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# Double Poisson brackets on free associative algebras

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*to the memory of S.V. Manakov*

**Abstract**

We discuss double Poisson structures in sense of M. Van den Bergh on free associative algebras focusing on the case of quadratic Poisson brackets. We establish their relations with an associative version of Young-Baxter equations, we study a bi-hamiltonian property of the linear-quadratic pencil of the double Poisson structure and propose a classification of the quadratic double Poisson brackets in the case of the algebra with two free generators. Many new examples of quadratic double Poisson brackets are proposed.

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# 1 Introduction

A Poisson structure on a commutative algebra $A$ is a Lie algebra structure on $A$ given by a Lie bracket

$$\{-,-\} : A \times A \mapsto A,$$

which is a derivation of $A$ i.e. satisfies the Leibniz rule

$$\{a, bc\} = \{a, b\}c + b\{a, c\}, \quad a, b, c \in A,$$

for the right (and, hence, for the left) argument.

It is well-known (see the discussion in [12]) that a naive translation of this definition to the case of a non-commutative associative algebra $A$ is not very interesting because of lack of examples different from the usual commutator (for prime rings it was shown in [13]).

It turns out [12, 17] that a natural generalization of Poisson structures on comutative associative algebras to a non-commutative case is a Lie structure on the vector space $H_0(A, A) = A/[A, A]$, where $[A, A]$ is the vector space spaned by all comutators $ab - ba$ where $a, b \in A$. The elements of this space are $0-$dimensional cyclic homology classes of $A$ and they are represented by "cyclic words" whose letters are the elements of $A$ [5]. In [12] such a structure was called an $H_0-$Poisson structure while in [10] the terminology a "non-abelian Poisson bracket" was used. Both names are somehow misleading to our mind (because one deals with a Lie structure with no multiplication structure on $H_0(A, A)$). Therefore we would suggest to call it a *trace bracket* by the following reason.

Let $A$ be a (unital) associative algebra over $\mathbb{C}$. For a fixed natural $n$ we denote by

$$\mathrm{Rep}_n(A) := \mathrm{Hom}(A, \mathrm{Mat_n}(\mathbb{C}))$$

the space of $n-$dimensional representations of $A$, and by $\mathbb{C}[\mathrm{Rep}_n(A)]$ the coordinate ring of this affine scheme $\mathrm{Rep}_n(A)$. Let $\mathrm{tr} : A \to \mathbb{C}[\mathrm{Rep}_n(A)]$ be the trace map. It is clear that $\mathrm{tr}(a)$ is a $GL_n(\mathbb{C})-$invariant element for any $a \in A$.

Since the map tr is well defined also on elements of $A/[A, A]$, any trace bracket $\{\}$ induces the following genuine Poisson bracket on the representation space function algebras:

$$\{\mathrm{tr}(a), \mathrm{tr}(b)\} = \mathrm{tr}(\{a, b\})$$

on Im(tr). This bracket (according to results from [16, 21]) can be extended to the subalgebra of all $GL_n(\mathbb{C})-$invariant elements of $\mathbb{C}[\mathrm{Rep}_n(A)]$ and, in certain important cases, even to the whole algebra $\mathbb{C}[\mathrm{Rep}_n(A)]$. We shall refer such brackets on $\mathbb{C}[\mathrm{Rep}_n(A)]$ and $\mathbb{C}[\mathrm{Rep}_n(A)]^{GL_n(\mathbb{C})}$ as *trace Poisson brackets.*

In this paper we shall consider as a basic example the case of free associative algebra $A = \mathbb{C} < x_1, \ldots, x_m >$ .

The coordinate algebra $\mathbb{C}[\mathrm{Rep}_n(A)]$ in this case is the polynomial ring of $mn^2$ variables $x^j_{i,\alpha}$, where

$$x_\alpha \to M_\alpha = \left( \begin{array}{ccc} x^1_{1,\alpha} & \cdot & x^n_{1,\alpha} \\ \cdot & \cdot & \cdot \\ x^1_{n,\alpha} & \cdot & x^n_{n,\alpha} \end{array} \right), \qquad 1 \le \alpha \le m.$$

The map tr gives the following interpretation of the variables $x^j_{i,\alpha}$ : if $E^j_i$ denotes the $(i,j)-$matrix unit (i.e. the $n \times n$ matrix with 0 everywhere except the $i-$th row and $j-$th column) then $x^j_{i,\alpha} = \mathrm{tr}(E^i_j M_\alpha)$.

The group $GL_n(\mathbb{C})$ acts on $M_1, ..., M_m$ by the conjugations. Any trace bracket on the free algebra $A$ is extended on $\mathbb{C}[\mathrm{Rep}_n(A)]$ and it yields a usual $GL_n(\mathbb{C})-$ invariant Poisson bracket such that the bracket between traces of any two matrix polynomials $P_i(M_1, ..., M_m)$, $i = 1, 2$ is a trace of some matrix polynomial $P_3$. Notice that not any $GL_n(\mathbb{C})-$invariant Poisson bracket on $\mathbb{C}[\mathrm{Rep}_n(A)]$ is a trace Poisson.

There are two different ways to represent explicitely the same trace brackets in the free algebra case. One is a standard way used in the Integrable System theory (see [8]), where brackets are given by

$$\{a, b\} =< \mathrm{grad}\, a,\, \Theta(\mathrm{grad}\, b) >, \qquad a, b \in A/[A, A],$$

for some Hamiltonian operator $\Theta$, a skew-symmetric operator expressed via left and right multiplication operators on $A$. The trace brackets define a Hamiltonian formalism for integrable models with matrix variables [8]. In particular, some of such models are bi-Hamiltonian with respect to compatible linear and quadratic trace Poisson brackets [10].

Another approach can be developped in terms of *double Poisson brackets* introduced in [17]. We shall remind their definition.

**Definition** (M. Van den Bergh). A double Poisson bracket on an associative algebra $A$ is a $\mathbb{C}$-linear map $\{\!\{,\}\!\} : A \otimes A \mapsto A \otimes A$ satisfying the following conditions:

$$\{\!\{u, v\}\!\} = -\{\!\{v, u\}\!\}^\circ, \tag{1.1}$$

$$\{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l + \sigma\{\!\{v, \{\!\{w, u\}\!\}\}\!\}_l + \sigma^2\{\!\{w, \{\!\{u, v\}\!\}\}\!\}_l = 0, \tag{1.2}$$

and

$$\{\!\{u, vw\}\!\} = (v \otimes 1)\{\!\{u, w\}\!\} + \{\!\{u, v\}\!\}(1 \otimes w). \tag{1.3}$$

Here $(u \otimes v)^\circ := v \otimes u$; $\{\!\{v_1, v_2 \otimes v_3\}\!\}_l := \{\!\{v_1, v_2\}\!\} \otimes v_3$ and $\sigma(v_1 \otimes v_2 \otimes v_3) := v_3 \otimes v_1 \otimes v_2$.

