Stability criteria and classification of singularities for equivariant lagrangian submanifolds

by

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§ 1. Introduction

Singularities of lagrangian submanifolds appeared as the natural objects in the study of the wave pattern with high-frequency waves coming from a point source and moving through a medium (cf. [14], [12]). The corresponding intensity of radiation is described by the asymptotics of the so-called rapidly oscilating integrals (cf. [7], [3]). Asymptotically (with high frequency) this intensity is infinite around the singularities (coustics) of lagrangian submanifolds generated by the appropriate phase functions (cf. [19], [7]). Thus the lagrangian submanifolds appeared initially as the spaces moddelizing the systems of rays in geometrical optics [3]. In the case of symmetries of the sources of radiation as well as when the boundary conditions (mirrors) exhibit some symmetry properties then the corresponding lagrangian submanifold describing the respective optical geometry of the system possesses also some symmetry properties (cf. [14], [17]). The similar problems with symmetric lagrangian submanifolds appeared also in variational calculus, nonlinear partial differential equations, and optimization (cf. [14], [24]).

An another domain where the singularities of lagrangian submanifolds play an important role is the symplectic bifurcation theory (cf. [23], [11], [10]) and the breaking of symmetry in mechanics and the structural phase transitions (cf. [17], [8], [9]). It was observed in [10] that the lagrangian submanifolds moddelize the space of equilibrium states of thermodynamical systems. In most of thermodynamic phase transitions in crystals (cf. [12]) the whole bifurcation picture can be described by an appropriate G-equivariant lagrangian submanifold in the corresponding phase space with the compact Lie group G of symmetry (cf. [11], [9]). The first step in the study of typical properties of constitutive sets in structural physics is the recognition and classification of stable G-equivariant germs of lagrangian submanifolds, which is the aim of the present paper.

In this paper we will study the infinitesemal stability and local stability criteria for the germs of equivariant lagrangian submanifolds near the fix-point of the symplectic action of the compact Lie group. Our purpose is twofold. First, we want to write down the algebraic criteria for the local G-stability. Secondly, we want to use this general method to investigate the normal forms of the stable G-equivariant lagrangian germs.

In [2], [22] there is a study of stable singularities of lagrangian submanifolds in the nonsymmetric case, and we will follow the notations and terminology used there. In Section 2 of our paper we provide the basic results and notation for further needs. In Section 3 we construct the infinitesemal stability conditions for G-invariant generating functions of G-equivariant lagrangian germs and show their effectiveness in calculations with the trivial, $\mathbb{Z}_2$, and D_m symplectic group actions. Section 4 is devoted to the complete calculation of stability criteria and classification of stable normal forms of equivariant lagrangian germs in the concrete $(\mathbb{Z}_2)^q$ group action. This action is motivated by the theory of phase transitions in uniaxial ferromagnets as well as in the all types of ferroelectrics. In Section 5, 6 we present the stability criteria in the Morse family (cf. [19]) approach. Here we derive the so-called linear infinitesemal stability condition and show its usefulness in some concrete symmetric problems. Following [9] we also give there the alternative approach to the study of G-equivariant lagrangian singularities in physical applications.

§ 2. Preliminaries

Let $\nu : G \to O(n)$ be an orthogonal representation of G in $\mathbb{R}^n$. By $C^\infty_\nu(n)$ we denote the set of smooth ν-invariant functions on $\mathbb{R}^n$ and by $\mathcal{E}_\nu(n)$ the set of all their germs at $0 \in \mathbb{R}^n$ (cf. [13]). We denote $\mathfrak{m}^k_\nu(n) = \mathfrak{m}^k(n) \cap \mathcal{E}_\nu(n)$, where $\mathfrak{m}^k(n)$ denotes the k-th power of the maximal ideal $\mathfrak{m}(n) \subset \mathcal{E}(n)$ (cf. [21]). For convenience we shall write also $\mathcal{E}_\nu(z)$, $\mathfrak{m}_\nu(z)$ etc. instead of $\mathcal{E}_\nu(n)$, $\mathfrak{m}_\nu(n)$, etc., where $z = (z_1,\ldots,z_n)$ denote the corresponding coordinates of $\mathbb{R}^n$. By $\mathcal{E}(n,\nu;m,\delta)$, where δ is an orthogonal representation of G in $\mathbb{R}^m$, we shall denote the set of germs (at $0 \in \mathbb{R}^n$) of equivariant mappings $\mathbb{R}^n \to \mathbb{R}^m$.

The foundational theory of equivariant singularities may be found in [13], [21]. Now we recall some of the basic facts needed for the development of the theory of equivariant lagrangian submanifolds.

Proposition 2.1. ([15], [21]). Let ν be an orthogonal representation of the compact Lie group G in $\mathbb{R}^n$.

a) There exists a polynomial mapping $\rho : \mathbb{R}^n \to \mathbb{R}^k$, called a Hilbert map, such that

$$\mathcal{E}_\nu(n) = \rho^*\mathcal{E}(k) .$$

The set $\rho(\mathbb{R}^n) \subset \mathbb{R}^k$ is semialgebraic.

b) If $\delta : G \to O(n)$ is an orthogonal representation of G in

$\mathbb{R}^m$ and $\mathbb{R}^{n+m} \ni (x,y) \to \mu(x,y) \in \mathbb{R}^r$ is the corresponding Hilbert map for $\nu \oplus \delta$, then the germs $\mathbb{R}^n \ni (x) \to$ $\to \frac{\partial \mu_i}{\partial y}(x,0)$, $1 \leq i \leq r$ generate the module $E(n,\nu;m,\delta)$ over $E_\nu(n)$.

Let us consider the cotangent bundle $T^*\mathbb{R}^n$ endowed with the standard symplectic structure (see [1]). We identify it with the lagrangian fibre bundle $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n$, $\pi' : (x,\xi) \to (x)$ endowed with the canonical symplectic structure $\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i$. The action ν of G on $\mathbb{R}^n$ can be canonically lifted to the symplectic action of G on $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$, say $T^*\nu : G \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. One can easily see that $T^*\nu \equiv \nu \oplus \nu$, where $(\nu \oplus \nu)_g(x,\xi) = (\nu_g x, \nu_g \xi)$ for $g \in G$, $(x,\xi) \in \mathbb{R}^{2n}$. An equivariant symplectomorphism $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which preserves the fibre bundle structure $\pi' : \mathbb{R}^{2n} \to \mathbb{R}^n$ will be called an equivariant lagrangian equivalence (ν-L-equivalence for short). By direct generalization of well-known results [19], [22] concerning of the nonequivariant case we obtain.

<u>Proposition 2.2</u>. Let $\Phi : (\mathbb{R}^{2n},0) \to (\mathbb{R}^{2n},0)$ be a germ of ν-L-equivalence, then there exists a diffeomorphism $\varphi \in \xi(n,\nu;n,\nu)$ and a smooth function $S \in E_\nu(n)$ such that

$$\Phi(x,\xi) = \varphi^*(x)(\xi + dS(x)) .$$

Let p be the ν-invariant point of $\mathbb{R}^{2n}$, by (L^G,p) we denote the germ of ν-invariant lagrangian submanifold in $(\mathbb{R}^{2n},\omega)$ (ν-L-germ for short). As we know by [9], any ν-L-germ $(L^G, p=(x_0,\xi_0))$ can be generated by the germ of the so-called Morse family $F : (\mathbb{R}^n \times \mathbb{R}^1, (x_0, 0)) \to \mathbb{R}$, $F \in \mathcal{E}_{\nu\oplus\delta}(n+1)$. Locally $(L^G, (x_0, \xi_0))$ can be written by the following equations

$$(2.1) \qquad \xi = \frac{\partial F}{\partial x}(x,\lambda) , \quad 0 = \frac{\partial F}{\partial \lambda}(x,\lambda) ,$$

where

$$(2.2) \qquad \operatorname{rank}\left(\frac{\partial^2 F}{\partial x \partial \lambda}, \frac{\partial^2 F}{\partial \lambda \partial \lambda}\right)(x_0, 0) = 1 .$$

Conversely, any germ $F \in \mathcal{E}_{\nu\oplus\delta}(n+1)$ satisfying (2.2) (G-Mf-germ) for short) defines the ν-L-germ via equations (2.1). A G-Mf-germ, generating (L,p) , with minimal number of parameters 1 is called a minimal G-Mf-germ (cf. [2], [7]). A minimal G-Mf-germ can be equivalently characterized by the requirements

$$\left(\frac{\partial^2 F}{\partial \lambda \partial \lambda}\right)(x_0, 0) = 0 .$$

The two G-Mf-germs $F' \in \mathcal{E}_{\nu\oplus\delta'}(n+1)$, $F \in \mathcal{E}_{\nu\oplus\delta}(n+1)$ are called G-L-equivalent if

$$F(x,\lambda) = F'(\varphi(x), \Lambda(x,\lambda)) + f(x) ,$$

where $(\Lambda, \varphi) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a diffeomorphism, and $\Lambda \in \mathcal{E}(n+1, \nu\oplus\delta; 1, \delta')$, $\varphi \in \mathcal{E}(n,\nu;n,\nu)$, $f \in \mathcal{E}_\nu(n)$. To be able to compare the various G-Mf-germs with different diemnsions of parameter spaces we introduce the notion of stable G-L-equivalence. We say that two G-Mf-germs $F_1 \in \mathcal{E}_{\nu\oplus\delta'}(n+1_1)$, $F_2 \in \mathcal{E}_{\nu\oplus\delta}(n+1_2)$ are stable G-L-equivalent if the corresponding

G-Mf-germs $F_1+Q_1 \in \mathcal{E}_{\nu\oplus\delta'\oplus id}(n+l_1+r_1)$, $F_2+Q_2 \in \mathcal{E}_{\nu\oplus\delta\oplus id}(n+l_2+r_2)$, where Q_1, Q_2 are the nondegenerate quadratic forms of the additional variables, are G-L-equivalent (cf. [2]). By straightforward generalization of [2], [22], [9] we obtain,

Proposition 2.3. Let $(L^G{}_1, p_1)$, $(L^G{}_2, p_2)$ be two ν-L-germs of $(T^*\mathbb{R}^n, \omega)$. They are ν-L-equivalent, i.e. there exists an ν-L-equivalence $\Phi: T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, such that $\Phi(p_1) = p_2$ and $\Phi(L_1^G) = L_2^G$, if and only if their G-Mf-germs are stable G-L-equivalent.

For the corresponding minimal G-Mf-germs we have the stronger result,

Proposition 2.4. The two ν-L-germs of $(T^*\mathbb{R}^n, \omega)$ are ν-L-equivalent if and only if their minimal G-Mf-germs are G-L-equivalent.

Correctness of these two equivalences is assured by the easily seen fact that any two G-Mf-germs generating ν-L-germ $(L^G, p) \subset T^*\mathbb{R}^n$ are stable G-L-equivalent.

Let $(L^G, 0) \subseteq (\mathbb{R}^{2n}, \omega)$ be a ν-L-germ. Let $k = \dim\ker D(\pi|_{L}G)(0)$, then the representation ν is reducible and can be written as the direct sum, at least, of two components $\nu = \nu_1 \oplus \nu_2$. The corresponding invariant subspaces for ν_1 and ν_2 respectively are indicated by Arnold's results (cf. [2], Theorem 10.6.), namely we can choose the numeration of coordinates in neighbourhood of $0 \in \mathbb{R}^{2n}$ in such a way that (x_I), (x_J) parametrize the invariant subspaces corresponding to the representations ν_1 and ν_2 respectively, $I = (i_1, \dots, i_k)$, $J = \{1, \dots, n\} - I$. The lifted representation $T^*\nu$ has a form

$\nu_1 \oplus \nu_2 \oplus \nu_1 \oplus \nu_2$, thus we can consider (ξ_I, x_J) as the new parametrization of the representation space for ν . On the basis of [9], [2], [22] there exists a generating function, say $(\xi_I, x_J) \to S(\xi_I, x_J)$, for $(L^G, 0)$ and $S \in \mathcal{E}_\nu(n)$. We will call this function a ν-IJ-germ generating the ν-L-germ $(L^G, 0)$ if L^G is defined near $0 \in \mathbb{R}^{2n}$ by the equations:

$$(2.3) \qquad \xi_J = \frac{\partial S}{\partial x_J}(\xi_I, x_J), \; x_I = -\frac{\partial S}{\partial \xi_I}(\xi_I, x_J) \; .$$

If $k = \dim \ker D(\pi|_L G)(0)$, then we have

$$\frac{\partial^2 S}{\partial \xi_I \partial \xi_I}(0) = 0$$

and the germ

$$(2.4) \qquad F : \mathbb{R}^{n+k} \ni (x, \lambda) \to S(\lambda, x_J) - \sum_{\alpha=1}^{k} \lambda_\alpha x_{i_\alpha}$$

is a minimal G-Mf-germ for $(L^G, 0)$, where the corresponding representation δ in the parameter space can be chosen as $\delta \cong \nu|_{\{x_J=0\}}$ (cf. [22]). Summarizing the above properties of $(L^G, 0)$ and repeating the genericity argument of [2] (Proposition 10.11) we obtain

<u>Proposition 2.5</u>. Generically, any ν-L-germ $(L^G, 0) \subseteq (T^* \mathbb{R}^n, \omega)$ has a ν-IJ-germ of generating function S with $J = \phi$ i.e. $\xi \to S(\xi)$, $S \in \mathfrak{m}_\nu^2(n)$.

