Group rings and division rings

by

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To the memory of Oberstudienrat Heinz-Joachim Dietrich

Abstract. Continuing the work in [11],[12] we study division algebras $D = k(G)$ over a field k which are generated by some polycyclic-by-finite subgroup G of the multiplicative group D^* of D. We discuss a specific class of examples of such division algebras that can be thought of as multiplicative analogs of the Weyl field. Furthermore, we show that the division algebras $D = k(G)$ always contain free subalgebras of rank ≥ 2, provided G is not abelian-by-finite. Finally, we discuss some open questions concerning commutative subfields and Lie commutators in $D = k(G)$.

INTRODUCTION

During the past decade, a considerable amount of work has been invested in the study of prime ideals in group algebras kG of polycyclic-by-finite groups G over a field k. After the pioneering work of Zalesskii [30], Roseblade's break-through in [22], and the finishing touches by Passman and the author [14], the subject has now reached a certain state of maturity: one has a detailed recipe for constructing all primes P of kG starting from prime ideals of group algebras kH, where H runs through special finite-by-abelian subquotients of G (see [13] or [14] for the precise formulation). The resulting class of algebras kG/P is a rich source of interesting examples of prime Noetherian rings whose fine structure is far from being well understood. For example, if $Q = Q(kG/P)$ denotes the classical ring of fractions of kG/P then, by Goldie's theorem, $Q = M_n(D)$ for some integer n and a suitable division k-algebra D both of which are in general quite mysterious to us. In the present note, continuing the work in [11], [12], we study the Goldie field D associated with a completely

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prime ideal P of kG. In other words, we study division k-algebras D generated by some polycyclic-by-finite group $G \leq D^*$, the multiplicative group of D. The restriction to completely prime ideals is partly justified by the following result, due to Zalesskii [30, Theorem 4] for primitive ideals and to Brown [2] in general:

Let P be a prime ideal of kG. Then there exists a characteristic subgroup G_o of G with G/G_o finite and such that $P \cap kG_o$ is a finite intersection of completely prime ideals of kG_o which are all conjugate under the action of G on kG_o.

In Section 1, we study a specific class of algebras B_λ ($\lambda \in k^*$) and their classical division rings of fractions E_λ. Each B_λ, and E_λ, is generated by a 2-generated nilpotent group of class 2 and can be viewed as a multiplicative analog of the Weyl algebra A_1, resp. the Weyl field $D_1 = Q(A_1)$ (see also [10]). Although the B_λ's are not isomorphic to A_1, and to each other, they share many of the well-known properties of A_1. The main result of Section 2 states that if $D = k(G)$ is a division algebra generated by some polycyclic-by-finite group $G \leq D^*$ and if G is not abelian-by-finite, then D contains a free k-subalgebra of rank at least 2. This result depends on recent work of L. Makar-Limanov [15]. Finally, in the last section, we briefly discuss commutative subfields and Lie commutators in division algebras $D = k(G)$ of the above type and mention a number of open questions.

<u>Notations and Conventions.</u> In this paper, k always denotes a commutative field. If D is a division k-algebra, then we use the notation $D = k(E)$ to indicate that D is generated, as division k-algebra, by the subset E of D. Furthermore, for any ring R (always with 1), R^* denotes the set of nonzero elements, $U(R)$ the group of units, $Z(R)$ the center, and $Q(R)$ the classical ring of fractions of R (if it exists). We use square brackets to denote group theoretical and Lie commutators. Thus $[x,y] = x^{-1}y^{-1}xy$ or $[x,y] = xy-yx$, depending on the context. Otherwise the notation is standard, and follows [18].

1. A CLASS OF EXAMPLES

Let $\lambda \in k^*$ be given and define $B_\lambda = B_\lambda(k)$ to be the k-algebra generated by two elements x and y together with their inverses x^{-1}, y^{-1} subject to the relation $xy = \lambda yx$. In short,

$$B_\lambda = k\{x^{\pm 1}, y^{\pm 1}\}/(xy-\lambda yx) \qquad (\lambda \in k^*).$$

Some results concerning these algebras and their tensor products have been announced in [10], even for k not necessarily a field. As far as I know, the B_λ's, or rather their power series analog, have made their first appearance in [7], with k being the field of rational numbers and $\lambda = 2$.

The algebra B_λ can be realized as the factor of the group algebra kG, with $G = \langle x,y \mid z=[x,y] \text{ central}\rangle$ the free nilpotent group of class 2 on 2 generators, modulo the ideal $(z-\lambda)kG$. B_λ can also be viewed as a twisted group algebra of the free abelian group of rank 2,

$$B_\lambda \cong k^t[Z \oplus Z],$$

or as an iterated Ore extension,

$$B_\lambda = k[X^{\pm 1}][Y^{\pm 1};\alpha] \quad \text{with} \quad X^\alpha = \lambda X.$$

In particular, B_λ is a Noetherian domain. We denote its classical division ring of fractions by $E_\lambda = E_\lambda(k)$. Note that $E_\lambda = k(\langle x,y\rangle)$, where $\langle x,y\rangle$ is nilpotent of class 2 (if $\lambda \neq 1$).

