Non-commutative deformations and Poisson brackets on projective spaces

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Introduction

The idea of the existence of the non-commutative algebraic geometry was mostly inspired by physics. This paper is aimed to present categorical approach to the problem and to compare different approaches to quantization from this point of view.

We will try to trace the route from general settings to concrete details on the example of projective spaces.

Here we consider non-commutative counterpart only of the classical algebraic geometry, that is, the geometry of smooth compact algebraic varieties over the field $\mathbf{C}$.

Briefly speaking non-commutative geometry is the following. Instead of smooth complete projective variety X one has to consider the (bounded) derived category $D^b_{coh}(X)$ of the category of coherent sheaves on X. Then, one needs to describe natural categorical properties of the class of all such categories.

The main principle of non-commutative algebraic geometry can be formulated as following:

A non-commutative algebraic variety is any category D, which satisfies all the natural properties of $D^b_{coh}(X)$ for commutative varieties X.

It is possible to put sufficiently many restrictions on categories D to obtain the classification problem of the same rigidity as in algebraic geometry, in the sense that only finitely dimensional families of these structures will arise. In the meantime, there seemingly exist no natural categorical restrictions to pose on a category in order to distinguish exactly the class of derived categories of coherent sheaves on smooth projective varieties. This is the main reason for non-commutative geometry to be mathematically justified.

Last remark about the general settings. We consider here categories without tensor structures. Objects like Yang-Baxter operators and quantum groups do not appear in this paper. The reason is that a variety is essentially determined by a triangulated structure of the derived category of coherent sheaves on it. Therefore, it is natural to divide the investigation of non-commutative geometry into two parts: homological properties of categories and proper tensor structures on them.

1 Non-commutative deformation procedures.

In this chapter we briefly outline main approaches to the problem of non-commutative deformations of algebraic varieties.

For functions on a usual variety the commutativity condition $fg = gf$ is valid. Variety itself can be locally considered as a spectrum of the algebra of functions:

$$X = \cup U_i, \text{ where } U_i = \mathrm{Spec}\mathcal{O}_{U_i}.$$

Here $\mathcal{O}_{U_i}$ are algebras of functions on open subsets U_i, which cover X.

The first idea to construct non-commutative variety was to substitute $\mathcal{O}_{U_i}$ by non-commutative algebras A_i, then, to consider non-commutative spectra 'Spec'A_i and to try to glue them together. Numerous problems arise on this way: it is complicated to construct the localizing classes, Noether properties are hard to prove for concrete algebras, etc.

In the case of non-commutative projective plane $\mathbf{P}^2_q$ there essentially exists only one 'open set' on it, which is a complement to an elliptic curve [BP].

Next idea was to consider the abelian category $Sh_{coh}X$ of coherent sheaves on X instead of $\mathcal{O}_X$.

If X is a smooth projective variety then there is Serre's description of $Sh_{coh}X$ by the following equivalence.

Let $f : X \to \mathbf{P}^n$ be a smooth embedding of a projective variety X into the projective space $\mathbf{P}^n$. Denote by $\mathcal{O}(1)$ antitautological sheaf on $\mathbf{P}^n$ and by $\mathcal{O}_X(1) = f^*\mathcal{O}(1)$ its restriction to X.

For a graded associative algebra A denote by Proj$A$ a quotient category of right finitely generated graded A-modules by the subcategory of finite modules.

Theorem(Serre).[S] Let $A = \sum H^0(X, \mathcal{O}_X(i))$ be the graded coordinate algebra of X(which is commutative).

Then the category $Sh_{coh}X$ is equivalent to the category Proj$A$.

One may consider a graded non-commutative algebra A with 'good' properties and regard Proj$A$ as a category of sheaves on a non-commutative algebraic variety.

