Shintani Functions and its application to Automorphic L-Functions on Classical Groups

I. The Case of Orthogonal Groups

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I. The case of Orthogonal Groups

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§0. Introduction

Let H be a connected reductive group with a faithful action on a vector space W. We suppose that H preserves a non-degenerate symmetric (or skew-symmetric or hermitian) form T of W. Let G be the stabilizer subgroup of an element w_0 of W in H. For a pair of automorphic forms F and f on H and G respectively, we define a function $\omega_{F,f}$ on $H(\mathbf{A})$ in the following manner:

$$(0.1) \qquad \omega_{F,f}(h) = \int_{G(\mathbf{Q})\backslash G(\mathbf{A})} F(gh)f(g)dg \quad , \qquad (h \in H(\mathbf{A})).$$

The object of this paper is to study this function, which we call *the global Shintani function associated with* F *and* f. Such a function was first introduced by Shintani ([Shi]; cf. [MS1]) for the case where H is the symplectic group of $W = \mathbf{Q}^{2(n+1)}$ equipped with the usual alternating form T and $w_0 = {}^t(1, 0, \cdots, 0) \in W$. Note that w_0 is an *isotropic* vector with respect to T and that G is the Jacobi group of degree n (a semi direct product of the Heisenberg group and Sp_n) in this case. Shintani made several interesting conjectures and gave an application of his function to the theory of automorphic L-functions of Siegel and Jacobi cusp forms.

In a series of the paper, we will study the Shintani function for the case where H is a classical group and w_0 is an *anisotropic* vector (hence G is a classical group of the same type as H). We investigate the orthogonal group case in this first part of the paper.

To explain our results, let $H = O(m+1)$ be the orthogonal group of a quadratic space (W, T) of dimension m+1 and G be the stabilizer subgroup of H of a suitable anisotropic vector of W. Then G is an orthogonal group O(m) of degree m. For a pair of cusp forms F and f on H(A) and G(A), we define $\omega_{F,f}$ by (0.1). Let $\mathcal{H} = \mathcal{H}(H(Q_p), H(Z_p))$ (resp. $\mathcal{H}' = \mathcal{H}(G(Q_p), G(Z_p))$) be the Hecke algebra of H (resp. G) at a finite prime p. Let $\omega_{F,f}^{(p)}$ denote the restriction of $\omega_{F,f}$ to $H(Q_p)$. If both F and f are Hecke eigenforms and if p is a *good* prime, then $\omega = \omega_{F,f}^{(p)}$ has the following property:

$$(0.2) \qquad (\varphi_p * \omega * \Phi_p)(h_p) := \int_{G_p} dx_p \int_{H_p} dy_p \ \varphi_p(x_p)\, \omega(x_p h_p y_p^{-1})\, \Phi_p(y_p)$$

$$= \lambda_p(\varphi_p)\, \Lambda_p(\Phi_p)\, \omega(h_p) \qquad (h_p \in H(Q_p), \Phi_p \in \mathcal{H}, \varphi_p \in \mathcal{H}').$$

Here $\Lambda_p \in \mathrm{Hom}_C(\mathcal{H}, C)$ (resp. $\lambda_p \in \mathrm{Hom}_C(\mathcal{H}', C)$) is determined by the local component Π_p (resp. π_p) of the automorphic representation Π (resp. π) assocated to F (resp. f). The space of C-valued functions on $G(Z_p)\backslash H(Q_p)/H(Z_p)$ satisfying (0.2) is denoted by $\Omega(\Lambda_p, \lambda_p)$ and called *the space of local Shintani functions attached to* Λ_p *and* λ_p. Then we may conjecture the following uniqueness of local Shintani functions:

(0.3) $\quad \dim_C \Omega(\Lambda_p, \lambda_p) \le 1$ for every Λ_p and λ_p?

A similar fact was conjectured by Shintani ([Shi]) in the case of the symplectic group. Furthermore he conjectured that the equality

$$(0.4)\qquad \int_{\mathbf{Q}_p^\times} \omega\left(\begin{pmatrix} t & & & \\ & 1_n & & \\ & & t^{-1} & \\ & & & 1 \end{pmatrix}\right) |t|^{s-n-1}\, d^\times t = \omega(1)\times \frac{L_p(\Lambda_p;\, s)}{L_p(\lambda_p;\, s+1/2)}$$

holds for $\omega \in \Omega(\Lambda_p, \lambda_p)$, where $L_p(\Lambda_p, s)$ and $L_p(\lambda_p, s)$ are the local standard zeta functions attached to Λ_p and λ_p. These two conjectures was proved in [MS1].

In this paper, we establish a similar formula to (0.4) for the case of orthogonal groups (Theorem 1.6), though the uniqueness problem (0.3) is still open in this case. Furthermore we introduce and study a certain Rankin-Selberg convolution attached to F and f. To be more precise, we let $G_1 = O(m+2)$ be a bigger orthogonal group containing H as the stabilizer subgroup of an anisotropic vector, and P_1 a maximal parabolic subgroup of G_1 whose Levi component is isomorphic to $GL(1) \times G$. Then we can construct an Eisenstein series $E(g_1, f; s)$ on G_1 attached to a cusp form f after Langlands ([L2]). The convolution of Rankin-Selberg type we study is given by

$$(0.5)\qquad Z_{F,f}(s) = \int_{H(\mathbf{Q})\backslash H(\mathbf{A})} F(h)\, E(h, f; s)\, dh.$$

Unwinding the Eisenstein series in (0.5), we obtain the "basic identity" between $Z_{F,f}(s)$ and a certain integral of the Shintani function $\omega_{F,f}$ (Theorem 1.5). Then the local result mentioned above implies that $Z_{F,f}(s)$ is equal to $\frac{L(F;\, s)}{L(f;\, s+1/2)}$ up to an elementary factor, where $L(F; s)$ (resp. $L(f; s)$) is the standard zeta function of F (resp. f) (Corollary 1.7). Therefore, at least when H is definite, we can describe the functional equation of $L(F; s)$

in terms of that of L(f; s) (Theorem 1.8). The proof will be carried out along a similar line as in [MS1]. However, in order to include not only unimodular quadratic forms but also *maximal* ones in our argument, we need various subtle facts of the arithmetic of quadratic forms.

It should be mentioned a similarity between our convolution and that of Gelbart and Piatetski-Shapiro for $O(2n) \times GL(n)$ ([GPSR]; see also the work of Piatetski-Shapiro, Rallis and Schiffmann for $G_2 \times GL(2)$ [PSRS]). The difference is that our method yields a quotient of two standard zeta functions of O(m+1) and O(m), though their construction gives the L-function $L(\Phi \times \varphi, \mathrm{std}\otimes\mathrm{std}; s)$, where $\Phi \times \varphi$ is a cusp form on $O(2n) \times GL(n)$ (or on $G_2 \times GL(2)$) and $\mathrm{std}\otimes\mathrm{std}$ is the tensor product of the standard representations of the L-groups. We should also note that our convolution may be considered as an example of "generalized Fourier coefficients of Eisenstein series", which are studied in generality by Furusawa and Shalika ([FS]).

We now explain a brief account of the exposition. In §1, after preparing several notation, we state our main results (§1.9). We show that these results are direct consequences of two key lemmas (Lemma A and B). In addition, we discuss several conjectures on analytic properties of the standard zeta functions of *definite* orthogonal groups. The next two sections are of preliminary nature. In §2, we construct embeddings of ε-hermitian spaces crucial in our argument and study its properties needed to establish the basic identity. In §3, we summarize several facts of the arithmetic of maximal lattices of ε-hermitian spaces to study the behavior of maximal open compact subgroups under the embeddings. In these two sections, we include the cases of unitary groups of ε-hermitian forms for future application. The last two sections are devoted to proofs of the key lemmas. The proof of Lemma A is straightforward and given in §4. The most difficult part is the proof of Lemma B, which can be seen as an analogue of Böcherer's result on Hecke series of Siegel modular forms (see [B]). In fact, we prove Lemma B in §5 by induction on the degree of orthogonal groups. We note that the proof of Lemma B uses Lemma A in an essential way.

As will be noted in §1, we may apply a similar method for classical groups of another type (the unitary groups, the quaternion unitary groups). We hope to investigate these cases in a forthcoming paper.

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§1. Main results

1.1 Embeddings of orthogonal groups Let T be a positive definite even integral symmetric matrix of rank $m+1$. Then the upper left $m \times m$ block S of T is also positive definite and even integral. We say that a non-degenerate even integral symmetric matrix S of rank m is *maximal* if $\mathbf{Z}^m$ is a maximal $\mathbf{Z}$-lattice with respect to S. In what follows, we suppose that both S and T are maximal. Then it is easy to see that $S_1 = \begin{bmatrix} & & 1 \\ & S & \\ 1 & & \end{bmatrix}$ is also maximal. Put

$$L = \mathbf{Z}^m, \ M = \begin{pmatrix} L \\ \mathbf{Z} \end{pmatrix} = \mathbf{Z}^{m+1}, L_1 = \begin{bmatrix} \mathbf{Z} \\ L \\ \mathbf{Z} \end{bmatrix} = \mathbf{Z}^{m+2},$$

$$V = L \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}^m, W = M \otimes_{\mathbf{Z}} \mathbf{Q} = \begin{pmatrix} V \\ \mathbf{Q} \end{pmatrix} = \mathbf{Q}^{m+1}, \ V_1 = L_1 \otimes_{\mathbf{Z}} \mathbf{Q} = \begin{bmatrix} \mathbf{Q} \\ V \\ \mathbf{Q} \end{bmatrix} = \mathbf{Q}^{m+2}.$$

The dual lattices of L, M and L_1 are denoted by $L^* = S^{-1}L$, $M^* = T^{-1}M$ and $L_1^* = S_1^{-1} L_1$. Let $G = O(V, S)$, $H = O(W, T)$ and $G_1 = O(V_1, S_1)$ be the orthogonal groups: $G(\mathbf{Q}) = \{g \in GL_m(\mathbf{Q}) \mid {}^t g S g = S\}$, $\cdots$ etc.

We write

$$T = \begin{pmatrix} S & -S\alpha \\ -{}^t\alpha S & -2a \end{pmatrix} \qquad (\alpha \in L^*, a \in \mathbf{Z})$$

and put $\eta = \begin{bmatrix} a \\ \alpha \\ 1 \end{bmatrix} \in L_1^*$. Then

$$(1.1) \qquad \Delta = S_1[\eta] = S[\alpha] + 2a < 0.$$

We define an embedding $j : W \to V_1$ by

$$(1.2)\qquad j\begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} -az - S(\alpha,y) \\ y \\ z \end{bmatrix} \qquad (y \in V, z \in \mathbf{Q}).$$

Then $V_1 = \mathbf{Q}\cdot\eta \oplus j(W)$ (direct orthogonal sum). Define an embedding $\iota : H \to G_1$ by

$$(1.3)\qquad \iota(h)\ (t\cdot\eta + j(w)) = t\cdot\eta + j(hw) \qquad (h \in H, t \in \mathbf{Q}, w \in W).$$

It is easy to see that $\iota(H)$ is the stabilizer subgroup of η in G_1: $\iota(H) = \{g_1 \in G_1 \mid g_1\eta = \eta\}$. Let P_1 be a maximal parabolic subgroup of G_1 given by

$$P_1(\mathbf{Q}) = \{\begin{bmatrix} t & * & * \\ 0 & g & * \\ 0 & 0 & t^{-1} \end{bmatrix} \mid t \in \mathbf{Q}^\times, g \in G(\mathbf{Q})\}.$$

Lemma 1.1 (cf. Lemma 2.2, Proposition 2.4)

(i) $G_1 = P_1\cdot\iota(H)$.

(ii) $P_1 \cap \iota(H) = \{\begin{bmatrix} 1 & -{}^t\alpha{}^t(g^{-1}-1)S & S((g^{-1}-1)\alpha,\alpha) \\ 0 & g & (1-g)\alpha \\ 0 & 0 & 1 \end{bmatrix} \mid g \in G\} \cong G.$

Let ι' be the embedding of G into H so that

$$(1.4)\qquad \iota(\iota'(g)) = \begin{bmatrix} 1 & -{}^t\alpha{}^t(g^{-1}-1)S & S((g^{-1}-1)\alpha,\alpha) \\ 0 & g & (1-g)\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

for $g \in G$. Then $\iota'(G)$ is the stabilizer subgroup of $\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \in W$ in H. In fact, we have

$$(1.5) \qquad \iota'(g) = \begin{pmatrix} g & (1-g)\alpha \\ 0 & 1 \end{pmatrix} \qquad (g \in G).$$

1.2 Maximal compact subgroups For a $\mathbf{Z}$-lattice X and a prime number p, we write $X_p = X \otimes_{\mathbf{Z}} \mathbf{Z}_p$. By maximality of L_p, $L_p' = \{x \in L_p^* \mid \frac{1}{2}S[x] \in p^{-1}\mathbf{Z}_p\}$ is a $\mathbf{Z}_p$-lattice containing L_p and L_p'/L_p forms a finite dimensional vector space over $\mathbf{Z}_p/p\mathbf{Z}_p = \mathbb{F}_p$. We set $\partial_p(S) = \dim_{\mathbb{F}_p} L_p'/L_p$. It is known that $0 \le \partial_p(S) \le 2$. The quantity $\partial_p(T)$ for T is similarly defined.

Let $K_p = G(\mathbf{Z}_p)$, $U_p = H(\mathbf{Z}_p)$ and $K_{1,p} = G_1(\mathbf{Z}_p)$ be maximal open compact subgroups of $G_p = G(\mathbf{Q}_p)$, $H_p = H(\mathbf{Q}_p)$ and $G_{1,p} = G_1(\mathbf{Q}_p)$ respectively.