The following lemma describes some basic properties of the B_λ's. Some of them have also been noted in [10]. We are mostly interested in the case where λ is not a root of unity. In this case the properties of B_λ closely mirror those of the Weyl algebra A_1 in characteristic 0, whereas the case where λ is a root of unity corresponds to the Weyl algebra in positive characteristics [21].

__Lemma 1.1.__ (a) Let $\lambda \in k^*$ be of finite (multiplicative) order n. Then B_λ is free of rank n^2 as a module over its center $Z(B_\lambda) = k[x^{\pm n}, y^{\pm n}]$. Moreover, for any ideal I of B, one has $I = (I \cap Z(B_\lambda))B_\lambda$.

(b) If $\lambda \in k^*$ has infinite order, then B_λ is a central-simple k-algebra of global and Krull dimension 1 and of Gelfand-Kirillov dimension 2.

__Proof.__ First suppose that λ has finite order n and set $C = k[x^{\pm n}, y^{\pm n}] \subset B_\lambda$. Then $C \subset Z(B_\lambda)$ and B_λ is free as a module over C, with basis $\{x^i y^j \mid 0 \leq i,j \leq n-1\}$. If $b = \Sigma_{i,j=0}^{n-1} c_{ij} x^i y^j \in Z(B_\lambda)$, with $c_{ij} \in C$, then

$$b = y^{-1} b y = \Sigma_{i,j=0}^{n-1} c_{ij} \lambda^i x^i y^j$$

and so $c_{ij} = c_{ij}\lambda^i$ for all i,j. Therefore, $c_{ij} = 0$ for $i \neq 0$. Similarly, for $j \neq 0$ one has $c_{ij} = 0$ which shows that $C = Z(B_\lambda)$. Now let I be an ideal of B_λ and set $I_1 = (I \cap C)B_\lambda \subset I$. It follows from the foregoing that B_λ/I_1 is free of rank n^2 over $K = C/I \cap C$, and a calculation as above shows that $K = Z(B_\lambda/I_1)$. B_λ, being an image of the group algebra of a finitely generated nilpotent group, is a polycentral ring [18, 11.3.12]. Thus, if I strictly contains I_1, then we must have $(I/I_1) \cap K \neq 0$ which is impossible. Therefore, $I = I_1$ and part (a) is proved.

As to part (b), the equality $Z(B_\lambda) = k$, for λ of infinite order, follows by a straightforward calculation as above. Simplicity of B_λ now is a consequence of polycentrality, or can be checked directly by the usual shortening trick. Finally, $\text{GK-dim}_k(B_\lambda) = 2$ follows from the fact that each monomial in the elements x,y and their inverses is a scalar multiple of an "ordered" monomial $x^i y^j$ $(i,j \in Z)$, and $\text{K-dim}(B_\lambda) = \text{gl.dim}(B_\lambda) = 1$ follows from [19, Theorem 4.5], e.g. (see also [10]). ∎

Corollary 1.2. (a) Let $\lambda \in k^*$ be of finite order n. Then $E_\lambda = Q(B_\lambda)$ is obtained by localizing B_λ at the nonzero elements of $Z(B_\lambda)$. Thus $Z(E_\lambda) = k(x^n,y^n)$ and $E_\lambda \cong Z(E_\lambda)^t[Z/nZ \oplus Z/nZ]$ is a twisted group algebra of $Z/nZ \oplus Z/nZ$ over $Z(E_\lambda)$.

(b) If λ is not a root of unity, then $Z(E_\lambda) = k$.

Proof. The assertions in (a) are immediate from Lemma 1.1(a). For (b) just note that for any simple or, more generally, polycentral ring R for which $Q(R)$ exists one has $Z(Q(R)) = Q(Z(R))$. ∎

It follows from [12, Corollary 2.2] that the Gelfand-Kirillov transcendence degree of E_λ over k equals 2. If λ has infinite order, then E_λ contains a free k-subalgebra (see Section 2). Hence E_λ has infinite Gelfand-Kirillov dimension in this case. On the other hand, for λ a root of unity, E_λ clearly has Gelfand-Kirillov dimension 2. Also, the commutative transcendence degree of E_λ in the sense of Resco [19] is clearly 2 if λ has finite order. For λ of infinite order, it equals 1, by [19, Theorem 4.3] (or see Section 3.A).