If $X = \mathbf{P}^n = \mathbf{P}(V)$ is a projective space, which is a projectivization of the vector space V, then $A = S^\bullet(V^*)$ is a symmetric algebra of the dual vector space. A is an algebra with quadratic commutativity relations. This relations can be considered as n(n+1)/2-dimensional subspace $S^2(V^*)$ in $(V^*) \otimes (V^*)$. Deforming this subspace in $(V^*) \otimes (V^*)$ one obtains non-commutative quadratic algebras, hence, non-commutative deformations of projective spaces.

The notion of 'good' algebras in this situation was introduced by Artin-Shelter[AS] under the name of regular algebras. In the paper [ATV] of Artin-Shelter-Van-den-Bergh regular algebras have been described in the case of 3 generators and 3 quadratic relations, which correspond to non-commutative projective planes.

Two questions naturally arise here:

i) does any non-commutative deformation of the category of coherent sheaves on the projective space can be obtained by this 'grading deformation' procedure?

ii) do different algebras A give different deformations of the category?

The answer on the first question is affirmative, at least infinitisimally. The point is that tangent vector to a one-parameter family of a non-commutative defoirmations of $D^b_{coh}(\mathbf{P}(V))$ is given by a global Poisson bracket on it. The tangent vector to a one-parameter family of 'good' formal quadratic deformations of the algebra $S^\bullet(V^*)$ is given by a quadratic Poisson bracket on V. We will show that any Poisson bracket on $\mathbf{P}(V)$ can be lifted up to a quadratic one on V.

The answer to the second question is already negative in the case of non-commutative projective planes. These planes compose a two-dimensional connected family, for which the commutative plane is a singular point (due to big automorphism group). For a generic point of the family there are 9 different regular algebras presenting the same category.

Next step towards abstract nonsense is to consider the derived category $D^b_{coh}(X)$ of coherent sheaves on X. It is a triangulated category. Deformation theory for triangulated categories gives the following description of infinitisimal deformations of a category D. The tangent vector to a 1-parameter family of deformations belongs to $\mathrm{Ext}^2_{D\to D}(\mathrm{id},\mathrm{id}) = 0$, where Ext-groups aretaken in the category of functors from D to itself, and id is the identity functor. All obstructions to formal deformations lie in the group $\mathrm{Ext}^3_{D\to D}(\mathrm{id},\mathrm{id})$. Precise formalism concerning extention groups in the category of functors (as well, as other invariants of triangulated categories) will be given elsewhere.

Computations for the category $D^b_{coh}(X)$ give the following result:

Theorem. Let $D = D^b_{coh}(X)$ be the derived category of coherent sheaves over a smooth projective algebraic variety X. Then:

$$\mathrm{Ext}^k_{D\to D}(\mathrm{id}, \mathrm{id}) = \oplus_{i+j=k} H^i(\Lambda^j T_X),$$

where $(\Lambda^j T_X)$ is the j-th exterior power of the tangent bundle T_X) on X.

In the case of projective spaces (quadrics, grassmanians and some other varieties) there exist an equivalence $D^b_{coh}(\mathbf{P}^n) \simeq D^b(\mathrm{mod} - A)$ with the derived category of right modules over a finite dimensional algebra A ([Ba],[Bon]).

This allows to construct deformations of the category by taking deformations of the algebra. Deformations of this kind give all formal deformations of the category (due to general deformation theory, briefly described above). As a result one obtains that the tangent cone to formal non-commutative deformations coincides with the set of Poisson brackets on the projective space.

2 Properties of categories

The following property of $D^b_{coh}(X)$ has been proved in [BK].

Theorem. If X is a smooth projective variety over $\mathbf{C}$, then $D^b_{coh}(X)$ is saturated, i.e. any exact functor from $D^b_{coh}(X)$ to the category $D^b(Vect)$, the derived category of vector spaces over $\mathbf{C}$, is representable.

The same property has the category $D^b(\mathrm{mod} - A)$, the derived category of right modules over an algebra A of finite projective dimension. Main characteristics of a variety (like Kodaira dimension) are defined by means of its canonical class. Its incarnation for the derived category of coherent sheaves is the so called Serre functor, introduced in [BK].

