots \wedge y_{n-2}$. Note that by definition $\mathrm{Coker}\delta = K_n^M(F)$. The homomorphism $\tilde{f}_{n+1}(n)$ is defined as follows:

$$\tilde{f}_{n+1}(n)(v_0, \cdots, v_{n+1}) := n![v_0, \cdots, v_{n+1}]$$

Lemma-construction 3.7 *For 2 affine m-flags in generic position in V^m :*

$$L_1^\bullet = (v_1, \cdots, v_m) \quad \text{and} \quad L_2^\bullet = (w_1, \cdots, w_m)$$

there are just 2 frames associated with both of them.

Proof: We have the following isomorphisms of 1-dimensional vector spaces:

$$s_1 : L_1^k / L_1^{k-1} \xrightarrow{\sim} L_1^k \cap L_2^{m-k+1}$$
$$s_2 : L_2^{m-k+1} / L_2^{m-k} \xrightarrow{\sim} L_1^k \cap L_2^{m-k+1}$$

Put $f_1^k := s_1(v_k)$, $f_2^{m-k+1} := s_2(w_{m-k+1})$. Then the frames $(f_1^1; \cdots ; f_1^m)$ and $(f_2^1; \cdots ; f_2^m)$ associated with both $L_1^\bullet$ and $L_2^\bullet$.

□

Let $f_1^k = \lambda_k \cdot f_2^k$, $\lambda_k \in F^*$, and

$$(v_1; \cdots ; v_m) = g \cdot (w_1, \cdots , w_m) , \quad q \in \mathrm{GL}_m(F) .$$

Then $\det g = \prod_{k=1}^{m} \lambda_k$ because $g = n_+ \cdot \lambda \cdot n_-$:

$$(\omega_i) \xrightarrow{n_-} \left(f_2^k\right) \xrightarrow{\lambda=(\lambda_k)} \left(f_1^k\right) \xrightarrow{n_+} (v_j)$$

where $n_-(n_+)$ is a lower (upper) triangular matrix and λ is a diagonal one with entries λ_k (the Gauss decomposition).

From the other hand the left-hand side in proposition 2.4 is equal to

$$f_1(1)\left(\sum_{k=1}^{m} \left(L_1^k \oplus L_2^{m-k} | l_1^k, l_2^{m-k+1}\right)\right) = f_1(1)\left(f_1^k, f_2^k\right) = \prod_{k=1}^{m} \lambda_k .$$

□

§ 4 The universal Chern class $c_n \in H^n\left(\mathrm{BGL}(m)_{\cdot}, \underline{\underline{K}}_n^M\right)$

1. **The Gersten resolution to Milnor's K-theory ([Ka]).** Let F be a field with a discrete valuation v and the residue class $\overline{F}_v (= \overline{F})$. The group of units U has a natural homomorphism $U \to \overline{F}^*$, $u \mapsto \overline{u}$. An element $\pi \in F^*$ is prime if $\mathrm{ord}_v(\pi) = 1$. There is a canonical homomorphism (see [M1]):

$$\partial : K_{n+1}^M(F) \to K_n^M\left(\overline{F}_v\right) \qquad (n \geq 0)$$

uniquely defined by properties $(u_i \in U)$

$$\begin{aligned} &1. \quad \partial(\{\pi, u_1, \cdots, u_n\}) = \{\overline{u}_1, \cdots, \overline{u}_n\} \\ &2. \quad \partial(\{u_1, \cdots, u_{n+1}\}) = 0 \end{aligned}$$

Let X be an excellent scheme (EGA [3] IV § 7), $X_{(i)}$ the set of all codimension i points x, $F(x)$ the field of functions corresponding to a point $x \in X_{(i)}$.
There is a sequence of group $\mathcal{K}(n)$. (Here $K_n^M(x) := K_n^M(F(x))$):

$$K_n^M(F(X)) \xrightarrow{\partial} \bigoplus_{x \in X_{(1)}} K_{n-1}^M(x) \xrightarrow{\partial} \bigoplus_{x \in X_{(2)}} K_{n-2}^M(x) \to \cdots \to \bigoplus_{x \in X_{(r)}} \mathbb{Z} \qquad (4.1)$$