Lemma 1.2 (cf. Lemma 3.6) *If* $\partial_p(T) = \partial_p(S)$, *we have*

$$\iota(U_p) = \iota(H_p) \cap K_{1,p},$$

$$\iota'(K_p) = \iota'(G_p) \cap U_p.$$

1.3 Hecke algebras In this subsection, we let $S \in M_m(\mathbf{Z}_p)$ be a non-degenerate maximal even integral symmetric matrix (in this case, "maximal" means that $\mathbf{Z}_p^m$ is a maximal $\mathbf{Z}_p$-lattice with respect to S). Put $G = O(S)$ and $K = G(\mathbf{Z}_p)$. Let $H(G_p, K_p)$ be the $\mathbf{C}$-algebra of compactly supported bi-K_p-invariant functions on G_p. Denote by $\nu_p = \nu_p(S)$ the Witt index of S at p. The Satake isomorphism Ψ_p gives an isomorphism of the Hecke algebra $H(G_p, K_p)$ onto $\mathbf{C}[T_1^{\pm 1}, \cdots, T_{\nu_p}^{\pm 1}]^{W_{\nu_p}}$, where $\mathbf{C}[T_1^{\pm 1}, \cdots, T_{\nu_p}^{\pm 1}]^{W_{\nu_p}}$ denotes the algebra of polynomials in $T_1^{\pm 1}, \cdots, T_{\nu_p}^{\pm 1}$ invariant under the subgroup W_{ν_p} of the automorphism group of $\mathbf{C}[T_1^{\pm}, \cdots, T_{\nu_p}^{\pm 1}]$ generated by the permutations of $T_1, \cdots, T_{\nu_p}$ and the involutions $T_i \to T_i^{-1}$ $(1 \le i \le \nu_p)$ ((see [Sa]). Thus the $\mathbf{C}$-algebra homomorphisms of $H(G_p, K_p)$ to $\mathbf{C}$ are parametrized by

$(\mathbf{C}^\times)^{\nu_p}/W_{\nu_p}$ (*Satake parameters*). For $\lambda = (\lambda_1, \cdots, \lambda_{\nu_p}) \in (\mathbf{C}^\times)^{\nu_p}/W_{\nu_p}$, we denote by $\lambda^\wedge$ the corresponding homomorphism of $H(G_p, K_p)$ to $\mathbf{C}$:

$$(1.6) \qquad \lambda^\wedge(\varphi) = \Psi_p(\varphi)(\lambda_1, \cdots, \lambda_{\nu_p}) \qquad (\varphi \in H(G_p, K_p)).$$

For $\lambda = (\lambda_1, \cdots, \lambda_{\nu_p}) \in (\mathbf{C}^\times)^{\nu_p}/W_{\nu_p}$, we define the L-factor $L_p(\lambda; s)$ as follows (cf. [Su], [MS2]):

$$(1.7) \qquad L_p(\lambda; s) = L_p^o(\lambda; s)\cdot A_{S,p}(s),$$

where

$$(1.8) \qquad L_p^o(\lambda; s) = \prod_{i=1}^{\nu_p(S)} \{(1 - \lambda_i p^{-s})(1 - \lambda_i^{-1} p^{-s})\}^{-1}$$

and

$$(1.9) \quad A_{S,p}(s) = \begin{cases} 1 & \text{if } (n_{o,p}(S), \partial_p(S)) = (0,0) \text{ or } (1,0) \\ (1 + p^{1/2-s}) & (1,1) \\ (1 - p^{-2s})^{-1} & (2,0) \\ (1 - p^{-s})^{-1} & (2,1) \\ (1 - p^{-s})^{-1}(1 + p^{1-s}) & (2,2) \\ (1 - p^{-1/2-s})^{-1} & (3,1) \\ (1 - p^{-1/2-s})^{-1}(1 + p^{1/2-s}) & (3,2) \\ (1 - p^{-s})^{-1}(1 - p^{-1-s})^{-1} & (4,2). \end{cases}$$

with $n_{o,p}(S) = m - 2\,\nu_p(S)$. It is well-known that $0 \le n_{o,p}(S) \le 4$.

1.4 Automorphic forms on definite orthogonal groups Going back to the notation of §1.1-2, we put $K_f = \prod_{p<\infty} K_p$. Let $S(K_f)$ be the space of automorphic forms on $G(\mathbf{A})$ given as follows:

$$S(K_f) = \{f : G(\mathbf{A}) \to \mathbf{C} \mid F(\gamma \cdot g \cdot g_\infty k_f) = f(g) \text{ for } \gamma \in G(\mathbf{Q}), g \in G(\mathbf{A}),$$
$$g_\infty \in G(\mathbf{R}) \text{ and } k_f \in K_f\},$$

where $\mathbf{A}$ stands for the adele ring of $\mathbf{Q}$. Note that $G(\mathbf{Q})\backslash G(\mathbf{A})/G(\mathbf{R})K_f$ is in fact a finte set since S is positive definite. We similarly define the space $S(U_f)$ of automorphic forms on $H(\mathbf{A})$ left invariant under $H(\mathbf{Q})$ and right invariant under $H(\mathbf{R})U_f$ with $U_f = \prod_{p<\infty} U_p$.

The Hecke algebra $H(G_f, K_f) = \otimes'_{p<\infty} H(G_p, K_p)$ (restricted tensor product) acts on $S(K_f)$ in a natural manner. Let $f \in S(K_f)$ be a Hecke eigenform on $G(\mathbf{A})$. This means that f is a common eigenform under the action of $H(G_f, K_f)$. Then, for each p, the Satake parameter $\lambda_{f,p} \in (\mathbf{C}^\times)^{V_p}/W_{V_p}$ is attached to f by $f * \varphi_p = \hat{\lambda}_{f,p}(\varphi_p) \cdot f$ for $\varphi_p \in H(G_p, K_p)$. We set

$$(1.10) \qquad L(f; s) = \prod_{p<\infty} L_p(f; s), \; L_p(f; s) = L_p(\lambda_{f,p}; s).$$

We call $L(f; s)$ *the standard zeta function attached to* f. The L-function $L(F; s)$ for a Hecke eigenform $F \in S(U_f)$ is defined in a similar manner. For the standard zeta functions of classical groups, refer to [GPSR] and [PSR].

1.5 The gamma factor of L(f; s) We set

(1.11) $A_{S,\infty}(s) =$

$$
\begin{cases}
(2\pi)^{-\rho s} (\det S)^{s/2} \prod_{j=1}^{\rho/2} \Gamma(s-\rho-1+2j)\, \Gamma(s-2+2j) & \text{if } m \equiv 0 \pmod 4 \\
(2\pi)^{-\rho s} (\det S)^{s/2} \prod_{j=1}^{(\rho+1)/2} \Gamma(s-\rho-1+2j) \prod_{j=1}^{(\rho-1)/2} \Gamma(s-1+2j) & \text{if } m \equiv 2 \pmod 4 \\
(2\pi)^{-\rho s} (2^{-1}\det S)^{s/2} \prod_{j=1}^{\rho} \Gamma(s-\rho-\frac{3}{2}+2j) & \text{if } m \text{ is odd.}
\end{cases}
$$

where $\rho = [\frac{m}{2}]$.

For a Hecke eigenform $f \in S(K_f)$, define

(1.12) $$\xi(f; s) = A_{S,\infty}(s)\, L(f; s).$$

It is known that $\xi(F; s)$ is continued to a meromorphic function of s on $\mathbf{C}$ (cf. Lemma 1.3). We put

(1.13) $$c(f; s) = \frac{\xi(f; s)}{\xi(f; 1-s)}$$

1.6 Conjectures on analytic properties of $\xi(f; s)$ In this subsection, we state several conjectures on $\xi(f; s)$ for a Hecke eigenform $f \in S(K_f)$ under the assumption that S is a maximal positive definite even integral symmetric matrix of rank m.

Conjecture 1 *The functional equation*

$$\xi(f; s) = \varepsilon_m\, \xi(f; 1-s)$$

holds. Here we put

$$\varepsilon_m = \begin{cases} -1 & \text{if } m \equiv \pm 3 \pmod 8 \\ 1 & \text{otherwise.} \end{cases}$$

Cojecture 2 *The poles of* $\xi(f; s)$ *are contained in the set* $\{s = \frac{m}{2} - k \mid 0 \le k \le m-1\}$. *Furthermore* $\xi(f; s)$ *has at most simple poles at* $s = \frac{m}{2}$ and $\frac{2-m}{2}$.

Conjecture 3 $\xi(f; s)$ *has a simple pole at* $s = \frac{m}{2}$ *if and only if* f *is a constant function.*

Remark. These conjectures are known to be true if $m \le 3$ or if f is constant (see [MS2]).

1.7 Eisenstein series In this subsection, we recall the definition of Eisenstein series on G_1 associated with $f \in S(K_f)$. We first define the action of $G_1(\mathbf{R})$ on $D = \mathbf{R}^m \times \mathbf{R}_+^\times$ ($\mathbf{R}_+^\times$ is the set of positive real numbers). For $X = (x, r) \in D$, put $X^\sim = \begin{bmatrix} -r - \frac{1}{2}S[x] \\ x \\ 1 \end{bmatrix} \in \mathbf{R}^{m+2}$. Then, for $(g, X) \in G(\mathbf{R}) \times D$, $g\langle X\rangle \in D$ is defined to be $g \cdot X^\sim = (g\langle X\rangle)^\sim \cdot j(g, X)$ with $j(g, X) \in \mathbf{R}^\times$. We let $K_{1,\infty} = \{g \in G_1(\mathbf{R}) \mid g\langle X_0\rangle = X_0\}$ be a maximal compact subgroup of $G_1(\mathbf{R})$ with $X_0 = (\alpha, -\frac{1}{2}\Delta) \in D$. We see that $\iota(H(\mathbf{R})) \subset K_{1,\infty}$, since $X_0^\sim = \eta$.

For $g_1 \in G_1(\mathbf{A})$, we fix an Iwasawa decomposition

$$g_1 = \begin{bmatrix} \alpha(g_1) & * & * \\ 0 & \beta(g_1) & * \\ 0 & 0 & \alpha(g_1)^{-1} \end{bmatrix} k_1(g_1)$$

where $\alpha(g_1) \in \mathbf{A}^\times$, $\beta(g_1) \in G(\mathbf{A})$ and $k_1(g_1) \in K_{1,\infty} \prod_{p<\infty} K_{1,p}$. For $f \in S(K_f)$ and $s \in \mathbf{C}$, Let $f(g_1; s)$ be a function on $G_1(\mathbf{A})$ given by

$$(1.14) \qquad f(g_1; s) = f(\beta(g_1)) \cdot |\alpha(g_1)|_{\mathbf{A}}^s \qquad (g_1 \in G_1(\mathbf{A})).$$

Here $|\cdot|_{\mathbf{A}}$ denotes the idele norm of $\mathbf{A}^{\times}$.

The Eisenstein series associated with f is defined by

$$(1.15) \qquad E(g_1, f; s) = \sum_{\gamma_1 \in P_1(Q)\backslash G_1(Q)} f(\gamma_1 g_1; s + \frac{m}{2}) \qquad (g_1 \in G_1(A)).$$

Put

$$(1.16) \qquad u_m(s) = \prod_{j=1}^{\sigma_m} (s + \frac{m}{2} + 1 - 2j)$$

where

$$(1.17) \qquad \sigma_m = [\frac{m+1}{4}]$$

For a Hecke eigenform $f \in S(K_f)$, we define the *normalized* Eisenstein series associated with f as follows:

$$(1.18)\ E^*(g, f; s) = \left|-\frac{\Delta}{2}\right|^{s/2} u_m(s)\, \xi(f; s+1)\, E(g, f; s) \times \begin{cases} 1 & \text{if m is even} \\ \xi(2s+1) & \text{if m is odd} \end{cases}$$

where $\xi(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$.

Lemma 1.3 ([L1]; see also [Su]) *Let* $f \in S(K_f)$ *be a Hecke eigenform. Then*

$$E^*(g, f; s) = (-1)^{\sigma_m} c(f; s)\, E^*(g, f; -s).$$

1.8 Rankin-Selberg convolution and Shintani functions In this paper, we study the following Rankin-Selberg convolution $Z_{F,f}(s)$ associated with $F \in S(U_f)$ and $f \in S(K_f)$:

$$(1.19) \qquad Z_{F,f}(s) = \int_{H(Q)\backslash H(A)} F(h)\, E^*(\iota(h), f; s - \frac{1}{2})\, dh.$$

From now on, we often see H (resp. G) as a subgroup of G_1 (resp. H) via the embedding ι (resp. ι'). By Lemma 1.3, we obtain

Proposition 1.4 *The integral $Z_{F,f}(s)$ can be continued to a meromorphic function of s on $\mathbf{C}$ and has a functional equation:*

$$Z_{F,f}(s) = (-1)^{\sigma_m} c(f; s)\, Z_{F,f}(1-s).$$

By unwinding the Eisenstein series in (1.19) and using Lemma 1.1, we get

$$Z_{F,f}(s) = d(f;s) \int_{G(\mathbf{A})\backslash H(\mathbf{A})} \int_{G(\mathbf{Q})\backslash G(\mathbf{A})} F(gh)\, f(\beta(gh))\, |\alpha(gh)|_{\mathbf{A}}^{s+\frac{m-1}{2}}\, dg\, dh$$

$$= d(f;s) \int_{G(\mathbf{A})\backslash H(\mathbf{A})} \int_{G(\mathbf{Q})\backslash G(\mathbf{A})} F(gh)\, f(g\cdot\beta(h))\, |\alpha(h)|_{\mathbf{A}}^{s+\frac{m-1}{2}}\, dg\, dh.$$

Here

(1.20) $$d(f;s) = \left|-\frac{\Delta}{2}\right|^{s/2-1/4} u_m\!\left(s-\frac{1}{2}\right) \xi\!\left(f; s+\frac{1}{2}\right) \times \begin{cases} 1 & \text{if } m \text{ is even} \\ \xi(2s) & \text{if } m \text{ is odd.} \end{cases}$$

Define

(1.21) $$\omega_{F,f}(h) = \int_{G(\mathbf{Q})\backslash G(\mathbf{A})} F(gh)\, f(g)\, dg \qquad (h \in H(\mathbf{A})).$$

We call $\omega_{F,f}$ the *Shintani function associated with* F *and* f. This function plays a central role in our paper. Note that $\omega_{F,f}(1) = \langle F|_{G(\mathbf{A})}, \overline{f} \rangle$ where $\langle\, ,\, \rangle$ is the usual Petersson inner product in $S(K_f)$. By changing the variable g into $g\,\beta(h)^{-1}$, we obtain

Theorem 1.5 (The basic identity)

$$Z_{F,f}(s) = d(f;\, s) \int_{G(\mathbf{A})\backslash H(\mathbf{A})} \omega_{F,f}(\beta(h)^{-1}h)\, |\alpha(h)|_{\mathbf{A}}^{s+\frac{m-1}{2}}\, dh.$$

Remark. In view of Proposition 2.4 and Lemma 2.2 in the next section, it is easy to see that a similar formula holds for *cusp* forms F and f on the unitary groups of (not necessarily definite) ε-hermitian forms.

1.9 Main results In what follows, we assume that $F \in S(U_f)$ and $f \in S(K_f)$ are Hecke eigenforms. Let $\Lambda_p \in (\mathbf{C}^\times)^{\nu_p(T)} / W_{\nu_p(T)}$ and $\lambda_p \in (\mathbf{C}^\times)^{\nu_p(S)} / W_{\nu_p(S)}$ be the Satake parameters corresponding to F and f. For $h' \in H(\mathbf{A})$ with the p-component = 1, the function $h_p \to \omega_{F,f}(h'h_p)$ on H_p belongs to the C-vector space

$$\Omega(\Lambda_p, \lambda_p) = \{\omega : H_p \to \mathbf{C} \mid \text{(i)}\ \ \omega(khu) = \omega(h) \quad (k \in K_p,\, h \in H_p,\, u_p \in U_p)$$
$$\text{(ii)}\ \ \varphi * \omega * \Phi = \lambda_p^{\wedge}(\varphi)\, \Lambda_p^{\wedge}(\Phi)\cdot\omega \quad (\varphi \in \mathcal{H}(G_p, K_p),\, \Phi \in \mathcal{H}(H_p, U_p))\}$$

where

$$(\varphi * \omega * \Phi)(h) = \int_{G_p} dx \int_{H_p} dy\, \varphi(x)\, \omega(xhy^{-1})\, \Phi(y).$$

We call $\Omega(\Lambda_p, \lambda_p)$ the space of *local Shintani functions associated with* Λ_p *and* λ_p. Such functions were first introduced by Shintani in his unpublished work ([Shi] ; see [MS1] for detail) in another situation ($G \to$ the Jacobi group of degree n, $H \to Sp_{n+1}$).

One of our main results is as follows:

Theorem 1.6 *Assume that* $\partial_p(T) = \partial_p(S)$. *For* $\omega \in \Omega(\Lambda_p, \lambda_p)$, *define*

$$Z_\omega(s) = \int_{G_p\backslash H_p} \omega(\beta(h)^{-1}h)\, |\alpha(h)|_p^{s+\frac{m-1}{2}}\, dh.$$

Then

$$Z_\omega(s) = \frac{L_p(\Lambda_p;\ s)}{L_p(\lambda_p;\ s+\frac{1}{2})}\ \omega(1) \times \begin{cases} 1 & \textit{if } m \textit{ is even} \\ \zeta_p(2s)^{-1} & \textit{if } m \textit{ is odd} \end{cases}$$

where $\zeta_p(s) = (1-p^{-s})^{-1}$.