Definition. Let D be a triangulated category with finite-dimensional Hom's. A covariant additive functor $F : \mathcal{D} \to \mathcal{D}$, which commutes with the translation functor, is called a *Serre functor* if it is a category equivalence and there are given bi-functorial isomorphisms

$$\alpha_{E,G} : \mathrm{Hom}(E, G) \simeq \mathrm{Hom}(G, F(E))^*$$

for any pair E, G of objects in D.

If X is a smooth projective algebraic variety, $n = \dim X$, $\mathcal{D} = D^b_{coh}(X)$ and K_X the canonical class of X, then the functor $(-) \otimes K_X[n]$ is a Serre functor on $\mathcal{D}$ in view of the Serre-Grothendieck duality:

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = \mathrm{Ext}^{n-i}(\mathcal{G}, \mathcal{F} \otimes K_X)^*.$$

Proposition.[BK] If a triangulated category is saturated, then a Serre functor exists and is unique up to canonical functor isomorphism.

Serre functor is a powerful tool for investigation of triangulated categories.

It is widely known that there exist abelian categories which have equivalent derived ones. Examples of such equivalence between the derived categories of coherent sheaves over some varieties and of right modules over some finite dimensional associative algebras can be constructed by means of the theory of exceptional collections (cf. [Bon]). Thus, one can construct a lot of t-structures (see [BBD]) in a triangulated category, by various identifications of the category with derived ones. But the t-structure, which appears as a result of identification with $D^b_{coh}(X)$ has an advantage of being compatible with the Serre functor in the category.

Definition. A t-structure $(D^{\leq 0}, D^{\geq 0})$ is called a *geometric t-structure* if there exist an integer n, such that the functor $S = F[-n]$ (a composition of the n-fold iteration of the translation functor with the Serre functor) preserves the t-structure:

$$SD^{\leq 0} \subset D^{\leq 0}, SD^{\geq 0} \subset D^{\geq 0}$$

Definition. A *non-commutative variety* is a saturated triangulated category D with a geometric t-structure. The number n from the definition of a geometric t-structure is called the dimension of D.

If $D = D^b_{coh}(X)$ for a smooth compact algebraic variety X, then S is the functor of twisting by canonical class K_X. Hence, it preserves the natural t-structure in D.

It would be nice to understand to which extend various geometric operations (like blowing up, ramifed covering, etc.), used for construction of new varieties, can be generalized to a non-commutative case.

It is interesting question even in the case of $D = D^b_{coh}(X)$ to describe all geometric t-structures on it. I will mention here only that there exist the unique, up to auto-equivalence of the category, geometric t-structure on a Fano variety, and it is not the case generally.

3 Quadratic deformations.

In this chapter we consider formal deformations of the polynomial algebra $A = S^\bullet(V^*)$ over a vector space V, $\dim V = n+1$ in the class of graded associative algebras:

$$A = A_0 \oplus A_1 \oplus A_2 \oplus ...,$$

$$A_0 = k, A_1 = V^*,$$

with *fixed* dimensions of the spaces A_i. Our purpose will be a description of the tangent cone to such deformations.

First, using the semi-continuity arguments for dimensions of A_i in the case of quadratic algebras, it is easy to show that locally near $S^\bullet(V^*)$ the algebras A should be quadratic up to the k-th component, for arbitrary k. Therefore, when A is a formal deformation of $S^\bullet(V^*)$, it has to be quadratic.

Let $A(t)$ be a formal quadratic deformation of the algebra $S^\bullet(V^*)$. Denote by $I(t)$ the space of quadratic relations for $A(t)$, $\dim I(t) = \frac{n(n+1)}{2}$, where $\dim V = n+1$.