We will follow [Ka] in the definition of ∂. Let us define for $y \in X_{(i)}$ and $x \in X_{i+1}$ a homomorphism

$$\partial_x^y : K_{*+1}^M(y) \to K_*^M(x)$$

as follows. Let Y be the normalisation of the reduced scheme $\{\overline{y}\}$. Set

$$\partial_x^y : \sum_{x'} N_{F(x')/F(x)} \circ \partial_{x'}$$

where x' ranges over all points of Y lying over x, $\partial_{x'} : K_{*+1}^M(y) \to K_*(x)$ is the tame symbol associated with the discrete valuation ring $\mathcal{O}_{Y,x'}$ and $N_{F(x')/F(x)}$ is the norm map $K_*^M(x') \to K_*^M(x)$ (see [BT], ch. I § 5 and [Ka], § 1.7). The coboundary ∂ is by definition the sum of these homomorphism ∂_x^y.

<u>**Proposition 4.1**</u> $\partial^2 = 0$.

Proof: See proof of proposition 1 in [Ka].

<u>**Theorem 4.2**</u> *The complex $\mathcal{K}(n)_\bullet$ is exact.*

□

2. **Explicit formula for a class $c \in H^n\left(\mathrm{BGL}(m)_\bullet, \underline{\underline{K}}_n^M\right)$.** Set $G := \underbrace{G \times \cdots \times G}_{n \text{ times}}$.
Recall that

$$BG_\bullet := pt \overset{s_0}{\underset{s_1}{\leftleftarrows}} G \overset{s_0}{\underset{s_2}{\leftleftarrows}} G^2 \cdots$$

is the symplicial scheme representing the classifying space for a group G. We will compute $H^n\left(BG_\bullet, \underline{\underline{K}}_n^M\right)$ using the Gersten resolution (4.1). So cochain we have to construct lives

Set

$$\tilde{\omega}_r := \sum_{j_0+\cdots+j_r=m-n} \tilde{\omega}_{j_0,\cdots,j_r} \in \bigoplus_{j_0+\cdots+j_r=m-n} K_r^M\Big(F\Big(\tilde{D}_{j_0,\cdots,j_r}\Big)\Big)$$

$$\omega_r := \sum_{j_0+\cdots+j_r=m-n} \omega_{j_0,\cdots,j_r;a} \in \bigoplus_{j_0+\cdots+j_r=m-n} K_r^M(F(D_{j_0,\cdots,j_r;a}))$$

Theorem 4.3 *Collection of elements* ω_r *defines a cocycle in the bicomplex (4.2).*

Proof: Choose a partition $i_0+\cdots+i_r=m-r$. Let $\tilde{\mathcal{E}}$ be a subvariety in the manifold of $(r+1)$-tuples of affine $(m-r+1)$-flags in V^m defined as follows:

$$\tilde{\mathcal{E}}_{i_0,\cdots i_r} := \left\{(L_0^\bullet,\cdots,L_r^\bullet)|\dim\left(\bigoplus_{p=0}^{r} L_p^{i_p}\right) = \left(\sum_{p=0}^{r} i_p\right) - 1\right\}$$

This is a codimension $n-r+1$ irreducible subvariety.

