ON WEAK LIE 2-ALGEBRAS

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Abstract. A Lie 2-algebra is a linear category equipped with a functorial bilinear operation satisfying skew-symmetry and Jacobi identity up to natural transformations which themselves obey coherence laws of their own. Functors and natural transformations between Lie 2-algebras can also be defined, yielding a 2-category. Passing to the normalized chain complex gives an equivalence of 2-categories between Lie 2-algebras and certain "up to homotopy" structures on the complex; for strictly skew-symmetric Lie 2-algebras these are L_∞-algebras, by a result of Baez and Crans. Lie 2-algebras appear naturally as infinitesimal symmetries of solutions of the Maurer–Cartan equation in some differential graded Lie algebras and L_∞-algebras. In particular, (quasi-) Poisson manifolds, (quasi-) Lie bialgebroids and Courant algebroids provide large classes of examples.

1. Introduction

The purpose of this note is to complete the categorification of the notion of Lie algebra started by Baez and Crans in [1], who introduced *semi-strict Lie 2-algebras*. By definition, such a structure is given by a bilinear bracket operation on a linear category, which is strictly skew-symmetric but obeys the Jacobi identity only up to a coherent trilinear natural transformation, called the Jacobiator. It was further shown that, upon passing to the normalized chain complex, such a structure is equivalent to a 2-term L_∞-algebra. Passing to cohomology and using homotopy invariance, one gets a classification of semistrict Lie 2-algebras in terms of 3rd Chevalley-Eilenberg cohomology of a Lie algebra with coefficients in a module.

From the point of view of category theory, this picture is somewhat incomplete, as the skew-symmetry holds strictly as an equation. Besides, there exist structures – Leibniz algebras and Courant algebroids – where the Jacobi holds strictly (in the form of a Leibniz identity), whereas it is the skew-symmetry that is weakened. It was shown in [10] for the case of Courant algebroids that skew-symmetrizing the bracket does lead to a 2-term L_∞-algebra; however, some information is lost in the process, and besides, it is not clear why this has to happen, conceptually. That is why, to have a better understanding of this phenomenon, we propose to work from the outset with *weak Lie 2-algebras*, where both the skew-symmetry and the Jacobi identity are allowed to hold only up to coherent natural transformations – the *skew-symmetrizator* and the *Jacobiator*. This structure resembles (and perhaps is Koszul dual to, in a sense yet to be made preceise) a linear braided monoidal category. Passing to the normalized chain complex leads to a new structure – *weak*

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L_∞-algebra. Weak Lie 2-algebras form a 2-category, and so do 2-term weak L_∞-algebras, and these 2-categories are equivalent (Theorem 2.21). We claim that weak Lie 2-algebras are the correct categorification of Lie algebras and henceforth refer to them simply as Lie 2-algebras. Lie 2-algebras with trivial skew-symmetrizator will be referred to as semistrict, as in [1]; those with trivial Jacobiator – hemistrict. These form full sub-2-categories. This is the content of the first section.

In the next section, we describe the skew-symmetrization functor, which is a projection onto the sub-2-category of semistrict Lie 2-algebras (Theorem 3.2).

In the following section, we discuss homotopy invariance of Lie 2-algebras and deduce a classification of skeletal Lie 2-algebras (Theorems 4.1 and 4.5). The classification uses what appears to be a new cohomology theory for Lie algebras, based on the fact that a Lie algebra is a Leibniz algebra which is also skew-symmetric: the Jacobiator defines a Loday-Pirashvili 3-cocycle whose behavior under permutations of the arguments is controlled by the skew-symmetrizator, and the fact that coboundaries are cocycles depends both on the skew-symmetry and Jacobi identity of the Lie algebra. This is similar to Eilenberg and MacLane's cohomology theory for abelian groups which uses both the associativity and commutativity of the group law. Skew-symmetrization induces a map from the new cohomology onto the 3rd Chevalley-Eilenberg cohomology (Theorem **??**); this map is an isomorphism if, and only if, the skew-symmetrizator is symmetric.

Finally, in the last section, we discuss applications of the theory to questions in deformation theory. Namely, given a solution of the Maurer-Cartan equation in a differential graded Lie algebra concentrated in degrees $(-3, +\infty)$, we construct the hemistrict Lie 2-algebra of its inner symmetries, mapping to the (ordinary) Lie algebra of infinitesimal symmetries, forming a categorified crossed module (Theorem 5.3). Such a dgla controls, for instance, the deformation theory of Courant algebroids [8] and hence, this theorem, combined with Theorem 3.2, generalizes the main result of [10]. The construction itself uses *derived brackets*, as described in [6]; in case of dgla's concentrated in degrees $(-2, +\infty)$, a similar but simpler construction is well known and yields the crossed module of infinitesimal symmetries.

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2. Lie 2-algebras and weak L_∞-algebras

2.1. **Categorified linear algebra.** Fix a ground field $\mathbf{k}$. It can be assumed arbitrary, except where indicated. We let **Vect** denote the category of vector spaces over $\mathbf{k}$. In what follows we shall freely use the notations and terminology of [1], with minor modifications. Thus, a *2-vector space* is a linear category, i.e. a category internal to **Vect**: objects and morphisms form vector spaces, and all the structure maps are linear. If V is such a category, we denote its space of objects by V_0, its space of arrows by V_1, the source and target maps, respectively, by $s, t : V_1 \to V_0$, the identity map by $1 : V_0 \to V_1$ (with $x \mapsto 1_x$), and the composite of the arrows $\mathbf{a} : x \to y$ and $\mathbf{b} : y \to z$ by $\mathbf{ba} : x \to z$. It is clear what it means for a functor between two linear categories, or a natural transformation between two such functors, to be linear. Given two linear functors $F : U \to V$ and $G : V \to W$, we denote their composition by $G * F : U \to W$. Given two linear natural transformations $\Phi : F \Rightarrow F'$ and $\Psi : G \Rightarrow G'$, with $F, F' : U \to V$ and $G, G' : V \to W$, we denote their horizontal composite by $\Psi * \Phi : G * F \Rightarrow G' * F'$; whiskering of a natural transformation by a functor is also denoted by $*$. Finally, given natural transformations $\Phi : F \Rightarrow G$ and $\Psi : G \Rightarrow H$, we denote their vertical composite by $\Psi \circ \Phi : F \Rightarrow H$. The definitions of these compositions are standard in category theory, and it is trivial to check that they preserve linearity. Thus we have the (strict) 2-category **2Vect**.

The standard constructions of linear algebra carry over to **2Vect** in an obvious manner. In particular, the tensor product of 2-vector spaces is defined "dimension-wise" and satisfies the usual universal property with respect to multilinear functors. This makes **2Vect** a symmetric monoidal 2-category; we denote the action of the transposition by $\sigma : V \otimes W \to W \otimes V$. In fact, **2Vect** is a *closed* symmetric monoidal 2-category which means, in particular, that $\mathrm{Hom}_{\mathbf{2Vect}}(V, W)$ is naturally a linear category, and the composition of linear functors is bilinear.