From the above local result, we obtain the following global one:

Corollary 1.7 *Assume that* $\partial_p(T) = \partial_p(S)$ *for every prime* p. *Let* $F \in S(U_f)$ *and* $f \in S(K_f)$ *be Hecke eigenforms. Then*

$$Z_{F,f}(s) = c\ \omega_{F,f}(1)\cdot\xi(F;\ s)\ u_m(s-\tfrac{1}{2})\ u_m(\tfrac{1}{2}-s) \times \begin{cases} \frac{1}{2}-s & \textit{if } m \equiv 2 \pmod 4 \\ 1 & \textit{otherwise} \end{cases}$$

with a non-zero constant c *independent of* F *and* f.

Combining Corollary 1.7 with Proposition 1.4, we get a relation between c(F; s) and c(f; s) (for the definition of c(f; s), see (1.13)).

Theorem 1.8 *Let the assumption be the same as in Corollary 1.7. Assume that* $\omega_{F,f}(1) \neq 0$. *Then*

$$c(F;\ s) = \chi_{MH,\infty}(T)\cdot c(f;\ s-\tfrac{1}{2})$$

where $\chi_{MH,\infty}(T)$ *is the Minkowski-Hasse character of* T *at the archimedian prime* :

$$\chi_{MH,\infty}(T) = \begin{cases} 1 & \text{if } m+1 \equiv 0, 1, 2, 7 \ (8) \\ -1 & \text{if } m+1 \equiv 3, 4, 5, 6 \ (8). \end{cases}$$

Corollary 1.9 *Let the assumption be the same as in Theorem 1.8. Then*

(i) If Conjecture 1 holds for f, *then so does for* F.

(ii) If Conjectures 1 and 2 hold for f, *then so do for* F.

(iii) If Conjectures 1, 2 and 3 hold for f, *then so do for* F.

Proof. The first part follows immediately from Theorem 1.8. As is well-known, all the poles of $E^*(g, f; s)$ are contained in those of its constant term

$$E_0^*(g, f; s) = \int_{Q^m \backslash A^m} E^*(\begin{bmatrix} 1 & -{}^tXS & -2^{-1}S[X] \\ & 1_m & X \\ & & 1 \end{bmatrix} g, f; s)\, dX.$$

By straightforward calculation, we have

$$(1.22) \qquad E_0^*(\begin{bmatrix} \alpha & & \\ & \beta & \\ & & \alpha \end{bmatrix}, f; s)$$

$$= \xi(f; s+1)\, u_m(s)\, |\alpha|_A^{s+m/2}\, |-\Delta/2|^{s/2}\, f(\beta) \times \begin{Bmatrix} 1 & \text{if m is even} \\ \xi(2s+1) & \text{if m is odd} \end{Bmatrix}$$

$$+ (-1)^{\sigma_m}\, \xi(f; s)\, u_m(-s)\, |\alpha|_A^{-s+m/2}\, |-\Delta/2|^{-s/2}\, f(\beta) \times \begin{Bmatrix} 1 & \text{if m is even} \\ \xi(2s) & \text{if m is odd} \end{Bmatrix}.$$

We assume that f satisfies Conjectures 1 and 2. Then the above formula implies that the poles of $E^*(g, f; s)$ are contained in the set $\{-\frac{m}{2} + k; 0 \le k \le m\}$ and the pole at $s = -\frac{m}{2}$ is at most simple. Finally we assume that f satisfies Conjectures 1, 2 and 3. By

(1.22) the residue of $E^*(g, f; s)$ at $s = \frac{m}{2}$ is equal to $c_1 f(1)$ with a non-zero constant c_1 if f is constant, and equal to 0 otherwise. Taking residues at $s = \frac{m+1}{2}$ of the both sides of the formula for $Z_{F,f}(s)$ in Corollary 1.7, we obtain

$$c_2 \operatorname*{Res}_{s=(m+1)/2} \xi(F; s) \langle F|_G, f\rangle_G = \begin{cases} \langle F, 1\rangle_H & \text{if } f \text{ is constant} \\ 0 & \text{otherwise} \end{cases},$$

where c_2 is a non-zero constant and $\langle , \rangle_G$ (resp. $\langle , \rangle_H$) stands for the Petersson inner product in $S(K_f)$ (resp. $S(U_f)$). Therfore Conjecture 3 holds for F under the assumption $\omega_{F,f}(1) \neq 0$. *q.e.d.*

Let S be a non-degenerate even integral symmetric matrix of rank m. We say that S has *the property (I)* if S is maximal and if S satisfies one of the following conditions:

(i) $m \leq 3$.

(ii) There exist $\gamma \in GL_m(\mathbf{Z})$ and an even integral symmetric matrix S' of rank $m-1$ with the property (I) such that ${}^t\gamma S \gamma = \begin{pmatrix} S' & -S'\beta \\ -{}^t\beta S' & -2b \end{pmatrix}$ and $\partial_p(S) = \partial_p(S')$ for every p.

Furthermore suppose that S is positive definite and let f be an automorphic form on $O(S)$ in the sense of §1.4.

Corollary 1.10 *Let* $S \in M_m(\mathbf{Z})$ *be a positive definite even integral symmetric matrix with the property (I). Let* f *be a Hecke eigenform on* $O(S)$. *If* $f(1) \neq 0$, *then Conjectures 1, 2 and 3 hold for* f.

1.10 First main lemma We need two lemmas (Lemma A and Lemma B) to prove Theorem 1.6. We let the notation be the same as in §1.1-2. Recall that $g_1 \in G_{1,p}$ is decompose into

$$\begin{bmatrix} \alpha(g_1) & * & * \\ 0 & \beta(g_1) & * \\ 0 & 0 & \alpha(g_1)^{-1} \end{bmatrix} k_1(g_1)$$

with $\alpha(g_1) \in \mathbf{Q}_p^\times$, $\beta(g_1) \in G_p$, $k_1(g_1) \in K_{1,p}$.

For $x \in M_n(\mathbf{Q}_p)$, we put $\mu_{n,p}(x) = \sum_{e_i<0} |e_i|$ where $\{p^{e_1}, \cdots, p^{e_r}, 0, \cdots, 0\}$ is a set of elementary divisors of x. For $s \in \mathbf{C}$, let $N_{G_p,s}$ be the function on G_p defined by

$$N_{G_p,s}(g) = p^{\mu_{m,p}(g)s} \qquad (g \in G_p). \tag{1.23}$$

Obviously $N_{G_p,s}$ is K_p-biinvariant. We define $N_{H_p,s}$ in a similar manner.

Lemma A *Assume that* $\partial_p(T) = \partial_p(S)$. *Then we have*

$$N_{H_p,s}(\iota'(g\beta(h)^{-1})\cdot h) = |\alpha(h)|_p^s \, N_{G_p,s}(g) \qquad (g \in G_p,\ h \in H_p).$$

1.11 Second main lemma Let $S \in M_m(\mathbf{Z}_p)$ be a non-degenerate maximal even integral symmetric matrix. Put $G = O(S)$ and $K = G(\mathbf{Z}_p)$. Then $\mathrm{Hom}_{\mathbf{C}}(H(G, K), \mathbf{C})$ is identified with $(\mathbf{C}^\times)^\nu/W_\nu$, where ν is the Witt index of S. For $\lambda \in (\mathbf{C}^\times)^\nu/W_\nu$, let $\Omega(\lambda)$ be the space of right K-invariant functions w on G satisfying $w * \varphi = \lambda^\wedge(\varphi)\, w$ for $\varphi \in H(G, K)$, where $\lambda^\wedge \in \mathrm{Hom}_{\mathbf{C}}(H(G, K), \mathbf{C})$ is defined by (1.6). Let $N_{G,s}$ be the function defined by (1.23).

Lemma B *For* $w \in \Omega(\lambda)$, *we have*

$$\int_G w(g)\, N_{G,s+m/2-1}(g)\, dg$$

$$= L_p^o(\lambda;\ s) \cdot \prod_{j=0}^{\nu-1} (1 - p^{-(s+j+n_o/2)})\ (1 + p^{-(s+j-\partial+n_o/2)}) \times w(1)$$

with $\partial = \partial_p(S)$ *and* $n_o = m - 2\nu$.

Remark. This result is an analogue of Böcherer's result ([B]) for orthogonal groups.

1.12 Proof of Theoem 1.6 We end this section by giving proof of Theorem 1.6 assuming Lemma A and Lemma B. For $\omega \in \Omega(\Lambda_p, \lambda_p)$, consider the integral

$$I_\omega(s) = \int_{H_p} \omega(h)\, N_{H_p,s+(m-1)/2}(h)\, dh.$$

By Lemma A, we have

$$I_\omega(s) = \int_{G_p\backslash H_p} dh \int_{G_p} dg\ \omega(gh)\, N_{H_p,s+(m-1)/2}(gh)$$

$$= \int_{G_p\backslash H_p} dh \int_{G_p} dg\ \omega(g\, \beta(h)^{-1}h)\ |\,\alpha(h)\,|_p^{s+(m-1)/2}\ N_{G_p,s+(m-1)/2}(g).$$

Applying Lemma B to the integral over G_p, we obtain

$$\int_{G_p} \omega(g\, \beta(h)^{-1}h)\, N_{G_p,s+(m-1)/2}(g)\, dg$$

$$= L_p^o(\lambda_p;\ s + \tfrac{1}{2}) \cdot \prod_{j=0}^{\nu_p(S)-1} (1 - p^{-(s+\frac{n_o(S)+1}{2}+j)})\ (1 + p^{-(s-\partial+\frac{n_o(S)+1}{2}+j)})$$

$$\times\ \omega(\beta(h)^{-1}h)$$

where $n_o(S) = m - 2\nu_p(S)$ and $\partial = \partial_p(S) = \partial_p(T)$. Thus we have

$$I_\omega(s) = Z_\omega(s)\cdot L_p^o(\lambda_p;\, s+\tfrac{1}{2})$$

$$\times \prod_{j=0}^{\nu_p(S)-1} \left(1 - p^{-(s+\frac{n_o(S)+1}{2}+j)}\right)\left(1 + p^{-(s-\partial+\frac{n_o(S)+1}{2}+j)}\right).$$

On the other hand, applying Lemma B to the integral over H_p in the definition of $I_\omega(s)$, we get

$$I_\omega(s) = L_p^o(\Lambda_p;\, s)\cdot \prod_{j=0}^{\nu_p(T)-1} \left(1 - p^{-(s+\frac{n_o(T)}{2}+j)}\right)\left(1 + p^{-(s-\partial+\frac{n_o(T)}{2}+j)}\right)\cdot\omega(1).$$

It remains to show that

$$(1.24)\qquad \frac{\displaystyle\prod_{j=0}^{\nu_p(T)-1} \left(1 - p^{-(s+\frac{n_o(T)}{2}+j)}\right)\left(1 + p^{-(s-\partial+\frac{n_o(T)}{2}+j)}\right)}{\displaystyle\prod_{j=0}^{\nu_p(S)-1} \left(1 - p^{-(s+\frac{n_o(S)+1}{2}+j)}\right)\left(1 + p^{-(s-\partial+\frac{n_o(S)+1}{2}+j)}\right)}$$

$$= \frac{A_{T,p}(s)}{A_{S,p}(s+\frac{1}{2})} \times \begin{cases} 1 & \text{if m is even} \\ \zeta_p(2s)^{-1} & \text{if m is odd.} \end{cases}$$

To prove this, observe that $n_o(T) = n_o(S) - 1 \Rightarrow \nu_p(T) = \nu_p(S) + 1$ and that $n_o(T) = n_o(S) + 1 \Rightarrow \nu_p(T) = \nu_p(S)$. This implies that the left-hand side of (1.24) is equal to

$$\begin{cases} \left(1 - p^{-(s+\frac{n_o(S)-1}{2})}\right)\left(1 + p^{-(s-\partial+\frac{n_o(S)-1}{2})}\right) & \text{if } n_o(T) = n_o(S) - 1 \\ 1 & \text{if } n_o(T) = n_o(S) + 1. \end{cases}$$

Then (1.24) is a straightforward consequence of the defintions of $A_{T,p}(s)$ and $A_{S,p}(s)$ in §1.3 in view of the following possible combination of $(n_o(S), n_o(T); \partial)$:

$$(n_o(S), n_o(T); \partial) = \quad (1, 0; 0), (2, 1; 0), (2, 1; 1), (3, 2; 1), (3, 2; 2), (4, 3; 2),$$

$$(0, 1; 0), (1, 2; 0), (1, 2; 1), (2, 3; 1), (2, 3; 2), (3, 4; 2).$$

q.e.d.

§2 Classical groups and embeddings

2.1 Classical groups Let k be a field whose characteristic is different from 2. Let K be a k-semisimple algebra that is one of the following three types:

$$K = \begin{cases} \text{k itself} & \text{(I)} \\ \text{a quadratic extention of k} & \text{(II)} \\ \text{a quaternion algebra over k} & \text{(III).} \end{cases}$$

Let $x \to \overline{x}$ be the involution of K given as follows:

$$\begin{cases} \text{the identity} & \text{in case (I)} \\ \text{the unique nontrivial automorphism of K of k} & \text{in case (II)} \\ \text{the main involution of K over k} & \text{in case (III).} \end{cases}$$

For $X \in M_{m,n}(K)$, put $X^* = {}^t\overline{X}$. For $\varepsilon = \pm 1$, we say that $S \in M_m(K)$ is an *ε-hermitian matrix* if $S^* = \varepsilon S$.

Let S be a non-degenerate ε-hermitian matrix of rank m. We define the unitary group $U(S)$ by $U(S)_k = \{g \in GL_m(K) \mid g^*Sg = S\}$. Let K^m denote the space of m-column vectors in K. For $x, y \in K^m$, we write $S(x, y) = x^*Sy$, $S[x] = x^*Sx$. An ε-hermitian matrix S is said to be *k-anisotropic* if $S[x] \neq 0$ for every $x \in K^m-\{0\}$ and *k-isotropic* otherwise. For $\xi \in K$, we put

$$\tau(\xi) = \xi + \varepsilon\,\overline{\xi}\,,\ N(\xi) = \xi\,\overline{\xi}\,,\ Tr(\xi) = \xi + \overline{\xi}\,.$$

Set $d = \dim_k K$ and $\kappa = \dim_k \mathrm{Ker}\,\tau$. There exist the following five cases:

$$\begin{cases} \text{(O)-case} & \text{(I)} & \varepsilon = 1 & \cdots & (d=1,\ \kappa=0) \\ \text{(Sp)-case} & \text{(I)} & \varepsilon = -1 & \cdots & (d=1,\ \kappa=1) \\ \text{(U)-case} & \text{(II)} & \varepsilon = \pm 1 & \cdots & (d=2,\ \kappa=1) \\ (\text{U}^+)\text{-case} & \text{(III)} & \varepsilon = 1 & \cdots & (d=4,\ \kappa=3) \\ (\text{U}^-)\text{-case} & \text{(III)} & \varepsilon = -1 & \cdots & (d=4,\ \kappa=1). \end{cases}$$

In what follows, the (Sp)-case is excluded for simplicity, though the algebraic group Sp_n appears as in (U^+)-case for $K = M_2(K)$.

