INDUCING STABILITY CONDITIONS

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Abstract. We study stability conditions induced by functors between triangulated categories. Given a finite group acting on a smooth projective variety we prove that the subset of invariant stability conditions embeds as a closed submanifold into the stability manifold of the equivariant derived category. As an application we examine stability conditions on Kummer and Enriques surfaces and we give a derived version of the Torelli Theorem for the latter surfaces. We also study the relationship between stability conditions on projective spaces and those on their canonical bundles.

1. Introduction

After the seminal paper [5], stability conditions have been investigated in various interesting geometric settings. For curves, the manifold parametrizing stability conditions has been explicitly described ([5, 30, 26]), while for A_n-singularities, a complete topological description of these spaces is available ([20, 31]). In particular, one can show that, in these cases, the stability manifold is connected and simply connected. For algebraic K3 surfaces, the situation is more intricate and neither the connectedness nor the simply connectedness has been proved. Nevertheless one distinguished connected component has been identified and related to the problem of describing the group of autoequivalences of the derived category of the surface ([6]). For smooth generic analytic K3 surfaces a complete description is given in [18].

A basic difficulty in the theory, however, remains: how to construct examples of stability conditions, at least when the geometry of the variety is well-understood? Naively, if two varieties X and Y are related in some intimate geometric way, we would expect to solve the previous problem for X once it is solved for Y. In other words the geometric liason between X and Y should produce some (maybe weak) relation between their stability manifolds.

In this paper, we develop a technique of inducing stability conditions via functors between triangulated categories with nice properties and show how, in certain geometric contexts, this procedure gives an answer to the problem mentioned above. In [35], Polishchuk proposed similar ideas from a somewhat different perspective (see the end of Section 2.2).

The first situation where this approach will be applied is the case of a smooth projective variety X with the action of a finite group G. Forgetting linearizations on equivariant complexes gives a natural faithful functor between the equivariant derived category $\mathrm{D}^{\mathrm{b}}_G(X)$ and $\mathrm{D}^{\mathrm{b}}(X)$. Since G acts in a natural way on the stability manifold $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ of $\mathrm{D}^{\mathrm{b}}(X)$, one can consider the subset of invariant stability conditions on the derived category $\mathrm{D}^{\mathrm{b}}(X)$. Our first result is now the following:

Theorem 1.1. *The subset of invariant stability conditions in* $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ *is a closed submanifold with a closed embedding into* $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_G(X))$ *via the forgetful functor.*

As we will point out, an analogous statement holds when we restrict to numerical stability conditions. Note that the existence of a bijection between the subset of invariant stability conditions in $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ and a certain subset of $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_G(X))$ was already observed in [35].

The interest of this theorem can be appreciated by considering two easy examples. In [12], the authors study the derived categories of certain weighted projective lines which are in fact stacks

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obtained as quotients of some plane cubics E by the action of a natural involution. Our result then realizes the stability manifold of E as a closed submanifold of the stability manifold of the derived category of each of these (stacky) weighted projective lines (see Example 2.19).

Even more interesting is the example of Kummer surfaces (see Example 2.20). In [6] Bridgeland describes the connected component of maximal dimension of the space $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ parametrizing numerical stability conditions on an abelian surface A. We show that this is embedded as a closed submanifold into a distinguished component of the space $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)))$ of numerical stability conditions on the Kummer surface $\mathrm{Km}(A)$. The latter component was also studied in [6] and its topology should be (at least conjecturally) related to the description of the group of autoequivalences of $\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A))$.

These two examples reflect the philosophical approach that we were mentioning at the beginning. Indeed the presence of a functor with good properties between the derived categories of projective varieties with a close geometric relationship allows one to compare certain features of the geometry of the corresponding stability manifolds.

This viewpoint appears clearly in the third example we are considering in the present paper: Enriques surfaces. Recall that an Enriques surface Y is a minimal smooth projective surface with 2-torsion canonical bundle ω_Y and $H^1(Y,\mathcal{O}_Y)=0$. The universal cover $\pi : X \to Y$ is a K3 surface and it carries a fixed-point-free involution $\iota : X \to X$ such that $Y = X/G$, where $G = \langle\iota\rangle$.

In [6], Bridgeland studied a distinguished connected component $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ of the stability manifold $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ parametrizing numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$. The idea here is to relate this component to some interesting connected component of $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$. Write $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G$ for the group of G-equivariant Hodge isometries of the Mukai lattice of X. Then our second main result gives a precise description of this relationship:

Theorem 1.2. *Let Y be an Enriques surface and $\pi : X \to Y = X/G$ its universal cover. Then there exist a connected component $\Sigma(Y)$ of $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$ naturally embedded into $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ as a closed submanifold and a natural homomorphism*

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(Y)) \longrightarrow \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G/G$$

whose image contains the index-2 subgroup of G-equivariant orientation preserving Hodge isometries quotiented by G.

Moreover, if Y is generic, the category $\mathrm{D}^{\mathrm{b}}(Y)$ does not contain spherical objects and $\Sigma(Y)$ is isomorphic to the distinguished connected component $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$.

As we will see in Proposition 3.15, in the generic case one can be even more explicit about the classification of rigid objects in $\mathrm{D}^{\mathrm{b}}(Y)$.

The relation between the connected component $\Sigma(Y)$ and the description of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(Y))$ is quite deep. Indeed we prove that there exists a covering map from $\Sigma(Y)$ onto some period domain and a subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(Y))$ acts as the group of deck transformations. So, the picture now is very similar to the one given by Bridgeland for K3 surfaces. In particular we state a conjecture (Conjecture 3.7) about $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(Y))$ which extends Conjecture 1.2 in [6] to the case of Enriques surfaces.

Finally we study two other geometric situations for which the procedure of inducing stability conditions via faithful functors can be exploited. First of all, we compare stability conditions on projective spaces with those on their canonical bundles. Our result in this direction is Theorem 4.5 whose main part can be summarized as follows:

Theorem 1.3. *An open subset of $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1))$ embeds into the stability manifold of the total space of the canonical bundle $\omega_{\mathbb{P}^1}$ as a fundamental domain for the action of the autoequivalences group.*

It is known by the work in [31, 20] that the space of stability conditions on the canonical bundle of the projective line is connected and simply connected. We provide a new simpler proof of these topological properties based on [18], on the way to proving the previous result.