Given a 2-vector space V, we define its *normalized cochain complex* $N(V)$ by

$$\begin{aligned} N(V)^0 &= V_0 \\ N(V)^{-1} &= \ker(s) \end{aligned}$$

with $d : N(V)^{-1} \to N(V)^0$ given by the restriction of t. It is easy to check (see [1]) that this defines a (strict) 2-functor from **2Vect** to the 2-category **2Term**, consisting of 2-term cochain complexes (concentrated in degrees $(-2, 0]$), chain maps and chain homotopies. In fact, the normalization functor has a quasi-inverse Γ, given on objects by

$$\begin{aligned} \Gamma(C)_0 &= C^0 \\ \Gamma(C)_1 &= C^0 \oplus C^{-1} \end{aligned}$$

with

$$\begin{aligned} s(x, a) &= x \\ t(x, a) &= x + da \\ 1_x &= (x, 0) \\ (y, b)(x, a) &= (x, a + b) \qquad \text{if} \quad y = x + da \end{aligned}$$

Theorem 2.1. *([1]) The 2-functors N and Γ give an equivalence of 2-categories* **2Vect** *and* **2Term**.

Remark 2.2. The functors N and Γ can be defined in a more general setting of simplicial vector spaces and non-positively graded cochain complexes. That they define an equivalence of these two categories is a classical theorem in homological algebra, due to Dold and Kan. The categorical equivalence in the theorem above follows from this by applying the nerve functor from linear categories to simplicial vector spaces. The souped-up 2-categorical version is due to Baez and Crans.

Remark 2.3. The simple observation underlying the above result is that any arrow $\mathbf{a} : x \to y$ in a linear category can be uniquely decomposed as $\mathbf{a} = 1_x + a$, where

$a = \mathbf{a} - 1_x \in \ker s$ is called the *arrow part* of $\mathbf{a}$. The linearity of composition then forces it to be just the addition of arrow parts. It also implies that any linear category is in fact a *groupoid*, for $(x + da, -a)(x, a) = (x, 0)$.

In the sequel, we shall freely use the canonical isomorphism $V \simeq \Gamma N(V)$ and write $\mathbf{a} = x + a$ when it is not likely to cause confusion.

2.2. **Multilinear operations.** Using the Dold-Kan correspondence, it is tempting to conclude right away that multilinear functorial operations on 2-vector spaces are in one-to-one correspondence with multilinear chain maps on their normalized complexes. The problem is that the normalization functor does not commute with tensor products; in fact, **2Term** is not even closed under the tensor product of cochain complexes. A proper treatment of this problem (certainly necessary if we want eventually to understand higher linear categories) requires a careful analysis of the behavior of the nerve and normalization functors with respect to tensor products and the Eilenberg-Zilber construction. This issue will be addressed elsewhere. Here we provide instead a quick fix based on the following observation.

Proposition 2.4. *Let $V_1, \dots, V_n, V$ be linear categories and $T : V_1 \otimes \cdots \otimes V_n \to V$ a linear functor. Then*

(1) *$\forall a_k \in N(V_k)^{-1}$ and $x_i \in N(V_i)^0$, $i \neq k$, $T(x_1, \dots, x_{k-1}, a_k, x_{k+1}, \dots, x_n) \in N(V)^{-1}$ and*

$$dT(x_1, \dots, x_{k-1}, a_k, x_{k+1}, \dots, x_n) = T(x_1, \dots, x_{k-1}, da_k, x_{k+1}, \dots, x_n)$$

(2) *For $a_i \in N(V_i)^{-1}$, $a_j \in N(V_j)^{-1}$ and arbitrary other arguments,*

$$T(\dots, a_i, \dots, a_j, \dots) = T(\dots, da_i, \dots, a_j, \dots) = T(\dots, a_i, \dots, da_j, \dots)$$

It follows that T is completely determined by its value on objects and on arrows of the form $x_1 \otimes \cdots \otimes x_{k-1} \otimes a_k \otimes x_{k+1} \otimes \cdots \otimes x_n$. More precisely,

Corollary 2.5. *The linear categories* $\mathrm{Hom}_{\mathbf{2Vect}}(V_1 \otimes \cdots \otimes V_n, V)$ *and* $\mathrm{Hom}_{\mathbf{Ch}}(N(V_1) \otimes \cdots \otimes N(V_n), N(V))$*, where* **Ch** *is the 2-category of non-positively graded cochain complexes, are canonically isomorphic.*

Let us spell out what the above proposition says in the case of a binary operation on a linear category V:

Proposition 2.6. *Let $[\cdot, \cdot] : V \otimes V \to V$ be a bilinear functor. Then*

$$[(x, a), (y, b)] = ([x, y], [x, b] + [a, y] + [a, b]) \tag{2.1}$$

and the following crossed module identities *hold:*

$$d[x, b] = [x, db] \tag{2.2}$$
$$d[a, y] = [da, y] \tag{2.3}$$
$$[da, b] = [a, b] = [a, db] \tag{2.4}$$
$$d[a, b] = [da, db] \tag{2.5}$$

The corresponding bracket $[\cdot, \cdot] : N(V) \otimes N(V) \to N(V)$, given by

$$[(x, a), (y, b)] = ([x, y], [x, b] + [a, y]), \tag{2.6}$$

is then a chain map. Conversely, any such operation on a 2-term chain complex C uniquely determines a bilinear functorial bracket on $\Gamma(C)$ by setting $[a,b]$ to be the *derived bracket*:

$$[a,b]=[da,b]=[a,db] \tag{2.7}$$

2.3. **The 2-category of Lie 2-algebras.** We are now ready to define weak Lie 2-algebras.

Definition 2.7. A *Lie 2-algebra* is a linear category L equipped with the following structure:

- a bilinear functor $[\cdot,\cdot]: L\otimes L\to L$, called the *bracket*;
- a bilinear natural transformation
$$S:[\cdot,\cdot]\Rightarrow -[\cdot,\cdot]*\sigma,$$
called the *skew-symmetrizator*;
- a trilinear natural transformation
$$J:[\cdot,[\cdot,\cdot]]\Rightarrow [[\cdot,\cdot],\cdot]+[\cdot,[\cdot,\cdot]]*\sigma_{12},$$
called the *Jacobiator*.