Lemma 2.1 *If S is k-anisotropic and not of type (U^-), then* K *is a division algebra.*

Proof. Write $S = (s_{ij})$. Since $\overline{s_{11}} = \varepsilon s_{11}$, s_{11} is in the center of K (note that the (U^-)-case is excluded here). If K is not division, there exists $x \in K - \{0\}$ such that $x\overline{x} = 0$. Then we have $S[\begin{pmatrix} x \\ 0 \end{pmatrix}] = \overline{x}\, s_{11} x = s_{11}\, \overline{x}\, x = 0$, which contradicts to the assumption that S is k-anisotropic. *q.e.d.*

2.2 Embeddings of ε-hermitian spaces and unitary groups In what follows, we fix a non-degenerate ε-hermitian matrix S of rank m. Then S defines an ε-hermitian form on the right K-module $V = K^m$. Put

$$S_1 = \begin{bmatrix} & & \varepsilon \\ & S & \\ 1 & & \end{bmatrix} \in GL_{m+2}(K)$$

and $V_1 = \begin{pmatrix} K \\ V \\ K \end{pmatrix}$. Choose and fix an element $\eta = \begin{bmatrix} a \\ \alpha \\ 1 \end{bmatrix}$ of V_1 ($a \in K, \alpha \in V$) so that

$$(2.1) \qquad \Delta := S_1[\eta] = \tau(a) + S[\alpha]$$

is invertible in K. Then $V_1 = \eta \cdot K \oplus \eta^\perp$ (orhtogonal sum with respect to S_1) where $\eta^\perp = \{X \in V_1 \mid S_1(\eta, X) = 0\}$. Define a right K-linear isomorphism j of $W = \begin{pmatrix} V \\ K \end{pmatrix} = K^{m+1}$ onto $\eta^\perp \subset V_1$ by

$$(2.2) \qquad j\begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} -\varepsilon\, \overline{a}\, z - S(\alpha, y) \\ y \\ z \end{bmatrix} \qquad (y \in V, z \in K).$$

Let T be an ε-hermitian matrix of rank m+1 given by

$$T[\begin{pmatrix} y \\ z \end{pmatrix}] = S_1[j(\begin{pmatrix} y \\ z \end{pmatrix})] \qquad (y \in V, z \in K).$$

Then we have

$$T = \begin{pmatrix} S & -S\alpha \\ -\alpha^*S & -\tau(a) \end{pmatrix}.$$

By the assumption $\Delta \in K^{\times}$, T is non-degenerate. We write G, H and G_1 for the unitary groups $U(S), U(T)$ and $U(S_1)$ respectively.

Define an embedding $\iota: H \to G_1$ by

(2.3) $$\iota(h)(\eta t + j(w)) = \eta t + j(hw) \qquad (h \in H, t \in K, w \in W).$$

It is easy to see that $\iota(H) = \{g_1 \in G_1 \mid g_1 \cdot \eta = \eta\}$. Let

$$P_1 = \{\begin{bmatrix} t & * & * \\ 0 & g & * \\ 0 & 0 & \bar{t}^{-1} \end{bmatrix} \mid t \in K^{\times}, g \in G\}$$

be a maximal parabolic subgroup of G_1. Then its unipotent radical is

$$N_1 = \{n_1(y, z) = \begin{bmatrix} 1 & -y^*S & z \\ 0 & 1_m & y \\ 0 & 0 & 1 \end{bmatrix} \mid y \in V, z \in K, \tau(z) + S[y] = 0\}.$$

Lemma 2.2 *We have*

$$P_1 \cap \iota(H) = \{p_1(g) = \begin{bmatrix} 1 & -\alpha^*(g^{-1} - 1)^*S & S((g^{-1} - 1)\alpha, \alpha) \\ 0 & g & (1 - g)\alpha \\ 0 & 0 & 1 \end{bmatrix} \mid g \in G\}$$

(*hence* $P_1 \cap \iota(H) \cong G$).

Proof. By (2.4), $p_1 = \begin{bmatrix} t & & \\ & g & \\ & & \bar{t}^{-1} \end{bmatrix} \cdot n_1(y, z) \in \iota(H)$ implies $p_1\eta = \eta$. It follows that

$$t(a - S(y, \alpha) + z) = a, \; g(\alpha + y) = \alpha, \; \bar{t}^{-1} = 1.$$

Solving these equations, we get

$$t = 1, \; y = (g^{-1} - 1)\alpha, \; z = S((g^{-1} - 1)\alpha, \alpha).$$

Since $\tau(z) + S[y] = 0$, we are done. *q.e.d.*

We define an embedding $\iota' : G \to H$ by

(2.4) $$\iota'(g) = \begin{pmatrix} g & (1-g)\alpha \\ 0 & 1 \end{pmatrix} \qquad (g \in G).$$

Then $\iota(\iota'(g)) = p_1(g)$ and $\iota'(G) = \{h \in H \mid h \cdot \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}\}$.

2.3 The orbit space $P_1\backslash G_1/\iota(H)$ We now study the structure of the orbit space $P_1\backslash G_1/\iota(H)$.

Lemma 2.3 *Assume that* K *is division. Then* T *is* k*-isotropic if and only if there exists* $x_o \in V$ *satisfying* $S[x_o] = \Delta$.

Proof. The "if" part is easy since $T[\begin{pmatrix} \alpha \\ 1 \end{pmatrix} + \begin{pmatrix} x_o \\ 0 \end{pmatrix}] = -\Delta + \Delta = 0$. Assume that T is k-isotropic. Then there exists a non-zero element y of W such that $T[y] = 0$. We write $y = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}\lambda + \begin{pmatrix} x \\ 0 \end{pmatrix}$ with $\lambda \in K$ and $x \in V$. Observe $S[x] = \bar{\lambda}\,\Delta\,\lambda$. If $\lambda \neq 0$, then $S[x\lambda^{-1}] = \Delta$. If $\lambda = 0$ and $x \neq 0$, we have $S[x] = 0$. This implies that S is k-isotropic and hence $S[V] = \tau(K)$. We are done. *q.e.d.*

Proposition 2.4 *Assume that* K *is division.*

(i) *If* T *is* k-*anisotropic, then* $G_1 = P_1 \cdot \iota(H)$.

(ii) *If* T *is* k-*isotropic, then* $G_1 = P_1 \cdot \iota(H) \cup P_1 \cdot \Upsilon_0 \cdot \iota(H)$ *where*

$$\Upsilon_0 = \begin{bmatrix} & & 1 \\ & 1_m & \\ \varepsilon & & \end{bmatrix} n_1(x_0 - \alpha, S(x_0 - \alpha, \alpha) - a)$$

and x_0 *is any element of* V *satisfying* $S[x_0] = \Delta$.

Proof. Let $g_1 \in G_1$ and put $g_1 \cdot \eta = \begin{bmatrix} a' \\ \alpha' \\ b' \end{bmatrix}$ $(a', b' \in K, \alpha' \in V)$. We first show that $g_1 \in P_1 \cdot \iota(H)$ if $b' \neq 0$. To prove this, we put $z = -a + \overline{b'}\, a' + S(\alpha' - \alpha, \alpha)$. It is clear that $\tau(z) + S[\alpha' - \alpha] = 0$ and $p_1 \eta = g_1 \eta$ with $p_1 = \begin{bmatrix} \overline{b'}^{-1} & & \\ & 1_m & \\ & & b' \end{bmatrix} \cdot n_1(\alpha' - \alpha, z) \in$ P_1. Thus our assertion follows. Assume that T is k-anisotropic. If $b' = 0$, we have $\tau(a) + S[\alpha] = S_1[\eta] = S_1[g_1\eta] = S[\alpha']$ and hence $T[\begin{pmatrix} \alpha+\alpha' \\ 1 \end{pmatrix}] = S[\alpha'] - S[\alpha] - \tau(a) =$ 0, which contradicts to the assumption. Thus (i) is proved. We next assume that T is k-isotropic. We claim that $g_1 \in P_1 \Upsilon_0 \iota(H)$ if $b' = 0$. Choose a pair $(y, z) \in V \times K$ so that $1 - S(y, x_0) = a'$, $\tau(z) + S[y] = 0$. Since $S[\alpha'] = \Delta$, there exists an element g of G such that $gx_0 = \alpha'$ by Witt's theorem. Thus we have $p_1 \Upsilon_0 \eta = \begin{bmatrix} a' \\ \alpha' \\ 0 \end{bmatrix}$ with $p_1 =$ $\begin{bmatrix} 1 & & \\ & g & \\ & & 1 \end{bmatrix} \cdot n_1(y, z)$, which proves the proposition. *q.e.d.*

2.4 The unipotent radicals of parabolic subgroups The content of this subsection will not be used in the paper, but we include it here for future application (see the remark of Theorem 1.5). Throughout this subsection, we suppose that K is division and that T is k-isotropic. Let $x_0 \in V$ be as in Proposition 2.4 and put $e = \begin{pmatrix} -x_0+\alpha \\ 1 \end{pmatrix}$ $\in W$, $e' = \begin{pmatrix} x_0+\alpha \\ 1 \end{pmatrix} \in W$. Then $T[e] = T[e'] = 0$ and $T(e, e') = -2\Delta \neq 0$. We see that $P' = \{h \in H \mid h\cdot e = e\cdot t \ (t \in K^\times)\}$ is a maximal parabolic subgroup of H.

Lemma 2.5 $\Upsilon_0^{-1} P_1 \Upsilon_0 \cap \iota(H) = \iota(P')$.

Proof. Let $h \in H$. Then $\iota(h) \in \Upsilon_0^{-1} P_1 \Upsilon_0$ if and only if $\iota(h)\Upsilon_0^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \Upsilon_0^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t$ for some $t \in K^\times$. Since $\Upsilon_0^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = j(e)$, we have $\iota(h)\,\Upsilon_0^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = j(h\cdot e)$. The lemma follows from this. *q.e.d.*

Let W' be the orthogonal compliment of $e\cdot K + e'\cdot K$ in W with respect to T. The unipotent radical N' of P' is given by $\{n' \in H \mid$ (i) $n'\cdot e = e$, (ii) for $w' \in W'$, $n'\cdot w' = e\lambda + w'$ for some $\lambda \in K\}$.

Lemma 2.6 $\Upsilon_0\, \iota(N')\, \Upsilon_0^{-1} \subset N_1$.

Proof. We first note that, if $g_1 \in G_1$ satisfies $g_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $g_1\begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ x \\ * \end{pmatrix}$, then we have $g_1 \in N_1$. Let $g_1 = \Upsilon_0\, \iota(n')\, \Upsilon_0^{-1}$ $(n' \in N')$. Since $\Upsilon_0^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = j(e)$, we have $g_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \Upsilon_0\, \iota(n')\, j(e) = \Upsilon_0\, j(e) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. A direct calculation shows that

$$\Upsilon_0^{-1}\begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} S(x_0-\alpha, x) \\ x \\ 0 \end{pmatrix} = \eta\,\mu + j\left(\begin{pmatrix} y \\ z \end{pmatrix}\right) \qquad (x \in V)$$

with $\mu = \Delta^{-1}S(x_0, x)$, $y = x - \alpha\Delta^{-1}S(x_0, x)$, $z = -\Delta^{-1}S(x_0, x)$ and hence

$$\iota(n')\Upsilon_0^{-1}\begin{pmatrix}0\\x\\0\end{pmatrix} = \eta\,\mu + j(n'\begin{pmatrix}y\\z\end{pmatrix}).$$

We denote the above vector by $\begin{pmatrix}a'\\b'\\c'\end{pmatrix}$. Then we have $g_1\begin{pmatrix}0\\x\\0\end{pmatrix} = \begin{pmatrix}c'\\b'+(x_0-\alpha)c'\\ *\end{pmatrix}$. It remains to verify

(2.5) $\qquad b' + (x_0 - \alpha)c' = x.$

To prove this, let $\begin{pmatrix}y\\z\end{pmatrix} = e\,\lambda + e'\,\lambda' + w'$ $(\lambda, \lambda' \in K, w' \in W')$. It is easy to see that $T(e, \begin{pmatrix}y\\z\end{pmatrix}) = 0$ and $T(e', \begin{pmatrix}y\\z\end{pmatrix}) = 2S(x_0, x)$ and hence we have $\lambda = -\Delta^{-1}S(x_0, x)$, $\lambda' = 0$, $w' = \begin{pmatrix}x - x_0\Delta^{-1}S(x_0,x)\\0\end{pmatrix}$. It follows that

$$n'\begin{pmatrix}y\\z\end{pmatrix} = n'e\lambda + n'w' = e\,(\lambda + \lambda'') + w' = \begin{pmatrix}(-x_0+\alpha)(\lambda+\lambda'')+x-x_0\Delta^{-1}S(x_0,x)\\ \lambda+\lambda''\end{pmatrix}$$

where $\lambda'' \in K$ is determined by $n'w' = e\,\lambda'' + w'$. It follows that $b' = x + (-x_0 + \alpha)\lambda''$ and $c' = \lambda''$,which implies (2.5). *q.e.d.*

§3. Maximal integral lattices of ε-hermitian spaces

In this section, we let k be a finite extention of $\mathbf{Q}_p$, o its maximal order, p a fixed prime element of o, K a semisimple algebra over k as in §2 and O a maximal order of K. We choose and fix a prime element π of O if K is a division algebra. We keep the notation of §2.

3.1 Integral lattices Let $S \in M_m(K)$ be a non-degenerate ε-hemitian matrix. An O-lattice L of $V = K^m$ is said to be *O-integral with respect to* S if $S(x, y) \in O$ and $S[x] \in \tau(O)$ for every $x, y \in L$. We say that $S = (s_{ij})$ is *integral* if O^m is O-integral with respect to S. This is equivalent to the assertion "$s_{ij} \in O$ and $s_{ii} \in \tau(O)$ $(1 \le i, j \le m)$".

Let $G = U(S) = \{g \in GL_m(K) \mid g^*Sg = S\}$ be the unitary group of an integral ε-hermitian matrix S. We put

$$(3.1) \qquad G_o = G \cap GL_m(O), G_o^* = \{ g \in G_o \mid (g-1) L^* \subset L\}.$$

where $L^* = S^{-1}L$ is the dual lattice of $L = O^m$ with respect to S. The following is easily verified.

Lemma 3.1

(i) $u \in G_o \Rightarrow u L = L, u L^* = L^*$.

(ii) G_o^* *is a normal subgroup of* G_o.

An O-integral lattice L is maximal with respect to S if and only if the following assertion holds: $Y \in L^*, S[Y] \in \tau(O) \Rightarrow Y \in L$. We say that S is *maximal* if S is integral and if O^m is maximal with respect to S.

Lemma 3.2 *Suppose that* S *is maximal.*

(i) *If* K *is division, then* S *is* $GL_m(O)$*-equivalent to* $\begin{bmatrix} & & \varepsilon J_v \\ & S_o & \\ J_v & & \end{bmatrix}$ *where* $J_v = \begin{pmatrix} 0 & & 1 \\ & \cdot & \\ & \cdot & \\ 1 & & 0 \end{pmatrix}$ $\in GL_v$ *and* $S_o \in M_{n_o}(O)$ *is a k-anisotropic* ε*-hermitian matrix. Furthermore* $O^{n_o} = \{x \in K^{n_o} \mid S_o[x] \in \tau(O)\}$.

(ii) *If* K *splits over* k *and* S *is of type* (U) *or* (U^+), *then* $S \in GL_m(O)$.