Now we introduce the fundamental notions necessary to obtain the finite classification of ν-invariant lagrangian submanifolds.

<u>Definition 2.6.</u> Let $L^G \subseteq (T^* \mathbb{R}^n, \omega)$ be a ν-invariant lagrangian submanifold. A ν-L-germ (L^G, p) is called stable if for an open ν-inv. neighbourhood U of p in $T^* \mathbb{R}^n$ and every smooth family L^G_t, $|t|<\varepsilon$, $(L^G_0, p) = (L^G, p)$ of ν-invariant lagrangian submanifolds there exist a smooth family Φ_t of ν-L-equivalences such that $\Phi_t(L^G_t \cap U) \supset L^G \cap V$, for some open ν-inv. neighbourhood V of p and sufficiently small t.

As was shown in [9] (cf. [2]) the standard notion of unfolding of singularity [20] can be adapted to represent the G-Mf-germs generating the germs of lagrangian submanifolds. Let $F \in \mathcal{E}_{\nu\oplus\delta}(n+1)$, we will call F the ν-unfolding of $f = F|_{\{0\} \times \mathbb{R}^1} \in \mathcal{E}_\delta(1)$ (cf. [9], [16]).

<u>Definition 2.7.</u> Let $\tilde{F} \in C^\infty_{\nu\oplus\delta}(n+1)$ be a representative of the germ of the ν-unfolding $F \in \mathcal{E}_{\nu\oplus\delta}(n+1)$. We say that F is stable if for any smooth family of functions $\tilde{F}_t \in C^\infty_{\nu\oplus\delta}(n+1)$, $|t|<\varepsilon$, $\tilde{F}_0 = \tilde{F}$, there exists a neighbourhood U of 0 in $\mathbb{R}^{n+1}$, family of diffeomorphisms $(\varphi_t, \Lambda_t) \in C^\infty(n,\nu;n,\nu) \oplus C^\infty(n+1,\nu\oplus\delta;1,\delta)$ and family of functions $f_t \in C^\infty_\nu(n)$ such that

$$F(x,\lambda) = F_t(\varphi_t(x), \Lambda_t(x,\lambda)) + f_t(x) ,$$

for $(x,\lambda) \in U$ and sufficiently small t.

According to the standard lines of the theory of stable singularities we can at first characterize the stable germs by the necessary infinitesemal condition so-called versality condition.

Definition 2.8. (cf. [16]) Let $F \in E_{\gamma\oplus\delta}(m+k)$ be a γ-unfolding of $f \in E_\delta(k)$. F is called the G-versal unfolding of f if for any orthogonal representation ν of G in $\mathbb{R}^n$ any ν-unfolding $\bar{F} \in E_{\nu\oplus\delta}(n+k)$ of f has the form

$$\bar{F}(x,\lambda) = F(\varphi(x), \Lambda(x,\lambda)) + \alpha(x)$$

where $\Lambda \in E(n+k, \nu\oplus\delta;\ k, \delta)$, $\varphi \in E(n, \nu;\ m, \gamma)$, $\alpha \in E_\nu(n)$.

On the basis of [7], [9], [18], [2] we know that the stable ν-L-germs (L^G,p) are effectively represented by the corresponding stable germs of ν-unfoldings. Our notion of ν-unfolding reduces to the standard notion of unfolding if we assume the trivial action of the group G . The corresponding theory is exhaustively presented in [24], [14]. For the symmetric case, following [2], [22], [7], we have the following elementary

Proposition 2.9. Let (L^G,p) be a ν-L-germ contained in $(T^*\mathbb{R}^n,\omega)$, let $F \in E_{\nu\oplus\delta}(n+k)$ be the corresponding G-Mf-germ, then the following properties are equivalent

a) (L^G,p) is stable ν-L-germ .

b) The G-Mf-germ F is stable as a ν-unfolding of $f = F|_{\{0\}\times\mathbb{R}^k} \in E_\delta(k)$.

Having the analytical representation of stable ν-L-germs, given in Proposition 2.9, we can characterize them by the infinitesemal stability property, i.e. versality of the correspomding G-Mf-germs as ν-unfoldings.

§ 3. Infinitesemal stability conditions for G-invariant generating functions

Let $(L_t^G,0)$ be a germ of the smooth family of ν-L-germs $L_t^G \subseteq T^*\mathbb{R}^n |t|<\varepsilon$. Up to the ν-L-equivalence (cf. [2], Proposition 10.11) we can represent this family in the following form

$$(3.1) \qquad L_t^G = \{ (x,\xi) \in T^* \mathbb{R}^n ;\ x = \frac{\partial S_t}{\partial \xi} (\xi) \} ,$$

where $t \to S_t(\xi) \in \mathcal{E}_\nu^\infty(n)$ is an appropriate family of generating functions (deformation of S_0). So we can reformulate the local stability of $(L_0^G,0)$ in terms of the smooth deformations S_t . If $(L_0^G,0)$ is stable and ε sufficiently small then there exists a smooth family Φ_t of ν-L-equivalences and an open neighbourhood U of $0 \in T^* \mathbb{R}^n$ such that

$$(3.2) \qquad \Phi_t(L_0 \cap U) \subset L_t .$$

Let us consider the vector field $X = \frac{d}{dt} \Phi_t\Big|_{t=0}$ on $T^* \mathbb{R}^n$. Since each Φ_t $(|t|<\varepsilon)$ is an equivariant symplectomorphism preserving the canonical fibration $T^* \mathbb{R}^n \to \mathbb{R}^n$, thus X must be the equivariant Hamiltonian vector field constant along the fibers of $T^* \mathbb{R}^n$, i.e. $X = - \frac{\partial H}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial H}{\partial \xi} \frac{\partial}{\partial x}$ where for $H \in C^\infty_{\nu\oplus\nu}(T^* \mathbb{R}^n)$ we can write

$$(3.3) \qquad H(x,\xi) = (A(x)\,|\,\xi) + B(x)$$

where $(.|.)$ denotes the canonical scalar product on $\mathbb{R}^n$, and from ν-invariance of H results $A \in C^{\infty}(n,\nu;n,\nu)$, $B \in C^{\infty}_{\nu}(n)$. Now, using the Hamilton-Jacobi theorem [1] for the family L_t^G we can write the equation

$$(3.4) \qquad \frac{\partial S_t}{\partial t}(\xi)\Big|_{t=0} = H\Big(\frac{\partial S_t}{\partial \xi}(\xi),\xi\Big)\Big|_{t=0}$$

near $0 \in \mathbb{R}^{n+1}$.

Note that to assure stability of $(L_0^G,0)$ the left hand side of (3.4) can be an arbitrary element of $C^{\infty}_{\nu}(n)$ satisfying the equation (3.4) with some ν-equivariant Hamiltonian H of the form (3.3).

Let us denote by H_{ν} the space of germs at $0 \in T^*\mathbb{R}^n$ of ν-invariant Hamiltonians $H : T^*\mathbb{R}^n \to \mathbb{R}$ of the form (3.3). Let $i_{L}G \in \xi(n,\nu;\ T^*\mathbb{R}^n, T^*\nu)$ be the lagrangian immersion $\xi \to (\frac{\partial S_0}{\partial \xi},\xi)$ corresponding to $(L^G,0)$.

<u>Lemma 3.1</u>. Let $(L_0^G,0)$ be a stable ν-L-germ, with a generating function $S_0 \in E_{\nu}(n)$ then we have

$$(3.5) \qquad E_{\nu}(n) = i^*_{L_0^G} H_{\nu}\ .$$

The proof of this lemma is obtained immediately on the basis of Definition 2.6. and [2] p.21.

Let π be the projection, $\pi(x,\xi) = x$, we denote $V_j(x,\xi) = (\xi|\varphi_j(x))$ where $\varphi_j(x) := \frac{\partial \mu_j}{\partial y}(x,0)$ and $\mu = (\mu_1,\ldots,\mu_b) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^b$ is the Hilbert map for the $\nu \oplus \nu$ action of G on $\mathbb{R}^n \times \mathbb{R}^n$.

Proposition 3.2. Let $(L_0^G, 0)$ be a stable ν-L-germ, with a generating function $S_0 \in \mathcal{E}_\nu(n)$, then the following infinitesemal stability condition is fulfilled:

$$(3.6) \qquad \mathcal{E}_\nu(n) = i_L^* G \langle V_1, \ldots, V_b, 1 \rangle_{\pi^* \mathcal{E}_\nu(n)}$$

where $\langle V_1, \ldots, V_b, 1 \rangle_{\pi^* \mathcal{E}_\nu(n)}$ is the submodule of $\mathcal{E}_{\nu \oplus \nu}(n+n)$ generated by $V_1, \ldots, V_b, 1$ over $\pi^* \mathcal{E}_\nu(n)$.

Proof. We know that $\mathcal{E}(n,\nu;n,\nu)$ is finitely generated over $\mathcal{E}_\nu(n)$ with generators $\varphi_j(x) = \frac{\partial \mu_j}{\partial y}(x,0)$ (see Proposition 2.1,b). Thus the right hand side of (3.5) we can write in the following way. Let $f \in \mathcal{E}_\nu(n)$ so on the basis of Lemma 3.1 we have

$$f(\xi) = (\xi | \sum_{i=1}^{b} c_i (\frac{\partial S_0}{\partial \xi}(\xi)) \varphi_i (\frac{\partial S_0}{\partial \xi}(\xi))) + B(\frac{\partial S_0}{\partial \xi}(\xi))$$

for some $c_i(x) \in \mathcal{E}_\nu(n)$, $B(x) \in \mathcal{E}_\nu(n)$, which gives exactly the infinitesemal stability condition (3.6).

Let $F \in \mathcal{E}_\nu(n)$, $(\xi_I, x_J) \to F(\xi_I, x_J)$ be the ν-IJ-germ generating for the ν-L-germ $(L^G, 0) \subseteq (T^* \mathbb{R}^n, \omega)$ (cf. § 2). The corresponding immersion of L^G , $i_{IJ} : \mathbb{R}^n \to T^* \mathbb{R}^n$, has a form

$$(3.7) \qquad i_{IJ}(\xi_I, x_J) = (\frac{\partial F}{\partial \xi_I}(\xi_I, x_J), x_J, \xi_I, -\frac{\partial F}{\partial x_J}(\xi_I, x_J)) \ .$$

Let us define for the ν-invariant germs $\rho \circ \pi \circ i_{IJ}$, $V_j \circ i_{IJ}$ $(j=1,\ldots,b)$ the following smooth mappings $\tilde{U} \in \mathcal{E}(a,a)$,

$\tilde{V}_j \in \mathcal{E}(a)$

$$(3.8) \qquad \tilde{U}\circ\rho = \rho\circ\pi\circ i_{IJ} \ , \ \tilde{V}_j\circ\rho = V_j\circ i_{IJ} \ , \ (j=1,\dots,b)$$

where $\rho : \mathbb{R}^n \to \mathbb{R}^a$ is the Hilbert map corresponding to the ν-action of G on $\mathbb{R}^n$.