The last example is concerned with the relationship between the spaces of stability conditions on Kleinian singularities and those of the corresponding quivers, with particular attention to the A_2-singularity case. The picture we get is completely similar to the one presented in the previous theorem.

The plan of the paper is as follows. In Section 2 we recall Bridgeland's construction of stability conditions and the metric properties of the stability manifold. We then show how a faithful functor induces stability conditions and apply these results to equivariant derived categories.

In Section 3 we study Enriques surfaces and show that a connected component of the stability manifold of such a surface embeds as a closed submanifold into the space of stability conditions of its universal cover. A derived Torelli Theorem is proved and we conjecturally relate the topology of the distinguished connected component to the group of autoequivalences. Some special properties of generic Enriques surfaces are then described.

Finally, Section 4 treats the case of projective spaces and of the corresponding cotangent bundles and of Kleinian singularities.

Unless otherwise stated all varieties considered in this paper are meant to be smooth projective and defined over the complex numbers.

2. Exact functors and invariant stability conditions

In this section we show how to induce stability conditions using exact functors with nice properties. This permits us to relate the stability manifolds of different projective varieties taking into account certain geometric relations between the geometry of the varieties themselves. Notice that a criterion for the existence of induced stability conditions in geometric contexts (including the ones we will consider) can be easily deduced from [35] (see Theorem 2.14).

As a first application, in Section 2.3 we prove Theorem 1.1 according to which, given a smooth projective variety with an action of a finite group, the closed submanifold of invariant stability conditions embeds into the stability manifold of the equivariant derived category. We use this to show that the unique connected component of maximal dimension of the stability manifold of an abelian surface embeds into Bridgeland's connected component of the stability manifold of the associated Kummer surface.

2.1. **Bridgeland's framework.**

In this section we recall a few results from [5] which will be used throughout this paper. For the moment, let $\mathbf{T}$ be an essentially small triangulated category.

Definition 2.1. A *stability condition* on $\mathbf{T}$ is a pair $\sigma = (Z, \mathcal{P})$ where $Z : K(\mathbf{T}) \to \mathbb{C}$ is a linear map (the *central charge*) while $\mathcal{P}(\phi) \subset \mathbf{T}$ are full additive subcategories for each $\phi \in \mathbb{R}$ satisfying the following conditions:

(a) If $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$.

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all ϕ.

(c) $\mathrm{Hom}_{\mathbf{T}}(E_1, E_2) = 0$ for all $E_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$.

(d) Any $0 \neq E \in \mathbf{T}$ admits a *Harder–Narasimhan filtration* (*HN-filtration* for short) given by a collection of distinguished triangles $E_{i-1} \to E_i \to A_i$ with $E_0 = 0$ and $E_n = E$ such that $A_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

In Bridgeland's terminology, the non-zero objects in the category $\mathcal{P}(\phi)$ are the *semistable* objects of phase ϕ while the objects A_i in (d) are the *semistable factors* of E. The minimal objects of $\mathcal{P}(\phi)$ are called *stable* of phase ϕ. Note that the HN-filtration is unique up to isomorphisms. We write

$\phi^+_\sigma(E) := \phi_1$, $\phi^-_\sigma(E) := \phi_n$, and $m_\sigma(E) := \sum_j |Z(A_j)|$. From the definition, each subcategory $\mathcal{P}(\phi)$ is extension-closed and abelian.

For any interval $I \subseteq \mathbb{R}$, $\mathcal{P}(I)$ is defined to be the extension-closed subcategory of $\mathbf{T}$ generated by the subcategories $\mathcal{P}(\phi)$, for $\phi \in I$. Bridgeland proved that, for all $\phi \in \mathbb{R}$, $\mathcal{P}((\phi, \phi+1])$ is the heart of a bounded t-structure on $\mathbf{T}$. The category $\mathcal{P}((0,1])$ is called the *heart* of σ.

Remark 2.2. As pointed out in [5, Prop. 5.3] to exhibit a stability condition on a triangulated category $\mathbf{T}$, it is enough to give a bounded t-structure on $\mathbf{T}$ with heart $\mathbf{A}$ and a group homomorphism $Z : K(\mathbf{A}) \to \mathbb{C}$ such that $Z(E) \in \mathbb{H}$, for all $0 \neq E \in \mathbf{A}$, and with the Harder–Narasimhan property (see [5, Sect. 5.2]). Recall that $\mathbb{H} := \{z \in \mathbb{C} : z = |z|\exp(i\pi\phi),\ 0 < \phi \leq 1\}$ and that the above homomorphism Z is called *stability function.*

As a special case, if $\mathbf{A} \subseteq \mathbf{T}$ is the heart of a bounded t-structure and moreover it is an abelian category of finite length (i.e. artinian and noetherian), then a group homomorphism $Z : K(\mathbf{A}) \to \mathbb{C}$ with $Z(E) \in \mathbb{H}$, for all minimal objects $E \in \mathbf{A}$, extends to a unique stability condition on $\mathbf{T}$.

A stability condition is called *locally-finite* if there exists some $\epsilon > 0$ such that each quasi-abelian subcategory $\mathcal{P}((\phi-\epsilon, \phi+\epsilon))$ is of finite length. In this case $\mathcal{P}(\phi)$ has finite length so that every object in $\mathcal{P}(\phi)$ has a finite *Jordan–Hölder filtration* (*JH-filtration* for short) into stable factors of the same phase. The set of stability conditions which are locally finite will be denoted by $\mathrm{Stab}(\mathbf{T})$.

By [5, Prop. 8.1] there is a natural topology on $\mathrm{Stab}(\mathbf{T})$ defined by the generalized metric

$$d(\sigma_1, \sigma_2) := \sup_{0 \neq E \in \mathbf{T}} \left\{ |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|, |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|, \left|\log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)}\right| \right\} \in [0, \infty]. \tag{2.1}$$

Remark 2.3. Bridgeland proved in [5, Lemma 8.2] that there are two groups which naturally act on $\mathrm{Stab}(\mathbf{T})$. The first one is the group of exact autoequivalences $\mathrm{Aut}(\mathbf{T})$ which, moreover, preserves the structure of generalized metric space just defined.