In addition, the following diagrams are required to commute:

$$\begin{tikzcd}
 & {[x,[y,[z,w]]]} \arrow[dl, "{[x,J_{y,z,w}]}"'] \arrow[dr, "{J_{x,y,[z,w]}}"] & \\
{[x,[[y,z],w]]+[x,[z,[y,w]]]} \arrow[d, "{J_{x,[y,z],w}+J_{x,z,[y,w]}}"'] & & {[[x,y],[z,w]]+[y,[x,[z,w]]]} \arrow[d, "{J_{[x,y],z,w}+[y,J_{x,z,w}]}"] \\
{\begin{array}{c}[[x,[y,z]],w]+[[y,z],[x,w]]+\\ +[[x,z],[y,w]]+[z,[x,[y,w]]]\end{array}} \arrow[dr, "{[J_{x,y,z},w]+1+1+[z,J_{x,y,w}]}"'] & & {\begin{array}{c}[[[x,y],z],w]+[z,[[x,y],w]]+\\ +[y,[[x,z],w]]+[y,[z,[x,w]]]\end{array}} \arrow[dl, "{1+1+J_{y,[x,z],w}+J_{y,z,[x,w]}}"] \\
 & {\begin{array}{c}[[[x,y],z],w]+[[y,[x,z]],w]+[[y,z],[x,w]]+\\ +[[x,z],[y,w]]+[z,[[x,y],w]]+[z,[y,[x,w]]]\end{array}} &
\end{tikzcd}$$

$$\begin{tikzcd}
{[[x,y],z]} \arrow[rr, "{[S_{x,y},z]}"] \arrow[dr, "{\hat{J}_{x,y,z}}"'] & & {-[[y,x],z]} \arrow[dl, "{-\hat{J}_{y,x,z}}"] \\
 & {[x,[y,z]]-[y,[x,z]]} &
\end{tikzcd}$$

$$\begin{tikzcd}
{[x,[y,z]]} \arrow[r, "{[x,S_{y,z}]}"] \arrow[d, "J_{x,y,z}"'] & {-[x,[z,y]]} \arrow[d, "-J_{x,z,y}"] \\
{[[x,y],z]+[y,[x,z]]} \arrow[r, "{S_{[x,y],z}+S_{y,[x,z]}}"] & {-[[x,z],y]-[z,[x,y]]}
\end{tikzcd}$$

$$\begin{tikzcd}
 & {[x,[y,z]]} \arrow[rr, "{S_{x,[y,z]}}"] \arrow[dl, "J_{x,y,z}"'] & & {-[[y,z],x]} \arrow[dr, "-\hat{J}_{y,z,x}"] & \\
{[[x,y],z]+[y,[x,z]]} \arrow[drr, "{[S_{x,y},z]+[y,S_{x,z}]}"'] & & & & {-[y,[z,x]]+[z,[y,x]]} \arrow[dll, "{1+S_{z,[y,x]}}"] \\
 & & {-[[y,x],z]-[y,[z,x]]} & &
\end{tikzcd}$$

Remark 2.8. The natural transformation $\hat{J}$ appearing in the diagrams is, essentially, the inverse of the Jacobiator:

$$\hat{J}_{x,y,z} = J^{-1}_{x,y,z} - 1_{[y,[x,z]]} : [[x,y],z] \to [x,[y,z]] - [y,[x,z]].$$

It carries the same information as J, but with a slight shift in emphasis: while J measures the failure of $ad(x) = [x,\cdot]$ to be a derivation of $[\cdot,\cdot]$, $\hat{J}$ measures the failure of ad to send $[\cdot,\cdot]$ to the commutator bracket of endomorphisms.

Remark 2.9. The coherence diagrams may look formidable at first, but in fact each of them just expresses the equality of two obvious ways of re-bracketing and permuting entries in a given expression. Notice that we do not require S to be symmetric in the sense that $-S_{y,x}S_{x,y} = 1_{[x,y]}$; however, if this is the case, the pentagonal diagram in the definition above is redundant. In fact, the triangle and square, in conjunction with

$$-S_{[y,z],x}S_{x,[y,z]} = 1_{[x,[y,z]]}$$

are equivalent to the triangle, square and pentagon. It is natural to wonder whether even this weakened symmetry assumption can be avoided, but it appears to be necessary for Theorem 3.2 to hold. In all examples we consider, S is symmetric.

Definition 2.10. A Lie 2-algebra L is called

- *semistrict* if $S = 1$;
- *hemistrict* if $J = 1$;
- *strict* if it is both hemistrict and semistrict

Definition 2.11. A morphism of Lie 2-algebras from $(L, [\cdot,\cdot], S, J)$ to $(L', [\cdot,\cdot]', S', J')$ consists of:

- a linear functor

$$F : L \to L'$$

- a linear natural transformation

$$F^2 : [\cdot,\cdot]' * (F \otimes F) \Rightarrow F * [\cdot,\cdot]$$

such that the following diagrams commute:

$$\begin{tikzcd}
{[F(x),F(y)]'} \arrow[r, "F^2_{x,y}"] \arrow[d, "S'_{F(x),F(y)}"'] & F([x,y]) \arrow[d, "F(S_{x,y})"] \\
-{[F(y),F(x)]'} \arrow[r, "-F^2_{y,x}"] & -F([y,x])
\end{tikzcd}$$

$$\begin{tikzcd}
{[F(x),[F(y),F(z)]']'} \arrow[r, "J'_{F(x),F(y),F(z)}"] \arrow[d, "{[1,F^2_{y,z}]'}"'] & {[[F(x),F(y)]',F(z)]' + [F(y),[F(x),F(z)]']'} \arrow[d, "{[F^2_{x,y},1]' + [1,F^2_{x,z}]'}"] \\
{[F(x),F([y,z])]'} \arrow[d, "F^2_{x,[y,z]}"'] & {[F([x,y]),F(z)]' + [F(y),F([x,z])]'} \arrow[d, "F^2_{[x,y],z} + F^2_{y,[x,z]}"] \\
F([x,[y,z]]) \arrow[r, "F(J_{x,y,z})"] & F([[x,y],z] + [y,[x,z]])
\end{tikzcd}$$

Definition 2.12. Given two morphisms $(F, F^2) : L \to L'$ and $(G, G^2) : L' \to L''$, their composite is defined to be $(G * F, (G * F)^2)$, where

$$(G * F)^2 = (G * F^2) \circ (G^2 * (F \otimes F))$$

Remark 2.13. The definition of $(G * F)^2$ is best understood as the total composite of the following pasting diagram:

$$\begin{tikzcd}
L \arrow[r, "F"] & L' \arrow[r, "G"] & L'' \\
L \otimes L \arrow[u, "{[\cdot,\cdot]}"] \arrow[r, "F \otimes F"'] \arrow[ur, phantom, "\Nwarrow F^2"] & L' \otimes L' \arrow[u, "{[\cdot,\cdot]'}"] \arrow[r, "G \otimes G"'] \arrow[ur, phantom, "\Nwarrow G^2"] & L'' \otimes L'' \arrow[u, "{[\cdot,\cdot]''}"']
\end{tikzcd}$$

which can be broken down as the vertical composite of the following two:

$$\begin{tikzcd}
 & L \arrow[dr, "F"] & & \\
L \otimes L \arrow[ur, "{[\cdot,\cdot]}"] \arrow[dr, "F \otimes F"'] & \Uparrow F^2 & L' \arrow[r, "G"] & L'' \\
 & L' \otimes L' \arrow[ur, "{[\cdot,\cdot]'}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & L' \arrow[dr, "G"] & \\
L \otimes L \arrow[r, "F \otimes F"] & L' \otimes L' \arrow[ur, "{[\cdot,\cdot]'}"] \arrow[dr, "G \otimes G"'] & \Uparrow G^2 & L'' \\
 & & L'' \otimes L'' \arrow[ur, "{[\cdot,\cdot]''}"'] &
\end{tikzcd}$$