Proposition 3.3. For a stable ν-L-germ $(L^G,0) \subseteq T^*\mathbb{R}^n$ and for its corresponding ν-IJ-germ $F \in \mathcal{E}_\nu(n)$ of generating function we have the following equivalent infinitesemal stability conditions

$$(3.9) \qquad \mathcal{E}_\nu(n) = \langle \rho^*\tilde{V}_1,\dots,\rho^*\tilde{V}_b, 1)_{(\pi\circ i_{IJ})^*\mathcal{E}_\nu(n)}$$

$$(3.10) \qquad \mathcal{E}_\nu(n) = \langle \rho^*\tilde{V}_1,\dots,\rho^*\tilde{V}_b,1\rangle_{(\tilde{U}\circ\rho)^*\mathcal{E}(a)}$$

$$(3.11) \qquad \mathcal{E}_\nu(n) = \langle \rho^*\tilde{V}_1,\dots,\rho^*\tilde{V}_b,1\rangle_{\mathbb{R}} + ((\pi\circ i_{IJ})^*\mathfrak{m}_\nu(n))\mathcal{E}_\nu(n)$$

$$(3.12) \qquad \mathcal{E}(a) = \langle \tilde{V}_1,\dots,\tilde{V}_b,1\rangle_{\mathbb{R}} + \langle \tilde{U}_1,\dots,\tilde{U}_a)_{\mathcal{E}(a)} + M_\rho(a)$$

where by $M_\rho(a) \subset \mathcal{E}(a)$ we denote all germs vanishing on $\rho(\mathbb{R}^n)$.

Proof. One can easily see that (3.9) results from (3.5), (3.6) and (3.7). By (3.8), conditions (3.9) and (3.10) are equivalent. Equivalence of (3.9) and (3.11) is a consequence of Equivariant Preparation Theorem (see [13] p.116). In fact $\mathcal{E}_\nu(n)/_{((\pi\circ i_{IJ})^*\mathfrak{m}_\nu(n))\mathcal{E}_\nu(n)}$ is a finite-dimensional vector space and its generators we can choose $\rho^*\tilde{V}_1,\ldots,\rho^*\tilde{V}_b,1$. Taking into account the equation $\tilde{U}\circ\rho = \rho\circ\pi\circ i_{IJ}$ we can rewrite (3.11) in the form (3.12). We need here only the fact that from equality $g\circ\rho = g'\circ\rho$, for some functions $g,g' \in E(a)$, results that $g-g' \in M_\rho(a)$.

Remark 3.4. Assume that ν is trivial, thus $\rho = \mathrm{id}_{\mathbb{R}^n}$ and $\rho^*\tilde{V}_I(\xi_I,x_J) = \xi_I$, $\rho^*\tilde{V}_J(\xi_I,x_J) = \frac{\partial F}{\partial x_J}(\xi_I,x_J)$, $E_\nu(n) = E(n)$ $M_\rho(n) = \{0\}$, $\tilde{U} = \mathrm{id}_{\mathbb{R}^n}$. Finally (3.9), (3.10) take the form

$$E(n) = <\frac{\partial F}{\partial x_J},\ \xi_I,\ 1)_{(\pi\circ i_{IJ})^*E(n)} ,$$

where

$$(\pi\circ i_{IJ})(\xi_I,x_J) = (\frac{\partial F}{\partial \xi_I}(\xi_I,x_J),\ x_J) ,$$

and for (3.11), (3.12) we have

$$E(n) = <\frac{\partial F}{\partial \xi_I},x_J>_{E(n)} + <\frac{\partial F}{\partial x_J},\ \xi_I,1)_{\mathbb{R}} .$$

Eliminating variables x_J by Preparation Theorem [6] we obtain

$$E(k) = <\frac{\partial F}{\partial \xi_I}\Big|_{x_J=0}>_{E(k)} + <\frac{\partial F}{\partial x_J}\Big|_{x_J=0},\xi_I,1>_{\mathbb{R}} , \quad k = \#I$$

which is exactly the standard versality condition for versal deformations [20], used by Arnold [2] in the classification theory of stable lagrangian singularities.

Example 3.5. (Infinitesemal stability condition for D_m-action). In many applications of equivariant singularity theory [8] we find the following irreducible representation of the group D_m :

$$\mu(g_1) : (x_1,x_2) \to (x_1,-x_2)$$

$$\mu(g_2) : (x_1,x_2) \to (x_1\cos{}^{2\pi}/m - x_2\sin{}^{2\pi}/m, x_1\sin{}^{2\pi}/m + x_2\cos{}^{2\pi}/m) ,$$

where, g_1,g_2 are generators of D_m . Let us write the corresponding infinitesemal stability conditions for D_m-equivariant singularities with corank at most two. In this case we consider the action

$$\nu : D_m \times \mathbb{R}^n \to \mathbb{R}^n, \ (g,(x_1,\dots,x_n)) \to (\mu(g)(x_1,x_2),x_3,\dots,x_n)$$

and the generating function

$$F(\xi_1,\xi_2,x_3,\dots,x_n) = \tilde{F}\circ\rho(\xi_1,\xi_2,x_3,\dots,x_n) ,$$

where the corresponding Hilbert map

$$\rho(x_1,\dots,x_n) = (z\bar{z}, z^m+\bar{z}^m, x_3,\dots,x_n), \ z = x_1+ix_2 .$$

Here $I = \{1,2\}$, $J = \{3,4,\dots,n\}$ and

$$i_{IJ}(\xi_I,x_J) = (\frac{\partial F}{\partial \xi_I}(\xi_I,x_J), x_J, \xi_I, -\frac{\partial F}{\partial x_J}(\xi_I,x_J)) .$$

We easily calculate

$$V_1(x,\xi) = \frac{1}{2}(\xi_I\bar{z}+\bar{\xi}_I z),\ V_2(x,\xi) = \frac{1}{2}(\xi_I z^{m-1}+\bar{\xi}_I\bar{z}^{m-1}) \quad ,$$

$$V_i(x,\xi) = \xi_i,\ i = 3,\dots,n$$

where we also denote

$$\xi_I = \xi_1 + i\xi_2 \ .$$

On the basis of (3.8) after straightforward calculations we obtain

$$\tilde{V}_1(u) = 2u_1\tilde{F}_{,1}(u) + m\, u_2\tilde{F}_{,2}(u),\ u = (u_1,u_2,\dots,u_n)$$

$$\tilde{V}_2(u) = 2^{m-2}\sum_{j=0}^{m-1}\binom{m-1}{j} m^j u_1{}^{m-j}\tilde{F}_{,1}(u)^{m-j-1}\tilde{F}_{,2}(u)^j w_{j-1}(u)$$

$$\tilde{V}_i(u) = -\tilde{F}_{,i}(u),\ 3 \le i \le n$$

where $w_{-1}(n) = u_2/u_1{}^m$ and the polynomials (of j-1-degree) $w_{j-1}(u) = (\bar{\xi}^m)^{j-1} + (\xi^m)^{j-1}$ are determined by the following recurrent formula:

$$(\bar{\xi}^m)^k + (\xi^m)^k = u_2^k - \sum_{i=1}^{[\frac{k-1}{2}]}\binom{k}{i}(u_1)^i((\bar{\xi}^m)^{k-2i} + (\xi^m)^{k-2i}) -$$

$$- \frac{1}{2}(1-(-1)^{k+1})\binom{k}{[\frac{k}{2}]}u_1{}^{[\frac{k}{2}]} \ .$$

Also for $\tilde{\tilde{U}}_i (i = 1,\dots,n)$ we obtain

$$\tilde{U}_1(u) = 4u_1\tilde{F}_{,1}^2(u)+4mu_2\tilde{F}_{,1}(u)\tilde{F}_{,2}(u)+4m^2u_1{}^{m-1}\tilde{F}_{,2}^2(u)$$

$$\tilde{U}_2(u) = 2^m \sum_{j=0}^{m} \binom{m}{j} m^j F_{,1}(u)^{m-j} F_{,2}(u)^j u_1^{m-j} w_{j-1}(u)$$

$$\tilde{U}_i(u) = u_i, \quad 3 \le i \le n .$$

Using the Malgrange preparation theorem, we find that (3.12) is equivalent to the following condition

$$\mathcal{E}(2) = \langle u_1\bar{F}_{,1}^2 + mu_2\bar{F}_{,1}\bar{F}_{,2} + m^2u_1^{m-1}\bar{F}_{,2}^2, \sum_{j=0}^{m}\binom{m}{j} m^j \bar{F}_{,1}^{m-j}\bar{F}_{,2}^{j} u_1^{m-j}\bar{w}_{j-1}\rangle_{\mathcal{E}(2)} +$$

$$+ \langle 2u_1\bar{F}_{,1} + mu_2\bar{F}_{,2}, \sum_{j=0}^{m-1}\binom{m-1}{j} m^j u_1^{m-j}\bar{F}_{,1}^{m-j-1}\bar{F}_{,2}^{j}\bar{w}_{j-1}, \bar{F}_{,3},\dots,\bar{F}_{,n}, 1\rangle_{\mathbb{R}} +$$

$$+ M_{\bar{\rho}}(2) \quad ,$$

where $\bar{F}_{,i}(u_1,u_2) = \frac{\partial \tilde{F}}{\partial u_k}(u_1,u_2,0)$, $\bar{w}_{j-1}(u_1,u_2) = w_{j-1}(u_1,u_2,0)$, and $M_{\bar{\rho}}(2)$ denotes the ideal of smooth function-germs vanishing on the set;

$$\{(u_1,u_2) : 4u_1^m - u_2^2 \ge 0, \; u_1 \ge 0\} .$$

That reduced formula for infinitesemal stability provides us the first step in indication of stable classes of ν-L-germs. The detailed analysis of this case we leave to the forthcoming paper. The classifying methods are the same as the ones presented in Section 4 for the $(\mathbb{Z}_2)^q$-action.

<u>Remark 3.6</u>. Let $\rho : \mathbb{R}^n \to \mathbb{R}^k$ be a Hilbert map for the ν-action of G on $\mathbb{R}^n$, so $\rho(\mathbb{R}^n) \subset \mathbb{R}^k$ is the semialgebraic set defined, say, by equations $f_1(u) = 0,\dots, f_r(u) = 0$ and inequalities

$h_1(u) \geq 0, \ldots, h_s(u) \geq 0$, where f_i, $h_j \in \mathbb{R}[u]$, $u \in \mathbb{R}^k$ are irreducible. Let us denote $M^*_\rho(k) = \langle f_1, \ldots, f_r \rangle_{E(k)}$ the ideal in $E(k)$ generated by $f_1, \ldots, f_r$. Obviously we have

$$(3.13) \qquad M^*_\rho(k) \subset M_\rho(k) .$$

However the equality in (3.13) usually does not hold, so we can not replace $M_\rho(k)$ by $M^*_\rho(k)$ in the condition (3.12). Nevertheless, by Nakayama's Lemma (cf.[6]), we can do it if

$$(3.14) \qquad M_\rho(k) - M^*_\rho(k) \subset \mathfrak{m}^\infty(k) .$$

Let us assume that (3.14) is fulfilled.

<u>Definition 3.7</u>. The equality

$$(3.15) \qquad E(a) = \langle \tilde{U}_1, \ldots, \tilde{U}_a \rangle_{E(a)} + \langle \tilde{V}_1, \ldots, \tilde{V}_b, 1 \rangle_{\mathbb{R}} + M^*_\rho(a)$$

is called the reduced condition for infinitesemal ν-L-stability.

<u>Remark 3.8</u>. Let us notice that the dependence of $\tilde{V}_i, \tilde{U}_i$ on $\frac{\partial \tilde{F}}{\partial u_j}$, in general, is not linear. In what follows we propose an equivalent approach to the classification problem of stable ν-L-germs using the Morse family notion. In that approach we derive the corresponding linear infinitesemal ν-L-stability condition. An equivalence of these two conditions results from the equivariant version of the Malgrange preparation theorem (cf.[13]).