Notice that very similar relations but with a different bi-module structure in (1.3) have appeared in [4].

The relations between double and trace Poisson brackets are established by M.Van den Bergh [17] as follows. Let $\mu$ denote the multiplication map $\mu : A \otimes A \to A$ i.e. $\mu(u \otimes v) = uv$. We define a $\mathbb{C}-$bilinear bracket operation in $A$ by $\{-, -\} := \mu(\{\!\{-, -\}\!\})$.

**Proposition 1.** Let $\{\!\{-,-\}\!\}$ be a double Poisson bracket on $A$. Then $\{-,-\}$ is a trace bracket on $A/[A,A]$ which is defined as

$$\{\bar{a},\bar{b}\} = \overline{\mu(\{\!\{a,b\}\!\}},\tag{1.4}$$

ou $\bar{a}$ means the image of $a \in A$ under the natural projection $A \to A/[A,A]$.

If $A = \mathbb{C} < x_1, \ldots, x_m >$ is the free associative algebra, then $\mathbb{C}[\mathrm{Rep}_n(A)] = \mathbb{C}[x^j_{i,\alpha}]$ where $1 \le \alpha \le m$.

If $\{\!\{x_\alpha, x_\beta\}\!\}$ is a double Poisson bracket on $A = \mathbb{C} < x_1, \ldots, x_m >$, then, using the Sweedler convention and drop the sign of sum, we obtain the trace Poisson brackets on $\mathbb{C}[\mathrm{Rep}_n(A)]$:

$$\{x^j_{i,\alpha}, x^l_{k,\beta}\} = \{\!\{x_\alpha, x_\beta\}\!\}'^{j}_{k} \{\!\{x_\alpha, x_\beta\}\!\}''^{l}_{i}$$

In this paper we shall consider linear and quadratic double Poisson brackets on free associative algebras. It turns out that linear double brackets are in one-to-one correspondence with $m$-dimensional associative algebra structures [18]. We establish relations between a class of quadratic double brackets and constant solutions of classical associative Yang-Baxter equation on $\mathrm{Mat}_m(\mathbb{C})$ introduced in [1]. The examples of double brackets related to non-constant solutions of various associative Yang-Baxter equations will be discussed in the forthcoming paper [11].

# 2 Quadratic double Poisson brackets

Let $A = \mathbb{C} < x_1, \ldots, x_m >$ be the free associative algebra. If double brackets $\{\!\{x_i, x_j\}\!\}$ between all generators are fixed, then the bracket between two arbitrary elements of $A$ is uniquely defined by identities (1.1) and (1.3). It follows from (1.1) that constant, linear, and quadratic double brackets are defined by

$$\{\!\{x_i, x_j\}\!\} = c_{ij} 1 \otimes 1, \qquad c_{i,j} = -c_{j,i},\tag{2.5}$$

$$\{\!\{x_i, x_j\}\!\} = b^k_{ij} x_k \otimes 1 - b^k_{ji} 1 \otimes x_k,\tag{2.6}$$

and

$$\{\!\{x_\alpha, x_\beta\}\!\} = r^{uv}_{\alpha\beta}\, x_u \otimes x_v + a^{vu}_{\alpha\beta}\, x_u x_v \otimes 1 - a^{uv}_{\beta\alpha}\, 1 \otimes x_v x_u,\tag{2.7}$$

where

$$r^{\sigma\epsilon}_{\alpha\beta} = -r^{\epsilon\sigma}_{\beta\alpha},\tag{2.8}$$

correspondingly. The summation with respect to repeated indexes is assumed.

It is easy to verify that the bracket (2.5) satisfies (1.2) for any skew-symmetric tensor $c_{ij}$. For the bracket (2.6) the condition (1.2) is equivalent to the identity

$$b^\mu_{\alpha\beta} b^\sigma_{\mu\gamma} = b^\sigma_{\alpha\mu} b^\mu_{\beta\gamma},\tag{2.9}$$

which means that $b^{\sigma}_{\alpha\beta}$ are structure constants of an associative algebra $A$.

**Proposition 2.** The bracket (2.7) satisfies (1.2) iff the following relations hold:

$$r^{\lambda\sigma}_{\alpha\beta} r^{\mu\nu}_{\sigma\tau} + r^{\mu\sigma}_{\beta\tau} r^{\nu\lambda}_{\sigma\alpha} + r^{\nu\sigma}_{\tau\alpha} r^{\lambda\mu}_{\sigma\beta} = 0, \tag{2.10}$$

$$a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\mu\sigma}_{\tau\alpha} a^{\nu\lambda}_{\sigma\beta}, \tag{2.11}$$

$$a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\sigma\tau} = a^{\mu\sigma}_{\alpha\beta} r^{\lambda\nu}_{\tau\sigma} + a^{\mu\nu}_{\alpha\sigma} r^{\sigma\lambda}_{\beta\tau} \tag{2.12}$$

and

$$a^{\lambda\sigma}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\sigma\nu}_{\alpha\beta} r^{\lambda\mu}_{\sigma\tau} + a^{\mu\nu}_{\sigma\beta} r^{\sigma\lambda}_{\tau\alpha}. \tag{2.13}$$

The trace Poisson bracket corresponding to any double Poisson bracket (2.7) can be defined on $\mathbb{C}[\mathrm{Rep}_n(A)]$ by the following way [10]:

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = r^{\gamma\epsilon}_{\alpha\beta} x^{j'}_{i,\gamma} x^{j}_{i',\epsilon} + a^{\gamma\epsilon}_{\alpha\beta} x^{k}_{i,\gamma} x^{j'}_{k,\epsilon} \delta^{j}_{i'} - a^{\gamma\epsilon}_{\beta\alpha} x^{k}_{i',\gamma} x^{j}_{k,\epsilon} \delta^{j'}_{i} \tag{2.14}$$

where $x^j_{i,\alpha}$ are entries of the matrix $x_\alpha$ and $\delta^j_i$ is the Kronecker delta-symbol. Relations (2.8), (2.10)-(2.13) hold iff (2.14) is a Poisson bracket.