Proof. The statement (i) is well-known (for example, see [Sa]). We give a proof of (ii) in the case of (U). The statement is similarly proved in the case of (U^+). Suppose that $L = O^m = o^m \oplus o^m$ is maximal with respect to $S = (S', \varepsilon {}^tS')$ $(S' \in GL_m(k))$. If $S' \notin GL_m(o)$, there exists $X' \in k^m - o^m$ such that $S'X' \in o^m$. Put $X = (X', 0) \in K^m - O^m$. Then $SX = (S'X', 0) \in L$ and $S[X] = (0, {}^tX')(S'X', 0) = 0 \in \tau(O)$, which contradicts to maximality of L. *q.e.d.*

3.2 Embedding of lattices In this subsection, we let S be a non-degenerate maximal ε-hermitian matrix of degree m. Then the lattice $L_1 = \begin{bmatrix} O \\ L \\ O \end{bmatrix} = O^{m+2}$ of $V_1 = \begin{bmatrix} K \\ V \\ K \end{bmatrix} = K^{m+2}$ is maximal with respect to $S_1 = \begin{bmatrix} & & \varepsilon \\ & S & \\ 1 & & \end{bmatrix}$. Let $(a, \alpha) \in O \times L^*$.

Then $T = \begin{pmatrix} S & -S\alpha \\ -\alpha^* S & -\tau(a) \end{pmatrix}$ is integral and defines an ε-hermitian structure on $W = \begin{pmatrix} V \\ K \end{pmatrix}$ $= K^{m+1}$. Let $j : W \to V_1$ be an embedding of ε-hermitian spaces given by

$$j\left(\begin{pmatrix} y \\ z \end{pmatrix}\right) = \begin{bmatrix} -\varepsilon \bar{a} z - S(\alpha,y) \\ y \\ z \end{bmatrix} \quad (y \in V, z \in K).$$

Then $j(M)$ is contained in L_1 where $M = \begin{pmatrix} L \\ O \end{pmatrix}$ is an O-integral lattice of W with respect to T. Let $\iota : H = U(T) \to G_1 = U(S_1)$ be as in §2: $\iota(h)\,(\eta\cdot t + j(X)) = \eta\cdot t + j(hX)$ $(h \in H, t \in K, X \in W)$. Note that $\eta = \begin{bmatrix} a \\ \alpha \\ 1 \end{bmatrix} \in L_1^*$.

Lemma 3.3 $\quad \iota(H) \cap G_{1,O}^* = \iota(H_O^*)$.

Remark. The inclusion $\iota(H) \cap G_{1,O} \supset \iota(H_O)$ does not always hold (see Proposition 3.7).

Proof. Let $X = \eta\cdot t + j\left(\begin{pmatrix} y \\ z \end{pmatrix}\right) \in V_1$ $(t \in K, y \in V, z \in K)$. We first show that $\begin{pmatrix} y \\ z \end{pmatrix} \in M^* = T^{-1}M$ if $X \in L_1^* = T_1^{-1}L_1$. Since $X = \begin{bmatrix} at - \varepsilon \bar{a} z - S(\alpha,y) \\ \alpha t + y \\ t + z \end{bmatrix}$, $X \in L_1^*$ implies $\alpha t + y \in L^*$ and $at - \varepsilon \bar{a} z - S(\alpha, y), t + z \in O$. Then we have

$$S\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} S(y-\alpha z) \\ -S(\alpha,y) - \tau(a)z \end{pmatrix}$$

$$= \begin{pmatrix} S(\alpha t+y) - S\alpha(t+z) \\ (at-\varepsilon \bar{a} z - S(\alpha,y)) - a(t+z) \end{pmatrix} \in M,$$

which proves our claim. Let $h \in H_O^*$. For $X = \eta\cdot t + j\left(\begin{pmatrix} y \\ z \end{pmatrix}\right) \in L_1^*$, we have

$$\iota(h)X = \eta t + j(\begin{pmatrix} y \\ z \end{pmatrix}) + j((h-1)\begin{pmatrix} y \\ z \end{pmatrix}) \equiv X \pmod{L_1},$$

since $(h-1)\begin{pmatrix} y \\ z \end{pmatrix} \in M$ by the above remark. This implies $\iota(h) \in G^*_{1,o}$. Next suppose that $h \in H$ satisfies $\iota(h) \in G^*_{1,o}$. For $\begin{pmatrix} y \\ z \end{pmatrix} \in M^*$, put

$$X_o = \eta\cdot(-z) + j(\begin{pmatrix} y \\ z \end{pmatrix}) = \begin{pmatrix} -S(\alpha,y)-\tau(a)z \\ y-\alpha z \\ 0 \end{pmatrix} \in L_1^*.$$

Since $j((h-1)\begin{pmatrix} y \\ z \end{pmatrix}) = (\iota(h)-1)X \in L_1$, we see $(h-1)\begin{pmatrix} y \\ z \end{pmatrix} \in M$ and hence $h \in H_o^*$.

q.e.d.

Lemma 3.4 *Suppose that* K *is division and let* π *be a prime element of* O. If $\zeta \in L^*_{1,\mathrm{prim}} = L_1^* - L_1^*\cdot\pi$, *then there exists an element* u *of* $G^*_{1,o}$ *such that* $u\zeta = \begin{bmatrix} * \\ * \\ 1 \end{bmatrix}$.

Proof. Let $\zeta = \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$ $(a, b \in O, \alpha \in L^*)$. If $b \in O^\times$, put $u = \begin{bmatrix} \bar{b} & & \\ & 1 & \\ & & b^{-1} \end{bmatrix} \in G^*_{1,o}$. If $b \in \pi O$ and $a \in O^\times$, put $u = \begin{bmatrix} \bar{a} & & \\ & 1 & \\ & & a^{-1} \end{bmatrix}\begin{bmatrix} & & \varepsilon \\ & 1 & \\ 1 & & \end{bmatrix}$. Finally suppose that $a, b \in \pi O$. Since $\alpha \in L^*_{\mathrm{prim}}$, we can find a pair $(y, z) \in \Lambda \times O$ so that $S(\alpha, y) \in O^\times$ and $\tau(z) + S[y] = 0$. Since

$$\begin{bmatrix} 1 & & \\ y & 1_m & \\ z & -\varepsilon y^*S & 1 \end{bmatrix}\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} * \\ * \\ ya-\varepsilon S(y,\alpha)+b \end{bmatrix},$$

the proof is reduced to the case where $b \in O^\times$. *q.e.d.*

3.3 We say that a pair $(a, \alpha) \in O \times L^*$ is *reduced* if $T[\begin{pmatrix} 1_m & X \\ 0 & t^{-1} \end{pmatrix}]$ is not integral for every $t \in O \cap K^\times - O^\times$ and every $X \in K^m$ with $T = \begin{pmatrix} S & -S\alpha \\ -\alpha^* S & -\tau(a) \end{pmatrix}$.

Lemma 3.5 *Suppose that* S *is maximal and exclude the case where* S *is of* (U^-)*-type and* K *splits over* k*. A pair* $(a, \alpha) \in O \times L^*$ *is reduced if and only if* $M = O^{m+1}$ *is maximal with respect to* $T = \begin{pmatrix} S & -S\alpha \\ -\alpha^* S & -\tau(a) \end{pmatrix}$.

To prove this, we need the following result.

Lemma 3.6 *Let the assumption be the same as in Lemma 3.5. If* $S[\alpha] - S[\beta t] \in \tau(O)$ *for* $\alpha, \beta \in L^*$ *and* $t \in O - O^\times$*, we have* $\alpha - \beta t \in L$.

Proof. If K splits, then we have $L^* = L$ so that the assertion is trivial. Thus we assume that K is division. Let l be the least non-negative integer satisfying $(\alpha - \beta t)\cdot\pi^l \in L$. Suppose that $l \geq 1$. Then

$$S[(\alpha - \beta t)\pi^{l-1}] = \overline{\pi}^{\,l-1}\cdot(S[\alpha] - \tau(S(\alpha, \beta t)) + S[\beta t])\cdot\pi^{l-1}$$

$$= \overline{\pi}^{\,l-1}\cdot(S[\alpha] - S[\beta t])\cdot\pi^{l-1} - \tau(\overline{\pi}^{\,-1}\cdot S((\alpha - \beta t)\pi^l, \beta t\pi^{l-2})\cdot\pi).$$

Observe that $S((\alpha - \beta t)\pi^l, \beta t\pi^{l-2}) \in O$ since $(\alpha - \beta t)\pi^l \in L$ and $\beta t\pi^{l-2} \in L^*$ (note that $t \in \pi O$ and $l - 2 \geq -1$). Since $\overline{\pi}^{\,-1}\cdot O\cdot\pi = O$, we have $S[(\alpha - \beta t)\pi^{l-1}] \in \tau(O)$. On the other hand, we have $(\alpha - \beta t)\pi^{l-1} \in L^*$. Since S is maximal, we have $(\alpha - \beta t)\pi^{l-1} \in L$, which is a contradiction. Thus $l = 0$ and we are done. *q.e.d.*

Proof of Lemma 3.5. First suppose that $(a, \alpha) \in O \times L^*$ is not reduced. Then there exists a pair $(t, X) \in (O \cap K^\times - O^\times) \times K^m$ such that $T[\begin{pmatrix} 1 & X \\ 0 & t^{-1} \end{pmatrix}] = \begin{pmatrix} S & S(X-\alpha t^{-1}) \\ (X-\alpha t^{-1})^* S & z_0 \end{pmatrix}$

is integral, where $z_0 = S[X] - \tau(S(X, \alpha)t^{-1}) - \overline{t}^{-1}\tau(a)t^{-1}$. Put $Y = \begin{pmatrix} X \\ t^{-1} \end{pmatrix}$. We show

that $TY \in M$ and $T[Y] \in \tau(O)$, which implies that M is not maximal. The second assertion is clear from $z_0 = T[Y]$. To prove the first one, observe

$TY = \begin{pmatrix} S(X-\alpha t^{-1}) \\ -S(\alpha,X)-\tau(a)t^{-1} \end{pmatrix}$. By assumption, we see that $S(X - \alpha t^{-1})$ is integral and it

remains to show that $b = -S(\alpha,X) - \tau(a)t^{-1} \in O$. Observe $Xt = (X - \alpha t^{-1}) \cdot t - (-\alpha)$ and $X - \alpha t^{-1}, -\alpha \in L^*$. We see that $S[Xt - \alpha] - S[-\alpha] = S[Xt] - \tau(S(Xt, \alpha)) = \overline{t}\, z_0 t + \tau(a) \in \tau(O)$. Applying Lemma 3.6, we see that $Xt \in L$ and hence $b \in O$. Next suppose that M is not maximal. Then we can find an element $Y = \begin{pmatrix} X \\ z \end{pmatrix}$ of $W - M$ $(X \in V, z \in K)$ so that $TY \in M$ and $T[Y] \in \tau(O)$. If $z \in O$, then $SX \in L$ and $S[X] \in \tau(O)$, which implies $X \in L$ by maximality of S. This contradicts to the choice of Y and hence we have $z \notin O$. If K is division, $z \notin O$ implies $z^{-1} \in O \cap K^\times - O^\times$ and hence that (a, α) is not reduced. Assume that K splits over k. We show that there exists $z' \in K^\times$ satisfying

(3.2) $\qquad z' \equiv z \pmod{O}$ and $z'^{-1} \in O$.

By a similar argument as above, the existence of such a z' implies that (a, α) is not reduced. First consider the case (II) so that $K = k \oplus k$ and $z = (z_1, z_2)$. We may assume that $z_1 \notin o$ and $z_2 \in o$. In this case, put $z' = (z_1, 1)$. Next consider the case

(III). In this case, $K = M_2(k)$. Then we may assume that $z = u_1 \begin{pmatrix} p^{\mu} & 0 \\ 0 & p^{\nu} \end{pmatrix} u_2$ with u_1, $u_2 \in O^{\times} = GL_2(o)$, $\mu < 0$ and $\nu \geq 0$. We put $z' = u_1 \begin{pmatrix} p^{\mu} & 0 \\ 0 & 1 \end{pmatrix} u_2$. In both cases, z' satisfies the condition (3.2). *q.e.d.*

3.4 For a maximal S, we now define an invariant $\partial(S)$ of the $GL_m(O)$-equivalence class of S. First suppose that K is division. Then $L' = \{X \in L^* \mid S[X] \in \tau(\pi^{-1}O)\}$ is an O-integral lattice containing L and L'/L forms a finite dimensional vector space over a finite field $O/\pi O$. Then $\partial(S)$ is defined to be

$$(3.3) \qquad \partial(S) = \dim_{O/\pi O} L'/L.$$

We set $\partial(S) = 0$ when K splits over k. It is known that $0 \leq \partial(S) \leq 2$. Let (a, α) be a reduced pair. Since $T = \begin{pmatrix} S & -S\alpha \\ -\alpha^* S & -\tau(a) \end{pmatrix}$ is maximal by Lemma 3.5, we can also define $\partial(T)$.

For the remainder of this section, we assume the following:

(3.4) $\varepsilon = 1$; moreover we exclude the case where $p \mid 2$ and K is a ramified extention of k.

In particular, the orthogonal group case is included.

Proposition 3.7 *Let* (a, α) *be a reduced pair and assume* (3.4). *If* $\partial(T) \leq \partial(S)$, *then we have* $\iota(H_O) = \iota(H) \cap G_{1,O}$.

Proof. To prove the proposition, we may assume that S is k-anisotropic in view of Lemma 3.2. Note that, if K splits over k, then S and T are unimodular and hence the lemma is trivial (see Lemma 3.2). Suppose that K is division. A classification of k-anisotropic ε-hrmitian matrices (up to $GL_m(O)$-equivalence) is available under the assumption (3.4) (see [HS]). Then we can check our assertion case by case. The verification is straightforward and we only list up the $GL_m(O)$-equivalence classes of ε-hermitian matrices in the case where S is of type (U) or (U^+) and K is division. The classification in the case of (O) may be found in [E].

(1) (U), K/k : an unramified quadratic extention

(1.a) $n_0 = 0, \partial = 0$

(1.b) $n_0 = 1, \partial = 0, S = (r), r \in o^\times$

(1.c) $n_0 = 1, \partial = 1, S = (pr), r \in o^\times$

(1.d) $n_0 = 2, \partial = 1, S = \begin{pmatrix} s & 0 \\ 0 & pr \end{pmatrix}, r, s \in o^\times$

(2) (U), K/k : a tamely ramified quadratic extention

(2.a) $n_0 = 0, \partial = 0$

(2.b) $n_0 = 1, \partial = 0, S = (r), r \in o^\times$

(2.c) $n_0 = 2, \partial = 0, S = \begin{pmatrix} s & 0 \\ 0 & r \end{pmatrix}, s, r \in o^\times, -sr \notin N_{K/k}(K).$

(3) (U^+), K : a division quaternion over k

(3.a) $n_0 = 0, \partial = 0$

(3.b) $n_0 = 1, \partial = 0, S = (r), r \in o^\times.$

q.e.d.

The following lemma is proved in a similar way as in the proof of the above proposition.

Lemma 3.8 *Assume* (3.4). *If* $\alpha, \beta \in L^*$ *satisfy* $S[\alpha] \equiv S[\beta] \pmod{\tau(O)}$, *then there exists an element* u *of* G_o *such that* $\beta \equiv u\alpha \pmod{L}$.