Example 3.9. Assume the representation ν of $G = \mathbb{Z}_2$ on $\mathbb{R}^n$ has the form

$$\mathbb{R}^n \ni (x_1,\dots,x_n) \to (\varepsilon x_1, x_2,\dots,x_n),\ \varepsilon \in G.$$

Let ν-L-germ $(L^G,0) \subset T^*\mathbb{R}^n$ has a ν-IJ-germ $S(\xi_1,x_2,\dots,x_n) = \tilde{S}\circ\rho(\xi_1,x_2,\dots,x_n)$, where $\rho : \mathbb{R}^n \to \mathbb{R}^n$, $\rho(\xi_1,x_2,\dots,x_n) = (\xi_1^2,x_2,\dots,x_n)$. In this case $M_\rho(n) \subset \mathfrak{M}^\infty(n)$, $M_\rho^*(n) = \{0\}$,

$$\tilde{V}_1(u) = -2u_1\tilde{S}_{,1}(u)$$

$$\tilde{V}_i(u) = \tilde{S}_{,i}(u),\ 2 \leq i \leq n$$

$$\tilde{U}_1(u) = u_1\tilde{S}^2_{,1}(u)$$

$$\tilde{U}_j(u) = u_j,\ 2 \leq j \leq n .$$

Thus we see that (3.12) is equivalent to the following condition

$$\mathcal{E}(n) = \langle u_1\tilde{S}^2_{,1}(u), u_2,\dots,u_n\rangle_{\mathcal{E}(n)} + \langle u_1\tilde{S}_{,1}(u), \tilde{S}_{,2}(u),\dots,\tilde{S}_{,n}(u),1\rangle_{\mathbb{R}} . \tag{3.16}$$

Using the Malgrange preparation theorem we obtain the following, suitable for further calculation, equivalent form of (3.16),

$$\mathcal{E}(1) = \langle u_1\bar{S}^2_{,1}(u_1)\rangle_{\mathcal{E}(1)} + \langle u_1\bar{S}_{,1}(u_1), \bar{S}_{,2}(u_1),\dots,\bar{S}_{,n}(u_1),1\rangle_{\mathbb{R}}$$

where

$$\mathcal{E}(1) \ni \bar{S},_{1}(u_{1}) = \frac{\partial \tilde{S}}{\partial u_{i}}(u_{1},0)\,, \quad i = 1,\ldots,n\ .$$

§ 4. Stable ν-L-germs with respect to the $(\mathbb{Z}_2)^q$ action

Now for the purposes of applications (cf.[11], [8]) we consider the following action of $G = (\mathbb{Z}_2)^q$

$$\nu : (\mathbb{Z}_2)^q \times \mathbb{R}^n \ni (\varepsilon_1, \ldots, \varepsilon_q, x) \to (x_1, \ldots, x_{n-q}, \varepsilon_1 x_{n-q+1}, \ldots, \varepsilon_q x_u) \in \mathbb{R}^n .$$

The corresponding Hilbert map (orbitmapping) for ν is defined by

$$\rho(x) = (x_1, \ldots, x_{n-q}, x_{n-q+1}^2, \ldots, x_n^2) .$$

Any ν-L-germ $(L^G,0) \subseteq T^* \mathbb{R}^n$ is ν-L-equivalent to the ν-L-germ, say $(L_1^G,0) \subseteq T^* \mathbb{R}^n$, which has the following generating function (see § 2).

$$(4.1) \qquad E_\nu(n) \ni S(\xi) = \tilde{S} \circ \rho(\xi) ,$$

where $\tilde{S} \in E(n)$.

Let us denote the partial derivatives $\frac{\partial S}{\partial \xi_i}$, $\frac{\partial^2 S}{\partial \xi_i \partial \xi_j}$, etc. of function S by $S_{,i}$, $S_{,ij}$, etc. and their values at 0 by a_i, a_{ij}, etc. On the basis of Proposition 3.3 after straightforward calculations we obtain immediately

<u>Proposition 4.1</u>. The ν-L-germ $(L^G,0) \subset T^* \mathbb{R}^n$, generated by the function $S = \tilde{S} \circ \rho$ is infinitesemally ν-L-stable if for every germ $\alpha \in E(n)$ there exists decomposition

$$(4.2)\qquad \alpha(z)=\sum_{i=1}^{n-q}\tilde{S}_{,i}(z)h_i(z)+c_0+\sum_{i=1}^{n-q}c_iz_i+\sum_{j=n-q+1}^{n}(z_j\tilde{S}^2_{,j}(z)h_j(z)+z_j\tilde{S}_{,j}(z)c_j)$$

where $h_k \in E(n)$ and $c_1 \in \mathbb{R}$.

To be more concrete and useful in some physical applications (cf.[9], [11]), without loosing of generality we concentrate now on the case $q = 2,\ n = 3$. The general case can be treated exactly in the same way, so we omit it here.

Definition 4.2. The function germ $\tilde{S} \in E(n)$, introduced in Proposition 4.1 and such that (4.2) is fulfilled is called an infinitesemally ν-L-stable germ.

Proposition 4.3. A function-germ $\tilde{S} \in E(3)$ is infinitesemally ν-L-stable if and only if the following conditions are satisfied:

(A_0) $\quad a_1 \neq 0$ (trivial case) or $a_1 = 0$, and

(A_1) $\quad a_2a_3a_{11} \neq 0$ or

(A_2) $\quad a_{11} = 0$ and $a_2a_3a_{111} \neq 0$ or

(A_3') $\quad a_2 = 0$ and $a_3a_{11}a_{12}(a_{12}^2-a_{11}a_{22}) \neq 0$ or

(A_3'') $\quad a_3 = 0$ and $a_2a_{11}a_{13}(a_{13}^2-a_{11}a_{33}) \neq 0$.

Proof. (Necessity) The above conditions arise as necessary for decomposition (4.2) mod $\mathfrak{m}^3(3)$.

(Sufficiency) For $\alpha \in E(3)$ we show how to define germs h_i and constants c_i satisfying (4.2) in the respective cases:

(A_0) : It is enough to take $h_1 = \alpha/\tilde{S}_{,1}$, $h_2 = h_3 = 0$, $c_i = 0$ for $i = 0,1,2,3$.

(A_1) : Now $a_1 = 0$ and let $a_2a_3a_{11} \neq 0$. We define $\alpha_i, u_i \in E(3)$ as follows $\alpha(x) = \alpha(0) + \sum_{i=1}^{3} x_i\alpha_i(x)$, $\tilde{S}_{,1}(x) = \sum_{i=1}^{3} x_i u_i(x)$. Then $u_1(0) = a_{11} \neq 0$, $\tilde{S}_{,2}(0) = a_2 \neq 0$, $\tilde{S}_{,3}(0) = a_3 \neq 0$. Hence we can take $c_0 = \alpha(0)$, $c_1 = c_2 = c_3 = 0$, $h_1 = \alpha_1/u_1$, $h_2 = (\alpha_2 - u_2h_1)/\tilde{S}_{,2}^2$, $h_3 = (\alpha_3 - u_3h_1)/\tilde{S}_{,3}^2$, which satisfy (4.2).

(A_2) : Let $a_1 = 0$, $a_{11} = 0$, $a_2a_3a_{111} \neq 0$. We define the new germs $\alpha_i, u_i \in E(3)$ $(i = 1,2,3)$, $\tilde{S}_{,1}(x) = x_1^2u_1(x) + x_2u_2(x) + x_3u_3(x)$, $\alpha(x) = \alpha(0) + x_1\alpha_{,1}(0) + x_1^2\beta_1(x) + x_2\alpha_2(x) + x_3\alpha_3(x)$. In this case $u_1(0) = a_{111} \neq 0$, $\tilde{S}_{,2}(0) = a_2 \neq 0$, $\tilde{S}_{,3}(0) = a_3 \neq 0$ thus it suffices to put $c_0 = \alpha(0)$, $c_1 = \alpha_{,1}(0)$, $h_1 = \beta_1/u_1$, $h_2 = (\alpha_2 - u_2h_1)/\tilde{S}_{,2}^2$, $h_3 = (\alpha_3 - u_3h_1)/\tilde{S}_{,3}^2$.

(A_3'), (for A_3'' we have the same procedure) : Assume $a_1 = 0$, $a_2 = 0$ and $a_3a_{11}a_{12}(a_{12}^2 - a_{11}a_{22}) \neq 0$. We see that the germ

$$\beta(x) = \alpha(x) - c_0 - c_1x_1 - c_2\tilde{S}_{,2}(x)x_2 + \tilde{S}_{,1}(x)(g_0 + g_1x_1 + g_2x_2)$$

belongs to the ideal $\langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_3\rangle$ provided that $c_0 = \alpha(0)$ and the constants c_1, c_2, g_0, g_1, g_2 satisfy the following system of linear equations (solvable iff $a_{11}a_{12}(a_{12}^2 - a_{11}a_{22}) \neq 0$) :

$$\alpha_{,1}(0) = a_{11}g_0 + c_1$$

$$\alpha_{,2}(0) = a_{12}g_0$$

$$\alpha_{,11}(0) = a_{111}g_0 + 2a_{11}g_{11}$$

$$\alpha_{,12}(0) = a_{112}g_0 + a_{12}g_1 + a_{11}g_2 + a_{12}c_2$$

$$\tfrac{1}{2}\alpha_{,22}(0) = a_{112}g_0 + a_{12}g_2 + a_{22}c_2 \ .$$

Now consider germs u_{ij}, β_i satisfying the following decompositions

$$\tilde{S}_{,i}(x) = x_1 u_{i1}(x_1) + x_2 u_{i2}(x_1,x_2) + x_3 u_{i3}(x_1,x_2,x_3) \ ,$$

for $i = 1,2$ and

$$\beta(x) = x_1^3\beta_1(x)+x_1^2x_2\beta_2(x)+x_1x_2^2\beta_3(x)+x_2^3\beta_4(x)+x_3\beta_5(x) \ .$$

Let germs $k_1,k_2,k_3,h_2 \in E(3)$ be the solutions to the following system of linear equations

$$\beta_1 = u_{11}k_1$$

$$\beta_2 = u_{12}k_1 + u_{11}k_3 + u_{21}^2h_2$$

$$\beta_3 = u_{11}k_2 + u_{12}k_3 + 2u_{21}u_{22}h_2$$

$$\beta_4 = u_{12}k_2 + u_{22}^2h_2 \ .$$

The above system is solvable since the system determinant at 0 is equal to $a_{11}(a_{12}^2-a_{11}a_{22})^2 \neq 0$. One can easily check that the germ

$$\gamma(x):=\beta(x)-\tilde{S}_{,1}(x_1)(x_1^2k_1(x)+x_2^2k_2(x)+x_1x_2k_3(x))-x_2\tilde{S}_{,2}^2(x)h_2(x)$$

belong to the ideal $\langle x_3\rangle$ in $\mathcal{E}(3)$, i.e. γ has the form $\gamma(x) = x_3\gamma'(x)$, where $\gamma' \in \mathcal{E}(3)$. Finally we observe that $c_0,c_1,c_3,h_2(x)$ defined as above, $c_3 := 0$ and

$$h_1(x):=g_0+g_1x_1+g_2x_2+x_1^2k_1(x)+x_2^2k_2(x)+x_1x_2k_3(x) \quad ,$$

$$h_3(x):=\gamma'(x)/\tilde{S}_{,3}^2(x)$$

satisfy (4.2). This completes the proof of Proposition 4.3.

Now we consider the recognition problem for the stable ν-L-germs. Let $J^2(\mathbb{R}^3,\mathbb{R}) \cong \mathbb{R}^3 \times J_0^2(\mathbb{R}^3,\mathbb{R})$ be the space of 2-jets of $C^\infty(3)$-functions (cf.[6]) with a coordinate system $(x_i;y,y_i,y_{ij})$. Let M_1,M_2,M_3,M_4 be submanifolds of $J_0^3(\mathbb{R}^3,\mathbb{R})$ defined by the following conditions
(A_1) : $M_1 = \{y_2y_3\cdot y_{11} \neq 0\}$, (A_2) : $M_2 = \{y_{11} = 0,\ y_2y_3 \neq 0\}$,
(A_3') : $M_3 = \{y_2 = 0,\ y_3y_{11}y_{12}(y_{12}^2-y_{11}y_{22}) \neq 0\}$,
(A_3'') : $M_4 = \{y_3 = 0,\ y_2y_{11}y_{13}(y_{13}^2-y_{11}y_{33}) \neq 0\}$. Their co-dimensions in $J_0^2(\mathbb{R}^3,\mathbb{R})$ are 0,1,1 and 1 respectively. The subset of those 2-jets, say at $x = (x_1,0,0)$, $x_1 \in \mathbb{R}$, which do not belong to $\bigcup_i M_i$ has codimension 2, i.e. it is a finite union of submanifolds of $J_0^2(\mathbb{R}^3,\mathbb{R})$ of codimension 2. Given $F \in C^\infty(3)$, let $j^2F : \mathbb{R}^3 \to J_0^2(\mathbb{R}^3,\mathbb{R})$ denotes the 2-jet extension of F (see e.g.[24], [20]). Thus on the basis

of Thom's tranversality theorem [13], [24] we obtain immediately

<u>Proposition 4.4</u>. For the generic function $F \in C^\infty(3)$ its all germs $(j^2F)(x_1,0,0)$ belong to $\bigcup_{i=1}^{4} M_i$.