Under a linear change of the generators $x_\alpha \to g^\beta_\alpha x_\beta$ the coefficients of tensors $r$ and $a$ are transformed in the standard way:

$$r^{\gamma\sigma}_{\alpha\beta} \to g^\lambda_\alpha g^\mu_\beta h^\gamma_\nu h^\sigma_\epsilon \, r^{\nu\epsilon}_{\lambda\mu}, \qquad a^{\gamma\sigma}_{\alpha\beta} \to g^\lambda_\alpha g^\mu_\beta h^\gamma_\nu h^\sigma_\epsilon \, a^{\nu\epsilon}_{\lambda\mu}, \tag{2.15}$$

Here $g^\beta_\alpha h^\gamma_\beta = \delta^\gamma_\alpha$.

The system of algebraic equations (2.8), (2.10)-(2.13) admits the following involution:

$$r^{\gamma\sigma}_{\alpha\beta} \to r^{\sigma\gamma}_{\alpha\beta}, \qquad a^{\gamma\sigma}_{\alpha\beta} \to -a^{\sigma\gamma}_{\beta\alpha}. \tag{2.16}$$

Given a solution $r$ of (2.8), (2.10), one can put $a^{ij}_{uv} = 0$ to satisfy equations (2.11)-(2.13). Note that the algebraic system of equations (2.8), (2.10) besides (2.16) admits the involution

$$r^{\gamma\sigma}_{\alpha\beta} \to r^{\alpha\beta}_{\gamma\sigma}. \tag{2.17}$$

Some examples of double Poisson brackets with zero tensor $a$ can be found using the one-to-one correspondence [1] between solutions of (2.8), (2.10) up to equivalence (2.15) and exact representations of anti-Frobenius algebras up to isomorphisms.

Recall that an *anti-Frobenius algebra* is an associative algebra $\mathcal{J}$ (not necessarily with unity) with non-degenerate anti-symmetric bilinear form ( , ) satisfying the following relation

$$(x, yz) + (y, zx) + (z, xy) = 0 \tag{2.18}$$

for all $x, y, z \in \mathcal{J}$.

**Construction.** Let $\mathcal{J}$ be a $p$-dimensional associative anti-Frobenius subalgebra in $\mathrm{Mat}_m(\mathbb{C})$ with a basis $y_i = (y^\alpha_{\gamma,i})$, $i = 1, ..., p$. Let $G = (g_{ij})$ be the matrix of the form. Then the tensor $r^{\alpha\beta}_{\gamma\delta} = g^{ij} y^\alpha_{\gamma,i} y^\beta_{\delta,j}$, where $G^{-1} = (g^{ij})$, satisfies (2.8), (2.10).

**Example 1**. Let $\mathcal{J}$ be the associative algebra of all $m \times m$ matrices with zero $m$-th row, $(x, y) = \mathrm{trace}([x, y]\, k^T)$, where $k \in \mathcal{J}$ is a fixed generic element. The corresponding bracket up to equivalence (2.15) is given by a tensor $r$ with the following non-zero components

$$r^{\alpha\beta}_{\alpha\beta} = r^{\beta\alpha}_{\alpha\beta} = r^{\alpha\alpha}_{\beta\alpha} = -r^{\alpha\alpha}_{\alpha\beta} = \frac{1}{\lambda_\alpha - \lambda_\beta}, \quad \alpha \neq \beta. \tag{2.19}$$

Here $\lambda_1, \ldots, \lambda_N$ are arbitrary pairwise distinct parameters. The generalization of (2.19) to the case $k$ zero rows, where $k$ is any divisor of $m$, can be found in [20]. □

It would be interesting to find an algebraic structure generalizing the anti-Frobenius algebras that corresponds to the whole set of relations (2.8), (2.10)-(2.13).

We may interpret the four index tensors $r$ and $a$ as:

1) operators on $V \otimes V$, where $V$ is an $m$-dimensional vector space;

2) elements of $\mathrm{Mat}_m(\mathbb{C}) \otimes \mathrm{Mat}_m(\mathbb{C})$;

3) operators on $\mathrm{Mat_m}(\mathbb{C})$.

For the first interpretation let $V$ be a linear space with a basis $e_\alpha$, $\alpha = 1, ..., m$. Define linear operators $r$, $a$ on the space $V \otimes V$ by

$$r(e_\alpha \otimes e_\beta) = r^{\sigma\epsilon}_{\alpha\beta} e_\sigma \otimes e_\epsilon, \qquad a(e_\alpha \otimes e_\beta) = a^{\sigma\epsilon}_{\alpha\beta} e_\sigma \otimes e_\epsilon.$$

Then the identities (2.8), (2.10)-(2.13) can be written as

$$\begin{gathered} r^{12} = -r^{21}, \quad r^{23} r^{12} + r^{31} r^{23} + r^{12} r^{31} = 0, \\ a^{12} a^{31} = a^{31} a^{12}, \\ \sigma^{23} a^{13} a^{12} = a^{12} r^{23} - r^{23} a^{12}, \\ a^{32} a^{12} = r^{13} a^{12} - a^{32} r^{13}. \end{gathered} \tag{2.20}$$

Here all operators act in $V \otimes V \otimes V$, $\sigma^{ij}$ means the transposition of $i$-th and $j$-th components of the tensor product, and $a^{ij}$, $r^{ij}$ mean operators $a$, $r$ acting in the product of the $i$-th and $j$-th components.

Note that first two relations mean that the tensor $r$ should be skew-symmetric solution of the classical associative Yang-Baxter equation [1].

In the second interpretation we consider the following elements from $\mathrm{Mat}_m(\mathbb{C}) \otimes \mathrm{Mat}_m(\mathbb{C})$: $r = r^{km}_{ij} e^i_k \otimes e^j_m$, $\quad a = a^{km}_{ij} e^i_k \otimes e^j_m$, where $e^i_j$ are the matrix unities: $e^j_i e^m_k = \delta^j_k e^m_i$. Then (2.8), (2.10)-(2.13) are equivalent to (2.20), where tensors belong to $\mathrm{Mat}_m(\mathbb{C})\otimes\mathrm{Mat}_m(\mathbb{C})\otimes\mathrm{Mat}_m(\mathbb{C})$. Namely, $r^{12}=r^{mk}_{ij}e^i_k\otimes e^j_m\otimes 1$ and so on. The element $\sigma$ is given by $\sigma=e^j_i\otimes e^i_j$.

