GENERALIZED ELLIPTIC GENERA

V. GRITSENKO (Lille/POMI)

In this talk we present a new link between the theory of automorphic forms and geometry. Let M be an (almost) complex compact manifold M of (complex) dimension d and E be a holomorphic vector bundle over M. We set $q=\exp(2\pi i\tau)$ and $y=\exp(2\pi iz)$ ($\tau\in\mathbb{H}_1$, $z\in\mathbb{C}$). One can define a formal series $\mathbf{E}_{q,y}\in K(M)[[q,y^{\pm 1}]]$ (see [W], [H1])

$$\mathbf{E}_{q,y}=\bigotimes_{n=0}^{\infty}\bigwedge\nolimits_{-y^{-1}q^n}E^*\otimes\bigotimes_{n=1}^{\infty}\bigwedge\nolimits_{-yq^n}E\otimes\bigotimes_{n=1}^{\infty}S_{q^n}T_M^*\otimes\bigotimes_{n=1}^{\infty}S_{q^n}T_M$$

where E^* is dual to E, T_M denotes the holomorphic tangent bundle of M, $\bigwedge_x E=\sum_{k\ge 0}(\wedge^k E)x^k$ and $S_xE=\sum_{k\ge 0}(S^kE)x^k$. We propose the following

Definition. *Modified Witten genus* (MWG) of $E\to M$ is a function

$$\chi(M,E;\tau,z)=q^{(r-d)/12}y^{r/2}\int_M\exp\Bigl(\frac{1}{2}\bigl(c_1(E)-c_1(T_M)\bigr)\Bigr)\cdot$$
$$\exp\Bigl(\bigl(p_1(E)-p_1(T_M)\bigr)\cdot G_2(\tau)\Bigr)\exp\Bigl(-\frac{c_1(E)}{2\pi i}\frac{\vartheta_z}{\vartheta}(\tau,z)\Bigr)\,\mathrm{ch}(\mathbf{E}_{q,y})\,\mathrm{td}(T_M)$$

where $c_1(E)$ and $p_1(E)$ are the first Chern and Pontryagin class of E, td is the Todd class, $\mathrm{ch}(\mathbf{E}_{q,y})$ is the Chern character applied to each coefficient of the formal power series and $\int_M$ denotes the evaluation of the top degree differential form on the fundamental cycle of the manifold. In the definition we use Jacobi theta-series of level two

$$\vartheta(\tau,z)=-q^{1/8}y^{-1/2}\prod_{n\ge 1}(1-q^{n-1}y)(1-q^ny^{-1})(1-q^n),\quad \vartheta_z(\tau,z)=\frac{\partial\vartheta}{\partial z}(\tau,z).$$

$G_2(\tau)=-\frac{1}{24}+\sum_{n=1}^{\infty}\sigma_1(n)\,q^n$ is a quasi-modular Eisenstein series of weight 2.

Witten genus. Let $E=0$ be the trivial vector bundle of rank 0. Let us assume that M admits a spin structure and $p_1(M)=0$. In this case

$$q^{d/12}\chi(M,0;\tau,z)=q^{-d/12}\int_M\mathrm{ch}\Bigl(\bigotimes_{n=1}^{\infty}S_{q^n}(T_M\oplus T_M{}^*)\Bigr)\hat{A}(M)=\frac{\text{Witten genus}\,(M,\tau)}{\eta(\tau)^{2d}}.$$

It is known that the Witten genus is a $SL_2(\mathbb{Z})$-modular form of weight $2k$.

Elliptic genus of a Calabi–Yau manifold. This case is of some interest in physics. Let M be an arbitrary complex compact manifold and $E=T_M$. The holomorphic Euler characteristic of $\mathbf{E}_{q,y}$

$$\phi_{ell}(M;\tau,z)=y^{d/2}\int_M\mathrm{ch}(\mathbf{E}_{q,y})\mathrm{td}(T_M)$$

is called (in physics) *elliptic genus* of M. One can see that the elliptic genus of M (as a function in τ, z) is equal to the zeroth term in a sum of $d+1$ summands of the modified Witten genus for $E = T_M$. These summands correspond to the powers of the first Chern class of M

$$\chi(M, T_M; \tau, z) = \phi_{ell}\,(M; \tau, z) + \sum_{n=1}^{d} \Big(\int_M c_1(M)^n (\dots) \Big).$$