Let us denote E_i, $i=1,2,3,4$ the subsets of all germs $F \in \mathcal{E}(3)$ satisfying conditions $(A_1),(A_2),(A_3'),(A_3'')$ of Proposition 4.3 respectively; together with $F(0) = F,_1(0) = 0$. These germs generate the corresponding ν-L-germs $(\{-\frac{\partial(F\circ\rho)}{\partial\xi}(\bar{\xi}),\xi)\},0)$. Using the appropriate canonical transformations we easily obtain:

<u>Proposition 4.5</u>. Let $F \in C^\infty(3)$, $x_0 = (x_1,0,0)$. If

$$(j^2F)(x_0) \in M_i \;,\; i = 1,2,3,4$$

then the germ (F,x_0) is ν-L-equivalent to a germ belonging to E_i .

Let us recall that two ν-inv. germs of generating functions are ν-L-equivalent iff the corresponding ν-L-germs are ν-L-equivalent (see § 2).

Now we try to find classes of ν-L-equivalent germs in E_i . For this purpose we introduce

<u>Definition 4.6</u>. Let $F(x,t) = F_t(x)$ be a smooth function on $\mathbb{R}^3 \times J$, where J is an open interval in $\mathbb{R}$. F is called inf-homotopy (and germs $(F_a,0)$, $(F_b,0)$, $a,b \in J$ are called inf-homotopic) if all germs $(F_t,0)$ belong to the same class E_i (we assume $F(0,t) = \partial F/_{\partial t}(0,t) = 0$ for any inf-homotopy $F(x,t)$)

Proposition 4.7. Any germ belonging to E_i $(i=1,\dots,4)$ is inf. -homotopic to one from the following list

(E_1) $\quad F(x_1,x_2,x_3) = \pm x_1^2 \pm x_2 \pm x_3$

(E_2) $\quad F(x_1,x_2,x_3) = \pm x_1^3 \pm x_2 \pm x_3$

(E_3) $\quad F(x_1,x_2,x_3) = \pm x_1^2 \pm (x_1 = x_2)^2 \pm x_3$

(E_4) $\quad F(x_1,x_2,x_3) = \pm x_3^2 \pm (x_1 \pm x_3)^2 \pm x_2$

Let us remark that the generating functions $F\circ\rho$, for F belonging to the respective classes (E_i) , correspond to the classification proved by Arnold in [2]. Thus this coincidence justify our notation $(A_1,(A_2),(A_3'),(A_3'')$.

Proof of Proposition 4.7. We consider only the case (E_3) . The conditions $\operatorname{sgn} a_{11} = \pm 1$, $\operatorname{sgn} a_2 = \pm 1$, $\operatorname{sgn} a_3 = \pm 1$, $\operatorname{sgn}(a_{12}^2 - a_{11}a_{22}) = \pm 1$, distinguish in the 4-dimensional space of coefficients $(a_{11},a_{12},a_3,a_{22}) = (F,_{11},F,_{12},F,_3,F,_{22},)(0)$ sixteen open convex regions. So, if germs $F', F'' \in E_3$ correspond to the same region, the following function

$$F(x,t) = t\,F'(x) + (1-t)F''(x)$$

is an inf-homotopy between them. Observing that the above forms of E_3 correspond to every of these regions completes the proof.

Proposition 4.8. Let $F(x,t)$, $(x,t) \in \mathbb{R}^3 \times J$ be an inf-homotopy, $S(x,t) := F(\rho(x),t)$ and $t_0 \in J$ be a fixed point. Then there exists an open neighbouhood $U \times I$ of $(0,t_0)$ and the smooth functions $a_i(x,t)$, $b(x,t)$ on $\mathbb{R}^3 \times \mathbb{R}$, with compact supports, such that

(i) $$a_1(x,t) = \frac{\partial b}{\partial x_1}(0,t) \ , \text{ for } t \in I \ ,$$

and

(ii) $$-\frac{\partial S}{\partial t}(x,t) = H(x,\frac{\partial S}{\partial x}(x,t),t) \ , \text{ for } (x,t) \in U \times I \ ,$$

where

$$H(x,y,t)=a_1(\rho(y),t)x_1+\sum_2^3 a_i(\rho(y),t)x_i y_i+b(\rho(y),t)$$

for $(x,y,t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$.

<u>Proof</u>. Assume $t_0=0$. From the proof of Proposition 4.3 it results that for any germ $\alpha \in \mathcal{E}(4)$ there exists decomposition

$$\alpha(x,t)=F,_1(x,t)h_1(x,t)+c_1(t)x_1+c_0(t)+\sum_2^3(x_i F,_i^2(x,t)h_i(x,t)+$$
$$+x_i F,_i(x,t)c_i(t))$$

with $c_i \in \mathcal{E}(1)$, $h_i \in \mathcal{E}(4)$. Substituting $c_i(t)=c_i(0)+\bar{c}_i(t)$, for $i=0,1,2,3$ and

$$h(x,t) = \bar{c}_0(t)+\bar{c}_1(t)x_1+\sum_2^3 x_i F,_i(x,t)\bar{c}_i(t)$$

we obtain

$$\alpha(x,t)=F,_1(x,t)h_1(x,t)+c_1(0)x_1+c_0(0)+\textstyle\sum_2^3(x_iF,_i^2(x,t)h_i(x,t)+$$

$$+x_iF,_i(x,t)c_i(0))+th(x,t) \ .$$

On the basis of Malgrange preparation theorem [20] applied to the germ $g : (\mathbb{R}^4,0) \to (\mathbb{R}^4,0)$,

$$g(x,t)=(F,_1(x,t),4x_2F,_2^2(x,t),4x_3F,_3^2(x,t),t) \ , \text{ for}$$

$$(x,t) \in \mathbb{R}^3 \times \mathbb{R}$$

we obtain the following decomposition:

$$\text{(iii)} \quad -\frac{\partial F}{\partial t}(x,t)=x_1a_1\circ g(x,t)+\textstyle\sum_2^3 2x_iF,_i(x,t)a_i\circ g(x,t)+b\circ g(x,t) \ ,$$

with $a_i, b \in \mathcal{E}(4)$ (we can take the representatives of these germs with compact supports).

Now if we consider (iii) at $(\rho(x),t)$ and such that $g(\rho(x),t) = (\rho(\frac{\partial S}{\partial x}(x,t),t)$ we easily get (ii).

In order to show (i) we have to consider the respective cases: In the case E_2 we have $F,_1(0,t)=F,_{11}(0,t)=0\neq F,_{111}(0,t)$. So, taking $\partial/\partial x_1$ and $\partial^2/\partial x_1^2$ of (iii) at $(0,t)$ we obtain $0=a_1(0,t)$ and $0=b,_1(0,t)F,_{111}(0,t)$. Thus (i) results. In the case E_3 we have $F,_1(0,t)=F,_2(0,t)=0\neq F,_{12}(0,t)$. Taking $\partial/\partial x_2$ of (ii) at $(0,t)$ we have $0=b,_1(0,t)F,_{12}(0,t)$, so $b,_1(0,t)=0$. Now by differentiation of (iii) with respect to x_1 at $(0,t)$ we obtain $0=a_1(0,t)$.

For E_1-case we have $F,_1(0,t)=0\neq F,_{11}(0,t)$, so taking $\partial/\partial x_1$ of (iii) at $(0,t)$ we get

$$0 = a_1(0,t) + b,_1(0,t)F,_{11}(0,t) .$$

Hence, if $a_1(0,t) = 0$, then $b,_1(0,t) = 0$. Thus it is enough to show that decomposition (iii) with $a_1(0,t) = 0$ is always possible. In fact as the Jacobian $(\partial g)/\partial(x,t)\neq 0$ at $(x,t)=(0,0)$ there exists $X_1 \in E(4)$ such that $x_1 = X_1 \circ g(x,t)$. If we set $\bar{a}_1(z,t) := a_1(z,t)-a_1(0,t)$ and $\bar{b}(z,t) := b(z,t)+a_1(0,t)X_1(z,t)$, we can substitute $\bar{a}_1,\bar{b}$ into (iii) for the place of a_1 and b respectively. But $\bar{a}_1(0,t)=0$, which completes the proof of Proposition 4.8.

Let $F(x,t)$, $S(x,t)$, $H(x,y,t)=H_t(x,y)$ be as in Proposition 4.8. We assume $t_0=0$, $I=(-\varepsilon,\varepsilon)$ for simplicity. Let us consider the time dependent Hamiltonian vector field on $T^*\mathbb{R}^3$

$$X_{H_t} = \sum_1^3\left(\frac{\partial H}{\partial y_i}(x,y,t)\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}(x,y,t)\frac{\partial}{\partial y_i}\right)$$

as well the vector field $\tilde{X}_H = \frac{\partial}{\partial t} + X_{H_t}$ on $T^*\mathbb{R}^3 \times \mathbb{R}$. X_{H_t} has the global flow g_t, $t \in \mathbb{R}$ (i.e. there exists the smooth mapping $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \ni (x,y,t) \to g_t(x,y,t) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $\frac{d}{dt}g_t(x,y) = X_{H_t}(g_t(x,y))$ and $g_0(x,y) = (x,y)$, for $(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3)$. This results from: (i) compactness of supports of a_i and b , (ii) the independence of "y"-component of X_{H_t} on x (so $y(t)$ can be found independently on x) , (iii) linearity of "x"-component of X_{H_t} with respect to x .

Lemma 4.9. g_t is a ν-L-equivalence for every $t \in \mathbb{R}$.

Proof. Since $\omega_H = \omega + dH \wedge dt$ is the invariant form of $\tilde{X}_H$ (see [1], then g_t is a symplectomorphism for every t . Take $\sigma \in G$. As $H_t(x,y) = H_t \circ \sigma(x,y)$, $\sigma \circ X_{H_t} = X_{H_t} \circ \sigma$ and $\frac{d}{dt}(\sigma \circ g_t - g_t \circ \sigma) = \sigma \circ X_{H_t} - X_{H_t} \circ \sigma = 0$, for every $t \in \mathbb{R}$. Hence $\sigma \circ g_t = g_t \circ \sigma$ holds for every $t \in \mathbb{R}$ since $g_0 = id_{T^* \mathbb{R}^3}$. Finally g_t preserves the fibration π (see § 2) because the "y"-component of X_{H_t} is independent of x . Thus the proof is completed.

Let us define the mapping $\Phi : \mathbb{R}^3 \times (-\varepsilon,\varepsilon) \to T^* \mathbb{R}^3$ as $\Phi(x,t) = \Phi_t(x) = (x, \frac{\partial S}{\partial x}(x,t))$ and let the ν-L-germ $\Phi(\mathbb{R}^3 \times \{t\}) = \{(x, \frac{\partial S}{\partial x}(x,t))\}$ be denoted by L_t^G .

Lemma 4.10. The global flow g_t forms the ν-L-equivalence of the ν-L-germs $(L_0^G, 0)$ and $(L_t^G, 0)$ for $|t| < \varepsilon$.