For the third interpretation, we shall define operators $r, a, \bar r, r^*, a^* : \mathrm{Mat}_m(\mathbb{C})\to\mathrm{Mat}_m$ by $r(x)^p_q=r^{mp}_{nq}x^n_m,\quad a(x)^p_q=a^{mp}_{nq}x^n_m,\ \bar r(x)^p_q=r^{pm}_{nq}x^n_m,\quad r^*(x)^p_q=r^{pm}_{qn}x^n_m,\quad a^*(x)^p_q=a^{pm}_{qn}x^n_m.$

Then (2.8), (2.10)-(2.13) provide the following operator identities:

$$
\begin{gathered}
r(x)=-r^*(x),\qquad r(x)r(y)=r(xr(y))+r(x)y),\\
\bar r(x)=-\bar r^*(x),\qquad \bar r(x)\bar r(y)=\bar r(x\bar r(y))+\bar r(x)y),\\
a(x)a^*(y)=a^*(y)a(x),\\
a^*(ya(x))=r(xa^*(y))-r(x)a^*(y),\\
a(x)a(y)=-a(r(y)x)-a(yr(x)),\\
a^*(a(x)y)=r(a^*(y)x)-a^*(y)r(x),\\
a(ya^*(x))=-\bar r(xa(y))+\bar r(x)a(y),\\
a^*(x)a^*(y)=a^*(\bar r(y)x)+a^*(y\bar r(x)),\\
a(a^*(x)y)=-\bar r(a(y)x)+a(y)\bar r(x)
\end{gathered}
$$

for any $x, y$. First two of these identities mean that operators $r$ and $\bar r$ satisfies the Rota-Baxter equation [7] and this fact implies also that the new matrix multiplications $\circ_r$ and $\circ_{\bar r}$ defined by

$$x\circ_r y=r(x)y+xr(y),\quad x\circ_{\bar r} y=\bar r(x)y+x\bar r(y)$$

are associative.

## 2.1 Examples and classification of low dimensional quadratic double Poisson brackets

It is easy to see that for $m=1$ non-zero quadratic double Poisson brackets does not exist. In the simplest non-trivial case $m=2$ the system of algebraic equations (2.8), (2.10)-(2.13) can be solved straightforwarly.

**Theorem 1.** Let $m=2$. Then the following Cases 1-7 form a complete list of quadratic double Poisson brackets up to equivalence (2.15). We present non-zero components of the tensors $r$ and $a$ only.

**Case 1.** $r^{21}_{22}=-r^{12}_{22}=1$. The corresponding (non-zero) double brackets read

$$\{\!\{v,v\}\!\}=v\otimes u-u\otimes v;$$

**Case 2.** $r^{21}_{22}=-r^{12}_{22}=1,\ a^{11}_{21}=a^{12}_{22}=1$. The corresponding (non-zero) double brackets:

$$\{\!\{v,v\}\!\}=v\otimes u-u\otimes v+vu\otimes 1-1\otimes vu,\ \{\!\{v,u\}\!\}=u^2\otimes 1,\ \{\!\{u,v\}\!\}=-1\otimes u^2;$$

**Case 3.** $r^{21}_{22} = -r^{12}_{22} = 1$, $a^{11}_{12} = a^{21}_{22} = 1$. The corresponding (non-zero) double brackets:

$$\{\!\{v, v\}\!\} = v \otimes u - u \otimes v + uv \otimes 1 - 1 \otimes uv,\ \{\!\{u, v\}\!\} = u^2 \otimes 1,\ \{\!\{v, u\}\!\} = -1 \otimes u^2;$$

**Case 4.** $r^{22}_{21} = -r^{22}_{12} = 1$. The corresponding (non-zero) double brackets:

$$\{\!\{v, u\}\!\} = v \otimes v,\ \{\!\{u, v\}\!\} = -v \otimes v;$$

**Case 5.** $r^{22}_{21} = -r^{22}_{12} = 1$,; $a^{21}_{11} = a^{22}_{12} = 1$. The corresponding (non-zero) double brackets:

$$\{\!\{v, u\}\!\} = v \otimes v - 1 \otimes v^2,\ \{\!\{u, v\}\!\} = -v \otimes v + v^2 \otimes 1,\ \{\!\{u, u\}\!\} = uv \otimes 1 - 1 \otimes uv;$$

**Case 6.** $r^{22}_{21} = -r^{22}_{12} = 1$,; $a^{12}_{11} = a^{22}_{21} = -1$. The corresponding (non-zero) double brackets:

$$\{\!\{v, u\}\!\} = v \otimes v - v^2 \otimes 1,\ \{\!\{u, v\}\!\} = -v \otimes v + 1 \otimes v^2,\ \{\!\{u, u\}\!\} = -vu \otimes 1 + 1 \otimes vu;$$

**Case 7.** $a^{11}_{22} = 1$. The corresponding (non-zero) double brackets:

$$\{\!\{v, v\}\!\} = u^2 \otimes 1 - 1 \otimes u^2.$$

**Proof.** Solving the system (2.10) for six components of the skew-symmetric tensor $r$, we obtain the following two solutions (we present non-zero components of the tensor $r$ only):

$$r^{21}_{22} = -r^{12}_{22} = x^2, \qquad r^{21}_{11} = -r^{12}_{11} = y^2, \qquad r^{21}_{12} = r^{21}_{21} = -r^{12}_{21} = -r^{12}_{12} = xy \tag{2.21}$$

and

$$r^{22}_{21} = -r^{22}_{12} = x^2, \qquad r^{11}_{21} = -r^{11}_{12} = y^2, \qquad r^{12}_{21} = r^{21}_{21} = -r^{21}_{12} = -r^{12}_{12} = xy, \tag{2.22}$$

where $x$ and $y$ are arbitrary parameters. Under the transformation (2.15) the parameters in (2.21) are changed as follows:

$$x \to \frac{1}{\Delta}(xg_{22} + yg_{12}), \qquad y \to \frac{1}{\Delta}(xg_{21} + yg_{11}),$$

where $\Delta = g_{22}g_{11} - g_{12}g_{21}$. For solution (2.22) we have

$$x \to \frac{1}{\Delta}(-xg_{11} + yg_{21}), \qquad y \to \frac{1}{\Delta}(xg_{12} - yg_{22}).$$

For non-zero solution (2.21) the remaining system (2.11)-(2.13) for the tensor $a$ besides for zero solution has the following two solutions:

$$a^{11}_{21} = a^{12}_{22} = x^2, \qquad a^{21}_{11} = a^{22}_{12} = -y^2, \qquad a^{11}_{11} = a^{12}_{12} = -a^{21}_{21} = -a^{22}_{22} = xy, \tag{2.23}$$

and

$$a^{11}_{12} = a^{21}_{22} = x^2, \qquad a^{12}_{11} = a^{22}_{21} = -y^2, \qquad a^{11}_{11} = a^{21}_{21} = -a^{12}_{12} = -a^{22}_{22} = xy. \tag{2.24}$$