Definition 2.14. Given two morphisms $(F, F^2), (G, G^2) : L \to L'$, a 2-morphism $\Theta : (F, F^2) \Rightarrow (G, G^2)$ is a linear natural transformation $\Theta : F \Rightarrow G$ making the following diagram commute:

$$\begin{tikzcd}
{[F(x), F(y)]'} \arrow[r, "F^2_{x,y}"] \arrow[d, "{[\Theta_x, \Theta_y]'}"'] & F([x, y]) \arrow[d, "\Theta_{[x,y]}"] \\
{[G(x), G(y)]'} \arrow[r, "G^2_{x,y}"] & G([x, y])
\end{tikzcd}$$

The horizontal and vertical composites of 2-morphisms are defined to be those of the corresponding natural transformations. With these definitions, it is now a matter of routine verification to obtain

Proposition 2.15. *Lie 2-algebras, their morphisms and 2-morphisms form a strict 2-category, denoted* **2Lie**.

2.4. **The 2-category of weak 2-term L_∞-algebras.** Let us now apply the normalization functor to the above construction. Given a Lie 2-algebra $(L, [\cdot,\cdot], S, J)$, denote the normalized cochain complex $N(L)$ of the underlying linear category L by $C = C^{-1} \xrightarrow{d} C^0$. The induced bracket on C is given by the formula 2.6 and is a chain map from $C \otimes C$ to C. Furthermore, writing

$$S_{x,y} = ([x, y], -\langle x, y\rangle) \tag{2.8}$$

$$J_{x,y,z} = ([x, [y, z]], -\langle x, y, z\rangle) \tag{2.9}$$

and using the naturality of S and J, we obtain

$$
\begin{aligned}
&(2.10) & [x,y]+[y,x] &= d\langle x,y\rangle \\
&(2.11) & [a,y]+[y,a] &= \langle da,y\rangle \\
&(2.12) & [x,b]+[b,x] &= \langle x,db\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&(2.13) & [x,[y,z]]-[[x,y],z]-[y,[x,z]] &= d\langle x,y,z\rangle \\
&(2.14) & [a,[y,z]]-[[a,y],z]-[y,[a,z]] &= \langle da,y,z\rangle \\
&(2.15) & [x,[b,z]]-[[x,b],z]-[b,[x,z]] &= \langle x,db,z\rangle \\
&(2.16) & [x,[y,c]]-[[x,y],c]-[y,[x,c]] &= \langle x,y,dc\rangle
\end{aligned}
$$

In other words, $\langle\cdot,\cdot\rangle : C\otimes C\to C[-1]$ is a chain homotopy

$$\langle\cdot,\cdot\rangle : [\cdot,\cdot]+[\cdot,\cdot]*\sigma \Rightarrow 0,$$

while $\langle\cdot,\cdot,\cdot\rangle : C\otimes C\otimes C\to C[-1]$ is a chain homotopy

$$\langle\cdot,\cdot,\cdot\rangle : [\cdot,[\cdot,\cdot]]-[[\cdot,\cdot],\cdot]-[\cdot,[\cdot,\cdot]]*\sigma_{12}\Rightarrow 0$$

The coherence conditions satisfied by S and J translate to equations involving $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot,\cdot\rangle$, defining a structure which we shall now describe.

Definition 2.16. A *2-term weak L_∞-algebra* is a 2-term cochain complex C, equipped with the following structure:

- a chain map $[\cdot,\cdot] : C\otimes C\to C$,
- a chain homotopy $\langle\cdot,\cdot\rangle : [\cdot,\cdot]+[\cdot,\cdot]*\sigma\Rightarrow 0$,
- a chain homotopy $\langle\cdot,\cdot,\cdot\rangle : [\cdot,[\cdot,\cdot]]-[[\cdot,\cdot],\cdot]-[\cdot,[\cdot,\cdot]]*\sigma_{12}\Rightarrow 0$,

such that the following equations hold:

$$
(2.17)\quad
\begin{aligned}
&[x,\langle y,z,w\rangle]+\langle x,[y,z],w\rangle+\langle x,z,[y,w]\rangle+[\langle x,y,z\rangle,w]+[z,\langle x,y,w\rangle]=\\
&=\langle x,y,[z,w]\rangle+\langle [x,y],z,w\rangle+[y,\langle x,z,w\rangle]+\langle y,[x,z],w\rangle+\langle y,z,[x,w]\rangle
\end{aligned}
$$

$$(2.18)\qquad \langle x,y,z\rangle+\langle y,x,z\rangle=-[\langle x,y\rangle,z]$$

$$(2.19)\qquad \langle x,y,z\rangle+\langle x,z,y\rangle=[x,\langle y,z\rangle]-\langle [x,y],z\rangle-\langle y,[x,z]\rangle$$

$$(2.20)\qquad \langle x,y,z\rangle-\langle y,z,x\rangle=\langle x,[y,z]\rangle+\langle z,[y,x]\rangle-[\langle x,y\rangle,z]-[y,\langle x,z\rangle]$$

Remark 2.17. We do not assume that $\langle\cdot,\cdot\rangle$ is symmetric, nor that $\langle\cdot,\cdot,\cdot\rangle$ is skew-symmetric: in fact, equations (2.18), (2.19) and (2.20) describe the symmetry properties of $\langle\cdot,\cdot,\cdot\rangle$. These equations, however, are easily seen to imply the following:

$$
\begin{aligned}
&(2.21) & [\langle x,y\rangle,z] &= [\langle y,x\rangle,z] \\
&(2.22) & [x,\langle y,z\rangle] &= [x,\langle z,y\rangle] \\
&(2.23) & \langle x,[y,z]\rangle &= \langle [y,z],x\rangle
\end{aligned}
$$

In fact, (2.18), (2.19) and (2.20) are equivalent to (2.18), (2.19) and (2.23). In addition, equations (2.10), (2.11) and (2.12) obviously imply

$$
\begin{aligned}
&(2.24) & d\langle x,y\rangle &= d\langle y,x\rangle \\
&(2.25) & \langle da,x\rangle &= \langle x,da\rangle
\end{aligned}
$$

Conversely, given a 2-term weak L_∞-algebra $(C, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \langle\cdot,\cdot,\cdot\rangle)$, we can define a bracket on $L = \Gamma(C)$ by the formula (2.1), with $[a, b]$ given by formula (2.7), and S and J defined by formulas (2.8) and (2.9), respectively. The derived bracket $[a, b]$ satisfies equations

$$[a, b] + [b, a] = \langle da, db\rangle \tag{2.26}$$

$$[a, [b, c]] - [[a, b], c] - [b, [a, c]] = \langle da, db, dc\rangle \tag{2.27}$$

and it is routine to check that the axioms of a Lie 2-algebra are satisfied.