Thus, $\chi(M, T_M; \tau, z) = \phi_{ell}\,(M; \tau, z)$ if $c_1(T_M) = 0$. For $E = T_M$ and $c_1(T_M) \neq 0$ the integral in the definition of MWG contains the only correction term

$$\exp\left(-\frac{c_1(T_M)}{2\pi i} \frac{\vartheta_z}{\vartheta}(\tau, z) \right).$$

It is known that the elliptic genus of a Calabi–Yau manifold is a weak Jacobi form of weight 0 and index $d/2$ (see [EOTY], [Hö], [KYY], [L]). *If $c_1(T_M) \neq 0$, then the elliptic genus of M is not a modular form in τ and z.* We add the three correction factors in the definition in order to obtain a function with a good behavior with respect to the modular transformations in τ and z. In general the elliptic genus is not an automorphic form in two variables but the modified Witten genus is.

Theorem 1. *Let $\chi(M, E; \tau, z)$ be the modified Witten genus. Then the product*

$$\chi(M, E; \tau, z) \left(\frac{\vartheta(\tau, z)}{\eta(\tau)} \right)^{d-r}$$

is a weak Jacobi form of weight 0 and index $d/2$.

We recall the definition of Jacobi forms. A holomorphic function $\phi(\tau, z)$ on $\mathbb{H}_1 \times \mathbb{C}$ is called a *nearly-holomorphic Jacobi form of weight k and index m* ($m \in \mathbb{Z}/2$) if it satisfies the functional equations

$$\phi(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}) = (c\tau + d)^k \, e^{2\pi i m \frac{cz^2}{c\tau+d}} \, \phi(\tau, z) \qquad (\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}))$$
$$\phi(\tau, z + \lambda\tau + \mu) = (-1)^{2t(\lambda+\mu)} \, e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \phi(\tau, z) \qquad (\lambda, \mu \in \mathbb{Z}).$$

The form $\phi(\tau, z)$ has the Fourier expansion of the type $\phi(\tau, z) = \sum_{n,\, l} f(n, l)\, q^n y^l$. If $a(n, l) \neq 0$ only for $n \geq 0$, then ϕ is called *weak*. We denote the space of all week Jacobi forms of weight k and index m by $J_{k,t}$. In particular, $\vartheta(\tau, z) \in J_{\frac{1}{2}, \frac{1}{2}}(v_\eta^3)$ (v_η is the multiplier system of the Dedekind η-function). The next lemma is very useful.

Lemma. *The following formula is valid*

$$\exp\left(-4\pi^2 G_2(\tau) \xi^2 - \frac{\vartheta_z}{\vartheta}(\tau, z)\xi \right) \vartheta(\tau, z + \xi) = \vartheta(\tau, z) \exp\left(-\sum_{n \geq 2} \wp^{(n-2)}(\tau, z) \frac{\xi^n}{n!} \right)$$

where $\wp(\tau, z) = z^{-2} + \sum_{\omega \in \mathbb{Z}\tau + \mathbb{Z}}^{\omega \neq 0} \big((z + \omega)^{-2} - \omega^{-2} \big) \in J_{2,0}^{mer}$ is the Weierstrass $\wp$-function and $\wp^{(n)}(\tau, z) = \dfrac{\partial^n}{\partial z^n} \wp(\tau, z) \in J_{n+2,0}^{mer}$.

(See [G1], [G2] for more details.)

Let $c(E)=\sum_{i=0}^r c_i(E)=\prod_{i=1}^r(1+x_i)$ be the total Chern class of E where x_i $(1\le i\le r)$ are the formal Chern roots of E. We denote by x'_j $(1\le j\le d)$ the Chern roots of T_M. Using the formula of the lemma we can split the Chern roots $\{x_i\}$ and $\{x'_j\}$ under the integral in MWG. More exactly, we have

$$\chi(M,E;\tau,z)=\frac{\vartheta^r}{\eta^{r+2d}}\int_M P(E;\tau,z)\cdot W(M;\tau). \tag{S}$$