Proposition 3.9 *Assume* (3.4) *and let* K *be a division algebra over* k. *If* (a, α) *is a reduced pair, then*

$$G_1 = \bigcup_{l \geq 0} \iota(H) \begin{bmatrix} \pi^{-l} & & \\ & 1_m & \\ & & \overline{\pi}^{l} \end{bmatrix} G_{1,o} \quad \textit{(disjoint union).}$$

Proof. Recall that $\iota(H)$ is the stabilizer subgroup of $\eta = \begin{bmatrix} a \\ \alpha \\ 1 \end{bmatrix}$ in G_1. Let $g_1 \in G_1$. Put $g_1^{-1}\eta = \zeta \cdot \overline{\pi}^{-l}$ with $\zeta \in L^*_{1,\mathrm{prim}}$ (note that l is uniquely determined). We first show $l \geq 0$. If $l < 0$, we have $g_1^{-1}\eta = \begin{bmatrix} a'' \\ \alpha'' \\ b'' \end{bmatrix} \cdot \pi$ with $\begin{bmatrix} a'' \\ \alpha'' \\ b'' \end{bmatrix} \in L_1^*$. Then

$$T[\begin{pmatrix} 1 & \alpha\pi^{-1} - \alpha'' \\ 0 & \pi^{-1} \end{pmatrix}] = \begin{pmatrix} S & -S\alpha'' \\ -\alpha^* S & z \end{pmatrix}$$

with $z = S[\alpha''] - \overline{\pi}^{-1}(S[\alpha] + \tau(a))\pi^{-1}$. Since $S_1[\eta] = S_1[g_1^{-1}\eta]$, we have $z = \tau(\overline{b''}\, a'') \in \tau(O)$ and hence that $T[\begin{pmatrix} 1 & \alpha\pi^{-1} - \alpha'' \\ 0 & \pi^{-1} \end{pmatrix}]$ is integral. This contradicts to the assumption that (a, α) is reduced. By Lemma 3.4, there exists an element u_1 of $G^*_{1,o}$ such that $g_1^{-1}\eta = u_1 \begin{bmatrix} a' \\ \alpha' \\ 1 \end{bmatrix} \overline{\pi}^{-l}$ with $\begin{bmatrix} a' \\ \alpha' \\ 1 \end{bmatrix} \in L_1^*$. It follows that $\tau(\pi^l a \overline{\pi}^l) +$

$S[\alpha \overline{\pi}^{l}] = \tau(a') + S[\alpha']$ and hence $S[\alpha \overline{\pi}^{l}] \equiv S[\alpha'] \pmod{\tau(O)}$. By Lemma 3.8, we can find an element u_o of G_O so that $u_o\alpha' = \alpha \overline{\pi}^{l} - \beta$ with $\beta \in L$. Choose an element b of O so that $\tau(b) + S[\beta] = 0$. Then

$$\begin{bmatrix} 1 & -\beta^* S & b \\ 0 & 1_m & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & u_o & \\ & & 1 \end{bmatrix} \begin{bmatrix} a' \\ \alpha' \\ 1 \end{bmatrix} = \begin{bmatrix} a'' \\ \alpha \overline{\pi}^{l} \\ 1 \end{bmatrix} \in L_1^* .$$

Put $y_o = \pi^l a \overline{\pi}^{l} - a''$. Then $y_o \in O \cap \mathrm{Ker}\,\tau$ and

$$\begin{bmatrix} 1 & 0 & y_o \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a'' \\ \alpha \overline{\pi}^{l} \\ 1 \end{bmatrix} = \begin{bmatrix} \pi^l & & \\ & 1 & \\ & & \overline{\pi}^{-l} \end{bmatrix} \eta \overline{\pi}^{l} .$$

Thus there exists an element u of $G_{1,O}$ such that $g_1^{-1}\eta = u \begin{bmatrix} \pi^l & & \\ & 1 & \\ & & \overline{\pi}^{-l} \end{bmatrix} \eta$. This proves the proposition. *q.e.d.*

§4. Proof of Lemma A

The object of this section is to prove Lemma A in §1.10. Let the notation and the assumption be as in §1. 10. In particular, T and S are supposed to be maximal. Throughout this section, we fix a rational prime p and write H, U, G, K, G_1 and K_1 for H_p, U_p, G_p, K_p, $G_{1,p}$ and $K_{1,p}$ respectively.

Let $\{p^{e_1}, \cdots, p^{e_r}, 0, \cdots, 0\}$ (0 appears $(n-r)$ times) be a set of elementary divisors of $x \in M_n(\mathbf{Q}_p)$. Recall that $\mu_{n,p}(x) = \mu_n(x)$ is the sum of $|e_i|$ with $e_i < 0$. Note that $p^{-\mu_n(x)}\mathbf{Z}_p$ coincides with the $\mathbf{Z}_p$-module generated by all the minors of x and $\mathbf{Z}_p$. The following is easily verified.

Lemma 4.1

(i) *For* $x, y \in M_n(\mathbf{Q}_p)$, *we have* $\mu_n(xy) \le \mu_n(x) + \mu_n(y)$.

(ii) *For* $x \in M_m(\mathbf{Q}_p)$ *and* $y \in M_n(\mathbf{Q}_p)$, *we have* $\mu_{m+n}\begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \ge \mu_m(x) + \mu_n(y)$.

Since Lemma A is trivial for a $\mathbf{Q}_p$-anisotropic T, we may assume that T is $\mathbf{Q}_p$-isotrpic. By taking suitable $\mathbf{Z}_p$-bases of M and L, we may suppose that S, T and S_1 have the following matrix forms:

$$(4.1) \quad S = \begin{pmatrix} R & -R\beta \\ -{}^t\beta R & -2b \end{pmatrix}, T = \begin{pmatrix} & & 1 \\ & R & \\ 1 & & \end{pmatrix}, S_1 = \begin{pmatrix} & & 1 \\ & S & \\ 1 & & \end{pmatrix}$$

($R \in M_{m-1}(\mathbf{Q}_p)$, $\beta \in \mathbf{Q}_p^{m-1}$, $b \in \mathbf{Q}_p$). Furthermore the embeddings $V = \mathbf{Q}_p^m \xrightarrow{j'} W = \mathbf{Q}_p^{m+1} \xrightarrow{j} V_1 = \mathbf{Q}_p^{m+2}$, $G \xrightarrow{\iota'} H \xrightarrow{\iota} G_1$ are given as follows: Put $\xi = \begin{pmatrix} b \\ \beta \\ 1 \end{pmatrix} \in$

$$M^* = T^{-1}M,\ \eta = \begin{pmatrix} 0 \\ \alpha \\ 1 \end{pmatrix} \in L_1^* = S_1^{-1}L_1 \text{ where } \alpha = D^{-1}\begin{pmatrix} \beta \\ 1 \end{pmatrix} \in L^* = S^{-1}L,\ D = T[\xi] =$$

$R[\beta] + 2b \in \mathbf{Q}_p$. Then

$$(4.2) \qquad j'\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} -by_2 - R(\beta, y_1) \\ y_1 \\ y_2 \end{pmatrix},$$

$$j(\xi t + j'(y)) = \begin{bmatrix} -S[\alpha] \\ \alpha \\ 1 \end{bmatrix} Dt + \begin{bmatrix} -S(\alpha, y) \\ y \\ 1 \end{bmatrix},$$

$$\iota'(g)(\xi t + j'(y)) = \xi t + j'(gy),$$

$$\iota(h)(\eta t + j(w)) = \xi t + j(hw)$$

$(y_1 \in \mathbf{Q}_p^{m-1},\ y_2 \in \mathbf{Q}_p,\ t \in \mathbf{Q}_p,\ y \in V,\ w \in W,\ g \in G,\ h \in H)$.

To prove Lemma A, we need an explicit form of ι, which will be also used in the next section.

Lemma 4.2

(i) *Let* $h' \in H' = O(R)$ *and* $t \in \mathbf{Q}_p^\times$. *Then we have*

$$\iota\left(\begin{pmatrix} t & & \\ & h' & \\ & & t^{-1} \end{pmatrix}\right) = \begin{bmatrix} t^{-1} & 0 & 0 & 0 \\ (h'-1)\beta & h' & (1-h')\beta & 0 \\ t^{-1}-1 & 0 & 1 & 0 \\ u_1 & u_2 & u_3 & t \end{bmatrix}$$

with $u_1 = b(-2 + t + t^{-1}) + R(\beta, (h'-1)\beta)$, $u_2 = {}^t\beta R(h'-t)$ *and* $u_3 = 2b(1-t) + R(\beta, (1-h')\beta)$.

(ii) *Let* $x \in \mathbf{Q}_p^{m-1}$. *Then we have*

$$\iota\left(\begin{pmatrix} 1 & -{}^t xR & -2^{-1}R[x] \\ 0 & 1_{m-1} & x \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1_{m-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2^{-1}R[x] & -{}^t xR & R(\beta,x) & 1 \end{bmatrix}.$$

(iii) *For* $g \in G$, *we have*

$$\iota(\iota'(g)) = \begin{bmatrix} 1 & {}^t\alpha S(1-g) & -S(\alpha,(1-g)\alpha) \\ 0 & g & (1-g)\alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof. This is proved by straightforward calculation. *q.e.d.*

Lemma 4.3 *If* $\partial_p(T) \le \partial_p(S)$, *then we have* $N_{H,s}(h) = N_{G_1,s}(\iota(h))$ *for* $h \in H$.

Proof. We may assume that $R = \begin{bmatrix} & & J_v \\ & R_o & \\ J_v & & \end{bmatrix}$ and $\beta = \begin{pmatrix} 0_v \\ \beta_o \\ 0_v \end{pmatrix}$ where v is the Witt index of R, R_o is $\mathbf{Q}_p$-anisotropic and $\beta_o \in \mathbf{Q}_p^{m-1-2v}$. Since $\iota(U) \subset K_1$ by the assumption and Proposition 3.7, we only have to check the assertion of the lemma for $h = \begin{pmatrix} p^{-r} & & \\ & h' & \\ & & p^r \end{pmatrix}$ with $h' = \mathrm{diag}(p^{r_1}, \cdots, p^{r_v}, 1_{m+1-2v}, p^{-r_v}, \cdots, p^{-r_1}) \in H'$ $(r, r_1, \cdots, r_v \ge 0)$. By Lemma 4.2, we have

$$\iota(h) = \begin{bmatrix} p^r & & & \\ & h' & & \\ & & 1 & \\ & & & p^{-r} \end{bmatrix} k_1,\ k_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p^r-1 & 0 & 1 & 0 \\ b(p^r-1)^2 & {}^t\beta_o T_o(p^r-1) & 2b(p^r-1) & 1 \end{bmatrix} \in K_1.$$

Thus $N_{G_1,s}(\iota(h)) = p^{-(r+r_1+\cdots+r_v)s} = N_{H,s}(h)$, which proves the lemma. *q.e.d.*

Proof of Lemma A. We first note that the assertion of lemma A is equivalent to

$$(4.3) \qquad N_{H,s}(h) = |\alpha(\iota(h))|^s \cdot N_{G,s}(\beta(\iota(h))) \quad (h \in H).$$

By Proposition 3.9, we get the decomposition $H = \bigcup_{l \geq 0} \iota'(G) h_l U$ with $h_l = \begin{pmatrix} p^{-l} & & \\ & 1_{m-1} & \\ & & p^l \end{pmatrix}$. Thus we only have to verify (4.3) for $h = \iota'(g) h_l$ ($g \in G$, $l \geq 0$). By Lemma 4.1 (i), we have $\mu_{m+1}(h) \leq \mu_{m+1}(\iota'(g)) + \mu_{m+1}(h_l) = \mu_{m+1}(\iota'(g)) + l$. On the other hand, by Lemma 4.3, Lemma 4.2 (iii) and Lemma 4.1 (ii), we obtain $\mu_{m+1}(h) = \mu_{m+2}(\iota(h)) = \mu_{m+2}(\iota(\iota'(g)h_l)) = \mu_{m+2}(\begin{bmatrix} p^l & * & * \\ 0 & g & * \\ 0 & 0 & p^{-l} \end{bmatrix}) \geq \mu_m(g) + l$, which implies $\mu_m(g) + l \leq \mu_{m+1}(h) \leq \mu_{m+1}(\iota'(g)) + l$. Applying Lemma 4.3 again (replace H and G_1 by G and H, respectively), we have $\mu_m(g) = \mu_{m+1}(\iota'(g))$ for $g \in G$. Thus we have $N_{H,s}(h) = N_{G,s}(g) \cdot p^{-ls}$. Since $\alpha(\iota(h)) = p^l$ and $\beta(\iota(h)) = g$, we are done. *q.e.d.*

§5. Proof of Lemma B

Throughout this section, we keep the notation and the assumption of §1.11. In particular, $S \in M_m(\mathbf{Z}_p)$ is a non-degenerate maximal even integral symmetric matrix, $G = O(S)$ and $K = G(\mathbf{Z}_p)$. We put $\partial = \partial_p(S)$. Let $G_1 = O(S_1)$ and $K_1 = G_1(\mathbf{Z}_p)$ with $S_1 = \begin{bmatrix} & & 1 \\ & S & \\ 1 & & \end{bmatrix}$. For $X \in \mathbf{Q}_p^m$, put $n_1(X) = \begin{bmatrix} 1 & -{}^tXS & -2^{-1}S[X] \\ 0 & 1_n & X \\ 0 & 0 & 1 \end{bmatrix} \in G_1$. The following result is a key to the proof of Lemma B.

Proposition 5.1 *For* $t \in \mathbf{Q}_p^\times$ *and* $g \in G$, *we have*

$$(5.1)\quad \int_{\mathbf{Q}_p^m} N_{G_1,s+m/2}(n_1(X)\begin{bmatrix} t & & \\ & g & \\ & & t^{-1} \end{bmatrix})\,dx$$

$$= |t|^{m/2}\,p^{-|\mathrm{ord}_p t|\cdot s}\,\frac{(1-p^{-(s+m/2)})(1+p^{-(s+m/2-\partial)})}{1-p^{-2s}}\;N_{G,s+m/2-1}(g).$$

We first demonstrate Lemma B assuming the above result. By Lemma 3.2, we may assume that S is of the form $S_\nu = \begin{bmatrix} & & J_\nu \\ & S_o & \\ J_\nu & & \end{bmatrix}$ where $J_\nu = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ and S_o is a maximal even integral $\mathbf{Q}_p$-anisotropic symmetric matrix of rank $n_o = m - 2\nu$. Note that $\partial_p(S_o) = \partial$. Put $G_\nu = O(S_\nu)$ and $K_\nu = G_\nu(\mathbf{Z}_p)$. For $\lambda \in (\mathbf{C}^\times)^\nu/W_\nu$, we define a function $\phi_{G_\nu,\lambda}$ on G_ν to be

$$(5.2)\quad \phi_{G_\nu,\lambda}(tuk) = \prod_{j=1}^{\nu} |t_j|^{n_o/2+\nu-j}\,\lambda_j^{\mathrm{ord}_p(t_j)}$$

where $t = \mathrm{diag}(t_1, \cdots, t_\nu, 1_{n_o}, t_\nu^{-1}, \cdots, t_1^{-1})$, $u \in G_\nu$ is an upper unipotent matrix of degree m and $k \in K_\nu$. Then Lemma B is equivalent to the following result.