The universal cover $\widetilde{\mathrm{Gl}}^+_2(\mathbb{R})$ of $\mathrm{Gl}^+_2(\mathbb{R})$ acts on the right in the following way. Let $(G, f) \in \widetilde{\mathrm{Gl}}^+_2(\mathbb{R})$, with $G \in \mathrm{Gl}^+_2(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is an increasing map, with $f(\phi+1) = f(\phi)+1$, the pair satisfying $G\exp(2i\pi\phi)/|G\exp(2i\pi\phi)| = \exp(2i\pi f(\phi))$, for all $\phi \in \mathbb{R}$. Then (G, f) maps $(Z, \mathcal{P}) \in \mathrm{Stab}(\mathbf{T})$ to $(G^{-1} \circ Z, \mathcal{P} \circ f)$.

The result from [5] that we will need for the rest of the paper is the following:

Theorem 2.4. **([5], Theorem 1.2.)** *For each connected component $\Sigma \subseteq \mathrm{Stab}(\mathbf{T})$ there is a linear subspace $V(\Sigma) \subseteq (K(\mathbf{T}) \otimes \mathbb{C})^\vee$ with a well-defined linear topology such that the natural map*

$$\mathcal{Z} : \Sigma \longrightarrow V(\Sigma), \qquad (Z, \mathcal{P}) \longmapsto Z$$

is a local homeomorphism. In particular, if $K(\mathbf{T}) \otimes \mathbb{C}$ is finite dimensional, Σ is a finite dimensional complex manifold.

The complex manifold $\mathrm{Stab}(\mathbf{T})$ will be called the *stability manifold* of $\mathbf{T}$. Recall that $V(\Sigma)$ is defined as the set of $W \in (K(\mathbf{T}) \otimes \mathbb{C})^\vee$ such that

$$\|W\|_\sigma := \sup\left\{ \frac{|W(E)|}{|Z(E)|} : E \text{ is } \sigma\text{-semistable} \right\} < \infty,$$

where $\sigma = (Z, \mathcal{P})$ is any stability condition in Σ. The linear topology on $V(\Sigma)$ is then given by this norm.

Suppose now that the category $\mathbf{T}$ is $\mathbb{C}$-linear and of finite type. The Euler–Poincaré form on $K(\mathbf{T})$

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i (E, F)^i,$$

where $(E, F)^i := \dim_\mathbb{C} \mathrm{Hom}_\mathbf{T}(E, F[i])$, allows us to define the *numerical Grothendieck group* $\mathcal{N}(\mathbf{T}) = K(\mathbf{T})/K(\mathbf{T})^\perp$ (the orthogonal is with respect to χ). To shorten notation, when $\mathbf{T} =$

$\mathrm{D^b}(X) := \mathrm{D^b}(\mathbf{Coh}(X))$ we will often write $\mathcal{N}(X)$ instead of $\mathcal{N}(\mathrm{D^b}(X))$. Notice that if X is a surface, then $\mathcal{N}(X) = H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z})$.

A stability condition $\sigma = (Z, \mathcal{P})$ such that Z factors through $K(\mathbf{T}) \twoheadrightarrow \mathcal{N}(\mathbf{T})$ is called *numerical*. We denote by $\mathrm{Stab}_{\mathcal{N}}(\mathbf{T})$ the complex manifold parametrizing numerical stability conditions. An analogue of Theorem 2.4 holds true also in the numerical setting (see [5, Cor. 1.3]).

We conclude this section with a discussion of two examples of stability conditions needed in the sequel.

Example 2.5. (K3 and abelian surfaces) We briefly recall the construction in [6] for abelian or K3 surfaces X. Fix $\omega, \beta \in \mathrm{NS}(X) \otimes \mathbb{Q}$ with ω in the ample cone and define the categories $\mathcal{T}(\omega, \beta)$ consisting of sheaves whose torsion-free part have μ_ω-semistable Harder–Narasimhan factors with slope greater than $\omega \cdot \beta$ and $\mathcal{F}(\omega, \beta)$ consisting of torsion-free sheaves whose μ_ω-semistable Harder–Narasimhan factors have slope smaller or equal to $\omega \cdot \beta$. Next consider the abelian category

$$\mathcal{A}(\omega, \beta) := \left\{ \mathcal{E} \in \mathrm{D^b}(X) : \begin{array}{l} \bullet\ \mathcal{H}^i(\mathcal{E}) = 0 \text{ for } i \notin \{-1, 0\}, \\ \bullet\ \mathcal{H}^{-1}(\mathcal{E}) \in \mathcal{F}(\omega, \beta), \\ \bullet\ \mathcal{H}^0(\mathcal{E}) \in \mathcal{T}(\omega, \beta) \end{array} \right\}.$$

and the $\mathbb{C}$-linear map

$$Z_{\omega,\beta} : \mathcal{N}(X) \longrightarrow \mathbb{C}, \qquad \mathcal{E} \longmapsto \langle \exp(\beta + i\omega), v(\mathcal{E}) \rangle,$$

where $v(\mathcal{E})$ is the Mukai vector of $\mathcal{E} \in \mathrm{D^b}(X)$ and $\langle -, - \rangle$ is the Mukai pairing (see [17, Ch. 10]). By [6, Lemma 6.2, Prop. 7.1], if $\omega \cdot \omega > 2$, the pair $(Z_{\omega,\beta}, \mathcal{A}(\omega, \beta))$ defines a stability condition (Remark 2.2).

In the rest of the paper we will be particularly interested in the connected component

$$\mathrm{Stab}^\dagger_{\mathcal{N}}(\mathrm{D^b}(X)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$$

extensively studied in [6]. It can be described as the connected component containing the stability conditions $(Z_{\omega,\beta}, \mathcal{A}(\omega, \beta))$ with ω and β as above.