Next, we apply normalization to a morphism $(F, F^2) : L \to L'$. The functor $F : L \to L'$ induces a chain map $f = (f^0, f^1) : C \to C'$, while F^2 can be written in the form

$$F^2_{x,y} = ([f^0(x), f^0(y)], -f^2(x, y)),$$

where $f^2 : C^0 \otimes C^0 \to C^{-1}$ satisfies the following:

$$[f^0(x), f^0(y)]' - f^0([x, y]) = d' f^2(x, y) \tag{2.28}$$

$$[f^1(a), f^0(y)]' - f^1([a, y]) = f^2(da, y) \tag{2.29}$$

$$[f^0(x), f^1(b)]' - f^1([x, b]) = f^2(x, db) \tag{2.30}$$

In other words, f^2 is a homotopy from $[\cdot,\cdot]' * (f \otimes f)$ to $f * [\cdot,\cdot]$

Definition 2.18. A *morphism* of weak L_∞-algebras from $(C, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \langle\cdot,\cdot,\cdot\rangle)$ to $(C', [\cdot,\cdot]', \langle\cdot,\cdot\rangle', \langle\cdot,\cdot,\cdot\rangle')$ consists of

- a chain map $f = (f^0, f^1) : C \to C'$
- a chain homotopy $f^2 : [\cdot,\cdot]' * (f \otimes f) \Rightarrow f * [\cdot,\cdot]$

such that the following equations hold:

$$\langle f^0(x), f^0(y)\rangle' - f^1(\langle x, y\rangle) = f^2(x, y) + f^2(y, x) \tag{2.31}$$

$$\begin{aligned}&\langle f^0(x), f^0(y), f^0(z)\rangle' - f^1(\langle x, y, z\rangle) =\\ &= [f^0(x), f^2(y, z)]' - [f^0(y), f^2(x, z)]' - [f^2(x, y), f^0(z)]' -\\ &- f^2([x, y], z) - f^2(y, [x, z]) + f^2(x, [y, z])\end{aligned} \tag{2.32}$$

Definition 2.19. The *composite* of two morphisms $(f, f^2) : C \to C'$ and $(g, g^2) : C' \to C''$ is defined to be $(gf, (g * f)^2)$, where

$$(g * f)^2(x, y) = g^2(f^0(x), f^0(y))) + g^1(f^2(x, y))$$

Similarly, writing a 2-morphism $\Theta : (F, F^2) \Rightarrow (G, G^2)$ as

$$\Theta_x = (F(x), -\theta(x)),$$

with θ a chain homotopy from f to g, leads to

Definition 2.20. A 2-*morphism* $\theta : f \Rightarrow g$ is a chain homotopy satisfying

$$f^2(x, y) - g^2(x, y) = [f^0(x), \theta(y)]' + [\theta(x), f^0(y)]' - \theta([x, y]) - [\theta(x), \theta(y)]',$$

where $[\theta(x), \theta(y)]'$ is given by the formula (2.7)

Conversely, morphisms and 2-morphisms of 2-term weak L_∞-algebras induce the same of the corresponding Lie 2-algebras. We summarize the above discussion in the following

Theorem 2.21. *Weak L_∞ algebras form a 2-category* $\mathbf{2TermWL}_\infty$ *with the structure just defined. The Dold-Kan correspondence induces an equivalence of 2-categories*

$$\mathbf{2Lie} \underset{\Gamma}{\overset{N}{\rightleftarrows}} \mathbf{2TermWL}_\infty$$

2.5. **Special cases.** Setting the skew-symmetrizator S to be the identity yields the notion of a semistrict Lie 2-algebra. It coincides with the one defined in [1], since the bracket and the Jacobiator are (forced to be) completely skew-symmetric in this case. Semi-strict Lie 2-algebras form a full sub-2-category corresponding to ordinary 2-term L_∞-algebras upon normalization, as already shown in [1].

On the other hand, setting the Jacobiator J to be the identity, we get a full sub-2-category of hemistrict Lie 2-algebras. The normalized complex $d : C^{-1} \longrightarrow C^0$ of such a 2-algebra inherits the structure of a differential graded *Leibniz* algebra, since the right hand sides of the equations 2.13–2.16 vanish. In particular, C^0 is a Leibniz algebra acting on C^{-1} on both sides. C^{-1} itself becomes a Leibniz algebra with respect to the derived bracket (defined, as usual, by formula (2.7)), making $d : C^{-1} \longrightarrow C^0$ a Leibniz algebra crossed module. Representations of Leibniz algebras and crossed modules were considered in [7]

But this is not all. In addition, we have a bilinear operation

$$\langle\cdot,\cdot\rangle : C^0 \otimes C^0 \to C^{-1}$$

which measures, via equations (2.10), (2.11) and (2.12), the failure of the Leibniz algebra C^0 to be a Lie algebra, as well as the failure of the representation C^{-1} of C^0 to be symmetric (in the terminology of [7]). It obeys the equations

$$\begin{aligned} [\langle x,y\rangle, z] &= 0 \\ [x, \langle y,z\rangle] &= \langle [x,y], z\rangle + \langle y, [x,z]\rangle \\ \langle x, [y,z]\rangle &= \langle [y,z], x\rangle \end{aligned}$$

Equations (2.22), (2.24) and (2.25) are implied, as in the general case. In particular, the image of $\langle\cdot,\cdot\rangle$ is an anti-symmetric submodule of C^{-1} (in the terminology of [7]), while the skew-symmetric part of $\langle\cdot,\cdot\rangle$ is annihilated by the action of C^0 on both sides and is contained in the kernel of d.

Example 2.22. Given a Leibniz algebra $\mathfrak{g}$, denote by $\mathfrak{g}^{\mathrm{ann}}$ the subspace of $\mathfrak{g}$ spanned by the elements of the form $[x,x]$, $x \in \mathfrak{g}$. It is in fact a two-sided ideal in $\mathfrak{g}$. Setting $C^0 = \mathfrak{g}$, $C^{-1} = \mathfrak{g}^{\mathrm{ann}}$ and d the inclusion map, we get a dg Leibniz algebra. Moreover, setting

$$\langle x,y\rangle = [x,y] + [y,x]$$

gives a skew-symmetrizator with all the required properties. Thus, *any* Leibniz algebra gives rise to a hemistrict Lie 2-algebra, albeit a rather special one: the skew-symmetrizator is symmetric, and C^{-1} is an anti-symmetric C^0-module. However, if the characteristic of the ground field is different from 2, any hemistrict Lie 2-algebra with surjective $\langle\cdot,\cdot\rangle$ and injective d is of this form.

Example 2.23. Let $\mathfrak{g}$ be a Lie algebra equipped with an *ad*-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$. Setting $C^0 = \mathfrak{g}$, C^{-1}=$\mathbf{k}$, $d = 0$ gives rise to a hemi-strict Lie 2-algebra with $[x,a] = -[a,x] = 0$ and the skew-symmetrizator given by $\langle\cdot,\cdot\rangle$.