Proposition 5.2

$$(5.3) \qquad \int_{G_\nu} \phi_{G_\nu,\lambda}(g)\, N_{G_\nu,s+m/2-1}(g)\, dg$$

$$= L_p^o(\lambda, s) \times \prod_{j=0}^{\nu-1} (1 - p^{-(s+j+n_o/2)})(1 + p^{-(s+j-\partial+n_o/2)}).$$

Proof of Proposition 5.2. We prove the assertion by induction on ν. The assertion is trivial if $\nu = 0$. Decompose $g \in G_\nu$ into $n(x)\begin{bmatrix} t & & \\ & g' & \\ & & t^{-1} \end{bmatrix}k$ $(x \in Q_p^{m-2}, t \in Q_p^\times, g' \in G_{\nu-1}, k \in K_\nu)$. Then a Haar measure dg on G_ν is given by $dg = |t|^{-(m-2)}\, dX\, d^\times t\, dg'\, dk$. Thus the left-hand side of (5.2) equals

$$(5.4) \qquad \int_{Q_p^{m-2}} dx \int_{Q_p^\times} |t|^{-m+2} d^\times t \int_{G_{\nu-1}} dg'\, \phi_{G_\nu,\lambda}\left(\begin{bmatrix} t & & \\ & g' & \\ & & t^{-1} \end{bmatrix}\right)$$

$$\times N_{G_\nu,s+m/2-1}\left(n(x)\begin{bmatrix} t & & \\ & g' & \\ & & t^{-1} \end{bmatrix}\right).$$

Observing $\phi_{G_\nu,\lambda}\left(\begin{bmatrix} t & & \\ & g' & \\ & & t^{-1} \end{bmatrix}\right) = |t|^{(m-2)/2} \lambda_1^{\mathrm{ord}_p t}\, \phi_{G_{\nu-1},\lambda'}(g')$ with $\lambda' = (\lambda_2, \cdots, \lambda_\nu) \in (C^\times)^{\nu-1}/W_{\nu-1}$ and applying Proposition 5.1, we see that (5.4) equals

$$(5.5) \qquad \frac{(1 - p^{-(s+(m-2)/2)})(1 + p^{-(s+(m-2)/2-\partial)})}{1 - p^{-2s}}$$

$$\times \int_{\mathbf{Q}_p^\times} \lambda_1^{\mathrm{ord}_p t}\, p^{-|\mathrm{ord}_p t|\cdot s}\, d^\times t \int_{G_{\nu-1}} \phi_{G_{\nu-1},\lambda'}(g')\, N_{G_{\nu-1},s+(m-2)/2-1}(g')\, dg'$$

The first integral in (5.5) is equal to $\dfrac{(1-p^{-2s})}{(1-\lambda_1 p^{-s})(1-\lambda_1^{-1}p^{-s})}$ and the indunctive

hypothesis asserts that the second one is equal to

$$L_p^o(\lambda', s) \prod_{j=0}^{\nu-2} (1 - p^{-(s+j+n_o/2)})(1 + p^{-(s+j-\partial+n_o/2)}).$$

These prove the proposition. *q.e.d.*

We now go back to proof of Proposition 5.1. We prove by induction on m. By the bi-K_1-invariance of $N_{G_1,s}$, we see that

$$N_{G_1,s+m/2}(n_1(X)\begin{bmatrix} t & & \\ & g & \\ & & t^{-1} \end{bmatrix})$$

$$= N_{G_1,s+m/2}(\begin{bmatrix} & & 1 \\ & 1_m & \\ 1 & & \end{bmatrix} n_1(X)\begin{bmatrix} t & & \\ & g & \\ & & t^{-1} \end{bmatrix}\begin{bmatrix} & & 1 \\ & 1_m & \\ 1 & & \end{bmatrix})$$

$$= N_{G_1,s+m/2}(n_1'(X)\begin{bmatrix} t^{-1} & & \\ & g & \\ & & t \end{bmatrix})$$

where

$$n_1'(X) = \begin{bmatrix} 1 & 0 & 0 \\ X & 1_m & 0 \\ -2^{-1}S[X] & -{}^tXS & 1 \end{bmatrix}.$$

Next observe that

$$\int_{\mathbf{Q}_p^m} N_{G_1,s+m/2}(n_1(X)\begin{bmatrix} t & & \\ & g & \\ & & t^{-1}\end{bmatrix}) \, dX = \int_{\mathbf{Q}_p^m} N_{G_1,s+m/2}(\begin{bmatrix} t^{-1} & & \\ & g^{-1} & \\ & & t\end{bmatrix} n_1(-X))dX$$

$$= |t|^m \int_{\mathbf{Q}_p^m} N_{G_1,s+m/2}(n_1(X)\begin{bmatrix} t^{-1} & & \\ & g^{-1} & \\ & & t\end{bmatrix}) \, dX$$

(note that $N_{G_1,s}(g_1^{-1}) = N_{G_1,s}(g_1)$ for $g_1 \in G_1$). These show that we only have to prove the following fact: For $t \in \mathbf{Q}_p^\times$ with $\mathrm{ord}_p(t) \le 0$, we have

(5.6) $$\int_{\mathbf{Q}_p^m} N_{G_1,s+m/2}(n_1'(X)\begin{bmatrix} t & & \\ & g & \\ & & t^{-1}\end{bmatrix}) \, dX$$

$$= |t|^{-(s+m/2)} N_{G,s+m/2-1}(g) \frac{(1-p^{-(s+m/2)})(1+p^{-(s+m/2-\partial)})}{1-p^{-2s}} .$$

We first verify (5.6) for the case where S is $\mathbf{Q}_p$-anisotropic. To do this, we collect several facts about the arithmetic of $\mathbf{Q}_p$-anisotropic quadratic forms. Let $V = \mathbf{Q}_p^m$, $L = \mathbf{Z}_p^m$ and put $z_X = 2^{-1}S[X]$ for $X \in V$.

Lemma 5.3 *Suppose that* S *is* $\mathbf{Q}_p$*-anisotropic.*

(i) $X \in L \Leftrightarrow z_X \in \mathbf{Z}_p$.

(ii) *If* $X \notin L$, *then* $z_X^{-1} X \in L$ *and* $1_m - z_X^{-1}X{}^tXS \in K$.

(iii) *We have*

$$1 + \int_{V-L} |z_X|^{-(s+m/2)} \, dX = \frac{(1-p^{-(s+m/2)})(1+p^{-(s+m/2-\partial)})}{1-p^{-2s}} .$$

Lemma 5.4 *The equality (5.6) holds if* S *is* $\mathbf{Q}_p$*-anisotropic.*

Proof. A straightforward calculation shows that, if $X \notin L$,

$$n'_1(X)\begin{bmatrix} t & & \\ & g & \\ & & t^{-1} \end{bmatrix}$$

$$= n_1(-z^{-1}X)\begin{bmatrix} z^{-1} & & \\ & 1_m & \\ & & z \end{bmatrix} n'_1(-z^{-1}X)\begin{bmatrix} & & -1 \\ & 1_m - z^{-1}X^tXS & \\ -1 & & \end{bmatrix}\begin{bmatrix} t & & \\ & g & \\ & & t^{-1} \end{bmatrix}$$

$$= k_1 \begin{bmatrix} (zt)^{-1} & & \\ & 1_m & \\ & & zt \end{bmatrix} k'_1$$

with $z = z_X = 2^{-1}S[X]$ and

$$k_1 = n_1(-z^{-1}X),\ k'_1 = n'_1(-z^{-1}Xt^{-1})\begin{bmatrix} & & -1 \\ & (1_m - z^{-1}X^tXS)g & \\ -1 & & \end{bmatrix}.$$

By Lemma 5.3, we have $k_1, k'_1 \in K_1$ and hence the left-hand side of (5.6) is equal to

$$|t|^{-(s+m/2)}\{1 + \int_{V-L} |z_X|^{-(s+m/2)}\, dX\}.$$

The lemma follows from this and Lemma 5.3 (iii). *q.e.d.*

From now on, we assume that S is $\mathbf{Q}_p$-isotropic. We may suppose that S is of the form $\begin{pmatrix} R & -R\beta \\ -{}^t\beta R & -2b \end{pmatrix}$ where R is a non-degenerate maximal even integral symmetric matrix of rank $m-1$ and $\partial = \partial_p(S) = \partial_p(R)$. Furthermore we may assume that $R = \begin{pmatrix} & & J_v \\ & R_0 & \\ J_v & & \end{pmatrix}$, $\beta = \begin{pmatrix} 0_v \\ \beta_0 \\ 0_v \end{pmatrix}$ where v is the Witt index of R, R_0 is $\mathbf{Q}_p$-anisotropic symmetric matrix of rank $n'_o = m - 1 - 2\nu$ and $\beta_o \in \mathbf{Q}_p^{n'_o}$.

Put $H = O(T)$ and $H' = O(R)$ with $T = \begin{pmatrix} & & 1 \\ & R & \\ 1 & & \end{pmatrix}$. We define embeddings

$H' \overset{\iota_o}{\to} G \overset{\iota'}{\to} H \overset{\iota}{\to} G_1$ as in §4, where we put $\iota_o(h') = \begin{pmatrix} h' & (1-h')\beta \\ 0 & 1 \end{pmatrix}$ $(h' \in H')$.

By definition, $\iota_o(H')$ (resp. $\iota'(G)$, $\iota(H)$) is the stabilizer subgroup of α (resp. ξ, η) in G (resp. H, G_1).

Let $K_1 = G_1(\mathbf{Z}_p)$, $K = G(\mathbf{Z}_p)$, $U = H(\mathbf{Z}_p)$ and $U' = H'(\mathbf{Z}_p)$. For $l \in \mathbf{Z}$, put $M_l = \begin{bmatrix} p^{-l} & & \\ & 1_m & \\ & & p^l \end{bmatrix} \in G_1$. By Proposition 3.9. we have the decomposition:

(5.7) $$G_1 = \bigcup_{l \geq 0} \iota(H) M_l K_1 \qquad \text{(disjoint union)}.$$

We need the following variant of Lemma A.

Lemma 5.5 If $g_1 = \iota(h) M_l k_1$ $(h \in H, l \geq 0, k_1 \in K_1)$,then

$$N_{G_1,s}(g_1) = p^{-ls} N_{H,s}(h).$$

Proof. Applying Lemma A for $H \overset{\iota}{\to} G_1 \overset{\iota_1}{\to} H_1 = O(\begin{pmatrix} & & 1 \\ & T & \\ 1 & & \end{pmatrix})$, we obtain

$$N_{G_1,s}(\iota(h\beta(g_1)^{-1}) \cdot g_1) = |\alpha(g_1)|_p^s N_{H,s}(h) \qquad (h \in H, g_1 \in G_1).$$

Since $\iota_1(g_1) = \begin{pmatrix} p^l & * & * \\ 0 & h & * \\ 0 & 0 & p^{-l} \end{pmatrix} \iota_1(k_1)$ and $\iota_1(k_1) \in H_1(\mathbf{Z}_p)$ (see Proposition 3.7), we have

$\alpha(g_1) = p^l$ and $\beta(g_1) = h$. This proves the lemma. *q.e.d.*

Since both sides of (5.6) are bi-K-invariant functions of $g \in G$, we may assume that $g = \iota_o(h_o)g_o$ where $h_o = \mathrm{diag}(t_1, \cdots, t_\nu, 1_{n'_o}, t_\nu^{-1}, \cdots, t_1^{-1}) \in H'$ and $g_o = \begin{pmatrix} 1_\nu & & & \\ & A & & B \\ & & 1_\nu & \\ & C & & D \end{pmatrix} \in G\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O\left(\begin{pmatrix} R_o & -R_o\beta_o \\ -{}^t\beta_o R_o & -2b \end{pmatrix}\right)\right)$. We note that $\iota_o(h_o) = \mathrm{diag}(t_1, \cdots, t_\nu, 1_{n'_o}, t_\nu^{-1}, \cdots, t_1^{-1}, 1)$.

Remark. If $\nu_p(S) = \nu_p(R)$, then $\begin{pmatrix} R_o & -R_o\beta_o \\ -{}^t\beta_o R_o & -2b \end{pmatrix}$ is $\mathbf{Q}_p$-anisotropic and we may take $g_o = 1$ in this case.

Fix $\omega \geq 0$. We calculate the following integral:

$$(5.8) \qquad I(s) = \int_{\mathbf{Q}_p^m} N_{G_1,s+m/2}(n'_1(X)\begin{bmatrix} p^{-\omega} & & \\ & \iota_o(h_o)g_o & \\ & & p^{\omega} \end{bmatrix})\, dX.$$

First observe that $I(s)$ equals

$$\int_{\mathbf{Q}_p^{m-1}} N_{G_1,s+m/2}(n'_1(\begin{pmatrix} x \\ 0 \end{pmatrix}))\begin{bmatrix} p^{-\omega} & & \\ & \iota_o(h_o)g_o & \\ & & p^{\omega} \end{bmatrix})\, dx$$

$$+ \sum_{\rho \geq 1} \int_{p^{-\rho}\mathbf{Q}_p^\times} du \int_{\mathbf{Q}_p^{m-1}} dx\; N_{G_1,s+m/2}(n'_1(\begin{pmatrix} x \\ 1-u \end{pmatrix}))\begin{bmatrix} p^{-\omega} & & \\ & \iota_o(h_o)g_o & \\ & & p^{\omega} \end{bmatrix}).$$

To calculate these integrals, we need three results from the arithmetic of local orthogonal groups. We postpone their proof until the last part of this section. Put $n_{g_o} = n_1((g_o^{-1}-1)\alpha)$. Note that $n_{g_o} = \begin{bmatrix} 1 & & \\ & g_o^{-1} & \\ & & 1 \end{bmatrix} \iota(\iota'(g_o))$ (Lemma 4.2 (iii)).

Lemma 5.6 *For* $\rho \geq 0$, *we have* $n_{g_o}^{-1} M_{\omega+\rho} \in \iota(h_\rho) M_{l_\rho} \cdot K_1$ *with some* $h_\rho \in H$. *Here* $l_\rho = \mathrm{Max}(\omega+\rho, \mu)$, $\mu = \mu_m(g_o)$ *(for the definition of* μ_m, *see §1.10).*

Remark. By the assumption of g_o, we may take $h_\rho = \begin{pmatrix} 1_v & & & \\ & A' & & B' \\ & & 1_v & \\ & C' & & D' \end{pmatrix}$ with $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in O(\begin{pmatrix} R_o & -R_o\beta_o \\ -{}^t\beta_o R_o & -2b \end{pmatrix})$.

Lemma 5.7 *For* $u \in p^{-\rho}\mathbf{Z}_p^{\times}$ $(\rho \geq 0)$ *and* $x \in \mathbf{Q}_p^{m-1}$, *we have*

$$n_1'(\begin{pmatrix} x \\ 1-u \end{pmatrix}) \begin{bmatrix} p^{-\omega} & & \\ & \iota_o(h_o)g_o & \\ & & p^{\omega} \end{bmatrix} \in \iota(n(x)\begin{pmatrix} u & & \\ & h_o & \\ & & u^{-1} \end{pmatrix}\iota'(g_o)h_\rho)\, M_{l_\rho} \cdot K_1,$$

where we put $n(x) = \begin{pmatrix} 1 & -{}^t xR & -2^{-1}R[x] \\ 0 & 1_{m-1} & x \\ 0 & 0 & 1 \end{pmatrix} \in H$ *for* $x \in \mathbf{Q}_p^{m-1}$.

Lemma 5.8 *We have* $\iota'(g_o)h_\rho \in \begin{pmatrix} p^{\tau_\rho} & * & * \\ 0 & 1_{m-1} & * \\ 0 & 0 & p^{-\tau_\rho} \end{pmatrix} U$ *with* $\tau_\rho = \omega + \rho - l_\rho + \mu$.