Example 2.6. (Projective spaces) Recall that an object E in a triangulated category $\mathbf{T}$ is *exceptional* if

$$\mathrm{Hom}^i_{\mathbf{T}}(E, E) \cong \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

An ordered collection of exceptional objects $E = \{E_0, \ldots, E_n\}$ is *strong exceptional* in $\mathbf{T}$ if $\mathrm{Hom}^k_{\mathbf{T}}(E_i, E_j) \neq 0$ only if $i \leq j$ and $k = 0$. A strong exceptional collection of two objects is a *strong exceptional pair*. Finally, a strong exceptional collection is *complete*, if E generates $\mathbf{T}$ by shifts and extensions.

By [2, 13] we know that $\mathrm{D^b}(\mathbb{P}^N)$ admits a complete strong exceptional collection given by $\{\mathcal{O}, \ldots, \mathcal{O}(N)\}$. Fix $E = \{\mathcal{E}_0, \ldots, \mathcal{E}_N\}$ to be a strong complete exceptional collection on $\mathrm{D^b}(\mathbb{P}^N)$. We construct some explicit examples of stability conditions associated to E in the following way (we use freely [25]).

By [25, Lemma 3.14], the extension-closed category $\langle E \rangle_{\overline{p}} := \langle \mathcal{E}_0[p_0], \ldots, \mathcal{E}_N[p_N] \rangle$, for $p_0 > p_1 > \ldots > p_N$ integrals numbers, is the heart of a bounded t-structure on $\mathrm{D^b}(\mathbb{P}^N)$. Then the Grothendieck group $K(\mathbb{P}^N)$ is a free abelian group of finite rank isomorphic to $\mathbb{Z}^{\oplus(N+1)}$ generated by the classes of $\mathcal{E}_0, \ldots, \mathcal{E}_N$. Fix $z_0, \ldots, z_N \in \mathbb{H}$ and define a stability function

$$Z_p : K(\langle \mathcal{E} \rangle_{\overline{p}}) \longrightarrow \mathbb{C}, \qquad \mathcal{E}_i[p_i] \longmapsto z_i,$$

for all i. By Remark 2.2 this extends to a unique locally finite stability condition on $\mathrm{D^b}(\mathbb{P}^N)$.

Define Θ_E as the subset of $\mathrm{Stab}(\mathbb{P}^N) := \mathrm{Stab}(\mathrm{D^b}(\mathbb{P}^N)) = \mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(\mathbb{P}^N))$ consisting of the stability conditions just constructed, up to the action of $\widetilde{\mathrm{Gl}}^+_2(\mathbb{R})$. By [25, Lemma 3.16], $\mathcal{E}_0, \ldots, \mathcal{E}_N$ are stable for all stability conditions in Θ_E. Moreover, for a stability condition having $\mathrm{rk}_{\mathbb{R}} Z_p = 1$ (thinking of Z_p as a map from $K(\mathbb{P}^N) \otimes \mathbb{R}$ to $\mathbb{C} \cong \mathbb{R}^2$), $\mathcal{E}_0, \ldots, \mathcal{E}_N$ are the only stable objects (up to shifts). Lemma 3.19 of [25] shows that $\Theta_E \subseteq \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N))$ is an open, connected and simply connected $(N+1)$-dimensional submanifold.

2.2. **Construction of induced stability conditions.**

In general stability, conditions do not behave well with respect to exact functors between triangulated categories. What we are going to show is that in the particular cases discussed below, it may be possible to induce stability conditions from one category to another (see [35] and also [26, 39]).

Let $F : \mathbf{T} \to \mathbf{T}'$ be an exact functor between two essentially small triangulated categories. Assume that F satisfies the following condition:

(Ind) $\qquad \mathrm{Hom}_{\mathbf{T}'}(F(A), F(B)) = 0$ implies $\mathrm{Hom}_{\mathbf{T}}(A, B) = 0, \qquad$ for any $A, B \in \mathbf{T}$.

For example, if F is faithful, condition (Ind) holds. Let $\sigma' = (Z', \mathcal{P}') \in \mathrm{Stab}(\mathbf{T}')$ and define $\sigma = F^{-1}\sigma' = (Z, \mathcal{P})$ by

$$\begin{aligned} &Z = Z' \circ F_*, \\ &\mathcal{P}(\phi) = \{E \in \mathbf{T} : F(E) \in \mathcal{P}'(\phi)\}, \end{aligned}$$

where $F_* : K(\mathbf{T}) \otimes \mathbb{C} \to K(\mathbf{T}') \otimes \mathbb{C}$ is the natural morphism induced by F. Later on we will denote by $F_*^\vee : (K(\mathbf{T}') \otimes \mathbb{C})^\vee \to (K(\mathbf{T}) \otimes \mathbb{C})^\vee$ the dual morphism of F_*.

Remark 2.7. (i) The categories $\mathcal{P}(\phi)$ are strongly full and additive. Moreover σ satisfies the first three properties of Definition 2.1. Since $\mathcal{P}((\phi - \epsilon, \phi + \epsilon)) \hookrightarrow \mathcal{P}'((\phi - \epsilon, \phi + \epsilon))$ is an exact functor of quasi-abelian categories, condition (Ind) yields that, in order to prove that σ is a locally-finite stability condition on $\mathbf{T}$, it will be sufficient to prove that HN-filtrations exist.

(ii) Let $\sigma' \in \mathrm{Stab}(\mathbf{T}')$ and suppose that $\sigma := F^{-1}\sigma' \in \mathrm{Stab}(\mathbf{T})$. Then $F^{-1}(\sigma' G) = \sigma G \in \mathrm{Stab}(\mathbf{T})$, for any $G \in \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$.