One has the following short exact sequence:

$$0 \to I(t) \to V^* \otimes V^* \to A_2(t) \to 0,$$

where the last map is defined by multiplication in $A(t)$.

For $t = 0$ it takes the form:

$$0 \to \Lambda^2 V^* \to V^* \otimes V^* \to S^2 V^* \to 0,$$

The tangent space to the grassmannian $G(\frac{n(n+1)}{2}, V^* \otimes V^*)$ at the point $\Lambda^2 V^*$ is well known to be isomorphic to $\mathrm{Hom}(\Lambda^2 V^*, S^2 V^*) = S^2 V^* \otimes \Lambda^2 V$. This space is naturally identified with quadratic bivector fields on V.

Theorem. Let A be a graded Koszul algebra [Man]. Then for any integer $s \geq 1$ and for any local deformation $A(t)$, with $A(0) = A$, $\mathrm{dim} A_i(t) = A_i$, for $i \leq 3$, there exist a neighbourhood U of the zero, such that $A(t)$, for $t \in U$, is a Koszul algebra up to s-th component, and $\mathrm{dim} A_i(t) = A_i$, for $i \leq s$.

Corrolary. Any formal graded deformation of a graded Koszul algebra, which preserves dimensions of the first three grading components, is Koszul and preserves dimensions of all the components.

Proof. If one consider formal deformations instead of local ones , one can pass to infinity by k in shrinking an open set U in the theorem.

Let us consider the algebraic subscheme Q of the Grassmanian $G(\frac{n(n+1)}{2}, V^* \otimes V^*)$, which set theoretically is a locus of subspaces I in $V^* \otimes V^*$ with the following equality satisfied:

$$\frac{\mathrm{dim} V^* \otimes V^* \otimes V^*}{(I \otimes V^*) + (V^* \otimes I)} \geq \mathrm{dim} S^3 V^*.$$

In view of the corollary the formal germ of this scheme at the point $\Lambda^2 V^*$ coincides with formal deformations of $S^\bullet V^*$.

Theorem. The local ring of the scheme Q at the point $\Lambda^2 V^*$ has no nilpotents. The tangent cone coincides with the set of Poisson brackets on V.

4 Poisson brackets on projective spaces.

Let X be a complete projective variety over a field of characteristic zero.

Consider the sheaf of graded algebras $\Lambda^\bullet T_X$, which is an exterior algebra of the tangent vector bundle T_X on X. Elements of $\Lambda^\bullet T_X$ are called multivectors. There is a graded Lie algebra structure on $\Lambda^\bullet T_X$ called Schouten-Nijenhuis bracket.

Let A, B be a pair of multivectors on X of degree n and m respectively.

Definition. A *Schouten-Nijenhuis bracket* $[A, B]$ of A and B is a multivector of degree $n+m-1$ defined by the formula:

$$i([A,B])\omega = (-1)^{nm+m} i(A) d i(B)\omega + (-1)^n i(B) d i(A)\omega - i(AB) d\omega,$$

where ω is an $(n+m-1)$-differential form, $i(\bullet)$ is an operator of internal product and d is the differential in the algebra of forms.

Definition. A *Poisson bracket* on X is a bivector $P \in \Lambda^2 T_X$, which satisfies the equation:

$$[P, P] = 0,$$

P determines a structure of a Poisson algebra on the sheaf of functions. If P is locally defined by the formula:

$$P = \sum p_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

then for a pair (f, g) of local functions on X one can define a Poisson bracket by the formula:

$$\{f, g\} = \sum p_{ij} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right).$$

Let us consider the projective space $\mathbf{P}^n = \mathbf{P}(V)$, where V is a complex vector space of dimension $n+1$ with coordinates ξ_i, $i = 0, ..., n$. Denote by $p : V \setminus 0 \to \mathbf{P}(V)$ the natural projection. The following is an invariant description of the 'computations in homogeneous coordinates'.