The first factor depends only on the vector bundle E

$$P(E;\tau,z)=\exp\Bigl(-\sum_{n\ge 2}\frac{\wp^{(n-2)}(\tau,z)}{(2\pi i)^n n!}\bigl(\sum_{i=1}^r x_i^n\bigr)\Bigr).$$

The second factor is the Witten factor

$$W(M;\tau)=\exp\Bigl(2\sum_{k\ge 2}\frac{G_{2k}(\tau)}{(2k)!}\bigl(\sum_{j=1}^d {x'_j}^{2k}\bigr)\Bigr)$$

which determines the Witten genus of the manifold M. Theorem 1 is a direct corollary of the representation (S) above. (See [G2] for more details.)

We note that the definition of MWG, Theorem 1 and its proof are valid *for an arbitrary compact Spin^c-manifold M and a Hermitian C^∞-vector bundle E over M.* Representation (S) is new also in the case of the elliptic genus of Calabi–Yai manifolds. It shows us that $\phi_{ell}(M,\tau,z)$ is a twisted version of the Witten genus. In order to define a generalization of the elliptic genus of Oshanine–Landweber as a function in τ and z one should use some other vector bundle instead of $\mathbf{E}_{q,y}$ above.

Examples for dimension $d=2,3,4$ and $d=6$, $r=2$. For $d=2$ and $d=3$ we consider an arbitrary E over M. It follows from (S) that

$$\chi(M_2,E_r;\tau,z)=\frac{1}{2(2\pi i)^2}\Bigl(\sum_{i=1}^r x_i^2\Bigr)[M]\frac{\vartheta^r}{\eta^{r+4}}\wp=\frac{1}{24}(c_1(E)^2-2c_2(E))\,\phi_{0,1}\Bigl(\frac{\vartheta}{\eta}\Bigr)^{r-2}$$

where $\phi_{0,1}(\tau,z)=(y+10+y^{-1})+q(\dots)$ is the unique, up to a constant, weak Jacobi form of weight 0 and index 1. Let $\phi_{0,\frac32}(\tau,z)=\vartheta(\tau,2z)/\vartheta(\tau,z)$. For $d=3$ we have

$$\chi(M_3,E_r;\tau,z)=\frac16\bigl(c_1(E)^3-3c_1(E)c_2(E)+3c_3(E)\bigr)\phi_{0,\frac32}(\tau,z)\Bigl(\frac{\vartheta}{\eta}\Bigr)^{r-3}.$$

Let us consider the case $d=4$, $r\ge 4$ with $p_1(E_4)=p_1(M_4)$ ($c_1(E_4)$ is arbitrary). We have the following formula for MWG

$$\chi(M_4,E_r;\tau,z)=\bigl[\hat A(M_4)\psi_{0,2}^{(2)}(\tau,z)+\bigl(r\hat A(M_4)-\hat A(M_4,E_r)\bigr)\phi_{0,2}(\tau,z)\bigr]\Bigl(\frac{\vartheta(\tau,z)}{\eta(\tau)}\Bigr)^{r-4},$$

where $\phi_{0,2}(\tau,z)=y+4+y^{-1}+q(\dots)$ is a Jacobi form related to a generalized Lorentzian Kac–Moody Lie algebra (see [GN1]) and $\psi_{0,2}^{(2)}=\phi_{0,1}^2-24\phi_{0,2}$.

Let us consider the case $d=6$, $r=2$, $p_1(E)=p_1(T_M)$. We assume also that MWG is holomorphic. (In the case under consideration this condition is equivalent to the relation $< c_1(E)^4c_2(E)-\frac{1}{6}c_1(E)^6 > [M]=0$). Then

$$\chi(M_6,E_2;\tau,z)=\big[\hat{A}(M)E_6(\tau)\phi_{-2,1}+(2\hat{A}(M)-\hat{A}(M,E))E_{4,1}(\tau,z)\big]\eta(\tau)^{-8},$$

where $\phi_{-2,1}=\vartheta^2/\eta^6$, $E_6(\tau)$ is the Eisenstein series of weight 6 and $E_{4,1}(\tau,z)$ is the Eisenstein–Jacobi series of weight 4 and index 1. ($E_{4,1}$ is equal to the Jacobi theta-series of the lattice $\mathbb{E}_8$.)