Proposition 4.4 *The component of* $\partial\tilde{\omega}_r$ *on* $\tilde{\mathcal{E}}_{i_0,\cdots,i_r}$ *is non zero if* $i_k=0$ *for some* k *but* $i_p>0$ *for* $p\neq k$ *. In this case it is equal to*

$$\overline{f}_r(r)\left(\bigoplus_{p\neq k} L_p^{i_p-1}|l_0^{i_0},\cdots,\widehat{l_k^{i_k}},\cdots,l_r^{i_r}\right) \tag{4.6}$$

Proof: Let $j_0+\cdots+j_r=m-n$ and

$$(l_0^1,\cdots,l_0^{m-n+1};\cdots;l_r^1,\cdots,l_r^{m-n+1}) \equiv (L_0^\bullet,\cdots,L_r^\bullet) \in \tilde{D}_{j_0,\cdots,j_r}$$

Choose a volume form in the codimension n-subspace $\left\langle l_0^1,\cdots,l_0^{j_0+1},\cdots,l_r^1,\cdots,l_r^{j_r+1}\right\rangle$. Then we can compute the determinant $\Delta(v_1,\cdots,v_{m-n+r})$ for any $m-n+r$ vectors in this subspace. Set

$$\Delta(j_{k+1}) := \Delta\left(l_0^1,\cdots,l_0^{j_0+1},\cdots,\widehat{l_k^{j_{k+1}}},\cdots,l_r^1,\cdots,l_r^{j_r+1}\right)$$

Then by definition

$$\tilde{\omega}_{j_0,\cdots,j_r} = \sum_{k=0}^{r}(-1)^k\Big\{\Delta(j_0+1),\cdots,\Delta(\widehat{j_k+1}),\cdots,\Delta(j_r+1)\Big\} \tag{4.7}$$

The coboundary $\partial\tilde{\omega}_{j_0,\cdots,j_r}$ can be nonzero on divisors $\Delta(j_{k+1})=0$ in $\tilde{D}_{j_0,\cdots,j_r}$ only. The component of $\partial\tilde{\omega}_{j_0,\cdots,j_r}$ on the divisor $\Delta(j_{k+1})=0$ is equal to

$$s\left(\bigoplus_{p=0}^{r} L_p^{j_p} \oplus l_k^{j_{k+1}}|l_0^{j_0+1},\cdots,\widehat{l_k^{j_{k+1}}},\cdots,l_r^{j_r+1}\right) \tag{4.8}$$

This formula implies immediately that the component of $\partial\tilde{\omega}_r$ on $\tilde{\mathcal{E}}_{i_0,\cdots i_r}$ is zero if $i_{k_1}=i_{k_2}=0$ for some $k_1\neq k_2$.

It follows from (4.8) that in the case $i_p>0$ for all p the component of $\partial\tilde{\omega}_r$ on $\tilde{\mathcal{E}}_{i_0,\cdots,i_r}$ is

$$f_r(r)\left(\sum_{k=0}^{r}(-1)^k\left(\bigoplus_{p=0}^{r} L_p^{i_p-1}+l_k^{i_k}|l_0^{i_0},\cdots,\widehat{l_k^{i_k}},\cdots,l_r^{i_r}\right)\right) \tag{4.9}$$

Note that $\left(\bigoplus_{p=0}^{r} L^{i_p-1}|l_0^{i_0},\cdots,l_r^{i_r}\right)$ is a configuration of $m+1$ vectors in an m-dimensional space (4.9) is equal to

$$f_r(r)\circ d'\left(\bigoplus_{p=0}^{r} L_p^{i_p-1}|l_0^{i_0},\cdots,l_r^{i_r}\right)$$

But this is equal to zero according to lemma 3.2.

Now suppose that $i_k=0$, $i_p\neq 0$ for $p\neq k$. Then (4.8) implies that the component of $\partial(\tilde{\omega}_r)$ on $\tilde{\mathcal{E}}_{i_0,\cdots,i_r}$ is exactly (4.6).