Lemma 2.8. *Assume $\sigma = F^{-1}\sigma'$ is a locally-finite stability condition on $\mathbf{T}$. Then there exist an open subset $U \subseteq \mathrm{Stab}(\mathbf{T}')$ containing σ' and a morphism $\varphi : U \to \mathrm{Stab}(\mathbf{T})$ such that $\varphi(\sigma') = \sigma$ and $\mathcal{Z} \circ \varphi = F_*^\vee \circ \mathcal{Z}'$.*

Proof. Let Σ' (respectively Σ) be the connected component of $\mathrm{Stab}(\mathbf{T}')$ (respectively $\mathrm{Stab}(\mathbf{T})$) containing σ' (respectively σ). Then the map $F_*^\vee : V(\Sigma') \to V(\Sigma)$ is continuous. Indeed, by definition,

$$\begin{aligned} \|F_*^\vee(W')\|_\sigma &= \sup\left\{\frac{|(F_*^\vee(W'))(E)|}{|Z(E)|} : E \text{ is } \sigma\text{-semistable}\right\} \\ &= \sup\left\{\frac{|W'(F(E))|}{|Z'(F(E))|} : E \text{ is } \sigma\text{-semistable}\right\} \\ &\leq \sup\left\{\frac{|W'(G)|}{|Z'(G)|} : G \text{ is } \sigma'\text{-semistable}\right\} \\ &= \|W'\|_{\sigma'}. \end{aligned}$$

The conclusion now follows from the fact that $\mathcal{Z}$ is a local homeomorphism (Theorem 2.4). □

Lemma 2.9. *The subset*

$$\mathrm{Dom}(F^{-1}) := \{\sigma' \in \mathrm{Stab}(\mathbf{T}') : \sigma = F^{-1}\sigma' \in \mathrm{Stab}(\mathbf{T})\}$$

is closed.

Proof. Assume $(\sigma')_s \to \overline{\sigma'} = (\overline{Z'}, \overline{\mathcal{P}'})$ in $\mathrm{Stab}(\mathbf{T}')$, with $(\sigma')_s \in \mathrm{Dom}(F^{-1})$. We want to prove that $\overline{\sigma} = F^{-1}\overline{\sigma'} = (\overline{Z}, \overline{\mathcal{P}})$ has HN-filtrations. If $0 \neq E \in \mathbf{T}$, then in $\overline{\sigma'}$ there exists a HN-filtration whose semistable factors are $A_1, \ldots, A_n$, with $\overline{\phi'}(A_1) > \ldots > \overline{\phi'}(A_n)$ and $\overline{\phi'}$ denotes the phase in $\overline{\sigma'}$.

Since $(\sigma')_s \to \overline{\sigma'}$, for $s \gg 0$ we can assume $(\phi')^{\pm}_s(A_1) > \ldots > (\phi')^{\pm}_s(A_n)$. By replacing each A_i by its HN-filtration in $(\sigma')_s$ we get the HN-filtration of $F(E)$ in $(\sigma')_s$. Since $(\sigma')_s \in \mathrm{Dom}(F^{-1})$, this HN-filtration is, up to isomorphism, the image via F of the HN-filtration of E in $\sigma_s = F^{-1}(\sigma')_s$. Note that we are using here the uniqueness of the semistable objects and of the morphisms in HN-filtrations, up to isomorphism. This means that $A_i \in F(\mathbf{T})$, for all $i \in \{1, \ldots, n\}$. By the uniqueness (up to isomorphism) of HN-filtrations, this proves that the HN-filtration just considered is the image of a HN-filtration in σ via the functor F. $\square$

We now show how to construct stability conditions using special exact functors and abelian categories. Similar existence results will be considered in Section 2.3.

Definition 2.10. Let $F : \mathbf{T} \to \mathbf{T}'$ be an exact functor. An abelian abelian category $\mathbf{A} \subseteq \mathbf{T}$ is *F-admissible* if

(i) $\mathbf{A}$ is the heart of a bounded t-structure;
(ii) $\mathrm{Hom}^{<0}_{\mathbf{T}'}(F(A), F(B)) = 0$, for all $A, B \in \mathbf{A}$;
(iii) F is full when restricted to $\mathbf{A}$.

For a subcategory $\mathbf{C}$ of $\mathbf{T}$, we denote by $\langle \mathbf{C} \rangle \subseteq \mathbf{T}$ the smallest extension-closed full subcategory containing $\mathbf{C}$. We now show how to produce hearts of bounded t-structures.

Lemma 2.11. *Let $F : \mathbf{T} \to \mathbf{T}'$ be an exact functor, and assume that $\langle F(\mathbf{T}) \rangle = \mathbf{T}'$. Let $\mathbf{A} \subseteq \mathbf{T}$ be an F-admissible abelian category. Then the subcategory $\mathbf{A}' := \langle F(\mathbf{A}) \rangle \subseteq \mathbf{T}'$ is the heart of a bounded t-structure on $\mathbf{T}'$.*

Proof. Since $\mathbf{T}'$ is generated by $F(\mathbf{T})$ by extensions, if $0 \neq E \in \mathbf{T}'$, there exist $M_1, \ldots, M_k \in \mathbf{T}$ such that E is generated by $F(M_1), \ldots, F(M_k)$. Moreover, due to the fact that $\mathbf{A}$ is the heart of a t-structure, we may assume that $M_i = A_k[i_k]$ with $A_1, \ldots, A_k \in \mathcal{A}$ and $i_1, \ldots, i_k$ integers. We want to prove that we can choose $A_1, \ldots, A_k$ such that $i_1 \geq \ldots \geq i_k$. Indeed, $\mathbf{A}'$ is a full additive subcategory of $\mathbf{T}'$ and $\mathrm{Hom}^{<0}_{\mathbf{A}'}(A', B') = 0$ if $A', B' \in \mathbf{A}'$. Hence by [5, Lemma 3.2] this suffices to imply that $\mathbf{A}'$ is the heart of a bounded t-structure on $\mathbf{T}$.