Before we turn to the isomorphism question for the division algebras E_λ, let us briefly recall a few general facts concerning twisted group algebras $k^t[H]$ of ordered groups H. By definition, $k^t[H]$ has a k-basis $\{\dot{x} \mid x \in H\}$, and multiplication in $k^t[H]$ is determined by the rule

$$\dot{x}\cdot\dot{y} = t(x,y)\dot{xy} \qquad (x,y \in H),$$

where $t\colon H \times H \to k^*$ is a 2-cocycle. The basis $\{\dot{x} \mid x \in H\}$ can always be normalized so that $\dot{e} = 1$ is the identity element of $k^t[H]$. Each $\dot{x}$ is a unit of $k^t[H]$ and, using the fact that H is ordered, it is easy to see that the group of units $U(k^t[H])$ of $k^t[H]$ consists precisely of the elements of the form $\alpha\dot{x}$ with $\alpha \in k^*$ and $x \in H$. Furthermore, we have a multiplicative map, the so-called lowest term map,

$$\ell\colon k^t[H] \to \{0\} \cup U(k^t[H])$$

defined by $\ell(0) = 0$ and $\ell(a) = \alpha_{x_0}\dot{x}_0$ for $0 \neq a = \Sigma_{x\in H}\, \alpha_x \dot{x}$ with $x_0 = \min\{x \in H \mid \alpha_x \neq 0\}$. If $S \subset k^t[H]$ is a (right) Ore set of regular elements, then ℓ extends to the localization $k^t[H]S^{-1}$ by setting $\ell(ab^{-1}) = \ell(a)\ell(b)^{-1}$. It is trivial to verify that ℓ is well-defined and remains multiplicative on $k^t[H]S^{-1}$.

The foregoing applies conveniently to B_λ. Indeed, if $x^{\pm 1}, y^{\pm 1}$ are the canonical generators of B_λ with $xy = \lambda yx$, then

$$B_\lambda = \bigoplus_{i,j\in Z} kx^i y^j \cong k^t[Z \oplus Z],$$

and $Z \oplus Z$ is of course an orderable group. Thus the corresponding lowest term map provides us with a multiplicative map

$$\ell\colon E_\mu^* \to U_\mu = \{\alpha x^i y^j \mid i,j \in Z,\ \alpha \in k^*\} = U(B_\mu)$$

which is the identity on U_μ, hence on $k^* \subset U_\mu$.

<u>Proposition 1.3.</u> Let $\lambda, \mu \in k^*$ be given. Then E_λ and E_μ are isomorphic as k-algebras if and only if $\lambda = \mu$ or $\lambda = \mu^{-1}$.

<u>Proof.</u> Since $B_\lambda = k\{x^{\pm 1}, y^{\pm 1}\}/(xy-\lambda yx) = k\{y^{\pm 1}, x^{\pm 1}\}/(yx-\lambda^{-1}xy) = B_{\lambda^{-1}}$, the condition is certainly sufficient. To prove necessity, let $\phi\colon E_\lambda \to E_\mu$ be a fixed k-algebra isomorphism, and let $x^{\pm 1}, y^{\pm 1} \in B_\lambda$ and $u^{\pm 1}, v^{\pm 1} \in B_\mu$ be the canonical generators with $xy = \lambda yx$, resp. $uv = \mu vu$. Let $\ell\colon E_\mu^* \to U_\mu = \{\alpha u^i v^j \mid i,j \in Z,\ \alpha \in k^*\}$ be the lowest term map with respect to a fixed ordering of $U_\mu/k^* \cong Z \oplus Z$. Then

$$f = \ell\circ\phi\colon E_\mu^* \to U_\mu$$

is a group homomorphism which is the identity on k^*. In particular, we obtain

$$\lambda = f(\lambda) = [f(x), f(y)],$$

and this belongs to the commutator subgroup $[U_\mu, U_\mu]$ of U_μ. But $[U_\mu, U_\mu] = \langle\mu\rangle$ and so we have $\lambda \in \langle\mu\rangle$. By symmetry, we conclude that $\langle\lambda\rangle = \langle\mu\rangle$ in k^*. Therefore, if λ has infinite order, then $\lambda = \mu$ or $\lambda = \mu^{-1}$ and we are done. Thus, in the following, we concentrate on the case where λ and μ have finite order n.

We show that f maps U_λ isomorphically onto U_μ. First note that f induces a map $\bar{f}\colon U_\lambda/k^* \to U_\mu/k^*$. Both groups are free abelian of rank 2 generated by $\bar{x}$ and $\bar{y}$, resp. $\bar{u}$ and $\bar{v}$, where we use overbars to denote images mod k^*. Suppose that $\bar{f}(\bar{x}^i\bar{y}^j) = 1$ with $0 \neq |i| + |j|$ minimal. Then

$$1 = [f(x^i y^j), f(y)] = \lambda^i$$

and so $i = ni_1$ for a suitable i_1. Similarly, $j = nj_1$ and hence

$$1 = \bar{f}(\bar{x}^i\bar{y}^j) = \bar{f}(\bar{x}^{i_1}\bar{y}^{j_1})^n .$$

Since U_μ/k^* is torsion-free, we conclude that $\bar{f}(\bar{x}^{i_1}\bar{y}^{j_1}) = 1$ which contradicts our minimality assumption. Therefore, $\bar{f}$ is injective, and hence the same is true for f on U_λ. To prove surjectivity note that, clearly, $\ell(E_\mu^*) = U_\mu$ and $\ell(E_\lambda^*) = f(B_\lambda^*)f(B_\lambda^*)$. Thus it suffices to show that $f(B_\lambda^*) = f(U_\lambda)$. But every $a \in B_\lambda^*$ can be written as a finite sum $a = \Sigma_i u_i$ with $u_i \in U_\lambda$ pairwise distinct mod k^*. Hence, by the above, the images $\overline{f(u_i)} \in U_\mu/k^*$ are distinct, say $\overline{f(u_1)}$ is the smallest with respect to the ordering of U_μ/k^*. Then we obtain that

$$f(a) = \ell(\Sigma_i \phi(u_i)) = \ell(\phi(u_1)) = f(u_1) ,$$

as required. Therefore, f on U_λ and $\bar{f}$ are isomorphisms.