□

3. **Relation to the classical construction of Chern cycles.** Suppose that a vector bundle E ver X s sufficiently many sections. Consider first of all the case when $\dim E=n$ and we are interested in $c_n(E)\in CH^n(X)$. Choose a section $s_0(x)\in\Gamma(X,E)$ that is transversal to the zero section of E. Then the subvariety

$$D_0 := \{x\in X|s_0(x)=0\}$$

has codimension n and represents the class $c_n(E)\in CH^n(X)$. Now let $s_1(x)$ be another generic section of E (i.e. it is transversal to the zero section of E too). Then

$$D_1 := \{x\in X|s_1(x)=0\}$$

should represent the same class in $CH^n(X)$. To see this let us consider a codimension $(n-1)$ subvariety

$$D_{01} := \{x\in X|\exists\lambda_0,\lambda_1\in\mathbb{C}\quad\text{such that}\quad\lambda_0 s_0(x)+\lambda_1 s_1(x)=0\}$$

There is a canonical rational function

$$\lambda_{01} := \frac{\lambda_0}{\lambda_1}\in F(D_{01})\quad\text{and}\quad\mathrm{Div}(\lambda_{01})=D_0-D_1$$

So D_0 and D_1 are canonically rationally equivalent cycles. Now let $s_2(x)$ be the third generic section of E. Put

$$D_{012}=\{x\in X|\dim\langle s_0(x),s_1(x),s_2(x)\rangle=2\}$$

Then $\operatorname{codim} D_{012}=n-2$ and there is a canonical element

$$\lambda_{012} := f_2(2)(s_0,s_1,s_2)\in K_2(F(D_{012}))$$
$$\partial(\lambda_{012})=\lambda_{01}-\lambda_{02}+\lambda_{12}$$

where $\partial : K_2(F(Y))\to\coprod_{y\in Y_{(1)}}F(y)^*$ is the tame symbol. Continuing this process we get for $r+1$ generic sections $s_0(x),\cdots,s_r(x)$ of E a codimension $(n-r)$ subvariety

$$D_{01\cdots r} := \{x\in X|\dim\langle s_0(x),\cdots,s_r(x)\rangle=r\}$$

Proof: See § 3 of [G2] or theorem 5. below.

In complexes (5.2) groups $B_n(F)$ placed in degree 1 and δ_n has degree $+1$.

The complex $B_F(2)$ is the well-known Bloch-Suslin complex.

2. **The motivic complexes $\Gamma(X;n)$ for a regular scheme X $(n \leq 3)$** . Let F be a field with a discrete valuation v and the residue class $\overline{F}_v$. Let us construct a canonical homomorphism of complexes

$$\partial_v : B_F(n) \to B_{\overline{F}_v}(n-1)[-1]$$

There is a homomorphism $\theta \; : \; \Lambda^n F^* \; \to \; \Lambda^{n-1}\overline{F}_v^*$ uniquely defined by the following properties ($u_i \in U$, $u \mapsto \overline{u}$ is the natural homomorphism $U \to \overline{F}_v^*$ and π is a prime: $\mathrm{ord}_v \pi = 1$) :

$$\begin{array}{ll} 1. & \theta(\pi \wedge u_1 \wedge \cdots \wedge u_{n-1}) = \overline{u}_1 \wedge \cdots \wedge \overline{u}_{n-1} \\ 2. & \theta(u_1 \wedge \cdots \wedge u_n) = 0 \end{array}$$

It clearly does not depend on the choice of π .

Let us define a homomorphism $s_v : \mathbf{Z}[P^1_F] \to \mathbf{Z}\left[P^1_{\overline{F}_v}\right]$ as follows

$$s_v\{x\} = \begin{cases} \{\overline{x}\} & \text{if } x \text{ is a unit} \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

<u>**Proposition 5.2**</u> *Homomorphism (5.4) induces a homomorphism*

$$s_v : B_n(F) \to B_n(\overline{F}_v) \ , \quad n = 2,3 \ .$$

Proof: Straightforward but tedious computations using explicit formula (3.17) from [G3] for generators of the subgroup $R_3(F)$.