For (2.22) the system (2.11)-(2.13) has the following two solutions:

$$a^{21}_{11} = a^{22}_{12} = x^2, \qquad a^{11}_{21} = a^{12}_{22} = -y^2, \qquad a^{11}_{11} = a^{12}_{12} = -a^{21}_{21} = -a^{22}_{22} = xy, \tag{2.25}$$

and

$$a^{12}_{11} = a^{22}_{21} = -x^2, \qquad a^{11}_{12} = a^{21}_{22} = y^2, \qquad a^{12}_{12} = a^{22}_{22} = -a^{11}_{11} = -a^{21}_{21} = xy. \tag{2.26}$$

In the case of zero tensor $r$ the remaining system (2.11)-(2.13) has the following solution:

$$a^{11}_{22} = x^4, \qquad a^{11}_{12} = a^{11}_{21} = -a^{12}_{22} = -a^{21}_{22} = x^3 y,$$
$$a^{11}_{11} = a^{22}_{22} = -a^{12}_{12} = -a^{12}_{21} = -a^{21}_{12} = -a^{21}_{21} = x^2 y^2,$$
$$a^{22}_{12} = a^{22}_{21} = -a^{12}_{11} = -a^{21}_{11} = xy^3, \qquad a^{22}_{11} = y^4$$

with the transformation rule

$$x \to \frac{1}{\Delta^2}(xg_{22} + yg_{12}), \qquad y \to \frac{1}{\Delta^2}(xg_{21} + yg_{11}).$$

Using (2.15), we normalize the solutions obtained above by $x = 1$, $y = 0$ and arrive at the statement of Theorem 1.

**Remark 1.** Cases 2 and 3 as well as Cases 5 and 6 are linked via involution (2.16).

**Remark 2.** Case 1 is equivalent to the double bracket from Example 1 with $m = 2$.

**Remark 3.** It is easy to verify (see [1]) that there exist only two non-isomorphic anti-Frobenius subalgebras in $\mathrm{Mat}_2(\mathbb{C})$. They are matrices with one zero column and matrices with one zero row. Cases 1 and 4 correspond to them.

**Remark 4.** Notice that the trace Poisson brackets for cases 2 and 4 are non-degenerate. Corresponding symplectic forms can be found in [2] (Example 5.7 and Lemma 7.1).

**Remark 5.** The corresponding Lie algebra structures on the trace space $A/[A, A]$ defining by 1.4 are trivial (abelian) in all cases, except the cases 2, 3 and 4 :

$$[\bar{u}, \bar{v}] = -\bar{u}^2 \quad (\text{Case 2}), \qquad [\bar{u}, \bar{v}] = \bar{u}^2 \quad (\text{Case 3}), \qquad [\bar{u}, \bar{v}] = -\bar{v}^2 \quad (\text{Case 4}).$$

This cases give the isomorphic Lie algebra structures on $A/[A, A]$ with respect to the involutions $u \to v, \quad v \to u$ and $u \to u \quad v \to -v$.

**Example 2.** Consider the trace Poisson bracket (2.14) corresponding to **Case 6**. Its Casimir functions are given by

$$\operatorname{tr} v^k, \qquad \operatorname{tr} uv^k, \qquad k = 0, 1, \ldots$$

where $u = x_1, v = x_2$. Functions $\operatorname{tr} u^i$ and $\operatorname{tr} vu^i$, where $i = 2, 3, ...$ commute each other with respect to this bracket.

The simplest integrable ODE system with matrix variables corresponds to the Hamiltonian $H = \frac{1}{2}\operatorname{tr} u^2$. This system has the following form

$$u_t = vu^2 - uvu, \qquad v_t = -uv^2 + vuv. \tag{2.27}$$

The matrix $v^{-1}u$ is an integral of motion for this system. The corresponding reduction $u = vC$, where $C$ is arbitrary constant matrix, gives rise to known integrable model [8]

$$v_t = v^2Cv - vCv^2.$$

The cyclic reduction of the latter equation yields the non-abelian modified Volterra equation.

To study symplectic leaves for this bracket we show that the bracket is equivalent to a pencil of compatible linear Poisson brackets.

Let

$$v = T\Lambda T^{-1}, \qquad u = TYT^{-1},$$

where $Y$ is a generic matrix, $\Lambda = diag(\lambda_1, ..., \lambda_m)$, where $\lambda_i \neq \lambda_j$ and $\lambda_i \neq 0$, and $T$ is a generic invertible matrix with $t_{1,j} = 1$. If we fix values of the Casimir functions $\operatorname{tr} v^k$ then $\lambda_i$ become constants.

Consider $y_{i,j}$ and $t_{i,j}$, $i > 1$ as coordinates on the corresponding $(2n^2 - n)$- dimensional Poisson submanifold. Then in this coordinates the restriction of the initial quadratic Poisson bracket $\{,\}$ has the form

$$\{,\} = \sum_{i=1}^{m} \lambda_i \{,\}_i,$$

where $\{,\}_i$ are some linear Poisson brackets.

Describe the structure of the Lie algebra $\mathcal{G}$ corresponding to the pencil. It turns out that

$$\mathcal{G} = \mathcal{Y} \oplus \mathcal{T},$$

where $[\mathcal{Y}, \mathcal{Y}] \subset \mathcal{Y}$, $[\mathcal{Y}, \mathcal{T}] \subset \mathcal{T}$, $[\mathcal{T}, \mathcal{T}] = \{0\}$. The subalgebra $\mathcal{Y}$ of dimension $n^2$ is generated by $y_{ij}$ and the $\mathcal{Y}$-module $\mathcal{T}$ of dimension $n(n-1)$ is generated by $t_{i,j}$, $i > 1$.

As an algebra $\mathcal{Y}$ can be considered as a trivial central extension of the algebra $\mathcal{Z}$ spanned by $z_{i,j} = y_{i,j} - y_{i,i}$, where $i \neq j$ by by $y_{1,1}, ..., y_{n,n}$.

The radical of $\mathcal{Z}$ is spanned by $r_i = \sum_{j \neq i} \frac{1}{\lambda_j} z_{j,i}$.

The centralizer $\mathcal{S}$ of $r_1$ is isomorphic to $\mathfrak{gl}_{n-1}(\mathbb{C})$ with $r_1$ being central. The isomorphism between $\mathcal{S}$ and $\mathrm{Mat}_{n-1}(\mathbb{C})$ is given by

$$e^i_j \to \frac{1}{\lambda_j}(z_{j+1,1} - z_{j+1,i+1}), \qquad i, j = 1, ..., n-1,$$

where $z_{k,k} = 0$ for any $k$. Here $e^i_j$ are the matrix unities.