We proceed by induction on k. For $k = 1$ there is nothing to prove and so assume $k > 1$. By induction hypothesis, we can suppose $i_1 \geq \ldots \geq i_{k-1}$. If $i_k = i_{k-1} + 1$, let A be a cone of $E_{k-2} \to E$. Then we have a diagram of exact triangles

$$\begin{tikzcd}
E_{k-2} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "\mathrm{id}"] & A \arrow[d] \\
E_{k-1} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F(A_k)[i_k] \arrow[d, "{f[i_k]}"] \\
F(A_{k-1})[i_k - 1] \arrow[r] & 0 \arrow[r] & F(A_{k-1})[i_k].
\end{tikzcd}$$

The condition that F is full when restricted to $\mathbf{A}$ yields $A \cong F(D)$, where D is a cone of $A_k[i_{k-1}] \to A_{k-1}[i_{k-1}]$. Since $\mathbf{A}$ is abelian, D is the extension

$$\ker(f)[i_k] \to D \to \mathrm{coker}(f)[i_k - 1].$$

So, the last part of the filtration becomes

$$\begin{tikzcd}
E_{k-3} \arrow[rr] & & E_{k-2} \arrow[rr] \arrow[dl] & & Q \arrow[rr] \arrow[dl] & & E, \arrow[dl] \\
& F(A_{k-2})[i_{k-2}] \arrow[ul, "{[1]}"] & & F(\mathrm{ker}(f))[i_k] \arrow[ul, "{[1]}"] & & F(\mathrm{coker}(f))[i_{k-1}] \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

where $Q[1]$ is a cone of the composite map $E \to A \to F(\mathrm{coker}(f))[i_{k-1}]$. Now, taking Q instead of E_{k-1} we have the desired filtration by induction.

Assume instead that $i_k > i_{k-1} + 1$. Then, as before, if A is a cone of $E_{k-2} \to E_k$, A is an extension of $F(A_k)[i_k]$ by $F(A_{k-1})[i_{k-1}]$. Since by hypothesis, there are no nontrivial extensions, $A \cong F(A_{i_k})[i_k] \oplus F(A_{k-1})[i_{k-1}]$ and we can filter E as

$$\begin{tikzcd}
E_{k-3} \arrow[rr] & & E_{k-2} \arrow[rr] \arrow[dl] & & Q' \arrow[rr] \arrow[dl] & & E, \arrow[dl] \\
& F(A_{k-2})[i_{k-2}] \arrow[ul, "{[1]}"] & & F(A_k)[i_k] \arrow[ul, "{[1]}"] & & F(A_{k-1})[i_{k-1}], \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

where $Q'[1]$ is a cone of the composite map $E \to A \to F(A_{k-1})[i_{k-1}]$. Induction then gives the desired filtration. □

Let us move to the problem of inducing stability conditions using an exact functor.

Proposition 2.12. *Let $F : \mathbf{T} \to \mathbf{T}'$ be an exact functor which satisfies* (Ind) *and assume that $\langle F(\mathbf{T})\rangle = \mathbf{T}'$. Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathbf{T})$ be such that its heart $\mathcal{P}((0, 1])$ is of finite length with a finite number of minimal objects. Assume furthermore that $\mathcal{P}((0, 1])$ is F-admissible. Define $\sigma' = F(\sigma) = (Z', \mathcal{P}')$, where $Z' = Z \circ F_*^{-1}$ and $\mathcal{P}'((0, 1]) = \langle F(\mathcal{P}((0, 1]))\rangle$. Then σ' is a locally finite stability condition on $\mathbf{T}'$. Moreover, $F^{-1}\sigma' = \sigma$.*

Proof. Arguing exactly as in the proof of the previous lemma, one can see that $\mathcal{P}'((0, 1])$ is of finite length and has a finite number of minimal objects, which are nothing but the images via F of the minimal objects of $\mathcal{P}((0, 1])$. So, $F_* : K(\mathbf{T}) \to K(\mathbf{T}')$ is an isomorphism and the definition of σ' has meaning.

Now the first part of the proposition follows form the previous lemma and from Remark 2.2, since $\mathcal{P}'((0, 1])$ is an abelian category of finite length. To prove that $F^{-1}\sigma' = \sigma$, we only have to show that if $E \in \mathbf{T}$ is σ-semistable, then $F(E)$ is σ'-semistable. This follows from Lemma 2.13. □

Lemma 2.13. *Let $F : \mathbf{A} \to \mathbf{A}'$ be a fully faithful exact functor between abelian categories. Assume that $\langle F(\mathbf{A})\rangle = \mathbf{A}'$. Then $F(\mathbf{A})$ is closed under subobjects and quotients.*

Proof. Let $0 \to M \xrightarrow{f} F(A)$, with $M \in \mathbf{A}'$. Since $\mathbf{A}'$ is generated by $F(\mathbf{A})$, there exists $A_1, \dots, A_k \in \mathbf{A}$ such that M is an extension of $F(A_1), \dots, F(A_k)$. Hence, by induction on the number of generators, we can assume that M fits in a short exact sequence

$$0 \longrightarrow F(B) \longrightarrow M \longrightarrow F(A_k) \longrightarrow 0$$

for some $B \in \mathbf{A}$. Since F is full, $\mathrm{coker}(f) \cong F(D)$, for some $D \in \mathbf{A}$. In particular, $M \cong F(E)$, for some $E \in \mathbf{A}$. □

The result in Proposition 2.12 will be applied in a geometric context (Section 4) where a weaker form of it would be enough. Notice that the latter is also a consequence of more general results in [35] and, for the convenience of the reader, we include the precise statement here (although it will not be explicitely used in the rest of this paper).

Let $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{T}}'$ be two triangulated categories in which all small coproducts exist and let $\mathbf{T} \subseteq \widetilde{\mathbf{T}}$ and $\mathbf{T}' \subseteq \widetilde{\mathbf{T}}'$ be full triangulated essentially small subcategories. Consider an exact functor $F : \widetilde{\mathbf{T}} \to \widetilde{\mathbf{T}}'$ such that:

- F commutes with small coproducts;
- F has a left adjoint $G : \widetilde{\mathbf{T}}' \to \widetilde{\mathbf{T}}$;
- $F(\mathbf{T}) \subseteq \mathbf{T}'$ and if $E \in \widetilde{\mathbf{T}}$ and $F(E) \in \mathbf{T}'$, then $E \in \mathbf{T}$;
- $G(\mathbf{T}') \subseteq \mathbf{T}$;
- the induced functor $F : \mathbf{T} \to \mathbf{T}'$ satisfies condition (Ind).