By Lemma 5.5 and Lemma 5.7, I(s) is equal to

$$\sum_{\rho=0}^{\infty} p^{\rho}(1-\delta(\rho\geq 1)p^{-1})\, p^{-l_\rho(s+m/2)} \times \int_{Q_p^{m-1}} N_{H,s+m/2}\left(n(x)\begin{pmatrix} p^{-\rho} & & \\ & h_o & \\ & & p^{\rho}\end{pmatrix}\iota'(g_o)h_\rho\right) dx$$

where $\delta(\rho\geq 1) = \begin{cases} 1 & \text{if } \rho \geq 1 \\ 0 & \text{if } \rho = 0. \end{cases}$ By Lemma 5.8, this is equal to

$$\sum_{\rho=0}^{\infty} p^{\rho}(1-\delta(\rho\geq 1)p^{-1})\, p^{-l_\rho(s+m/2)} \times \int_{Q_p^{m-1}} N_{H,s+m/2}\left(n(x)\begin{pmatrix} p^{\tau_\rho-\rho} & & \\ & h_o & \\ & & p^{-(\tau_\rho-\rho)}\end{pmatrix}\right) dx.$$

By inductive hypothesis, I(s) equals

$$\sum_{\rho=0}^{\infty} p^{\rho}(1-\delta(\rho\geq 1)p^{-1})\, p^{-l_\rho(s+m/2)-(\tau_\rho-\rho)\cdot\frac{m-1}{2}-|\tau_\rho-\rho|(s+1/2)}$$

$$\times \frac{(1-p^{-(s+m/2)})(1+p^{-(s+m/2-\partial)})}{1-p^{-2(s+1/2)}}\, N_{H',s+m/2-1}(h_o).$$

Since $N_{G,s+m/2-1}(g) = p^{-\mu(s+m/2-1)}\, N_{H',s+m/2-1}(h_o)$ for $g = \iota_o(h_o)g_o$, we get

$$I(s) = J(s) \times \frac{(1-p^{-(s+m/2)})(1+p^{-(s+m/2-\partial)})}{1-p^{-2(s+1/2)}}\, N_{G,s+m/2-1}(g)$$

where $J(s)$ is the sum

$$\sum_{\rho=0}^{\infty} p^{\rho}(1-\delta(\rho\geq 1)p^{-1})\, p^{-l_\rho(s+m/2)-(\tau_\rho-\rho)\cdot\frac{m-1}{2}-|\tau_\rho-\rho|(s+1/2)+\mu(s+m/2-1)}.$$

To evaluate this sum, recall that $l_\rho = \mathrm{Max}(\omega+\rho, \mu)$, $\tau_\rho = \omega+\rho-l_\rho+\mu =$

$\mathrm{Min}(\omega+\rho, \mu)$.

Lemma 5.9

(i) $\rho \leq \tau_\rho \Leftrightarrow \rho \leq \mu$.

(ii) *If* $\rho > \mu$, *then we have* $\tau_\rho = \mu$ *and* $l_\rho = \omega + \rho$.

Proof. Observe that $\tau_\rho - \rho = \omega + \mu - \mathrm{Max}(\omega + \rho, \mu)$. If $\rho > \mu$, then $\mathrm{Max}(\omega + \rho, \mu)$ $= \omega + \rho$, which implies $\tau_\rho = \mu$, $l_\rho = \omega + \rho$ and $\tau_\rho \leq \rho$. Next suppose that $\rho \leq \mu$. Then $\tau_\rho - \rho = \begin{cases} \mu - \rho & \text{if } \omega+\rho \geq \mu \\ \omega & \text{if } \omega+\rho < \mu \end{cases}$ and hence $\tau_\rho \geq \rho$. *q.e.d.*

By Lemma 5.9, J(s) is equal to

$$\sum_{\rho=0}^{\mu} (1 - \delta(\rho\geq 1)p^{-1})\, p^{\rho-\omega(s+m/2)-\mu}$$

$$+ \sum_{\rho=\mu+1}^{\infty} (1 - p^{-1})\, p^{\rho-(\omega+\rho)(s+m/2)-(\mu-\rho)(m/2-1-s)+\mu(s+m/2-1)}$$

$$= p^{-\omega(s+m/2)} + p^{-\omega(s+m/2)} \frac{(1-p^{-1})p^{-2s}}{1-p^{-2s}}$$

$$= p^{-\omega(s+m/2)} \frac{1-p^{-2(s+1/2)}}{1-p^{-2s}}.$$

This implies that

$$I(s) = p^{-\omega(s+m/2)} \frac{(1-p^{-(s+m/2)})(1+p^{-(s+m/2-\partial)})}{1-p^{-2s}} N_{G,s+m/2-1}(g),$$

which completes the proof of (5.6).

It now remains to show Lemma 5.6 – Lemma 5.8. We first prove Lemma 5.7 assuming Lemma 5.6. Put $g_1 = n_1'\left(\begin{pmatrix} x \\ 1-p^{-\rho} \end{pmatrix}\right) \cdot \begin{bmatrix} p^{-\omega} & & \\ & \iota_o(h_o)g_o & \\ & & p^{\omega} \end{bmatrix}$ with $\rho \geq 0$, $x \in \mathbf{Q}_p^{m-1}$. By Lemma 4.2, we have

$$g_1 = n_1'\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) n_1'\left(\begin{pmatrix} 0 \\ 1-p^{-\rho} \end{pmatrix}\right) \begin{bmatrix} p^{\rho} & & \\ & 1_m & \\ & & p^{-\rho} \end{bmatrix} \begin{bmatrix} p^{-(\omega+\rho)} & & \\ & \iota_o(h_o)g_o & \\ & & p^{\omega+\rho} \end{bmatrix}$$

$$= \iota(n(x) \begin{bmatrix} p^{-\rho} & & \\ & h_o & \\ & & p^{\rho} \end{bmatrix} \iota'(g_o)) \cdot n_{g_o}^{-1} M_{\omega+\rho}.$$

By Lemma 5.6, g_1 is in $\iota(n(x) \begin{bmatrix} p^{-\rho} & & \\ & h_o & \\ & & p^{\rho} \end{bmatrix} \iota'(g_o)h_\rho) M_{l_\rho} \cdot K_1$, which proves Lemma 5.7.

To prove the remaining lemmas, first consider the case $\nu_p(S) = \nu_p(R)$. As was noted before, we may take $g_o = 1$ so that Lemma 5.6 and Lemma 5.8 are trivial in this case. In what follows we suppose that $\nu_p(S) = \nu_p(R) + 1$. In view of the remark after Lemma 5.6, we may assume that $\nu = \nu_p(R) = 0$ (that is, R is $\mathbf{Q}_p$-anisotropic).

Lemma 5.10 *Let* R *and* $S = \begin{pmatrix} R & -R\beta \\ -{}^t\beta R & -2b \end{pmatrix}$ *be maximal even integral symmetric matrices of rank* $m-1$ *and* m, *respectively. Assume that* $\partial_p(R) = \partial_p(S)$, $\nu_p(R) = 0$ *and* $\nu_p(S) = 1$. *Put* $\alpha = D^{-1}\begin{pmatrix} \beta \\ 1 \end{pmatrix}$ *with* $D = R[\beta] + 2b$. *For* $g \in G = O(S)$, *put* $\mu = \mu_m(g)$. *Then*

(i) $p^{\mu}g^{-1}\alpha \in L^*_{prim}$.

(ii) $n_g = n_1((g^{-1} - 1)\alpha) \in \begin{bmatrix} p^{-\mu} & 0 & 0 \\ * & * & 0 \\ * & * & p^{\mu} \end{bmatrix} \cdot K_1$.

Proof. The assertion (i) and (ii) are trivial if $\mu = 0$. From now on we assume that $\mu > 0$. For simplicity, we put $L = \mathbf{Z}_p^m$, $M = \mathbf{Z}_p^{m-1}$, $L^* = S^{-1}L$ and $M^* = R^{-1}M$. By maximality of M with respect to R, $M' = \{x \in M^* \mid 2^{-1}R[x] \in p^{-1}\mathbf{Z}_p\}$ forms a $\mathbf{Z}_p$-lattice (see §3.4). Under our assumption $\partial(R) = \partial(S)$, we have $\beta \in M'^* = R^{-1}M'$ (see the remark in the proof of Theorem 2.6 in [Su]).

Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be the block decomposition corresponding to $m = (m-1) + 1$. By definition of μ, $p^{\mu}g$ is a primitive element of $M_m(\mathbf{Z}_p)$. Since $Sg_0^{-1}\alpha = -\begin{pmatrix} {}^tC \\ {}^tD \end{pmatrix}$, it is sufficient to show $p^{\mu}\begin{pmatrix} {}^tC \\ {}^tD \end{pmatrix} \in L_{prim}$ to prove the assertion (i). Suppose that $p^{\mu}\begin{pmatrix} {}^tC \\ {}^tD \end{pmatrix} \in p{\cdot}L$. We claim that, for every $x \in M$, we have $p^{\mu}Ax \in p{\cdot}M$, which implies $p^{\mu}A \in p{\cdot}M_{m-1}(\mathbf{Z}_p)$. Observe that

$$2^{-1}R[xp^{\mu}] = 2^{-1}S[\begin{pmatrix} x \\ 0 \end{pmatrix}p^{\mu}] = 2^{-1}S[g\begin{pmatrix} x \\ 0 \end{pmatrix}p^{\mu}] = 2^{-1}S[\begin{pmatrix} Ax \\ Cx \end{pmatrix}p^{\mu}]$$

$$= 2^{-1}R[Axp^{\mu}] - R(p^{\mu}Ax, \beta){\cdot}Cxp^{\mu} - b(Cxp^{\mu})^2.$$

Since $2^{-1}R[xp^{\mu}]$, $b(Cxp^{\mu})^2 \in p^2\mathbf{Z}_p$, $R(p^{\mu}Ax, \beta) \in \mathbf{Z}_p$ and $Cxp^{\mu} \in p\mathbf{Z}_p$, we get $2^{-1}R[Axp^{\mu}] \in p\mathbf{Z}_p$. It follows that $Axp^{\mu} \in p{\cdot}M'$ and hence that $R(p^{\mu}Ax, \beta) \in p\mathbf{Z}_p$ by the above remarks. This implies $2^{-1}R[Axp^{\mu}] \in p^2\mathbf{Z}_p$. Then Lemma 3.2 (i) shows our claim. We can prove $p^{\mu}B \in pM$ in a similar way. Thus we get $p^{\mu}g \in p{\cdot}M_m(\mathbf{Z}_p)$, which is a contadiction. The assertion (i) has been proved. To prove (ii), it is sufficient to show that the first row of $\begin{bmatrix} p^{\mu} & 0 & 0 \\ * & * & 0 \\ * & * & p^{-\mu} \end{bmatrix} \cdot n_g$ is integral and primitive. Observe that

the first row is given by $(p^{\mu}, -p^{\mu} \cdot {}^t(S(g^{-1}-1)\alpha), -2^{-1}p^{\mu} \cdot S[(g^{-1}-1)\alpha])$. We have proved that $p^{\mu} \cdot S(g^{-1}-1)\alpha$ is integral and primitive. It remains to show that $2^{-1}p^{\mu} \cdot S[(g^{-1}-1)\alpha]$ is integral. Take the least non-negative integer l satisfying $p^{l} \cdot (g^{-1}-1)\alpha \in L$. By (i), we have $l \geq \mu$. Assume that $l > \mu$. Then $2^{-1}S[(g^{-1}-1)\alpha p^{l-1})] = -S((g^{-1}-1)\alpha p^{l}, \alpha p^{l-2}) \in \mathbf{Z}_p$ since $(g^{-1}-1)\alpha p^{l} \in L$ and $\alpha p^{l-2} \in L^*$ (note that $l \geq 2$). On the other hand, we see that $(g^{-1}-1)\alpha p^{l-1} \in L^*$ by (i). Then the maximality of S implies $(g^{-1}-1)\alpha p^{l-1} \in L$, which contradicts to the definition of l. This shows that $l = \mu$. Then the assertion (ii) follows from $2^{-1}p^{\mu} \cdot S[(g^{-1}-1)\alpha] = -S(p^{\mu}(g^{-1}-1)\alpha, \alpha)$. *q.e.d.*

Proof of Lemma 5.6. Recall that $\iota(H)$ is the stabilizer subgroup of $\eta = \begin{bmatrix} 0 \\ \alpha \\ 1 \end{bmatrix}$ in G_1.

Put $\eta' = M_{\omega+\rho}^{-1} n_{g_o} \cdot \eta = \begin{bmatrix} 0 \\ g_o^{-1}\alpha \\ p^{-\omega-\rho} \end{bmatrix}$. By Lemma 5.10 (i), we have $p^{l_\rho} \cdot \eta' \in L_{1,prim}^*$.

Choose $X \in \mathbf{Z}_p^m$ so that $-S(X, g_o^{-1}\alpha p^{l_\rho}) + p^{l_\rho - \omega - \rho} = 1$. Then $n_1'(X)\eta' \cdot p^{l_\rho}$ $= \begin{bmatrix} 0 \\ g_o^{-1}\alpha p^{l_\rho} \\ 1 \end{bmatrix}$. We can find an element κ of K so that $Y = \kappa\alpha p^{l_\rho} - g_o^{-1}\alpha p^{l_\rho} \in \mathbf{Z}_p^m$.

Then

$$n_1(Y)\, n_1'(X) \cdot p^{l_\rho}\eta' = \begin{bmatrix} 0 \\ \kappa\alpha p^{l_\rho} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \kappa & \\ & & 1 \end{bmatrix} M_{l_\rho}^{-1} \cdot p^{l_\rho}\eta,$$

which implies $M_{l_\rho}\begin{bmatrix}1 & & \\ & \kappa^{-1} & \\ & & 1\end{bmatrix} n_1(Y)n_1'(X)M_{\omega+\rho}^{-1}\, n_{g_0} \in \iota(H)$. We are done. *q.e.d.*

Proof of Lemma 5.8. The proof of Lemma 5.6 shows that we can take $h_\rho \in H$ so that

$$n_{g_o}\iota(h_\rho) = M_{\omega+\rho}\, n_1'(-X)n_1(-Y)\, M_{l_\rho}^{-1}\begin{pmatrix}1 & & \\ & \kappa & \\ & & 1\end{pmatrix}$$

with $X, Y \in \mathbf{Z}_p^m$ and $\kappa \in K$. Hence

$$\iota(\iota'(g_o)h_\rho) = \begin{pmatrix}1 & & \\ & g_o & \\ & & 1\end{pmatrix} M_{\omega+\rho}\, n_1'(-X)\, M_{l_\rho}^{-1}\, n_1(-Yp^{-l_\rho})\begin{pmatrix}1 & & \\ & \kappa & \\ & & 1\end{pmatrix}.$$

Since $-Yp^{-l_\rho} = -\kappa\alpha + g_o^{-1}\alpha = (g_o^{-1} - 1)\alpha - (\kappa - 1)\alpha$, we have $n_1(Yp^{-l_\rho}) =$ $n_{g_o}\cdot n_1(-(\kappa-1)\alpha)$. Note that $n_1(-(\kappa-1)\alpha) = \iota(\iota'(\kappa))\begin{pmatrix}1 & & \\ & \kappa^{-1} & \\ & & 1\end{pmatrix} \in K_1$ and $n_{g_o} \in \begin{bmatrix}p^{-\mu} & 0 & 0\\ * & 1_m & 0\\ * & * & p^{\mu}\end{bmatrix}$ K_1 (see Lemma 5.10 (ii)). Thus $\iota(\iota'(g_o)h_\rho) \in \begin{bmatrix}p^{-\tau_\rho} & 0 & 0\\ * & 1_m & 0\\ * & * & p^{\tau_\rho}\end{bmatrix} K_1$, which proves the lemma in view of Lemma 5.4. *q.e.d.*

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