The radical of $\mathcal{Z}$ is the direct sum of two commutative $\mathcal{S}$-modules of dimensions $n-1$ and 1. The first one is spanned by $v_i = r_i - r_1$. The second is generated by $r_1$. The commutator relations between the modules is given by $[r_1, v_i] = v_i$.

The module $\mathcal{T}$ is a direct sum of $n$-dimensional submodules $\mathcal{T}_i$ spanned by $t_{i,k}$, $i > 1$. The commutator relations are

$$[y_{i,j}, t_{k,l}] = \delta^i_l \lambda_i (t_{k,i} - t_{k,j}). \qquad \square$$

A complete classification in the case $m = 3$ based on a straightforward analysis of equations (2.8), (2.10)-(2.13) seems to be a solvable but very tedious task. However, additional assumptions that are equivalent to a system of linear equations for components of tensors $r$ and $a$ simplifies the problem. For example, we can easily obtain several new examples of double Poisson brackets assuming that $\operatorname{tr} x_2^k$ and $\operatorname{tr} x_3^k$, where $k = 1, 2, ...$ are Cazimir functions. One of such brackets is given by

$$r^{22}_{21} = r^{23}_{31} = r^{32}_{31} = -r^{22}_{12} = -r^{32}_{13} = -r^{23}_{13} = 1, \qquad a^{12}_{11} = a^{22}_{21} = a^{23}_{11} = a^{32}_{31} = -a^{23}_{13} = -1.$$

The corresponding (non-zero) double Poisson brackets:

$$\{\!\{y, x\}\!\} = y \otimes y - y^2 \otimes 1, \qquad \{\!\{x, y\}\!\} = -y \otimes y + 1 \otimes y^2,$$

$$\{\!\{z, x\}\!\} = y \otimes z + z \otimes y - zy \otimes 1 - 1 \otimes yz, \quad \{\!\{x, z\}\!\} = -y \otimes z - z \otimes y + yz \otimes 1 + 1 \otimes zy;$$

$$\{\!\{x, x\}\!\} = -yx \otimes 1 + 1 \otimes yx - zy \otimes 1 + 1 \otimes zy.$$

Taking $H = \frac{1}{2} \operatorname{tr} x_1^2$ as a Hamiltonian for the corresponding trace Poisson bracket (2.14), we arrive at an integrable system

$$u_t = vu^2 - uvu + wvu - uwv, \qquad v_t = -uv^2 + vuv, \qquad w_t = [w, [u, v]],$$

where $u = x_1$, $v = x_2$, $w = x_3$. After the reduction $w = 0$ this system coincides with (2.27).

Another way to construct new examples is to consider brackets homogenious with respect to any rescaling $x_i \to \mu_i x_i, \quad \mu_i \in \mathbb{C}$. Notice that all canonical forms **Case 1**-**Case 7** in Theorem 1 are homogeneous. When $m = 3$ one of the simplest homogeneous brackets is given by

$$r^{31}_{22} = -r^{13}_{22} = \alpha, \qquad a^{13}_{22} = \beta, \qquad a^{31}_{22} = \gamma.$$

for some constant $\alpha, \beta, \gamma$. The corresponding family of (non-zero) double Poisson brackets reads as:

$$\{\!\{y, y\}\!\} = \alpha(z \otimes x - x \otimes z) + \beta(xz \otimes 1 - 1 \otimes xz) + \gamma(zx \otimes 1 - 1 \otimes zx).$$

# 3 Compatible linear and quadratic double Poisson bracket

The bi-Hamiltonian approach to integrability has been developed by F.Magri and his group [3]. It is based on the notion of compatible Poisson brackets. By analogy we define compatible double Poisson brackets as follows.

**Definition.** Double Poisson brackets $\{\!\{u,v\}\!\}_1$ and $\{\!\{u,v\}\!\}_2$ on an associative $\mathbb{C}-$ algebra $A$ are called compatible if

$$\{\!\{u,v\}\!\}_1 + \lambda\{\!\{u,v\}\!\}_2$$

is a double Poisson bracket on $A$ for any $\lambda \in \mathbb{C}$

The compatibility criteria for a pair of double Poisson brackets is quite similar to the usual one:

$$\{\!\{u,\{\!\{v,w\}\!\}_2\}\!\}_1 + \sigma\{\!\{v,\{\!\{w,u\}\!\}_2\}\!\}_1 + \sigma^2\{\!\{w,\{\!\{u,v\}\!\}_2\}\!\}_1 +$$
$$+\{\!\{u,\{\!\{v,w\}\!\}_1\}\!\}_2 + \sigma\{\!\{v,\{\!\{w,u\}\!\}_1\}\!\}_2 + \sigma^2\{\!\{w,\{\!\{u,v\}\!\}_1\}\!\}_2 = 0.$$

It is clear that compatible double Poisson brackets induce (see Proposition 1) compatible trace Poisson brackets.

Consider the case when one of the brackets is a linear double bracket and another is a quadratic.

**Proposition 3.** Let $A = \mathbb{C} < x_1, \ldots, x_n >$. Consider the linear and the quadratic double Poisson brackets given by the (2.6) and (2.7). Then their compatibility conditions have the following form:

$$b^s_{\alpha\gamma}a^{vu}_{s\beta} - b^s_{\gamma\beta}a^{vu}_{\alpha s} + b^u_{s\beta}a^{vs}_{\alpha\gamma} - b^v_{\alpha s}a^{su}_{\gamma\beta} = 0 \tag{3.28}$$

$$b^s_{\beta\alpha}r^{uv}_{s\gamma} - b^u_{\beta s}r^{sv}_{\alpha\gamma} - b^v_{s\alpha}r^{us}_{\beta\gamma} - b^v_{\gamma s}a^{us}_{\beta\alpha} + b^u_{s\gamma}a^{sv}_{\beta\alpha} = 0. \tag{3.29}$$

**Proof.** Straightforward verification.

Let $A$ be an $m$-dimensional associative algebra with the multiplication law $e_i e_j = b^k_{ij}e_k$. Define linear operators $r,\ a$ on the space $A \otimes A$ by

$$r(e_\alpha \otimes e_\beta) = r^{\sigma\epsilon}_{\alpha\beta}e_\sigma \otimes e_\epsilon, \qquad a(e_\alpha \otimes e_\beta) = a^{\sigma\epsilon}_{\alpha\beta}e_\sigma \otimes e_\epsilon.$$

In terms of these operators acting on $A$ the compatibility conditions (3.28), (3.29) can be rewritten as

$$a(xz \otimes y) - a(x \otimes zy) + a(x \otimes z)(1 \otimes y) - (x \otimes 1)a(z \otimes y) = 0, \tag{3.30}$$

and

$$r(yx \otimes z) - (y \otimes 1)r(x \otimes z) - r(y \otimes z)(1 \otimes x) - (1 \otimes z)a(y \otimes x) + a(y \otimes x)(z \otimes 1) = 0. \tag{3.31}$$

The relation (3.30) is nothing but the cocycle condition for the Hochschild cochains $C^2(A, A \otimes A)$. Here we consider the outer bimodule structure in $A \otimes A$.