We can now state the following:

Theorem 2.14. ([35]) *Let $\sigma' = (Z', \mathcal{P}') \in \mathrm{Stab}(\mathbf{T}')$ be such that $FG(\mathcal{P}'(\phi)) \subseteq \mathcal{P}'((\phi, +\infty))$, for all $\phi \in \mathbb{R}$. Then $F^{-1}\sigma' \in \mathrm{Stab}(\mathbf{T})$.*

Proof. Due to what we have already observed, it is enough to show that, in $\sigma := F^{-1}\sigma'$, every $E \in \mathbf{T}$ admits a HN-filtration.

Let $A' \to F(E) \to B'$ be the last triangle in the HN-filtration of $F(E)$ in σ', with $A' \in \mathcal{P}'((\phi, +\infty))$ and $B' \in \mathcal{P}'(\phi)$. Let $I \subseteq \mathbb{R}$ be an interval. Define

$$\mathcal{P}(I) := \{C \in \mathbf{T} : F(C) \in \mathcal{P}'(I)\}.$$

Under our assumptions [35, Thm. 2.1.2] applies. Thus the pair $(\mathcal{P}((\phi, +\infty)), \mathcal{P}((-\infty, \phi]))$ defines a t-structure on $\mathbf{T}$. Hence there exists a triangle $A \to E \to B$ in $\mathbf{T}$, where $A \in \mathcal{P}((\phi, +\infty))$ and $B \in \mathcal{P}((-\infty, \phi])$. Applying the functor F, we get $F(A) \to F(E) \to F(B)$ in $\mathbf{T}'$, with $F(A) \in \mathcal{P}'((\phi, +\infty))$ and $F(B) \in \mathcal{P}'((-\infty, \phi])$.

By uniqueness of the HN-filtration, $A' \cong F(A)$ and $B' \cong F(B)$. Hence $B \in \mathcal{P}(\phi)$ and, proceeding further with A, we get the existence of the HN-filtration for E in σ. □

2.3. Invariant stability conditions.

Let X be a smooth projective variety over $\mathbb{C}$ with an action of a finite group G. We denote by $\mathbf{Coh}_G(X)$ the abelian category of G-equivariant coherent sheaves on X, i.e. the category whose objects are pairs $(\mathcal{E}, \{\lambda_g\}_{g\in G})$, where $\mathcal{E} \in \mathbf{Coh}(X)$ and, for any $g_1, g_2 \in G$, $\lambda_{g_i} : \mathcal{E} \xrightarrow{\sim} g_i^*\mathcal{E}$ is an isomorphism such that $\lambda_{g_1g_2} = g_2^*(\lambda_{g_1}) \circ \lambda_{g_2}$. The set of these isomorphisms is a *G-linearization* of $\mathcal{E}$ (very often a G-linearization will be simply denoted by λ). The morphisms in $\mathbf{Coh}_G(X)$ are just the morphisms of coherent sheaves compatible with the G-linearizations (for more details see, for example, [3, 10]). We put $\mathrm{D}^{\mathrm{b}}_G(X) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}_G(X))$. Since G is finite, when needed, $\mathrm{D}^{\mathrm{b}}_G(X)$ can equivalently be described in terms of G-equivariant objects in $\mathrm{D}^{\mathrm{b}}(X)$.

The main aim of this section is to prove Theorem 1.1. To this end, consider the functors

$$\mathrm{Forg}_G : \mathrm{D}^{\mathrm{b}}_G(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X)$$

which forgets the G-linearization, and

$$\mathrm{Inf}_G : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_G(X)$$

defined by

$$\mathrm{Inf}_G(\mathcal{E}) := \left(\bigoplus_{g\in G} g^*\mathcal{E}, \lambda_{\mathrm{nat}}\right),$$

where λ_{nat} is the natural G-linearization. These functors are clearly adjoint:

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_G(X)}(\mathrm{Inf}_G(\mathcal{E}), (\mathcal{F}, \beta)) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E}, \mathrm{Forg}_G((\mathcal{F}, \beta))).$$

Since both functors are faithful, the results of Section 2.2 apply and they induce stability conditions on $\mathrm{D}^{\mathrm{b}}_G(X)$ and $\mathrm{D}^{\mathrm{b}}(X)$.

The group G acts on $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ and $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ in the obvious manner via the autoequivalences $\{g^* : g :\in G\}$. Hence we can define the subset

$$\Gamma_X := \{\sigma \in \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)) : g^*\sigma = \sigma, \text{ for any } g \in G\}.$$

Lemma 2.15. *Γ_X is a closed submanifold of $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ such that the diagram*

$$\begin{tikzcd} \Gamma_X \cap \Sigma \arrow[r, hook] \arrow[d, "\mathcal{Z}|_{\Gamma_X}"'] & \Sigma \arrow[d, "\mathcal{Z}"] \\ (V(\Sigma))_G \arrow[r, hook] & V(\Sigma) \end{tikzcd}$$

commutes and $\mathcal{Z}|_{\Gamma_X}$ is a local homeomorphism. Here $(-)_G$ is the G-invariant part and Σ is a connected component of $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$.

Proof. The subset Γ_X is closed because $\Gamma_X = \bigcap_{g\in G}(g,\mathrm{id})^{-1}(\Delta) = (\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)))_G$, where Δ is the diagonal in $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)) \times \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$. Consider now the map $\mathcal{Z}|_{\Gamma_X} : \Gamma_X \to (V(\Sigma))_G$ and let $\sigma \in \Gamma_X$. It is enough to prove that $\mathcal{Z}|_{\Gamma_X}$ is a local homeomorphism in a neighbourhood of σ.

Take the open subset $U := \{\sigma' \in \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)) : d(\sigma',\sigma) < 1/2\}$ where $\mathcal{Z}|_U$ maps onto $V \subseteq V(\Sigma)$. Then $\Gamma_X \cap U$ goes homeomorphically into $V \cap (V(\Sigma))_G$. Indeed, $\mathcal{Z}(U \cap \Gamma_X) \subseteq V \cap (V(\Sigma))_G$ and, if $Z' \in V \cap (V(\Sigma))_G$, take $\sigma' \in U$ such that $\mathcal{Z}(\sigma') = Z'$. We now have

$$d(\sigma', g^*\sigma') \leq d(\sigma',\sigma) + d(g^*\sigma',\sigma) = d(\sigma',\sigma) + d(\sigma',\sigma) < 1,$$

where the last equality holds since σ is in Γ_X and G acts by isometries on the stability manifold. Moreover $\mathcal{Z}(g^*\sigma') = Z' \circ (g^*)^{-1} = Z' = \mathcal{Z}(\sigma')$. Applying [5, Lemma 6.4], we get $\sigma' = g^*\sigma'$ so that $\sigma' \in U \cap \Gamma_X$. $\square$

The following lemma is proved in [35] in a more general form and it follows from Theorem 2.14. For the convenience of the reader we outline the proof in our special case. We keep the notation of Section 2.2.