Let T be the tangent vector bundle on $\mathbf{P}(V)$ (which we identify with the coherent sheaf of its sections), let $E = \sum \xi_i \frac{\partial}{\partial \xi_i}$ be the Euler vector field on $V \setminus 0$, let $\mathcal{T}_0$ be the sheaf vector fields on $V \setminus 0$, which are homogeneous of degree zero (in other words, fields which commute with the Euler field).

Let us consider the pull backs $p^{-1}\mathcal{O}$, $p^{-1}T$ along p of the sheaves of functions and vector fields on the projective space (the pull back is taken as one of the sheaves of abelian groups, not coherent sheaves).

We have the following short exact sequence of sheaves on $V \setminus 0$

$$0 \to p^{-1}\mathcal{O} \to \mathcal{T}_0 \to p^{-1}T \to 0 \tag{1}$$

Here an imbedding $p^{-1}\mathcal{O} \to \mathcal{T}_0$ is defined by the exterior multiplication with the Euler field E. The projection $\mathcal{T}_0 \to p^{-1}T$ exists, because $\mathcal{T}_0$ consists of fields, which are constant along the fibers of p.

Pushing forward the sequence (1) along p to $\mathbf{P}(V)$ one obtains the well known Euler short exact sequence of coherent sheaves on $\mathbf{P}(V)$:

$$0 \to \mathcal{O} \to V \bigotimes \mathcal{O}(1) \to T \to 0 \tag{2}$$

Thus, (1) may be concidered as a homogeneous incarnation of the Euler sequence.

Taking exterior powers of (1) one obtains the exact sequence:

$$0 \to p^{-1}(\Lambda^{i-1}T) \to (\Lambda^i\mathcal{T})_0 \to p^{-1}(\Lambda^i T) \to 0,$$

where $\Lambda^i E$ is the i-th exterior power of a vector bundle E, and $(E)_0$ is a sheaf of zero degree sections of a vector bundle on $V \setminus 0$ with a natural action of the Euler field.

Let P be a Poisson bracket on an algebraic variety X, i.e. P is the global section of the $\Lambda^2 T_X$, with $[P, P] = 0$ as a global section of $\Lambda^3 T_X$.

By Koszul [Kos] the Schouten-Nijenhuis bracket can be locally defined by the formula:

$$[u, v] = (-1)^p(D(uv) - D(u)v - (-1)^p uD(v)), \deg u = p, \tag{3}$$

where D is a differential operator on $\Lambda^\bullet T_X$ of degree -1 and order 2. This operator D is determined by a torsion free connection ∇ on T_X. After fixing ∇ the operator has a form:

$$D = \sum dx_i \otimes \nabla_{\frac{\partial}{\partial x_i}}$$

in arbitrary local coordinates $\{x_i\}$. In fact, D is determined by means of the action of ∇ on the higher exterior power $\Lambda^m T_X$, where $m = \dim X$.

For $X = V \setminus 0$ one has natural trivialization of the tangent bundle. For this trivialization the operator D is defined in the homogeneous coordinates by the formula:

$$D = \sum \frac{\partial}{\partial \xi_i} \otimes d\xi_i$$

$\frac{\partial}{\partial \xi_i}$ acts on $\alpha = \sum f_I(\xi)\frac{\partial}{\partial \xi_I}$ (I is a multiindex) by derivation of f_I, and $d\xi_i$ acts by convolution with $\frac{\partial}{\partial \xi_I}$(here I is a multyindex).

For a Poisson bracket $\tilde{P}$ on $V \setminus 0$ one has the identity:

$$D(\tilde{P} \wedge \tilde{P}) - 2D(\tilde{P}) \wedge \tilde{P} = 0. \tag{4}$$

Now we would like to compare Poisson brackets on $\mathbf{P}(V)$ with quadratic Poisson bracket on V, that is, Poisson brackets with coeffitients being a homogeneous quadratic functions on ξ_i. This brackets are exactly those of degree zero with respect to the action of the Euler field.