Consider the class of associative algebras $A$ such that the first and second Hochschild cohomologies with coefficients in the outer bimodule $A \otimes A$ are trivial. In particular, semi-simple associative algebras belong to this class. If $H^2(A, A \otimes A) = 0$, then

$$a(x \otimes y) = \phi(xy) - (x \otimes 1)\phi(y) - \phi(x)(1 \otimes y) \tag{3.32}$$

for some $\phi : A \to A \otimes A$. The operator $\phi$ is defined up to the double derivations

$$d_s : x \longrightarrow (x \otimes 1)\, s - s\,(1 \otimes x),$$

where $s \in A \otimes A$ is an arbitrary element.

**Proposition 4.** Suppose that the tensor $a$ is defined by (3.32). If $H^1(A, A \otimes A) = 0$, then any solution of (3.31) has the form

$$r(x \otimes y) = (x \otimes 1)\psi(y) - \psi(y)(1 \otimes x) + (1 \otimes y)\phi(x) - \phi(x)(y \otimes 1) \tag{3.33}$$

for some $\psi : A \to A \otimes A$.

**Proof.** Let

$$r(x \otimes y) = \tilde{r}(x \otimes y) + (1 \otimes y)\phi(x) - \phi(x)(y \otimes 1).$$

It follows directly from (3.31) and (3.32) that

$$\tilde{r}(yx \otimes z) - (y \otimes 1)\tilde{r}(x \otimes z) - \tilde{r}(y \otimes z)(1 \otimes x) = 0$$

If $H^1(A, A \otimes A) = 0$, then

$$\tilde{r}(x \otimes y) = (x \otimes 1)\psi(y) - \psi(y)(1 \otimes x)$$

for some $\psi : A \to A \otimes A$.

We denote as usual by $\sigma$ the flip $\sigma(x \otimes y) = y \otimes x$. It follows from $r(x \otimes y) = -\sigma \circ r(y \otimes x)$ that

$$(1 \otimes y)\mu(x) - \mu(x)(y \otimes 1) + (x \otimes 1)(\sigma \circ \mu(y)) - (\sigma \circ \mu(y))(1 \otimes x) = 0, \tag{3.34}$$

where $\mu(x) = \phi(x) + \sigma \circ \psi(x)$.

We are searching all candidates for $\mu : A \to A \otimes A$ to be a solution of 3.34 for any $x, y \in A$. The trivial solution $\mu = 0$ and hence, $\psi = -\sigma\phi$ implies the solution for $r(x \otimes y)$ in the form

$$r(x \otimes y) = (\sigma \circ \phi(y))(1 \otimes x) - (x \otimes 1)(\sigma \circ \phi(y)) + (1 \otimes y)\phi(x) - \phi(x)(y \otimes 1). \tag{3.35}$$

If we take $\mu$ in the form

$$\mu(x) = (x \otimes 1)s - s(1 \otimes x),$$

where $s \in A \otimes A$ is an arbitrary skew-symmetric element : $\sigma(s) = -s$. Then we can straightforwardly verify that $\mu(x)$ is a solution of 3.34.

In this case

$$\phi + \sigma\psi = (x \otimes 1)s - s(1 \otimes x)$$

and we can choose $\tilde{\phi} = \phi + (x \otimes 1)s - s(1 \otimes x)$ such that $\psi = -\sigma\tilde{\phi}$ and the answer for $r(x \otimes y)$ is again given by 3.35.

**Conjecture.** If $A$ is a finite unital associative algebra such that $H^1(A, A \otimes A) = H^2(A, A \otimes A) = 0$, then all solutions of 3.34 have the form $(x \otimes 1)s - s(1 \otimes x)$ for some $s \in \Lambda^2(A)$.

We have checked the conjecture in the case of matrix associative algebra.

The case $a(x \otimes y) = 0$ corresponds to

$$\phi : x \to (x \otimes 1)\, s - s\, (1 \otimes x), \tag{3.36}$$

where $s \in A \otimes A$ is any fixed element. Define a tensor $r$ by formulas (3.35) and (3.36). Explicitely, up to a constant multiplier,

$$r(x \otimes y) = s(y \otimes x) + (x \otimes y)s - (1 \otimes y)s(1 \otimes x) - (x \otimes 1)s(y \otimes 1). \tag{3.37}$$

**Theorem 2.** Ler $r$ is defined by (3.37) and $s \in A \otimes A$ satisfies the associative Yang-Baxter equation on $A$:

$$s^{12} = -s^{21}, \qquad s^{23}s^{12} + s^{31}s^{23} + s^{12}s^{31} = 0. \tag{3.38}$$

Then

$$\{\!\{x_\alpha, x_\beta\}\!\} = r^{uv}_{\alpha\beta}\, x_u \otimes x_v$$

is a quadratic double Poisson bracket on $T(A) = \mathbb{C} < x_1, \ldots, x_m >$ compatible with the linear bracket

$$\{\!\{x_i, x_j\}\!\} = b^k_{ij} x_k \otimes 1 - b^k_{ji} 1 \otimes x_k,$$

where $r^{uv}_{\alpha\beta}$ are components of $\sigma r$ and $b^k_{ij}$ are structure constants of $A$.

**Remark 1.** We observe that in the case $a = 0$ the condition 3.31 is the outer bimodule derivation property in the *first argument*. That is why the quadratic double Poisson bracket from the theorem 2 can be written in following way:

$$\{\!\{u, v\}\!\} = \sigma r(u \otimes v), \quad u, v \in A. \tag{3.39}$$

Then $\sigma r$ obviously satisfies the outer bimodule derivation property in the *second argument* which guaranties the Leibniz property 1.3 for the double bracket defined by 3.39. In other words the tensor $R := \sigma r : A \otimes A \to A \otimes A$ satisfies to T. Schedler conditions ([19]):

- $R(u \otimes v) = -\sigma R \sigma(u \otimes v)$;
- $R^{12}R^{13} + R^{13}R^{23} - R^{23}R^{12} = 0$;

- $R$ can be considered as a derivation of $A^e \otimes A^e-$action on $(A \otimes A)_{l,r}$ with values in $(A \otimes A)_{i,o}$ where $(A \otimes A)_{l,r}$ means that $A^e \otimes A^e$ acts on the left factor of $A \otimes A$ by the first (left) $A^e$ and on the right factor - by the second $A^e$ :

$$(u \otimes u^o) \otimes (v \otimes v^o)(a \otimes b) = (uau^o) \otimes (vbv^o).$$

Analogously, $(A \otimes A)_{i,o}$ means that $A^e \otimes A^e$ acts on the left factor of $A \otimes A$ by the inner action and on the right factor - by the outer action:

$$(u \otimes u^o) \otimes (v \otimes v^o)(a \otimes b) = (vau^o) \otimes (ubv^o).$$

**Remark 2.** The conditions of the Theorem 2 are satisfied for the case of finitely dimenisonal *quasi-triangular coboundary infinitesimal bialgebra* ([1]). The conditions 3.38 mean that the algebra $A$ has also a compatible coalgebra sructure $\Delta_s : A \to A \otimes A$ such that $\Delta_s(x) = (x \otimes 1)s - s(1 \otimes x)$ for $s \in \Lambda^2(A)$.