We conclude that $\overline{f(x)} = \bar{u}^i\bar{v}^j$, $\overline{f(y)} = \bar{u}^r\bar{v}^s$ with $is-jr = \pm 1$. Consequently,

$$\lambda = [f(x),f(y)] = [u^iv^j,u^rv^s] = \mu^{is-jr} = \mu^{\pm 1} ,$$

and the proposition is proved. ∎

Proposition 1.3 extends [10, Theorem 1] which states that (for k a domain) B_λ and B_μ are isomorphic as k-algebras iff $\lambda = \mu^{\pm 1}$. The above argument essentially follows the lines of the proof of the isomorphism theorem [11, Theorem 4.1]. In fact, for λ of infinite order, Proposition 1.3 could have been deduced from that result.

It can be shown that in E_λ the identity element cannot be written as a sum of Lie commutators (Section 3.B). In particular, the E_λ's are all distinct from the Weyl field $D_1 = Q(A_1)$.

We close this section with a few facts concerning projective and injective modules for B_λ, $\lambda \in k^*$ of infinite order. The corresponding assertions for the Weyl algebra A_1 in characteristic 0 are well-known. We therefore restrict ourselves to a few indications and refer to the literature whenever possible.

<u>Proposition 1.4.</u> Let $\lambda \in k^*$ be of infinite order.
(a) For any nonzero right ideal I of B_λ, $B_\lambda \oplus I \cong B_\lambda \oplus B_\lambda$. In particular, every right ideal is generated by at most 2 elements.
(b) A finitely generated projective right B_λ-module is either free or isomorphic to a right ideal of B_λ.
(c) For $n > 1$, the matrix ring $M_n(B_\lambda)$ is a principal right (and left) ideal ring, whereas B_λ is not a principal ideal ring.
(d) B_λ has no nonzero finitely generated injective modules.

Proof. (a) We follow Webber [29]. Write B_λ as an Ore extension, $B_\lambda = R[Y^{\pm 1};\alpha]$ with $R = k[X^{\pm 1}]$ and $X^\alpha = \lambda X$. Then $S = R^*$ is an Ore set in B_λ and $B_\lambda S^{-1} \cong k(X)[Y^{\pm 1};\alpha]$ is a principal ideal domain. Therefore, for any nonzero right ideal I of B_λ there exists $0 \neq d_0 \in I$ such that $\bar{I} = I/d_0 B_\lambda$ is R-torsion. If $\bar{I} \neq 0$ choose $0 \neq \bar{d}_1 \in \bar{I}$ such that $M_1 = \mathrm{ann}_R(\bar{d}_1)$ is maximal among the annihilators of nonzero elements of $\bar{I}$. Then M_1 is a maximal ideal ideal of R , and this easily implies that $M_1 B_\lambda$ is a maximal right ideal of B_λ. Therefore, $\bar{d}_1 B_\lambda \cong B_\lambda/p_1 B_\lambda$ where $M_1 = (p_1)$. Continuing this way and using the fact that B_λ is Noetherian we can write $I = \Sigma_{i=0}^{m} d_i B_\lambda$ with

$$\Sigma_{i=0}^{r+1} d_i B_\lambda / \Sigma_{i=0}^{r} d_i B_\lambda \cong B_\lambda / p_{r+1} B_\lambda \quad \text{for } r = 0,1,\dots,m-1.$$

Applying Schanuel's Lemma we obtain

$$B_\lambda \oplus \Sigma_{i=0}^{r} d_i B_\lambda \cong p_{r+1} B_\lambda \oplus \Sigma_{i=0}^{r+1} d_i B_\lambda \cong B_\lambda \oplus \Sigma_{i=0}^{r+1} d_i B_\lambda$$

and so, inductively, $B_\lambda \oplus I \cong B_\lambda \oplus d_0 B_\lambda \cong B_\lambda \oplus B_\lambda$.

(b) In view of part (a), this follows from [29, Theorem 1].

(c) The first assertion follows from (b) and [3, Theorem 7] and the second is immediate from [27, Corollary 1.8].

(d) This can be shown as in the case of the Weyl algebra A_1 [16, Theorem 5.5]. We omit the details. ∎

2. FREE SUBALGEBRAS

In this section, we consider division k-algebras D generated by an arbitrary polycyclic-by-finite group $G \leq D^*$. The proof of our main result (Theorem 2.3) depends upon two major ingredients which we now describe.

Let A be a finitely generated free abelian group and let H be a group acting on A. The action of H on A is said to be rationally irreducible if $A \otimes_Z Q$ is an irreducible module for the rational group algebra QH or, equivalently, if H normalizes no proper pure subgroup of A. The following result is due to G. Bergman [1a] (cf. also [18, 9.3.9] and [5]).