First, quadratic Poisson brackets on V are in one-to-one correspondence with quadratic Poisson brackets on $V \setminus 0$. The correspondence is given by its restriction on $V \setminus 0$ in one direction and by trivial extention on V in the opposite direction.

Let $P \in H^0(\Lambda^2 T)$ be a global vector field on $\mathbf{P}(V)$. It may be considered as a section of $p^{-1}(\Lambda^2 T)$ on $V \setminus 0$. There exists a lifting of P along the mapping

$$(\Lambda^2 \mathcal{T})_0 \to p^{-1}(\Lambda^2 T).$$

Indeed, by pushing forward along p to $\mathbf{P}(V)$ one has:

$$H^1(V \setminus 0, p^{-1}(T)) = H^1(\mathbf{P}(V), T) = 0.$$

Let $\tilde{P} \in H^0(\Lambda^2 \mathcal{T})_0$ be a lifting of P. In homogeneous coordinates $\tilde{P}$ takes a form:

$$\tilde{P} = \sum a_{ij}^{kl} \xi_k \xi_l \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j}$$

P is a Poisson bracket if and only if

$$p_{3*}[\tilde{P}, \tilde{P}] = 0,$$

where $p_{3*} : (\Lambda^3 \mathcal{T})_0 \to \Lambda^3 T$ is a natural projection. It follows from the following fact, which can be easily checked locally:

Lemma. Let $f : U \to V$ be a smooth map of two smooth varieties. Let A, B be a pair of multivectors on V such that there exist liftings $\tilde{A}, \tilde{B}$ of them along f. Then Schouten bracket commutes with f in the following sense:

$$[A, B]_V = f_*[\tilde{A}, \tilde{B}]_U$$

(obviously, f_* exists).

The kernel of p_{3*} is generated by fields of the form:

$$F = E \wedge G,$$

where E is the Euler field, G belongs to $H^0(\Lambda^2 \mathcal{T})_0$.

Short exact sequences (4) can be glued in a long exact sequence of sheaves(and, analogously, of its global sections):

$$0 \to \mathcal{O}_0 \to \mathcal{T}_0 \to \Lambda^2\mathcal{T}_0 \to \Lambda^3\mathcal{T}_0 \to ... \qquad (5)$$

where differential D_E is the multiplication by the Euler field:$D_E(P) = E \wedge P$.

Thus, the restriction on $\tilde{P}$ which defines a Poisson bracket on $\mathbf{P}(V)$, has the form:

$$D_E(D(\tilde{P} \wedge \tilde{P}) - 2\tilde{P} \wedge D(\tilde{P})) = 0 \qquad (6)$$

Lemma. In the complex (5) one has:

$$D_E D + D D_E = (n+1)\mathrm{id},$$

where id is the identity operator.

Certainly, any quadratic Poisson bracket on $V \setminus 0$ projects into a Poisson bracket on $\mathbf{P}(V)$. This fact is simply a specification of symplectic reducton for Poisson brackets in the case of the map $p : V \setminus 0 \to \mathbf{P}(V)$.

Infinitisimal version of the first question posed in the introduction sounds as follows:

Does there exist a lifting of a Poisson bracket on the projective space to a quadratic Poisson bracket on V?

Theorem. Let $P \in \Lambda^2 T$ be a Poisson bracket on $\mathbf{P}(V)$. There exists a unique lifting $\tilde{P}$ of it to a Poisson bracket on V with properties:

i) $L_E \tilde{P} = 0$,

ii) $D\tilde{P} = 0$

(L_E is a Lie derivative along E).

Proof. Consider the complex of the global sections of the sequence (5) :

$$0 \to k \to V^* \otimes V \to S^2 V^* \otimes \Lambda^2 V \to ... \qquad (7)$$

which is an exact sequence. Recall that the operator D is also a differential in (7) ($D^2 = 0$), which decreases the grading, i.e. acts in the opposite direction to D_E.