Finally, if both the skew-symmetrizator and the Jacobiator are trivial (i.e. the Lie 2-algebra is strict), we get a differential graded Lie algebra on the normalized complex. The derived bracket (2.7) is then a Lie bracket, yielding a Lie algebra crossed module.

3. Skew-symmetrization

In this section we assume that the characteristic of the ground field $\mathbf{k}$ is different from 2 or 3.

Suppose $(V, [\cdot,\cdot], S, J)$ is a Lie 2-algebra, $(C, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \langle,\cdot,\cdot,\cdot\rangle)$ – the corresponding weak L_∞-algebra. Define multilinear skew-symmetric maps $\{\cdot,\cdot\} : \bigwedge^2 C \to C$ and $\{\cdot,\cdot,\cdot\} : \bigwedge^3 C \to C[-1]$ as follows:

$$\begin{aligned}
\{x,y\} &= \frac{1}{2}([x,y]-[y,x]) \\
\{x,a\} &= \frac{1}{2}([x,a]-[a,x]) = -\{a,x\} \\
\{x,y,z\} &= [x,y,z] - T(x,y,z),
\end{aligned}$$

where

$$\begin{aligned}
[x_1,x_2,x_3] &= \frac{1}{6}\sum_{\sigma\in S_3}(-1)^\sigma\langle x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\rangle \\
T(x,y,z) &= \frac{1}{6}(\langle\{x,y\},z\rangle + \langle\{y,z\},x\rangle + \langle\{z,x\},y\rangle)
\end{aligned}$$

Then we can prove the following

Proposition 3.1. *$(C, \{\cdot,\cdot\}, \{\cdot,\cdot,\cdot\})$ is an L_∞-algebra.*

which can be expanded to

Theorem 3.2. *Skew-symmetrization defines a projection 2-functor*

$$SS : \mathbf{2Lie} \to \mathbf{SS2Lie}$$

onto the 2-category of semistrict Lie 2-algebras.

Corollary 3.3. *In particular, there is a skew-symmetrization 2-functor*

$$SS : \mathbf{HS2Lie} \to \mathbf{SS2Lie}$$

from hemistrict to semistrict Lie 2-algebras with

$$\{x,y,z\} = -T(x,y,z)$$

Example 3.4. Applying this to the hemistrict Lie 2-algebra of Example 2.23 yields a semistrict Lie 2-algebra with the same underlying category and brackets, with the Jacobiator given by

$$\langle\cdot,\cdot,\cdot\rangle = -\frac{1}{2}\langle[\cdot,\cdot],\cdot\rangle$$

In the case when $\mathfrak{g}$ is semisimple, with $k = \langle\cdot,\cdot\rangle$ the Killing form, the resulting Lie 2-algebra is the string Lie 2-algebra denoted by $\mathfrak{g}_{-\frac{1}{2}}$ in [1] and $\mathfrak{str}_{-\frac{1}{2}}(\mathfrak{g})$ in [4].

4. Categorical and Homotopy invariance

Recall that an equivalence of linear categories consists of a pair of linear functors $F : V \to V'$, $G : V' \to V$, together with linear natural transformations $\Phi : F * G \Rightarrow 1'_V$ and $\Psi : G * F \Rightarrow 1_V$. It is a standard fact in category theory that a functor F induces an equivalence of categories if and only if it is fully faithful and essentially surjective; its quasi-inverse G is then unique up to natural isomorphism. This carries over to the linear case with obvious modifications.

Using the Dold-Kan correspondence, it is easy to deduce that a linear functor F is fully faithful and essentially surjective if and only if $f = N(F)$ is a quasi-isomorphism (a chain map inducing isomorphism in cohomology), and if G is a quasi-inverse to F, then $g = N(G)$ is a homotopy inverse to f, and vice versa. In particular, if C is a cochain complex of vector spaces and H is its cohomology, viewed as a complex with zero differential, then there exists a homotopy equivalence $C \rightleftarrows H$ (a Hodge decomposition).

A categorically invariant algebraic structure is, heuristically, a structure that can be transferred along categorical equivalences. This means that if V is a category equipped with this structure and $F : V \to V'$ is fully faithful and essentially surjective, then there exists the same type of structure on V', unique up to equivalence, such that F induces an equivalence of categories with the structure. Making this precise in full generality requires a categorification of the notion of operad. We shall not attempt this here, taking advantage instead of the Dold-Kan correspondence in order to transfer everything to chain complexes, where the similar notion of a homotopy-invariant algebraic structure is well-known.

Recall also that a category is called *skeletal* if isomorphic objects are equal. It follows that a linear category is skeletal if and only if its normalized complex has zero differential. By Hodge decomposition, every linear category is equivalent to a skeletal one. We call a Lie 2-algebra skeletal if its underlying linear category is.

We have the following result.

Theorem 4.1. *The structure of 2-term weak L_∞-algebra is homotopy-invariant; equivalently, the structure of a Lie 2-algebra is categorically invariant. In particular, every Lie 2-algebra is equivalent, as a Lie 2-algebra, to a skeletal one.*

Remark 4.2. That semistrict Lie 2-algebras are categorically invariant follows from the well-known fact that L_∞-algebras are homotopy-invariant. However, hemistrict Lie 2-algebras are *not* categorically invariant.

It remains to determine what skeletal Lie 2-algebras look like, and to classify them up to equivalence.

So, let L be a skeletal Lie 2-algebra, C its normalized complex, with $d = 0$. Because of this last fact, C^0 is an honest Lie algebra, acting on C^{-1}, with $[x, a] = -[a, x]$. The Jacobiator $\langle \cdot, \cdot, \cdot \rangle$, which we shall here rename j, obeys the equation (2.17), which can be rewritten in the form

$$\begin{aligned}&[x, j(y, z, w)] - [y, j(x, z, w)] + [z, j(x, y, w)] + [j(x, y, z), w]- \\ &-j([x, y], z, w) - j(y, [x, z], w) - j(y, z, [x, w])+ \\ &+j(x, [y, z], w) + j(x, z, [y, w]) - j(x, y, [z, w]) = 0\end{aligned} \tag{4.1}$$

The reader can recognize this equation as saying that j is a 3-cocycle in the Loday-Pirashvili complex for C^0, viewed as a Leibniz algebra, with coefficients in the (symmetric) representation C^{-1}, as defined in [7]. It is not, however, a Chevalley-Eilenberg cocycle, for lack of skew-symemtry. In fact, with the skew-symmetrizator denoted by s, equations (2.18), (2.19) and (2.23) translate in this case to

$$j(x,y,z) + j(y,x,z) = [z, s(x,y)] \tag{4.2}$$
$$j(x,y,z) + j(x,z,y) = [x, s(y,z)] - s([x,y],z) - s(y,[x,z]) \tag{4.3}$$
$$s([x,y],z) = s(z,[x,y]) \tag{4.4}$$