To avoid such computations one can consider subgroups $\mathcal{R}_n(F) \subset \mathbf{Z}[P^1_F]$ defined in s. 4 of § 1 in [G3]. Then more or less by definition $s_v(\mathcal{R}_n(F)) = \mathcal{R}_n(\overline{F}_v)$ and $\delta(R_n(F)) = 0$. So there are corresponding groups $\mathcal{B}_n(F) := \frac{\mathbf{Z}[P^1_F]}{\mathcal{R}_n(F)}$ together with homomorphisms s_v : $\mathcal{B}_n(F) \to \mathcal{B}_n(\overline{F}_v)$.

□

Set

$$\partial_v := s_v \otimes \theta : B_k(F) \otimes \Lambda^{n-k} F^* \to B_k(\overline{F}_v) \otimes \Lambda^{n-k-1}\overline{F}_v^* \tag{5.5}$$

<u>**Lemma 5.3**</u> *The homomorphism ∂_v commutes with the coboundary δ and hence defines a homomorphism of complexes (5.3).*

Proof: Straightforward computation. See also s. 14 of § 1 in [G2] where the corresponding fact proved for groups $\mathcal{B}_n(F)$.

□

Now let X be an arbitrary regular scheme, $X_{(i)}$ the set of all codimension i points of X, $F(x)$ the field of functions corresponding to a point $x \in X_{(i)}$. We define the motivic complexes $\Gamma(X,n)$ as the total complexes associated with the following bicomplexes:

$$\Gamma(X,1): \quad F(X)^* \xrightarrow{\partial_1} \coprod_{x \in X_{(1)}} \mathbf{Z}$$

$$\Gamma(X,2): \quad \begin{array}{ccccc} \Lambda^2 F(X)^* & \xrightarrow{\partial_1} & \coprod_{x \in X_{(1)}} F(x)^* & \xrightarrow{\partial_2} & \coprod_{x \in X_{(2)}} \mathbf{Z} \\ \uparrow \delta & & & & \\ B_2(F(X)) & & & & \end{array}$$

$$\Gamma(X;3): \quad \begin{array}{ccccccc} \Lambda^3 F(X)^* & \xrightarrow{\partial_1} & \coprod_{x \in X_{(1)}} \Lambda^2 F(x)^* & \xrightarrow{\partial_2} & \coprod_{x \in X_{(2)}} F(x)^* & \xrightarrow{\partial_3} & \coprod_{x \in X_{(3)}} \mathbf{Z} \\ \uparrow \delta & & \uparrow \delta & & & & \\ B_2(F(X)) \otimes F(X)^* & \xrightarrow{\partial_1} & B_2(F(X)) & & & & \\ \uparrow \delta & & & & & & \\ B_3(F(X)) & & & & & & \end{array}$$

where $B_n(F(X))$ is placed in degree 1 and coboundaries have degree $+1$.

The coboundaries ∂_i are defined as follows. $\partial_1 := \coprod_{x \in X_{(1)}} \partial_{v_x}$. The others are a little bit more complicated. Let $x \in X_{(k)}$ and $v_1(y), \cdots, v_m(y)$ be all discrete valuations of the field $F(x)$ over a point $y \in X_{(k+1)}$, $y \in \overline{x}$. Then $\overline{F(x)}_i := \overline{F(x)}_{v_i(y)} \supset F(y)$. (If $\overline{x}$ is nonsingular at the point y, then $\overline{F(x)}_i = F(y)$ and $m = 1$). Let us define a homomorphism $\partial_2 : \Lambda^2 F(x) \to F(y)^*$ as the composition

$$\Lambda^2 F(x)^* \xrightarrow{\oplus \partial_{v_i(y)}} \bigoplus_{i=1}^{m} \overline{F(x)}_i^* \xrightarrow{\oplus N_{\overline{F(x)}_i / F(y)}} F(y)^*$$

and $F(x)^* \xrightarrow{\oplus \partial_{v_i}} \bigoplus_{i=1}^{m} \mathbf{Z} \xrightarrow{\Sigma} \mathbf{Z}$.