Theorem 2.1 (Bergman). Let A be a finitely generated free abelian group and let H be a group acting on A. Suppose that H and all its subgroups of finite index act rationally irreducibly on A. If I is a proper H-invariant ideal of the group algebra kA, then either $I = 0$ or kA/I is finite dimensional over k.

The second result that we will need is due to L. Makar-Limanov [15]. Strictly speaking, he considers the group algebra kH of the discrete Heisenberg group $H = \langle x,y \mid z=[x,y] \text{ central}\rangle$, and its division ring of fractions. His methods, however, can easily be adapted to deal with the algebras $B_\lambda(k)$ and $E_\lambda(k)$, where $\lambda \in k^*$ has infinite order (Section 1). Note that $Q(kH)$ can in fact be written as $Q(kH) = E_z(k(z))$.

Theorem 2.2 (Makar-Limanov). Let $\lambda \in k^*$ be of infinite order. Then $E_\lambda(k)$ contains a free k-subalgebra of rank 2.

The following result extends this to division algebras generated by arbitrary polycyclic-by-finite groups.

Theorem 2.3. Let $D = k(G)$ be a division k-algebra generated by some polycyclic-by-finite group $G \leq D^*$. Then D contains a free k-subalgebra of rank ≥ 2 if and only if G is not abelian-by-finite.

Proof. If G is abelian-by-finite, then D is finite dimensional over its center, and hence D does not contain free subalgebras. If G is nilpotent-by-finite but not abelian-by-finite, then after dropping to a subgroup of finite index, we may assume that G is non-abelian torsion-free nilpotent. Then G contains elements x and y whose commutator $z = [x,y]$ is $\neq 1$ and commutes with x and y. Set $K = k(z) \subset D$ and consider the K-algebra $B \subset D$ generated by x and y and their inverses. Clearly, B is an image of $B_z(K)$. Since $z \in K^*$ has infinite order, $B_z(K)$ is simple and we do in fact have isomorphisms $B \cong B_z(K)$ and $Q(B) \cong E_z(K)$. The existence of a free subalgebra in D now follows from Theorem 2.2, because the embedding $B \subset D$ extends to an embedding $Q(B) \subset D$.

Thus, in the following, assume that G is not nilpotent-by-finite and that all its nilpotent subgroups are abelian-by-finite. We will proceed in three steps.

Step 1. G contains a subgroup H which is a semidirect product $H = A \rtimes \langle z\rangle$ with A free abelian of rank at least 2 and with z and all its powers acting rationally irreducibly on A.
Proof. After dropping to a subgroup of finite index, we may assume that the Fitting radical $B = \mathrm{Fitt}(G)$ and G/B are both free abelian $\neq \langle 1\rangle$ (use [18, 12.1.5]). Fix $z \in G$, $z \notin B$ and set $V = B \otimes_Z Q$. Replacing z by a suitable power if necessary, we may assume that all irreducible $Q\langle z\rangle$-submodules of V remain irreducible for $Q\langle z^n\rangle$ for all $n \geq 1$ (choose n so that the composition length of the $Q\langle z^n\rangle$- socle of V is maximal) and, moreover, that z acts trivially on the 1-dimensional $Q\langle z\rangle$-submodules of V (their intersections with B are infinite cyclic groups normalized by z so z^2 acts trivially). Then V must contain an irreducible $Q\langle z\rangle$-submodule U of dimension at least 2, for otherwise the minimal polynomial of z on V would be of the form $(z-1)^r$ for some r

and $\langle B,z\rangle$ would be nilpotent and normal in G, contradicting the fact that $B = \mathrm{Fitt}(G)$. Thus we can take $A = U \cap B$ and $H = \langle A,z\rangle = A \rtimes \langle z\rangle$.

<u>Step 2.</u> H contains a free semigroup on two generators.
<u>Proof.</u> This is a consequence of more general work of Rosenblatt [23]. The present special case however can quickly be dealt with using an idea of H. Bass [1]. Let ζ be the automorphism induced by z^{-1} on A and let $K = \mathbb{Q}[\zeta] \subset \mathrm{End}_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$. Then K is a finite field extension of $\mathbb{Q}$ and ζ is not a root of unity. By a theorem of Kronecker, there exists a $\mathbb{Q}$-embedding, σ of K into the complex numbers, $\mathbb{C}$, with $|\zeta^{\sigma}| > 1$ [9 , p. 215]. After replacing z by a suitable power, we may assume that $|\zeta^{\sigma}| > 2$. We show that for any $a \in A$, $a \neq 1$, the semigroup generated by z and az is free. Indeed, suppose that

$$z^{i_o}az^{i_1}\cdots az^{i_r} = z^{j_o}az^{j_1}\cdots az^{j_s}$$

is a nontrivial relation, with $r,s \geq 0$, $i_1,\ldots,i_r,j_1,\ldots,j_s \geq 1$, and $i_o,j_o \geq 0$. Rewriting this relation as