One can obtain many others exact formulae. The fact that MWG of a vector bundle is a Jacobi modular form has some number of applications in the theory of vector bundles (relations between $\hat{A}$-genera and between other cohomological invariants, some obstructions if rank<dim, arithmetic of special values). As one can see from the definition, the coefficients of MWG are rational with bounded denominators. In the next theorem we determine $\mathbb{Z}$-structure of the graded ring $J_{0,*}^{\mathbb{Z}}$ of all weak Jacobi forms of weight 0 with integral Fourier coefficients. We denote by $J_{0,m}^{\mathbb{Z}}(q)$ the subspace of integral Jacobi form with q^0-term equal to 0.

Theorem 2. 1. *$J_{0,m}^{\mathbb{Z}}(q)=\xi_{0,6}J_{0,m-6}^{\mathbb{Z}}$, where $\xi_{0,6}(\tau,z)=(\vartheta(\tau,z)/\eta(\tau))^{12}$.*

2. *$J_{0,m}^{\mathbb{Z}}/J_{0,m}^{\mathbb{Z}}(q)=\mathbb{Z}\,[\psi_{0,m}^{(1)},\dots,\psi_{0,m}^{(m)}]$ is a free $\mathbb{Z}$-module of rank m. Moreover we can chose a basis with the following q^0-terms*

$$[\psi_{0,m}^{(n)}(\tau,z)]_{q^0}=y^n+n^2y+(2n^2-2)+n^2y^{-1}+y^{-n}\qquad\qquad (2\le n\le m),$$

$$[\psi_{0,m}^{(1)}]_{q^0}=\frac{1}{(12,m)}\big(my+(12-2t)+my^{-1}\big),$$

where $(12,m)$ is the greatest common divisor of 12 and m.

3. *The graded ring of all weak Jacobi forms of weight 0 with integral coefficients is finitely generated*

$$J_{0,*}^{\mathbb{Z}}=\bigoplus_m J_{0,m}^{\mathbb{Z}}=\mathbb{Z}\,[\phi_{0,1},\phi_{0,2},\phi_{0,3},\phi_{0,4}]$$

where $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}=\phi_{0,\frac{3}{2}}^2$ (defined in examples above) are algebraicly independent and $4\phi_{0,4}=\phi_{0,1}\phi_{0,3}-\phi_{0,2}^2$.

The proof of the theorem see in [G1]. We remark that $\xi_{0,6}$ is MWG of a rank 12 vector bundle over a point. One can consider the generators $\phi_{0,n}$ as analytic analogies of the Tate's parameters b_2, b_4, b_6, b_8 of the elliptic curve (see [T]). Then $\xi_{0,6}$ is the discriminant of the curve.

One of the direct corollaries of the last theorem is the following property of Euler number of a Calabi–Yau manifold. Let us consider the case

$$d=r,\quad p_1(E)=p_1(T_M),\quad c_1(E)=0\ (\text{over }\mathbb{Z}).$$

We consider spin bundle $\Delta^+(E)-\Delta^-(E)$ such that $\mathrm{ch}(\Delta^+(E)-\Delta^-(E))=\prod_{i=1}^r(e^{x_i/2}-e^{-x_i/2})$. Then

$$\hat{A}(M,\Delta^+(E)-\Delta^-(E))\equiv\begin{cases}0 \mod 8 & d\equiv 1,2,5 \mod 8\\ 0 \mod 4 & d\equiv 6,7 \mod 8\\ 0 \mod 2 & d\equiv 3,4 \mod 8\\ 0 \mod 3 & d\not\equiv 0 \mod 3.\end{cases}$$

The same relations are true, in particular, for the Euler number e of a Calabi–Yau manifold. For example, $e(CY_5) \equiv 0 \mod 24$ and $e(CY_{10}) \equiv 0 \mod 24$. The divisibily of the Euler number by 3 was proved by Hirzebruch in 1960. After my talk at MPI on this subject Prof. Hirzebruch informed me that the properties of $e(CY_d)$ modulo 8 were known for him (non-published, see [H2]). We note also that $e(CY_d)/24$ is a special obstruction in the string theory. (See [SVW], where $e(CY_4) \equiv 0 \mod 6$ was proved.)