Proof. First we show that $g_t(L_0^G) = L_t^G$. By straightforward calculations it can be checked that the vector field

$$A(x,t) := \frac{d}{dt}\Phi_t(x) - X_{H_t}(\Phi_t(x)) = \sum_{i,j} \frac{\partial H_t}{\partial y_i}(\Phi_t(x)) \left(\frac{\partial}{\partial x_i} + \frac{\partial^2 S_t}{\partial x_i \partial x_j}(x) \frac{\partial}{\partial y_j}\right)$$

is tangent to L_t^G at the point $\Phi_t(x)$ for every $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$. Let $B(x,t)$ be a smooth vector field on $\mathbb{R}^3 \times \mathbb{R}$ and $\varepsilon' \in \mathbb{R}$, $0 < \varepsilon' < \varepsilon$, be such that

$$\Phi_*(B,(x,t)) = A(x,t) \text{ , for } (x,t) \in \mathbb{R}^3 \times (-\varepsilon',\varepsilon') \text{ ,}$$

where Φ_* denotes the corresponding tangent map (cf. [1]). Denote by h_s the flow of $-B(x,t) + \frac{\partial}{\partial t}$ on $\mathbb{R}^3 \times \mathbb{R}$ (assumed to be defined globally, for simplicity). Then

$$h_s(\mathbb{R}^3 \times \{t\}) = \mathbb{R}^3 \times \{t+s\} \ , \text{ for } \ s,t \in \mathbb{R} \ .$$

Let us define $k_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in \mathbb{R}$, by the formula

$$k_t(x) = \Phi(h_t(x,0)) \ .$$

It is easily seen that $\frac{d}{dt}k_t(x) = X_{H_t}(k_t(x))$ and $k_0(\mathbb{R}^3) = L_0^G$. Hence, by the uniqueness theorem for the first order differential equations we obtain $k_t = g_t$ and $g_t(L_0^G) = k_t(\mathbb{R}^3) = L_t^G$, for $|t|<\varepsilon'$. To complete the proof it suffices to notice that $g_t(0,0) = (0,0)$ since $X_{H_t}(0,0) = 0$, by (i) and (ii) of Proposition 4.8 and $g_0(0,0) = (0,0)$ which completes the proof.

By the above two lemmas we obtain immediately

<u>Proposition 4.11</u>. Any two inf-homotopic germs belonging to $\mathcal{E}(3)$ are ν-L-equivalent.

It is easily verified that for any $F \in C^\infty(3)$ the mapping $j^2F : \mathbb{R}^3 \to J_0^2(\mathbb{R}^3, \mathbb{R})$ is transversal to M_i , (i=1,2,3,4) . Hence if $j^2F(x) \in M_i$, for every function $F_0 \in C^\infty(3)$ sufficiently close to F there exists point $x_0 \in \mathbb{R}^3$ close to x such that $j^2F_0(x_0) \in M_i$. Hence (F,x) and (F_0,x_0) are ν-L-equivalent to two inf-homotopic germs from E_i , so they are ν-L-equivalent. Thus we obtain

<u>Proposition 4.12</u>. Let $F \in C^{\infty}(3)$. Any germ (F,x) where $x = (x_1,0,0)$ and $j^2F(x) \in \cup M_i$ is ν-L-stable germ.

Now we can formulate the classification theorem for the normal forms of ν-L-stable germs of generating functions.

<u>Proposition 4.13</u>. Any ν-L-stable germ (F,x_0) , where $F \in C^{\infty}(3)$ and $x_0 = (x_{01},0,0)$, is ν-L-equivalent to the germ at $0 \in \mathbb{R}^3$ of one of the following normal forms:

(A_1) $\qquad F(x_1,x_2,x_3) = x_1^2 + x_2 + x_3$

(A_2) $\qquad F(x_1,x_2,x_3) = x_1^3 + x_2 + x_3$

(A_3) $\qquad F(x_1,x_2,x_3) = \pm x_2^2 \pm (x_2 + x_1)^2 + x_3$

<u>Proof</u>. By Propositions 4.4, 4.5, 4.7, 4.11, 4.12, it is enough to construct the ν-L-equivalences which reduces the normal forms of Proposition 4.7 to the normal forms listed above. But it is easy achieved by the ν-L-equivalences of the form $(x,y) \to (\alpha_i x_i + \beta_i y_i,\ y_i)$ for appropriate $\alpha_i, \beta_i \in \{-1,0,1\}$. Which completes the proof of Proposition 4.13.

§ 5. Stability conditions for G-invariant Morse families

Now using the Morse families local formalism (cf. [19] we derive the corresponding linear infinitesemal stability conditions for ν-L-germs. Consider a smooth family $(L_t^G,0)$, $|t|<\varepsilon$ of ν-L-germs with the corresponding smooth family F_t, $|t|<\varepsilon$ of G-Mf-germs. For simplicity we denote F_0, $(L_0^G,0)$ by F, $(L^G,0)$ resp. and assume that all Morse families of the family F_t are minimal (see § 2). Let $(L^G,0)$ be the stable ν-L-germ. Thus for sufficiently small ε_1, by Proposition 2.4, $F_t(|t|<\varepsilon_1)$ is locally trivial, i.e.

$$(5.1) \qquad F_t(x,\lambda) = F(\varphi_t(x),\ \Lambda_t(x,\lambda)) + f_t(x) \ ,$$

where $\Lambda_t \in \mathcal{E}(n+1,\ \nu\oplus\sigma;\ 1,\sigma)$, $\varphi_t \in \mathcal{E}(n,\nu;n,\nu)$, $f_t \in \mathcal{E}_\nu(n)$ and $(\varphi_t,\Lambda_t) \in \mathcal{E}(n+1,\ \nu\oplus\sigma;\ n+1,\ \nu\oplus\sigma)$ is the local family of diffeomorphisms.

By $\mathcal{M}$ we denote the space of minimal G-Mf-germs

$$\mathcal{M} = \{F \in \mathcal{E}_{\nu\oplus\sigma}(n+1);\ (\partial^2 F/\partial\lambda_i\partial\lambda_j)(0) = 0\} \ .$$

On the basis of (5.1) and theorems of Section 2 we have

<u>Proposition 5.1</u>. Let $(L^G,0)$ be the stable ν-L-germ. Then the necessary condition for the restricted local G-L-stability of the corresponding G-Mf-germ, F, is following:

$$(5.2)\qquad \mathcal{M} \subset \left(\frac{\partial F}{\partial \lambda} \mid \mathfrak{m}(n+1)\mathcal{E}(n+1,\ \nu\oplus\sigma;\ 1,\sigma)\right) + \left(\frac{\partial F}{\partial x} \mid \pi_n^{*}\mathcal{E}(n,\nu;\ n,\nu)\right) + \pi_n^{*}\mathcal{E}_{\nu}(n)\ ,$$

where the first and second terms are submodules of $\mathcal{E}_{\nu\oplus\sigma}(n+1)$ defined by the standard scalar products $(.|.)$ on $\mathbb{R}^1$ and $\mathbb{R}^n$ respectively, $\pi_n : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$.

Let $\mu' : \mathbb{R}^{n+1} \times \mathbb{R}^1 \to \mathbb{R}^b$, $\rho' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^a$ be the Hilbert maps for $\nu \oplus \sigma \oplus \sigma$ and $\nu \oplus \nu$ respectively. Let us denote

$$\varphi_i(x,\lambda) = \frac{\partial \mu'_i}{\partial \lambda'}(x,\lambda,0)\ ,\ (i=1,\ldots,b)$$

$$\psi_j(x) = \frac{\partial \rho'_j}{\partial x'}(x,0)\ ,\ (j=1,\ldots,a)\ .$$

Thus on the basis of Proposition 2.1 and condition (5.2) we have immediately

<u>Corollary 5.2</u>. In terms of the generators of the modules $\mathcal{E}(n+1,\ \nu\oplus\sigma;\ 1,\sigma)$, $\mathcal{E}(n,\nu;\ n,\nu)$, the condition (5.2) of Proposition 5.1, can be rewritten in the following form:

$$(5.3)\qquad \mathcal{M} \subset \left\langle \left(\frac{\partial F}{\partial \lambda} \mid \varphi_1\right),\ldots,\left(\frac{\partial F}{\partial \lambda} \mid \varphi_b\right)\right\rangle_{\mathcal{E}_{\nu\oplus\sigma}(n+1)} + \left\langle \left(\frac{\partial F}{\partial x} \mid \pi_n^{*}\psi_1\right),\ldots,\left(\frac{\partial F}{\partial x} \mid \pi_n^{*}\psi_a\right),1\right\rangle_{\pi_n^{*}\mathcal{E}_{\nu}(n)}\ .$$

When a physical system with symmetry exhibits the structural phase transitions then the notion of "order parameter" is well established (cf. [12]) and its dimensionality is rather stable feature of the system. This is a reason for the restricted stability condition introduced in Proposition 5.1. However, from the point a view of the standard singularity theory of lagrangian submanifolds [7] the corresponding deformation space is $\mathcal{E}_{\nu\oplus\sigma}(n+1)$. Thus, at first, we consider the stronger condition of infinitesemal G-L-stability

$$\mathcal{E}_{\nu\oplus\sigma}(n+1) = <(\tfrac{\partial F}{\partial\lambda}|\varphi_1),\dots,(\tfrac{\partial F}{\partial\lambda}|\varphi_b)>_{\mathcal{E}_{\nu\oplus\sigma}(n+1)} + <(\tfrac{\partial F}{\partial x}|\pi_n^*\psi_1),\dots \tag{5.4}$$

$$\dots,(\tfrac{\partial F}{\partial x}|\pi_n^*\psi_a),1>_{\pi_n^*\mathcal{E}_\nu(n)} .$$

This condition immediately follows from the ν-stability of the corresponding ν-L-germ $(L_0^G,0)$, introduced in Section 2. Let $\bar{\mu} : \mathbb{R}^{n+1} \to \mathbb{R}^k$ and $\bar{\rho} : \mathbb{R}^n \to \mathbb{R}^r$ be the Hilbert mappings for $\nu\oplus\sigma$ and ν actions respectively. For further use we define the new Hilbert map for the $\nu\oplus\sigma$-action,

$$\mu = (\bar{\mu},\bar{\rho}\circ\pi_n) : \mathbb{R}^{n+1} \to \mathbb{R}^k \times \mathbb{R}^r .$$

As we know the germs $(\frac{\partial F}{\partial\lambda}|\varphi_i)$, $(\frac{\partial F}{\partial x}|\psi_j)$, $1 \leq i \leq b$, $1 \leq j \leq a$ are $\nu\oplus\sigma$-invariant, thus we can have their smooth preimages by the Schwarz [15] homomorphism:

$$\tilde{H}_i \circ \mu = (\tfrac{\partial F}{\partial\lambda}|\varphi_i), \ 1 \leq i \leq b \tag{5.5}$$

$$\tilde{E}_j \circ \mu = (\tfrac{\partial F}{\partial x}|\psi_j), \ 1 \leq j \leq a ,$$

where $\tilde{H}_i, \tilde{E}_j \in E(k+r)$.

Proposition 5.3. Let $(L^G,0) \subset (T^* \mathbb{R}^n, \tilde{\omega})$ be the stable ν-L-germ. Then the necessary infinitesemal G-L-stability condition for the corresponding G-Mf-germ F can be written in the following form

$$(5.6) \quad E(k) = \langle H_1,\dots,H_b\rangle_{E(k)} + \langle E_1,\dots,E_a,1\rangle_{\mathbb{R}} + M_\mu(k+r|k)$$

where $H_i = \tilde{H}_i|_{\mathbb{R}^k\times\{0\}}$, $(i=1,\dots,b)$, $E_j = \tilde{E}_j|_{\mathbb{R}^k\times\{0\}}$, $(j=1,\dots,a)$ and $M_\mu(k+r|k)$ is the restriction of $M_\mu(k+r)$ to $\mathbb{R}^k\times\{0\}$.