To see when two skeletal Lie 2-algebras are equivalent, we first remark that skeletal categories are equivalent if and only if they are strictly isomorphic, and that a morphism between skeletal Lie 2-algebras is, in particular, a strict homomorphism of the underlying Lie algebras and representations. Because of this, it is necessary and sufficient to determine when

$$(1,f) : (C, [\cdot,\cdot], s, j) \longrightarrow (C, [\cdot,\cdot], s', j')$$

is a morphism of weak L_∞-algebras, with $f : C^0 \otimes C^0 \to C^{-1}$. The condition is given by equations (2.31) and (2.32) which read in this case:

$$s'(x,y) - s(x,y) = f(x,y) + f(y,x) \tag{4.5}$$
$$j'(x,y,z) - j(x,y,z) = \begin{array}{l} [x, f(y,z)] - [y, f(x,z)] - [f(x,y), z] - \\ -f([x,y],z) - f(y,[x,z]) + f(x,[y,z]) \end{array} \tag{4.6}$$

The second equation says that Loday-Pirashvili cocycles j' and j are cohomologous. Setting

$$g(x,y) = -f(x,y) + [x, \theta(y)] + [\theta(x), y] - \theta([x,y])$$

with an arbitrary $\theta : C^0 \to C^{-1}$ then gives a morphism $(1,g)$ the other way, with $\theta : (1,g) * (1,f) \Rightarrow (1,0)$.

In general, given a Lie algebra $\mathfrak{g}$ and a representation M, define $ZL^3_{\mathrm{Lie}}(\mathfrak{g}, M)$ to be the space consisting of pairs (s,j), where $s : \mathfrak{g}^{\otimes^2} \to M$ and $j : \mathfrak{g}^{\otimes^3} \to M$ satisfy equations (4.1-4.4); define $BL^3_{\mathrm{Lie}}(\mathfrak{g}, M)$ to be the space of pairs (s,j) such that there exists an $f : \mathfrak{g}^{\otimes^2} \to M$ making equations (4.5) and (4.6) hold with s (resp. j) on the left hand side. We can prove an easy

Lemma 4.3. $BL^3_{\mathrm{Lie}}(\mathfrak{g}, M) \subseteq ZL^3_{\mathrm{Lie}}(\mathfrak{g}, M)$

Remark 4.4. The proof of the lemma depends not only on the equations (2.13-2.16) (with vanishing right hand sides), as for the differential in the Loday-Pirashvili complex, but also on the skew-symmetry

$$[x,y] = -[y,x]$$
$$[x,a] = -[a,x]$$

Lie algebras in this context are to be viewed as Leibniz algebras which are also skew-symmetric.

We define $HL^3_{\mathrm{Lie}}(\mathfrak{g}, M) = ZL^3_{\mathrm{Lie}}(\mathfrak{g}, M)/BL^3_{\mathrm{Lie}}(\mathfrak{g}, M)$. With this we have obtained the following

Theorem 4.5. *Skeletal Lie 2-algebras are classified up to equivalence by the following data:*

- *a Lie algebra* $\mathfrak{g}$
- *a representation* M *of* $\mathfrak{g}$
- *a class* $[(s,j)] \in HL^3_{\mathrm{Lie}}(\mathfrak{g}, M)$

Remark 4.6. This classification generalized the classification of semistrict skeletal Lie 2-algebras due to Baez and Crans [1], for our cohomology space then reduces to the Chevalley-Eilenberg cohomology. But it is also remarkably similar to the classification of skeletal braided categorical groups, due to Joyal and Street (Proposition 3.1 of [5]). In that classification, a very similar construction, due to Eilenberg and MacLane and going back to 1950, was used to obtain a group $H^3_{\mathrm{ab}}(G, M)$ for an abelian group G with coefficients in a representation M. We suspect that Lie 2-algebras may be related to linear braided monoidal categories by a a sort of categorified Koszul duality, yet to be described.

Theorem 4.7. *The skew-symmetrization functor induces a map*

$$\mathrm{ss} : HL^3_{\mathrm{Lie}}(\mathfrak{g}, M) \longrightarrow H^3(\mathfrak{g}, M)$$

onto the Chevalley-Eilenberg cohomology, fitting into the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\wedge^2 \mathfrak{a}, M) \xrightarrow{\iota} HL^3_{\mathrm{Lie}}(\mathfrak{g}, M) \xrightarrow{\mathrm{ss}} H^3(\mathfrak{g}, M) \longrightarrow 0$$

where

$$\mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$$

is the abelianization, and $\iota(a) = [(a, 0)]$. A canonical splitting is given by $\phi \mapsto [(0, \phi)]$.

Example 4.8. Let $\mathfrak{g}$ be a semisimple Lie algebra, with Killing form k and Cartan tensor $\phi = k([\cdot, \cdot], \cdot)$. Since semi-simple Lie algebras are perfect, $\mathfrak{a} = 0$, hence ss is an isomorphism. It sends the class of $(k, 0)$ to that of $-\frac{1}{2}\phi$ (Example 3.4). Notice that ϕ is the Loday-Pirashvili coboundary of k, hence $(k, 0)$ is cohomologous to $(0, -\frac{1}{2}\phi)$. The corresponding hemistrict and semistrict Lie 2-algebras are equivalent as Lie 2-algebras and, since the cohomology space is one-dimensional, any Lie 2-algebra structure with this underlying Lie algebra and module is equivalent to a multiple of either.

5. Applications

In this section we assume that the characteristic of the ground field $\mathbf{k}$ is zero.

Recall that, in deformation theory, one considers L_∞-algebras of the form

$$(L = \bigoplus_{k \in (-n, +\infty)} L^k, \delta = [\cdot], [\cdot, \cdot], [\cdot, \cdot, \cdot], \dots)$$

where the k-nary bracket is of degree $2 - k$ and n is a non-negative integer. One is then interested in the space of solutions of the (generalized) Maurer-Cartan equation

$$\mathrm{MC}(L) = \{\gamma \in \mathrm{Ł}^1 | \sum_{k \geq 1} \frac{1}{k!} [\gamma, \dots, \gamma] = 0\}$$

where the kth summand is the k-nary bracket of γ with itself (in particular, $[\gamma] = \delta(\gamma)$ is the differential). The equation describes the type of algebraic structure one wishes to study. To make sense of it, one either renders L nilpotent by tensoring with a nilpotent commutative algebra, or assumes that all but a finite number of k-nary brackets are zero; in fact, in most cases occurring in practice only the differential and the binary bracket are nontrivial, making L a differential graded Lie algebra (dgla).

There is an equivalence relation on $\mathrm{MC}(L)$ induced by the infinitesimal action of $L^0 \ni x \mapsto \bar{x}$ where

$$\bar{x}(\gamma) = \delta(x) + [\gamma, x] + \frac{1}{2}[\gamma, \gamma, x] + \cdots$$

If L is a dgla with no components of negative degree ($n = 1$), L^0 is a Lie algebra; integrating its action (for nilpotent L) gives rise to an action groupoid, known as the Deligne groupoid of L. This groupoid presents the moduli stack of L, which is the main object of study in deformation theory.