**Remark 3.** We observe that there is a natural class of skew-symmetric 2-tensors $s \in \Lambda^2(A)$. Namely, M. Van den Bergh [17] had introduce a notion of a "momentum" map in the case of double Poisson brackets. Let us remind that there is a distinguish double derivation $\Delta : A \to A \otimes A$ such that $\Delta(a) = a \otimes 1 - 1 \otimes a$ for any $a \in A$. Then the *moment map* for $A$ is an element $m \in A$ such that $\{\!\{m, a\}\!\} = \Delta(a)$. Sometimes the double derivation $H_m := \{\!\{m, -\}\!\}$ is called a *Hamiltonian* double vector field. The image of the moment map is evidently a skew-symmetric tensor so we can take as a particular case of the previous remark the solution

$$\mu_m(x) = (x \otimes 1)\{\!\{m, b\}\!\} - \{\!\{m, b\}\!\}(1 \otimes x) = (x \otimes 1)H_m(b) - H_m(b)(1 \otimes x)$$

for any $b \in A$.

**Example 3.** Let $A = \mathrm{Mat}_2(\mathbb{C}) = \mathbb{C} < x, y, z, t >$ . Then there exists a unique (up to equivalence) quadratic bracket with $a = 0$ compatible with the corresponding linear one. This bracket has the following form:

$$r^{12}_{23} = r^{13}_{33} = r^{14}_{43} = r^{22}_{12} = r^{22}_{24} = r^{41}_{31} = r^{42}_{32} = r^{43}_{33} = 1.$$

The remaining non-zero components of tensor $r$ are defined by the skew-symmetricity of $r$ : $r^{ij}_{pq} = -r^{ji}_{qp}$.

The corresponding (non-zero) double Poisson brackets can be expressed as

$$\{\!\{x, y\}\!\} = y \otimes y; \quad \{\!\{x, z\}\!\} = -x \otimes t; \quad \{\!\{y, z\}\!\} = x \otimes y - y \otimes t;$$

$$\{\!\{y, t\}\!\} = y \otimes y; \quad \{\!\{z, z\}\!\} = x \otimes z + t \otimes z - z \otimes x - z \otimes t;$$

$$\{\!\{z, t\}\!\} = -t \otimes x.$$

It is a straightforward verification that a Casimir element is given by $C = x + t$ but it is impossible to restrict the brackets to the "Casimir zero level" (the traceless matrices in the representation $A = \mathrm{Mat}_2(\mathbb{C}) = \{\begin{pmatrix} x & y \\ z & t \end{pmatrix}\}$ :

$$\{\!\{x, y\}\!\} = y \otimes y; \quad \{\!\{x, z\}\!\} = x \otimes x; \quad \{\!\{y, z\}\!\} = x \otimes y + y \otimes x;$$

$$\{\!\{y, x\}\!\} = y \otimes y; \quad \{\!\{z, z\}\!\} = 0; \quad \{\!\{z, x\}\!\} = x \otimes x$$

(the "restricted" brackets are not skew-symmetric).

# 4 Conclusions and perspectives

We have discussed an analogue of the Lenard- Magri compatibility for linear and quadratic double Poisson brackets in free associative algebras. We have interpreted this conditions in terms of Hochchild cochains and we have proposed few examples of solutions to these conditions. We have classified all double Poisson brackets in the case of the free associative algebra with two generators. Our interest to the double Poisson structures was initially motivated by some examples of a non-commutative integrability dicussed previously in [8] and [10]. We are going to review a version of non-commutative Hamiltonian formalism connected the trace and double Poisson brackets with the initial approach of [8, 14] in the forthcoming publications.

There are still many other interesting questions which deserve to be discussed. The natural question of a quantization the Van den Bergh construction was posed by D. Calaque (private comunication and see also http://mathoverflow.net/questions/29543/what-is-a-double-star-product). Our theorem 2 gives an idea of such a quantization for the tensor algebra associative $r-$matrix $R$ using a quantization (if it is known ) of the associative skew-symmetric $r-$matrix $s$ in 3.38. The latter can be quantized using the ideas of [6].

We have focused in this paper on the case of the free associative algebra. But the construction of double brackets was widely studied in the framework of the non-commutative symplectic geometry ([17, 5, 12]) aiming to describe a trace Poisson structure on quiver path algebra representations. The paper [2] proposes some $r-matrix$ constructions to such quadratic structures. Some of examples from [2] are coincided with our examples. We want to stress that cited paper doesn't study general quadratic double Poisson brackets and the compatibility with their linear counterparts.

The original Van den Bergh construction contains also many other interesting structures and one of them is a *Quasi-Poisson* double structure ( when the double analog of the Jacobi identity 1.3 is "violated" or in other words the "triple product" $\{\!\{u, v, w\}\!\} \in A \otimes A \otimes A$ is no more equal to zero but is somehow under a control). See the details in [17]. Recently an interesting paper [15] had discussed the Quasi-Poisson double structures with the analogs of trace brackets on representations of the group algebra $A = K(\pi)$ where the group $\pi$ is the fundamental group of a surface. The relations with the Goldman bracket, skein algebra and

Fox multiplication were discussed. It would be interestiong to compare our tensor approach to the results of [15].

Finally, the last but not the least interesting subject concerns to general ( not necessary constant ) solutions of various associative Yang-Baxter equations. The paper in progress ([11]) contains some preliminary results in classification of parameter-dependent double Poisson brackets and some of new examples of such brackets.

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When this paper was prepared to a submission we were informed that S.V. Manakov, a famous researcher who had invested a lot in the modern Integrable Systems theory, had passed away. Many of his works were a source of inspiration for us. We are dedicating this paper to his memory.

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