Proof. Inserting the expressions (5.5) to the condition (5.4) and taking the surjective homomorphism $\mu^* : E(k+r) \to E_{\nu\oplus\sigma}(n+1)$ we obtain the equivalent condition:

$$\mu^*E(k+r) = \langle\mu^*\tilde{H}_1,\dots,\mu^*\tilde{H}_b\rangle_{\mu^*E(k+r)} + \langle\mu^*\tilde{E},\dots,\mu^*\tilde{E}_a,1\rangle_{\mu^*\pi_r^*E(r)} ,$$

where $\pi_r : \mathbb{R}^{k+r} \to \mathbb{R}^r$, $(z,y) \to (y)$ is the canonical projection. Thus we can take (5.4) in the following equivalent from

$$(5.7) \quad E(k+r) = \langle\tilde{H}_1,\dots,\tilde{H}_b\rangle_{E(k+r)} + \langle\tilde{E}_1,\dots,\tilde{E}_a,1\rangle_{\pi_r^*E(r)} + M_\mu(k+r) ,$$

where $M_\mu(k+r)$ is defined in § 3. Let A be the finite generated $E(k+r)$-module,

$$A = E(k+r)/\langle\tilde{H}_1,\dots,\tilde{H}_b\rangle E(k+r) + M_\mu(k+r) .$$

On the basis of (5.7) we have

$$\text{(ii)} \qquad A/_{\pi_r^*(\mathfrak{m}(r))A} = [\tilde{E}_1, \ldots, \tilde{E}_e, 1]_{\mathbb{R}} .$$

Thus applying the Malgrange preparation theorem we see that the condition (5.7) is equivalent to (5.6). This completes the proof of Proposition 5.3.

Let us notice that the functions H_i, E_j depend linearly on F, which give some advantage of the Morse family approach comparing to the generating functions method presented in the preceding sections. These two approaches are equivalent, however the direct method of description of lagrangian singularities by generating functions is convenient from the point of view of physical applications where the generating functions, usually, have a physical meaning of the equilibrum potentials (see Appendix).

Similarly as in Section 3, the condition

$$\text{(5.8)} \qquad \mathcal{E}(k) = \langle H_1, \ldots, H_b \rangle_{\mathcal{E}(k)} + \langle E_1, \ldots, E_a, 1 \rangle_{\mathbb{R}} + M_\mu^*(k+r|k)$$

will be called a linear condition of infinitesemal G-L-stability. If we assume that $M_\mu(k+r|k) - M_\mu^*(k+r|k) \subset \mathfrak{m}^\infty(k)$ then by the Nakayama's Lemma [20] we obtain equivalence of the two conditions (5.8) and (5.6).

Example 5.4. Assume that $\nu : G \to O(n)$ is trivial. Let $(\xi_I, x_J) \to S(\xi_I, x_J)$ be a IJ-germ for $(L,0) \subset T^* \mathbb{R}^n$ and the corresponding Morse family $F \in \mathcal{E}(n+k)$ be given by (2.4), where $k = \#I$. In this case we can put $\mu = \mathrm{id}_{\mathbb{R}^{n+k}}$. We also find easily that (5.7) takes the form

$$\mathcal{E}(n+k) = \langle (\frac{\partial S}{\partial \lambda_1}(\lambda_I, x_J) - x_1), \ldots, (\frac{\partial S}{\partial \lambda_k}(\lambda_I, x_J) - x_K) \rangle_{\mathcal{E}(n+k)} +$$

$$+ \langle \frac{\partial S}{\partial x_{J_1}}(\lambda_I, x_J), \ldots, \frac{\partial S}{\partial x_{J_{n-k}}}(\lambda_I, x_J), \lambda_1, \ldots, \lambda_k, 1 \rangle_{\mathcal{E}(n)} .$$

And equivalently, (5.6) we can write in the following form

$$\mathcal{E}(\xi_I, x_J) = \langle \frac{\partial S}{\partial \xi_I}(\xi_I, x_J), x_J \rangle_{\mathcal{E}(\xi_I, x_J)} + \langle \frac{\partial S}{\partial x_J}(\xi_I, x_J), \xi_I, 1 \rangle_{\mathbb{R}} .$$

We can write for (5.6) even more reduced form:

$$\mathcal{E}(\xi_I) = \langle \frac{\partial S}{\partial \xi_I}(\xi_I, 0) \rangle_{\mathcal{E}(\xi_I)} + \langle \frac{\partial S}{\partial x_J}(\xi_I, 0), \xi_I, 1 \rangle_{\mathbb{R}}$$

which is exactly the standard condition for versality (infinitesemal stability) of unfoldings of singularity $\eta = S|_{\mathbb{R}^I \times \{0\}}$ (cf. [2], [20]).

Example 5.5. Let us take $G = \mathbb{Z}_2$ and its action on $\mathbb{R}^n$ is defined as follows:

$$\nu_\varepsilon(x_1, x_2, \ldots, x_n) = (\varepsilon x_1, x_2, \ldots, x_n), \quad \varepsilon \in \mathbb{Z}_2 = \{\pm 1\}, \quad x \in \mathbb{R}^n .$$

Let a ν-L-germ $(L^G, 0)$ has the following ν-IJ-germ of generating function

$$S(\xi_1, x_2, \ldots, x_n) = \tilde{S} \circ \rho(\xi_1, x_2, \ldots, x_n) ,$$

where $\rho : \mathbb{R}^n \to \mathbb{R}^n$, $(\xi_1, x_2, \ldots, x_n) \to (\xi_1^2, x_2, \ldots, x_n)$. The corresponding Morse family:

$$(5.9) \qquad F(x,\lambda) = S(\lambda, x, \ldots, x_n) - \lambda x_1$$

and the corresponding representation σ has the form

$$\sigma_\varepsilon(\lambda) = \varepsilon\lambda \ .$$

Define a Hilbert map $\bar{\mu} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ for $\nu \oplus \sigma$ as

$$\bar{\mu}(x,\lambda) = (\lambda^2, \lambda x_1, x_1^2, x_2, \ldots, x_n) \ .$$

We find that $M_{\bar{\mu}}(n+2)$ is the set of smooth function-germs vanishing on the set

$$(5.10) \qquad \bar{\mu}(\mathbb{R}^{n+1}) = \{(y_1, \ldots, y_{n+2});\ y_2^2 - y_1 y_3 = 0\}$$

and also we have

$$M^*_{\bar{\mu}}(n+2|2) = y_2^2 E(y_1, y_2) \ .$$

After straightforward calculations we obtain

$$\tilde{H}_1(y) = 2y_1 \tilde{S},_1(y_1, y_4, \ldots, y_{n+2}) - y_2$$

$$\tilde{H}_2(y) = 2y_2 \tilde{S},_1(y_1, y_4, \ldots, y_{n+2}) - y_3$$

$$\tilde{E}_1(y) = -y_2, \tilde{E}_i(y) = \tilde{S},_i(y'), i = 2, \ldots, n \ .$$

Substituting them to (5.7) we get the condition

$$\mathcal{E}(y) = \langle 2y_1\tilde{S},_1(y')-y_2, 2y_2\tilde{S},_1(y')-y_3\rangle_{\mathcal{E}(y)} + \\ + \langle y_2, \tilde{S},_2(y'),\dots,\tilde{S},_n(y')\rangle_{\mathcal{E}(y'')} + \mathcal{M}_\mu(n+2)$$

where $y'=(y_1,y_4,\dots,y_{n+2})$, $y''=(y_3,\dots,y_{n+2})$ and as k,r in Proposition 5.3 we put $k=2$, $r=n$.

Thus the infinitesemal ν-L-stability condition for the ν-L-germ $(L^G,0)$ has the form

$$\mathcal{E}(y_1,y_2) = \langle 2y_1\bar{S},_1(y_1)-y_2, y_2\bar{S},_1(y_1), y_2^2\rangle_{\mathcal{E}(y_1,y_2)} + \\ + \langle y_2, \bar{S},_2(y_1),\dots,\bar{S},_n(y_1), 1\rangle_{\mathbb{R}}, \tag{5.11}$$

where $\bar{S},_\alpha(y_1) := \tilde{S},_\alpha(y_1,0,\dots,0)$.

From the decomposition

$$\mathcal{E}(y_1,y_2) = \mathcal{E}(y_1)+y_2\mathcal{E}(y_1)+y_2^2\mathcal{E}(y_1,y_2),$$

on the basis of (5.11) we obtain:

$$\mathcal{E}(y_1)+y_2\mathcal{E}(y_1) = (2y_1\bar{S},_1(y_1)-y_2)\mathcal{E}(y_1)+y_2\bar{S},_1(y_1)\mathcal{E}(y_1) + \\ + \langle y_2, \bar{S},_2(y_1),\dots,\bar{S},n(y_1), 1\rangle_{\mathbb{R}}.$$

In other words, for every $a(y_1), b(y_1) \in \mathcal{E}(y_1)$ exist $h_1,h_2 \in \mathcal{E}(y_1)$ and constants $c_0,\dots,c_n \in \mathbb{R}$ such that

$$(5.12)\quad \begin{cases} a(y_1)=2y_1\bar{S},_1(y_1)h_1(y_1)+\bar{S},_2(y_1)c_2+\ldots+\bar{S},_n(y_1)c_n+c_0 \\ \\ b(y_1)=-h_1(y_1)+\bar{S},_1(y_1)h_2(y_1)+c_1 \ . \end{cases}$$

Eliminating h_1 in these equations we get an equivalent condition:

$$(5.13)\quad a(y_1)+2y_1\bar{S},_1(y_1)b(y_1)=y_1\bar{S}^2,_1(y_1)h_2(y_1)+y_1\bar{S},_1(y_1)c_1+,\ldots,+$$
$$+\ \bar{S},_n(y_1)c_n+c_0\ .$$

We easily see that (5.13) can be written in the form

$$(5.14)\quad E(y_1)=\langle y_1\bar{S}^2,_1(y_1)\rangle_{E(y_1)}+\langle y_1\bar{S},_1(y_1),\bar{S},_2(y_1),\ldots,\bar{S},_n(y_1),1\rangle_{\mathbb{R}}$$

which gives an another form for infinitesemal ν-L-stability of the ν-L-germ $(L^G,0)\subset T^*\mathbb{R}^n$.

Remark 5.6. We derived the condition (5.14) in Section 3 (see formula (3.16)), in a quite different way. In Example 5.5 we showed the equivalence of these two approaches to the classification problem of stable ν-L-germs of lagrangian submanifolds. It seems that the Morse family approach is very useful in explicit calculations because of the linearity of the corresponding infinitesemal stability conditions.

§ 6. Versality and stability of ν-L-germs

In the preceding sections we characterize the infinitesemal stability of ν-L-germs through the corresponding infinitesemal stability conditions for their G-Morse family germs. To have an adequate approach to local stability of ν-L-germs by the corresponding locally stable generating families we have to introduce the modified notion of G-unfolding (cf. [16]) and adapt this notion to be used in the standard Morse family approach (cf. [9]).

Let $\eta \in \mathcal{E}_\sigma(k)$ for some orthogonal representation σ of G. The pair (ν,f), where $\nu : G \to O_n(\mathbb{R})$ is a representation of G, and $f \in \mathcal{E}_{\nu\oplus\sigma}(n+k)$ such that $f|_{\{0\}\times\mathbb{R}^k} = \eta$ is called an n-parametric G-unfolding of η with respect to the representation ν. Let σ be fixed for all G-unfoldings of the germ η.

Let γ be an orthogonal representation of G in $\mathbb{R}^s$. A morphism of G-unfoldings $(\Phi,\alpha) : (\gamma,h) \to (\nu,f)$ of the germ η is defined by the following maps

i) $\Phi=(\varphi,\psi) \in \mathcal{E}(s+k,\gamma\oplus\sigma;k,\sigma) \oplus \mathcal{E}(s,\gamma;n,\nu)$

ii) $\alpha \in \mathcal{E}_\gamma(s)$

and the following condition

$$h = f \circ \Phi + \alpha \circ \pi_s$$

where $\pi_s : \mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R}^s$ is the canonical projection.