$$a^{\zeta^{i_o}+\zeta^{i_o+i_1}+\cdots+\zeta^{i_o+\cdots+i_{r-1}}}z^{\Sigma_{l=0}^{r} i_l}$$
$$= a^{\zeta^{j_o}+\zeta^{j_o+j_1}+\cdots+\zeta^{j_o+\cdots+j_{s-1}}}z^{\Sigma_{l=0}^{s} j_l}$$

we see that $\Sigma_{l=0}^{r} i_l = \Sigma_{l=0}^{s} j_l$. Using the fact that a generates $A \otimes_{\mathbb{Z}} \mathbb{Q}$ as a K-vector space, we further deduce that

$$\zeta^{i_o}+\zeta^{i_o+i_1}+\cdots+\zeta^{i_o+\cdots+i_{r-1}} = \zeta^{j_o}+\zeta^{j_o+j_1}+\cdots+\zeta^{j_o+\cdots+j_{s-1}}.$$

Since $(i_o,i_1,\ldots,i_r) \neq (j_o,j_1,\ldots,j_s)$, it follows from the above that ζ satisfies a nontrivial polynomial $f(X) = \Sigma_{l=0}^{n} r_l X^l$ with coefficients $r_l \in \{0,\pm 1\}$, $r_n \neq 0$. Therefore, in $\mathbb{C}$ we have $0 = \Sigma_{l=0}^{n} r_l(\zeta^{\sigma})^l$ and so

$$|\zeta^{\sigma}|^n \leq \Sigma_{i=0}^{n-1} |\zeta^{\sigma}|^i = \frac{|\zeta^{\sigma}|^n - 1}{|\zeta^{\sigma}| - 1}$$

which contradicts $|\zeta^{\sigma}| > 2$.

<u>Step 3.</u> The canonical k-algebra map $kH \to D$ given by the inclusion $H \subset D^*$ is an embedding.
<u>Proof.</u> Let I denote the kernel of this map and suppose that I is nonzero. Then I is completely prime and $I \cap kA$ is also nonzero. For, every $\alpha \in kH$ can be uniquely expressed as $\alpha = \Sigma_{i=p}^{q} \alpha_i z^i$ with $\alpha_i \in kA$. Choose $0 \neq \alpha \in I$ of minimal length $q-p$ and with $p=0$. If $q \neq 0$, then $\beta^{z^{-q}} \neq \beta$ for some $\beta \in kA$, and

$$\beta\alpha - \alpha\beta = \Sigma_{i=1}^{q} \alpha_i(\beta - \beta^{z^{-i}})z^i \in I$$

is nonzero and shorter than α. Therefore, $q = 0$ and hence $\alpha \in I \cap kA$. Since $I \cap kA$ is z-stable, Theorem 2.1 implies that $F = kA/I \cap kA$ is a finite dimensional field extension of k, isomorphic to the k-subalgebra of D generated by A. Now z acts on F by k-automorphisms and so some power of z must act trivially on F, hence on A. However, this contradicts our construction of H so that I must be zero.

The theorem now follows, because Step 2 shows that kH contains a free k-algebra and hence so does D, by Step 3. ∎

Corollary 2.4. Let $D = k(G)$ be a division k-algebra with $G \leq D^*$ polycyclic-by-finite. Then D has finite Gelfand-Kirillov dimension over k if and only if G is abelian-by-finite.

3. MISCELLANY

A) COMMUTATIVE SUBFIELDS

The following lemma is extracted from [20]. Here, Kdim denotes (Rentschler-Gabriel-) Krull dimension.

Lemma 3.1 (Resco, Small, Wadsworth). Let A be an absolutely Noetherian k-algebra (i.e., $A \otimes_k K$ is Noetherian for all field extensions K/k) and let $R = AS^{-1}$ be the localization of A with respect to a right Ore set $S \subset A$. Then every commutative subfield L of R with $k \subset L$ is finitely generated over k. Moreover, if $\mathrm{Kdim}(A \otimes_k K) \leq n$ for all field extensions K/k, then $\mathrm{tdeg}_k L \leq n$.

Proof. $R \otimes_k L$ is obtained by localizing $A \otimes_k L$ with respect to $S \otimes_k 1$, and hence $R \otimes_k L$ is Noetherian as $A \otimes_k L$ is. Moreover, $R \otimes_k L$ is free as a module over $L \otimes_k L$ and so $L \otimes_k L$ must also be Noetherian. By 28 , L is finitely generated over k. Finally, $\mathrm{tdeg}_k L = \mathrm{Kdim}(L \otimes_k L) \leq \mathrm{Kdim}(R \otimes_k L) \leq \mathrm{Kdim}(A \otimes_k L)$, where the first inequality follows from the freeness of $R \otimes_k L$ over $L \otimes_k L$ and the second holds, since $R \otimes_k L$ is a localization of $A \otimes_k L$. ∎

The lemma applies to the case where $A = kG/I$ is an image of a group algebra kG with G polycyclic-by-finite. In this case, by [25], an upper bound for the Krull dimensions of $A \otimes_k K$ is given by the Hirsch number h(G) of G. In particular, we have the following

Corollary 3.2. If $D = k(G)$ is a division k-algebra generated by some polycyclic-by-finite group G, then all commutative subfields

$L \supset k$ of D are finitely generated over k, and $\operatorname{tdeg}_k L \leq h(G)$.