3. **Motivic Chern classes** $c_n \in H^{2n}_{\mathcal{M}}(BGL_m(F)_\bullet, \mathbf{Z}(n))$, $n \leq 3$. Recall that

$$BG_\cdot := pt \overset{s_0}{\underset{s_1}{\leftleftarrows}} G \overset{s_0}{\underset{s_2}{\Lleftarrow}} G^2 \cdots$$

We have to construct a 2n-cocycle c_n in the bicomplex

$$\Gamma(G;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^n;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma\left(G^{2n-1};n\right) \tag{4.7}$$

where $s^* = \Sigma(-1)^i s_i$. Its components in

$$\Gamma(G;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^n;n) \tag{128}$$

□

So we have defined a homomorphism $f(r) : BC_*(r) \to B_F(2)$. It is non-zero only on the Grassmannian subcomplex $C_*(2) \subset BC_*(2)$.

c) $n = 3$. We have to define a homomorphism from the total complex associated with the bicomplex

$$\begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow \\ \to & C_6(4) & \to & C_5(4) & \to & C_4(4) \\ & \downarrow & & \downarrow & & \downarrow \\ \to & C_5(3) & \to & C_4(3) & \to & C_3(3) \end{array}$$

to the complex

$$B_3(F) \to B_2(F) \otimes F^* \to \Lambda^3 F^*$$

A homomorphism $f_3(3) : C_3(3) \to \Lambda^3 F^*$ was defined by formula (3.2). Set

$$\begin{aligned} &f_4(3) : C_4(3) \to B_2(F) \otimes F^* \\ &f_4(3) : (l_0, \cdots, l_4) \mapsto \frac{1}{2}\mathrm{Alt}\{r(\bar{l}_0|\bar{l}_1, \cdots, \bar{l}_u)\}_2 \otimes \Delta(l_0, l_1, l_2) \end{aligned} \tag{5.15}$$

Proposition 5.5 $f_4(3)$ *does not depend on the choice of the volume form* $\omega_3 \in \Lambda^3 (V^3)^*$ *that we need for the definition of* $\Delta(l_0, l_1, l_2)$.

Proof: The difference between the right-hand sides of (5.15) computed using $\lambda \cdot \omega_3$ and ω_3 is proportional to (right-hand side of (5.14) $\otimes\lambda$. So it is zero by lemma 5.4.

□

Proposition 5.6 $f_3(3) \circ d = \delta \circ f_4(3)$

Proof: Direct calculation using the formula

$$r(\bar{l}_1, \cdots, \bar{l}_u) = \frac{\Delta(l_1, l_3) \cdot \Delta(l_2, l_4)}{\Delta(l_1, l_4) \cdot \Delta(l_2, l_3)}$$

□

Now set

$$\begin{aligned} &f_5(3) : C_5(3) \to B_3(F) \\ &f_5(3) : (l_0, \cdots, l_5) \mapsto \mathrm{Alt}\left\{\frac{\Delta(l_0, l_1, l_3) \cdot \Delta(l_1, l_2, l_4) \cdot \Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4) \cdot \Delta(l_1, l_2, l_5) \cdot \Delta(l_2, l_0, l_3)}\right\}_3 \end{aligned} \tag{5.16}$$

Theorem 5.7 $f_4(3) \circ d = \delta \circ f_5(3)$

Proof: See proof of theorem 3.10 in [G3].

□

Proposition 5.8 $f_k(3) \circ d' = 0$ *for* $k = 3, 4, 5$.

Proof: For $k = 3$ this is lemma 3.2. For $k = 4, 5$ see theorem 3.12 in [G3].

□

Proposition 5.9 $f_5(3) \circ d = 0$ *in* $B_3(F)$.