If ψ is a diffeomorphism, then (Φ,α) is called an isomorphism of G-unfoldings. We say that a G-unfolding (ν,f) of the germ η is G-versal if for any other G-unfolding (γ,h) of η there exists a morphism $(\Phi,\alpha) : (\gamma,h) \to (\nu,f)$. The G-versal unfolding of η is called G-miniversal if a dimension of a basis n of the unfolding is a possible smallest number (cf. [6]). We see that the above introduced isomorphism of G-unfoldings (Φ,α) defines the lagrangian equivalence of γ-L-germ $(L_1^G,0)$ generated by h and the ν-L-germ $(L_2^G,0)$ generated by f , i.e. there is the G-equivaraint symplectomorphism $R^G : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ preserving the fibre structure $\pi_{\mathbb{R}^n} : T^*\mathbb{R}^n \to \mathbb{R}^n$ and such that $R^G(L_1^G) = L_2^G$, $R^G(b) = 0$. We know (see [9], [6], p.269) that R^G can be locally written as follows

$$(6.1)\quad (x,\xi) \to (\psi(x),\ {}^tD\psi(x)^{-1}(\xi+d\alpha(x))) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$$

with $\psi \in \mathrm{Diff}(n,\gamma;n,\nu)$, $\alpha \in \mathcal{E}_\gamma(n)$.
The converse statement is also true, i.e. if the γ-L-germ $(L_1^G,(x_0,\xi_0)) \subset T^*\mathbb{R}^n$ is mapped into the ν-L-germ $(L_2^G,(\bar{x}_0,\bar{\xi}_0)) \subset T^*\mathbb{R}^n$ by a germ of G-equivariant symplectomorphism $(T^*\mathbb{R}^n,(x_0,\xi_0)) \to (T^*\mathbb{R}^n,(\bar{x}_0,\bar{\xi}_0))$ of the form (6.1) then the corresponding G-unfoldings, say h and f , generating $(L_1^G,(x_0,\xi_0))$ and $(L_2^G,(\bar{x}_0,\bar{\xi}_0))$ respectively are isomorphic as G-unfoldings (cf. [22]).

Let $S : (\mathbb{R}^n \times \mathbb{R}^k,0) \to \mathbb{R}$, $S \in \mathcal{E}_{\nu\oplus\sigma}(n+k)$ be a germ of a generating family for the ν-L-germ $(L^G,0) \subset T^*\mathbb{R}^n$

Definition 6.1. A G-invariant lagrangian submanifold $L^G \subset T^* \mathbb{R}^n$ is called G-versal at $0 \in L^G$ if a germ S of a generating family of $(L^G, 0)$ is G-versal unfolding if the germ $\eta = S|_{\{0\} \times \mathbb{R}^k} \in \mathcal{E}_\sigma(k)$.

Let us endow the space of G-unfoldings $C^\infty_{\nu\oplus\sigma}(n+k)$ and the space of G-equivariant lagrangian immersions $I(n,\rho;n+n,\nu\oplus\nu)$ with the induced C^∞-Whitney topology, then the G-versal ν-L-germ $(L^G,0)$ is locally stable, i.e. for every G-invariant neighbourhood V of 0 in $T^* \mathbb{R}^n$ there exists an open neighbourhood U of the G-equivariant lagrangian immersion $i_{L^G} : (\mathbb{R}^n, 0) \to (T^* \mathbb{R}^n, 0)$ in $I(n,\rho;n+n,\nu\oplus\nu)$ (where ρ is the linearised representation $\nu\oplus\nu|_{L^G}$) that for every $i \in U$ there exists $p \in \text{Image } i \subset V$ such that the ν-L-germs $(L^G,0)$ and $(\text{Image } i,\ p)$ are ν-L-equivalent (or $(i_{L^G},0)$, $(i, i^{-1}(p))$ are G-equivalent as immersions [7]). Thus the local ν-L-stability of ν-L-germs has an adjoint formulation in terms of the stable G-unfoldings of invariant singularities (cf. [24], [2]).

Let $\eta \in \mathfrak{m}^2_\sigma(k)$, by $J(\eta)$ we denote the Jacobi ideal of η generated by the partial derivatives $\partial\eta/\partial\lambda_1, \ldots, \partial\eta/\partial\lambda_k$. $J(\eta)$ is a G-submodule of the G-module $\mathcal{E}(k)$. Following [9] (see also [16]) we have the main result concerning the G-versal ν-L-germs.

Proposition 6.2. Let $\sigma : G \to O_k(\mathbb{R})$ be a fixed representation of G in $\mathbb{R}^k$, let (ν, S) be a G-unfolding of a germ $\eta = S|_{\{0\} \times \mathbb{R}^k}$ which generates the ν-L-germ $(L^G,0) \subset T^* \mathbb{R}^n$, we set $n = \dim_{\mathbb{R}} \mathfrak{m}(k)/_{J(\eta)} < \infty$. Let γ be the representation of G in the vector space $\mathfrak{m}(k)/_{J(\eta)} \cong \mathbb{R}^n$ and $r : \mathfrak{m}(k)/_{J(\eta)} \to$

$\to \mathfrak{m}(k)$ an equivariant splitting of the exact sequence of G-modules $0 \to J(\eta) \to \mathfrak{m}(k) \overset{r}{\underset{\longrightarrow}{\longleftarrow}} \mathfrak{m}(k)/_{J(\eta)} \to 0$ such that the function $f : \mathfrak{m}(k)/_{J(\eta)} \oplus \mathbb{R}^k \to \mathbb{R}$, $f(x,\gamma) = \eta(\lambda) + r(x)(\lambda)$ is a Morse family, then

i) f is a generating family for the G-versal γ-L-germ $(L_1^G,0)$.

ii) The ν-L-germ $(L^G,0)$ with the generating family (ν,S) is G-versal if and only if a morphism of G-unfoldings $(\Phi,\alpha) : (\nu,S) \to (\gamma,f)$ is an isomorphism.

Proof of this proposition can be found in [9] (p.187).

The main tool in proving Proposition 6.2 as well as to classify the corresponding normal forms for G-versal ν-L-germs is the infinitesemal versality notion (cf. [24], [16]).

Let $\eta \in E_\sigma(k)$ and $f \in E_{\nu\oplus\sigma}(n+k)$ be a G-unfolding of η. Thus $df \in E(n+k)\otimes(X\oplus\Lambda)^*$ (where we denote $\Lambda \cong \mathbb{R}^k$, $X \cong \mathbb{R}^n$) has the two components $d_1f \in E(n+k)\otimes\Lambda^*$, and $d_2f \in E(n+k)\otimes X^*$. Let us consider the second component and the sequence of homomorphisms (cf. [16])

$$E(n+k) \to E(n+k)\otimes X^* \to E(k)\otimes X^* \to E(k)/_{J(\eta)}\otimes X^*$$

(6.2)

$$f \quad \to \quad d_2f \quad \to d_2f|_\Lambda \to \overline{d_2f}|_\Lambda = \delta f .$$

We see that δf is G-invariant, i.e. $\delta f \in (E(k)/_{J(\eta)}\otimes X^*)^G$, δf is identified also to G-equivariant homomorphism $X \to E(k)/J(\eta)$. If the homomorphism δf is surjective we say that the G-unfolding (ν,f) is infinitesemally versal. It is proved in [16] that the two notions: infinitesemal versality and versality, are equivalent.

We can adapt the above notions to the symplectic objects and write down, for G-Mf-germs, the corresponding sequence

$$\mathfrak{m}(k)\mathcal{E}(n+k)+\mathfrak{m}^2(n) \to (\mathfrak{m}(k)\mathcal{E}(n+k)+\mathfrak{m}(n))\otimes(\mathbb{R}^n)^* \to \mathfrak{m}(k)\otimes(\mathbb{R}^n)^* \to$$

$$\to \mathfrak{m}(k)/J(\eta)\otimes(\mathbb{R}^n)^*$$

$$f \to \delta f \in \mathrm{Hom}_G(\mathbb{R}^n, \mathfrak{m}(k)/J(\eta)) \ .$$

<u>Definition 6.3</u>. Let (ν,f) be G-Mf-germ for the ν-L-germ $(L^G,0) \subset T^*\mathbb{R}^n$. We say that $(L^G,0)$ is an infinitesemally G-versal if the corresponding G-homomorphism δf is surjective.

<u>Proposition 6.4</u>. The ν-L-germ $(L^G,0)$ is G-versal if and only if $(L^G,0)$ is infinitesemally G-versal.

<u>Proof</u>. On the basis of Proposition 6.2 and Corollary 3.7 in [16] (cf. [9]).

Following the standard lines of lagrangian singularity theory (see [2], [9], [7], [18]) we can summarize the stability theory of invariant lagrangian submanifolds in the following

<u>Proposition 6.5</u>. Let $i_L G : (L^G,0) \to (T^*\mathbb{R}^n,0)$ be a germ of G-equivariant lagrangian immersion. Let $S : (\mathbb{R}^n \times \mathbb{R}^k,0) \to \mathbb{R}$ be a corresponding generating family for $(L^G,0)$. Then the following conditions are equivalent:

(i) $(i_L G, 0)$ is locally stable

(ii) $(i_L G, 0)$ is infinitesemally stable

(iii) $(S,0)$ is a versal G-unfolding of the germ $\eta = S|_{\{0\} \times \mathbb{R}^k}$

(iv) $(S,0)$ is an infinitesemally versal G-unfolding of the germ $\eta = S|_{\{0\} \times \mathbb{R}^k}$.

Proof. The equivalence of (i) and (ii) results immediately by equivariant local version of the Theorem 5.1.3 in [6]. By [22] Theorem 4 and the previous results we obtain an equivalence of conditions (i), (iii). The equivalence of the notion of infinitesemal stability for lagrangian G-immersions and infinitesemal versality for generating G-invariant Morse families follow from the corresponding equivariant reformulation of standard arguments in [24](see also [7]).

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References

[1] R.Abraham, J.E.Marsden, Foundations of Mechanics (2nd ed.), Benjamin (Cummings, Reading) 1978.

[2] V.I.Arnold, Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities, Functional Anal. Appl.6 (1972) 254-272.

[3] V.I.Arnold, S.M.Gusein-Zade, A.N.Varchenko, Singularities of Differentiable Maps, Vol.I, Birkhäuser 1985.

[4] E.Bierstone, Local properties of smooth maps equivariant with respect to finite group actions, J.Diff. Geometry 10 (1975), 523-540.

[5] S.Bochner, Compact groups of differentiable transformations, Ann.of Math.46 (1945), 372-381.

[6] Th.Bröcker, L.Lander, Differentiable Germs and Catastrophes, London Math.Soc.Lecture Notes 17, Cambridge Univ. Press 1975.

[7] J.Duistermaat, Oscilatory integrals, Lagrange immersions and unfoldings of singularities, Comm.Pure Appl. Math.27 (1974), 207-281.

[8] M.Golubitsky, D.Schaeffer, "Singularities and groups in bifurcation theory", Vol.1", Springer-Verlag, New York, 1984.

[9] S.Janeczko, On G-versal lagrangian submanifolds, Bull. of the Polish Academy of Sciences, Math. 31 (1983),183-190.

[10] ———————, Geometrical approach to phase transitions and singularities of lagrangian submanifolds, Demonstratio Math.16 (1983), 487-502.

[11] S.Janeczko, A.Kowalczyk, Equivariant singularities of lagrangian manifolds and uniaxial ferromagnet, to appear in SIAM Journ.of Appl.Anal.

[12] L.D.Landau, E.M.Lifshitz, Electrodynamics of continous media, Pergamon, 1963.

[13] V.Poènaru, Singularities C^∞ en présence de symétrie, Lecture Notes in Math.510, Springer 1976.

[14] T.Poston, I.Stewart, Catastrophe theory and its applications, Pitman 1978.

[15] G.Schwarz, Smooth functions invariant under the action of a compact Lie group, Topology 14, (1975), 63-68.

[16] P.Slodowy, Einige Bemerkungen zur Entfaltung symmetrischer Funktionen, Math.Z.158 (1978), 157-170.

[17] R.Thom, Symmetries gained and lost, in "Math.Physics and Physical Mathematics" ed.K.Maurin, R.Raczka, PWN-Polish Scientific Publishers, Warszawa 1976, 293-320.

[18] C.T.C.Wall, Geometric properties of generic differentiable manifolds, Lecture Notes in Math. 597, (Berlin: Springer 1976), 707-774.

[19] A.Weinstein, Lectures on symplectic manifolds, C.B.M.S. Conf.Series, A.M.S., 29 (1977).

[20] G.Wassermann, Stability of Unfoldings, Lecture Notes in Math.393, (berlin: Springer 1974).

[21] ————————, Classification of singularities with compact abelian symmetry, Regensburger Math.Schriften Nr.1 (1977).

[22] V.M.Zakalyukin, On lagrangian and legendrian singularities, Funct.Anal.Appl.10 (1976), 23-31.

[23] V.M.Zakalynkin, Bifurcations of wave-fronts depending on one parameter, Funct.Anal.Appl.10 (1976), 69-70.

[24] E.C.Zeeman, Catastrophe Theory, selected papers 1972-1977 Addison-Wesley 1977.