It easily follows from the lemma that any element $P \in H^0((\Lambda^2\mathcal{T})_0) = S^2V^* \otimes \Lambda^2 V$ can be uniquely decomposed into a sum: $P = Q + R$, where $DQ = 0, D_E R = 0$. A lifting of any bivector field on a projective space is determined up to adding by an element R, such that $D_E R = 0$. Hence, for any $P \in H^0(\mathbf{P}(V), \Lambda^2 T)$ there exists unique $\tilde{P}$, such that $\tilde{P}$ is projecting into P and $D\tilde{P} = 0$.

Since P is a Poisson bracket one deduces from (6):

$$D_E D(\tilde{P} \wedge \tilde{P}) = 0.$$

Applying the operator D to this equality and using the lemma one obtains:

$$DD_E D(\tilde{P} \wedge \tilde{P}) = (n+1)D(\tilde{P} \wedge \tilde{P}) - D_E D^2(\tilde{P} \wedge \tilde{P}) = (n+1)D(\tilde{P} \wedge \tilde{P}) = 0.$$

Therefore,

$$[\tilde{P}, \tilde{P}] = D(\tilde{P} \wedge \tilde{P}) - 2D(\tilde{P}) \wedge \tilde{P} = 0.$$

QED

Quadratic Poisson bracket is an element of $S^2V^* \otimes \Lambda^2 V$. As a $GL(V)$-module this space can be decomposed (if $n \geq 2$) into a direct sum of two components - indecomposable representations of $GL(V)$ with Young diagrams (3,1) and (2,1,1). The decomposition, which was mentioned in the proof of the theorem coincides with the decomposition of a tensor into these two invariant summands (though it also exists on a local level).

In the case of $\mathbf{P}^1$ there is only one trivial Poisson bracket on it and a 3-dimensional space of the quadratic ones:

$$\tilde{P} = (a\xi^2 + b\xi\eta + c\eta^2)\frac{\partial}{\partial\xi} \wedge \frac{\partial}{\partial\eta}.$$

In the case of $\mathbf{P}^2$, any global section of $\Lambda^2 T \simeq \mathcal{O}(3)$ (thus, a cubic form) determines a Poisson bracket on $\mathbf{P}^2$, since $\Lambda^3 T = 0$.

Quadratic Poisson bracket $\tilde{P}$ can be decomposed into the sum : $P = Q + R$, where $DQ = 0, D_E R = 0$. Put $G = DR$, $f = D_E Q$. G is a non-divergent linear vector field and f, is a cubic form. One can easily see that the Poisson equations (4) is equivalent in this case to the property that f is vanished by the Lie action of the field G. There is no such a fields for a generic cubic form on three variables. This means, that for a generic Poisson bracket on $\mathbf{P}^2$ there exist the only lifting to a quadratic Poisson bracket, namely, the one described in the theorem.

Non-trivial Poisson brackets on $\mathbf{P}^3$ are of rank 2 in the sense that rank of a bracket as a bilenear form on the cotangent space at a generic point is equal to 2.

It is usuful to study Poisson brackets of rank 2 by means of algebraic geometry. One can corresponde to a rank 2 Poisson bracket a rank 2 reflexive sheaf on $\mathbf{P}^n$. In this way one obtains an effective description of rank 2 Poisson brackets.

Let me state at the end a conjecture about degeneracy loci of a Poisson bracket on a Fano variety.

Conjecture. Let X be a Fano variety. Denote by X_k:

$$X_k = \{x \in X, (rank\ of\ the\ bracket\ at\ x) = k\}.$$

Then X_k has a component of dimension more than k.

For the case $k = n - 3$, n odd, where $n = \dim X$, this fact can be proved with using of Bott's theorem on homological obstructions to integrability of a subbundle in the tangent bundle. There are also some evidence of examples, which come from physics (Sklyanin algebras).

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