In general, $h(G)$ is a very crude bound. For example, if $\lambda \in k^*$ has infinite order, then $B_\lambda(k) \otimes_k K = B_\lambda(K)$ has Krull dimension 1, by Lemma 1.1. Therefore, Lemma 3.1 shows that commutative subfields of $E_\lambda(k)$ have transcendence degree at most 1 over k, whereas any division algebra $D = k(G)$ with $h(G) = 1$ is finite dimensional over its center. R. Resco has conjectured that if G is torsion-free polycyclic-by-finite and $D = Q(kG)$ is the division ring of fractions of kG, which is a domain by [4],[6], then all commutative subfields $L \supset k$ of D have transcendence degree bounded by

$$c(G) = \max \{h(A) \mid A \text{ an abelian subgroup of } G\}.$$

We cannot prove this, but the following discussion should shed some light on this problem. For the rest of this subsection, we keep the following notation:

> G is torsion-free polycyclic-by-finite, and
> D is the division ring of fractions of the group algebra kG.
> For any subgroup $H \leq G$ we set $D_G(H) = \{g \in G \mid g$ has only finitely many H-conjugates$\}$, and $C_R(H)$ denotes the centralizer of H in R (for given R).

The main content of the following lemma is due to M. Smith [24].

<u>Lemma 3.3.</u> For any subgroup H of G, $C_D(H) = Q(C_{kG}(H)) \subset Q(kD_G(H))$ and $Q(kD_G(H))$ is finite dimensional over $C_D(H)$.

<u>Proof.</u> Set $R = kD_G(H)$ and note that $C_{kG}(H) \subset R$. More precisely, the action of H on R by conjugation factors through some finite image $\bar{H}$ of H, and $C_{kG}(H)$ is the fixed subring of R under this action. Therefore, since R is a Noetherian domain, $C_{kG}(H)$ is a Goldie domain and $Q(C_{kG}(H))$ is the fixed subring of $Q(R)$ under the action of $\bar{H}$ ([17, Theorem 5.5], e.g.). Also, $Q(R)$ is finite dimensional over $Q(C_{kG}(H))$ ([17, Lemma 2.18]). Finally, The proof of [24, Theorem 6] shows that $C_D(H) \subset Q(R)$. Hence, clearly, $C_D(H) = C_{Q(R)}(H) = Q(C_{kG}(H))$ and the lemma is proved. ∎

We will call a commutative subfield $L \supset k$ of D <u>almost maximal</u> if L is not contained in a commutative subfield of D having larger transcendence degree over k than L or, equivalently, if $C_D(L)/L$ is algebraic. The following is immediate from Lemma 3.3.

<u>Corollary 3.4.</u> Let A be an abelian subgroup of G. If $A = D_G(A)$ (A has finite index in $D_G(A)$), then $k(A) = Q(kA)$ is a maximal (resp., almost maximal) commutative subfield of D.

We conclude this subsection by mentioning a few instances where the corollary applies.

Examples 3.5. (a) Suppose that A is a maximal abelian subgroup of G and A is subnormal in $D_G(A)$. Then we do in fact have equality, $A = D_G(A)$, and so $k(A)$ is maximal. To see this, choose a subnormal series $A = D_n \lhd D_{n-1} \lhd \dots \lhd D_o = D_G(A)$. Then any n-fold commutator $[a_n,[a_{n-1},\dots,[a_1,d]]\dots]$ with $a_i \in A$, $d \in D_o$ belongs to A. Consider an (n-1)-fold commutator $c = [a_{n-1},\dots,[a_1,d]\dots]$. Let $a \in A$ be arbitrary and choose m so that a^m is central in D_o. Then, since A is abelian and $[a,c] \in A$, we have

$$1 = [a^m,c] = [a,c]^m,$$

and hence $[a,c] = 1$. Therefore, $c \in C_G(A) = A$. By induction, we obtain $D_o = A$, as we have claimed.

Since subnormality is automatic if G is nilpotent, we recover M. Smith's result [24, Corollary 8]. In general, however, maximal abelian subgroups A of G need not satisfy $A = D_G(A)$ (e.g., take $G = \langle x,y \mid y^{-1}xy = x^{-1}\rangle$ and $A = \langle y\rangle$) and so $k(A)$ need not be maximal.

(b) If $A \leq G$ is abelian and satisfies $h(A) = c(G)$, then it is trivial to verify that A has finite index in $D_G(A)$. Thus $k(A)$ is at least almost maximal in this case.

B) LIE COMMUTATORS

Let $D = k(G)$ be a division k-algebra generated by some polycyclic-by-finite group G and assume that char $k = 0$. It would be interesting to know whether the identity element $1 \in D$ can be written as a sum of Lie commutators in D. If not, then this fact would distinguish division algebras of the above type from division algebras E generated by finite dimensional Lie subalgebras of $E_{[\ ,\ ]}$ (i.e., E with Lie bracket $[a,b] = ab - ba$), at least if k is algebraically closed. This follows from the following simple observation.