We finish the talk with some possible strange "periodicity" for MWG. If $p_1(E) = p_1(M)$ and $c_1(E) = 0$, then MWG is the partition function for $(0,2)$-symmetric non-linear sigma-model (see [KM]). Let us assume that $\chi(M_1, E_2, \tau, z) \equiv \chi(M_1, E_2, \tau, z)$. Then Theorem 1 implies that $r_1 = r_2$ and $d_1 \equiv d_2 \mod 12$.

Question. To construct a vector bundle E of rank 2 over a complex manifold M of dim $= 14$ such that $p_1(E) = p_1(M)$, $c_1(E) = 0$, $\hat{A}(M) = 0$, $\hat{A}(M, E) = -2$ and $\hat{A}(M, E \otimes T_M) = -68$. (The last condition is equivalent to $\hat{A}(M, S^2E) + 2\hat{A}(M, T_M) = 0$.) If such a vector bundle exist then its Second Quantized Elliptic Genus is equal, up to a simple factor, to the automorphic correction of the simplest hyperbolic Kac–Moody algebra (see [D], [G1], [GN2]).

References

[EOTY] T. Eguchi, H. Ooguri, A. Taormina, S.-K. Yang,, *Superconformal Algebras and String Compactification on Manifolds with $SU(N)$ Holonomy*, Nucl. Phys. **B315** (1989), 193–221.

[D] R. Dijkgraaf, *The Mathematics of Fivebranes*, Documenta Mathem. ICM-1998 (1998).

[G1] V. Gritsenko, *Elliptic genus of Calabi–Yau manifolds and Jacobi and Siegel modular forms*, Algebra i Analyz (St. Petersburg Math. Jour.) **11:5** (1999).

[G2] V. Gritsenko, *Complex vector bundles and Jacobi forms*, Proc. of Symposium "Automorphic forms and L-functions", Kyoto, 1999.

[G3] V. Gritsenko, *Modified Witten genus*, Course of lectures at RIMS, Kyoto, 1998.

[GN1] V.A. Gritsenko, V.V. Nikulin, *Siegel automorphic form correction of some Lorentzian Kac–Moody Lie algebras*, Amer. J. Math. **119** (1997), 181–224.

[GN2] V.A. Gritsenko, V.V. Nikulin, *The Igusa modular forms and "the simplest" Lorentzian Kac–Moody algebras*, Matem. Sbornik **187** (1996), 1601–1643.

[H1] F. Hirzebruch, *Elliptic genera of level N for complex manifolds*, Differential geometrical Methods in Theoretical Physics (K. Bleuler, M. Werner, eds.), Kluwer Acad. Publ., 1988, pp. 37–63.

[H2] F. Hirzebruch, *Letter to V. Gritsenko from 11 August 1997.*

[Hö] G. Höhn, *Komplex elliptische Geschlecter und S^1-äquivariante Kobordismustheorie*, Diplomarbeit (1991), Bonn.

[K] I. Krichever, *Generalized elliptic genera and Baker–Akhiezer functions*, Mat. Zametki **47** (1990), 34–45.

[KYY] T. Kawai, Y. Yamada, S.-K. Yang, *Elliptic Genera and N=2 Superconformal Field Theory*, Nucl. Phys. **B414** (1994), 191-212.

[KM] T. Kawai, K. Mohri, *Geometry of* $(0,2)$ *Landau–Ginzburg orbifolds*, Nucl. Phys. **B425** (1994), 191-216.

[L] K. Liu, *On elliptic genera and theta-functions*, Topology **35** (1996), 617–640.

[SVW] S. Sethi, C. Vafa, E. Witten, *Constraints on low-dimensional string compactifications*, Nucl. Phys. **B480** (1996), 213–224.

[T] J. T. Tate, *The arithmetic of elliptic curves*, Invent. Math. **23** (1974), 179–206.

[W] E. Witten, *The index of the Dirac operator in loop space*, Elliptic Curves and Modular Forms in Algebraic Topology, Springer-Verlag, 1988, pp. 161–181.

Département de Mathématique Université Lille I
59655 Vileneuve d'Asq Cedex, France