However, the presence of negative degrees leads to a richer structure involving higher symmetries. This was first noticed by Deligne who constructed, in an unpublished letter to L. Breen, a strict 2-groupoid over $\mathrm{MC}(L)$ for a dgla L with $n = 2$. This construction was rediscovered by Getzler [3]. It uses the derived bracket on L^{-1} parametrized by $\gamma \in \mathrm{MC}(L)$. In particular, the 2-group of automorphisms of γ is obtained by integrating the corresponding Lie algebra crossed module under the derived bracket (i.e. a strict Lie 2-algebra in our sense).

It was Getzler [2] who generalized this construction to L_∞-algebras and higher values of n. The result is a weak n-groupoid which, by definition, is a Kan complex with unique fillers for horns in dimension higher than n. However, much remains to be understood about the structure of higher symmetries even at the infinitesimal level. Here we present a construction, for a dgla with $n = 3$, of a kind of crossed module involving Lie 2-algebras of our kind.

To begin, notice that there is a family of L_∞-algebras parametrized by $\mathrm{MC}(L)$:

$$\begin{aligned}
\delta_\gamma &= \delta + [\gamma, \cdot] + \frac{1}{2}[\gamma, \gamma, \cdot] + \cdots \\
[\cdot, \cdot]_\gamma &= [\cdot, \cdot] + [\gamma, \cdot, \cdot] + \frac{1}{2}[\gamma, \gamma, \cdot, \cdot] + \cdots \\
[\cdot, \cdot, \cdot]_\gamma &= [\cdot, \cdot, \cdot] + [\gamma, \cdot, \cdot, \cdot] + \cdots \\
\cdots & \quad \cdots
\end{aligned}$$

and that truncation

$$L^{-n+1} \overset{\delta_\gamma}{\to} \cdots \overset{\delta_\gamma}{\to} L^{-1} \overset{\delta_\gamma}{\to} \bar{L}^0$$

where $\bar{L}^0 = \ker \delta_\gamma$ defines an L_∞-subalgebra, the algebra of infinitesimal automorphisms of γ. It is this algebra that we shall presently study.

5.1. **Case $n = 2$.** Here we have a 2-term L_∞-algebra of the form

$$L^{-1} \xrightarrow{d} \bar{L}^0$$

with only $d = \delta_\gamma$, $[\cdot, \cdot]_\gamma$ and $[\cdot, \cdot, \cdot]_\gamma$ nontrivial. This gives rise to a semistrict Lie 2-algebra, with the derived bracket, defined by formula (2.7), giving the crossed module structure.

Example 5.1. Let M be a smooth manifold. Set $L = \Gamma(\wedge^\cdot TM)[1]$, the algebra of smooth multivector fields. It is a dgla under the Schouten bracket and zero differential. A solution γ of the Maurer-Cartan equation is, by definition, a Poisson structure on M. The strict Lie 2-algebra of infinitesimal automorphisms of γ has $L^{-1} = C^\infty(M)$, $\bar{L}^0$ the space of Poisson vector fields, and $d = \delta_\gamma = [\gamma, \cdot]$ the Lichnerowicz differential, sending a function a to its Hamiltonian vector field. The derived bracket on L^{-1} is just the Poisson bracket of functions determined by γ.

This is the main example whose integration was given in [3]. It can be generalized to any Lie bialgebroid.

Example 5.2. With L as above, let H be a 3-form on M. It extends by the Leibniz rule to define a trilinear operation $[\cdot,\cdot,\cdot]$ on L of degree -1. Together with the Schouten bracket, it defines an L_∞-structure if and only if H is closed. A solution γ of the Maurer-Cartan equation is an H-twisted Poisson structure ([11], [9]). The construction of the Lie 2-algebra of infinitesimal automorphism proceeds as in the previous example, except now it is only semistrict. This example generalizes to any quasi-Lie bialgebroid [9].

5.2. **Case $n = 3$, dgla.** In this case the truncated dgla is 3-term:

$$L^{-2} \xrightarrow{d} L^{-1} \xrightarrow{d} \bar{L}^0$$

with $d = \delta_\gamma = \delta + \{\gamma, \cdot\}$ and $\{\cdot,\cdot\}$ the bracket on L. In particular, $\bar{L}^0$ is a Lie algebra acting on L^{-1} and L^{-2} in a way compatible with d, but there is also a *symmetric* bilinear map

$$\{\cdot,\cdot\} : L^{-1} \otimes L^{-1} \longrightarrow L^{-2}$$

which will play the role of a skew-symmetrizator, so let us denote it by $\langle\cdot,\cdot\rangle$ from now on. Set $C^i = L^{i-1}$, $i = -1, 0$, and introduce the derived brackets on C as follows:

$$\begin{aligned} [x, y] &= \{dx, y\} \\ [x, a] &= \{dx, a\} \\ [a, x] &= 0 \end{aligned}$$

for $x, y \in C^0$, $a \in C^{-1}$.

Furthermore, viewing $\bar{L}^0$ as a cochain complex concentrated in degree 0 or, equivalently, as a linear category with only identity arrows, we get a chain map

$$(d, 0) : (C^0, C^{-1}) \longrightarrow (\bar{L}^0, \{0\}),$$

so that $\partial = \Gamma(d, 0) : \Gamma(C) \longrightarrow \bar{L}^0$ is a linear functor.

Lastly, since the action of $\bar{L}^0$ (given by $T \mapsto \{T, \cdot\} = -\{\cdot, T\}$) commutes with d, it induces a functorial action of $\bar{L}^0$ on the linear category $\Gamma(C)$. We have the following

Theorem 5.3. *For any dgla L concentrated in degrees $(-3, 0]$,*

- *$\Gamma(C)$ is a hemistrict Lie 2-algebra;*
- *$\partial : \Gamma(C) \longrightarrow \bar{L}^0$ is a morphism of Lie 2-algebras;*
- *$\bar{L}^0$ acts on $\Gamma(C)$ by strict derivations.*

In addition, the following crossed module identities hold:

$$\begin{aligned} \partial\{T, \mathbf{f}\} &= \{T, \partial\mathbf{f}\} \\ \{\partial\mathbf{f}, \mathbf{g}\} &= [\mathbf{f}, \mathbf{g}] \end{aligned}$$

where $T \in \bar{L}^0$, $\mathbf{f}, \mathbf{g} \in \Gamma(C)$.

Example 5.4. Let $E \to M$ be a vector bundle with a fiberwise smooth inner product $\langle\cdot,\cdot\rangle$. In [8] we constructed a dgla L as above, such that the solutions of the Maurer-Cartan equation are precisely Courant algebroid structures on E, with the Courant bracket defined as the derived bracket. Theorems 5.3 and 3.2 combine to yield the main result of [10] as an immediate corollary.

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