Lemma 3.6. Let E be a division algebra over k. If $E_{[\ ,\ ]}$ contains a non-abelian nilpotent Lie algebra, or a non-abelian finite dimensional Lie algebra over k and k is algebraically closed, then there exist elements $a,b \in E$ with $ab - ba = 1$.

Proof. Suppose $\underline{g} \subset E_{[\ ,\ ]}$ is a nilpotent Lie algebra which is not commutative. Then there exists an element $c \in \underline{g}$ such that $[c,\underline{g}]$ is nonzero and is contained in the center of $\underline{g}$. Choose $b \in \underline{g}$ with $[c,b] = cb - bc \neq 0$ and set $a = c[c,b]^{-1} \in E$. Then

$[a,b] = 1$ in E.

If, on the other hand, $\underline{g} \subset E[\ ,\ ]$ is finite dimensional non-nilpotent, then Engel's theorem [8, Sec. 3.2] implies that there exists $a \in \underline{g}$ such that $\operatorname{ad}(a) \in \operatorname{End}(\underline{g})$ is not nilpotent. Let $0 \neq c \in \underline{g}$ be an eigenvector for $\operatorname{ad}(a)$ with nonzero eigenvalue $\gamma \in k$ and set $b = \gamma^{-1}c \in E$. Then $[ac^{-1},b] = 1$ in E. ∎

Algebraic closure of k is definitely required in the above. For example, the standard basis $\{1,i,j,k\}$ of the real quaternions H spans a Lie subalgebra of H, but $1 \in H$ is not even a sum of Lie commutators (use the embedding $H \subset M_2(C)$, or the following proposition).

We now return to division algebras generated by polycyclic-by-finite groups. The following result extends, and uses, [11, Lemma 2.3].

<u>Proposition 3.7.</u> Let G be a finitely generated nilpotent-by-finite group and let char $k = 0$. Let P be a prime ideal of kG and set $R = Q(kG/P)$. Then $1 \notin [R,R]$, the space of Lie commutators in R.

<u>Proof.</u> For G finitely generated nilpotent, this follows from [11, Lemma 2.3]. In general, choose G_o to be a nilpotent normal subgroup of finite index in G. Then $P \cap kG_o$ is a finite intersection of pairwise incomparable prime ideals P_i, $i=1,2,\ldots,l$, of kG_o. Moreover, it is routine to check that the Ore set S of regular elements of $kG_o/P\cap kG_o$ remains Ore and regular in kG/P. Therefore,

$$R = (kG/P)S^{-1} = \bigoplus_x xR_o ,$$

where $R_o = (kG_o/P\cap kG_o)S^{-1}$ and x runs through a transversal for G_o in G. Now R_o is the direct product of the rings $R_i = Q(kG_o/P_i)$, $i=1,2,\ldots,l$, and so

$$R \subset \operatorname{End}(R_{R_o}) \cong M_n(R_o) \cong \prod_{i=1}^{l} M_n(R_i),$$

where n is the order of G/G_o. Thus it suffices to establish the assertion for the matrix rings $M_n(R_i)$. But we know it is true for each R_i. Hence the canonical map of k-spaces $T\colon R_i \to R_i/[R_i,R_i]$ does not vanish on 1. T can be lifted to a map $T'\colon M_n(R_i) \to R_i/[R_i,R_i]$ by setting $T'([r_{st}]) = \Sigma_{s=1}^{n} T(r_{ss})$. Since T' inherits k-linearity and the trace property $T'(AB) = T'(BA)$ from T and maps the identity $1 \in M_n(R_i)$ to the nonzero element $n\cdot T(1)$, we conclude that $1 \notin [M_n(R_i),M_n(R_i)]$ as required. ∎

The proposition applies in particular to division algebras $D = k(G)$ generated by nilpotent-by-finite groups G. Sometimes the char 0 assumption is superfluous here. For example, if $D = k(G)$ with G torsion-free nilpotent, then $1 \notin [D,D]$ holds in any characteristic. This follows from [11, Sec. 2], where explicit traces are constructed for so-called Hilbert-Neumann algebras. The same construction also applies to the division rings $E_\lambda(k)$ and, more generally, to the division rings of fractions of twisted group algebras $k^t[G]$ with G an ordered group. For general polycyclic-by-finite groups, however, char 0 is definitely needed. For example, consider the Weyl algebra $A_1 = k\{x,y\}$, $xy - yx = 1$, with char $k = p > 0$. Then $D_1 = Q(A_1)$ is generated, as division algebra, by the elements $a = xy$ and x which satisfy $x^{-1}ax = a - 1$. Therefore, the generating group $G = \langle a,x \rangle$ is polycyclic and is in fact isomorphic to a semidirect product of the form $Z^{(p)} \rtimes Z$. On the other hand, R. Snider [26] has shown that if char $k = 0$ and $G \cong Z^{(r)} \rtimes Z$ for some r, then $D = Q(kG)$ satisfies $1 \notin [D,D]$.

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