Lemma 2.16. *If $\sigma = (Z,\mathcal{P}) \in \Gamma_X$, then $\mathrm{Forg}_G^{-1}\sigma \in \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_G(X))$.*

Proof. As we pointed out in Remark 2.7, it is sufficient to show that in $\mathrm{Forg}_G^{-1}\sigma$ any object has a HN-filtration. So, let $(\mathcal{E},\lambda) \in \mathrm{D}^{\mathrm{b}}_G(X)$ and take

$$\mathcal{E}' \xrightarrow{s} \mathcal{E} \xrightarrow{t} \mathcal{A}$$

to be the last triangle in the HN-filtration of $\mathcal{E}(= \mathrm{Forg}_G((\mathcal{E},\lambda))) \in \mathrm{D}^{\mathrm{b}}(X)$ in σ, where $\mathcal{A} \in \mathcal{P}(\phi)$ and $\mathcal{E}' \in \mathcal{P}(>\phi)$.

Since, by assumption, for any $g \in G$, we have $g^*\mathcal{E}' \in \mathcal{P}(>\phi)$ and $g^*\mathcal{A} \in \mathcal{P}(\phi)$, it follows that $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E}', g^*\mathcal{A}) = 0$. Hence, for any $g \in G$, there exist unique morphisms $\beta_g^1 : \mathcal{E}' \xrightarrow{\sim} g^*\mathcal{E}'$ and $\beta_g^2 : \mathcal{A} \xrightarrow{\sim} g^*\mathcal{A}$, making the diagram

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "s"] \arrow[d, "\beta_g^1"'] & \mathcal{E} \arrow[r, "t"] \arrow[d, "\lambda_g"'] & \mathcal{A} \arrow[d, "\beta_g^2"'] \\ g^*\mathcal{E}' \arrow[r, "g^*s"] & g^*\mathcal{E} \arrow[r, "g^*t"] & g^*\mathcal{A} \end{tikzcd}$$

commutative. Notice that, by uniqueness, $\beta^i_{hg} = h^*(\beta^i_g) \circ \beta^i_h$ and $\beta^i_{\mathrm{id}} = \mathrm{id}$, for any $g, h \in G$ and $i \in \{1,2\}$. Moreover, the same argument shows that β^i_g is an isomorphism.

Since $(\mathcal{A},\beta^2) \in \mathrm{Forg}_G^{-1}\mathcal{P}(\phi)$, proceeding further with the object $(\mathcal{E}',\beta^1)$ we get a HN-filtration for $(\mathcal{E},\lambda)$. $\square$

This allows us to introduce a map $\mathrm{Forg}_G^{-1} : \Gamma_X \to \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_G(X))$.

Proposition 2.17. *The morphism Forg_G^{-1} just defined is continuous and the subset $\mathrm{Forg}_G^{-1}(\Gamma_X)$ is a closed embedded submanifold.*

Proof. The first part of the statement is obvious from Lemma 2.8 and Lemma 2.16.

To prove the second assertion, consider the functor Inf_G. If $\sigma = (Z, \mathcal{P}) \in \Gamma_X$ and $\sigma' = \mathrm{Forg}_G^{-1}\sigma$, then $\mathrm{Inf}_G^{-1}\sigma' = (|G| \cdot \mathrm{id}_{\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})})\sigma \in \mathrm{Stab}(\mathrm{D^b}(X))$. This can be easily shown by recalling that, by definition, for any $e \in K(X) \otimes \mathbb{C}$, the following series of equalities holds true

$$(\mathrm{Inf}_G^{-1} \circ \mathrm{Forg}_G^{-1})Z(e) = Z(\mathrm{Forg}_{G*}(\mathrm{Inf}_{G*}(e))) = Z\left(\sum_{g\in G} g^* e\right) = |G|Z(e).$$

Moreover, on the level of slicings, we get the following description

$$(\mathrm{Inf}_G^{-1} \circ \mathrm{Forg}_G^{-1})(\mathcal{P})(\phi) = \left\{\mathcal{E} \in \mathrm{D^b}(X) : \bigoplus_{g\in G} g^*\mathcal{E} \in \mathcal{P}(\phi)\right\},$$

for any $\phi \in \mathbb{R}$. Since $\mathcal{P}(\phi)$ is closed under direct summands, $(\mathrm{Inf}_G^{-1} \circ \mathrm{Forg}_G^{-1})(\mathcal{P})(\phi) \subseteq \mathcal{P}(\phi)$. On the other hand, the fact that $g^*\sigma = \sigma$ implies that if $\mathcal{E} \in \mathcal{P}(\phi)$, then $\bigoplus_{g\in G} g^*\mathcal{E} \in \mathcal{P}(\phi)$. Hence $\mathrm{Inf}_G^{-1}\sigma' = (|G| \cdot \mathrm{id}_{\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})})\sigma \in \mathrm{Stab}(\mathrm{D^b}(X))$. See also [35, Prop. 2.2.3].

Using this fact and Lemma 2.8, we get that Inf_G^{-1} is continuous in its domain and that the morphism $\mathrm{Inf}_G^{-1} \circ \mathrm{Forg}_G^{-1} : \Gamma_X \to \Gamma_X$ is an isomorphism. In particular $\mathrm{Forg}_G^{-1}(\Gamma_X)$ is an embedded submanifold.

To prove that $\mathrm{Forg}_G^{-1}(\Gamma_X)$ is also closed, take a sequence $\sigma_n' := \mathrm{Forg}_G^{-1}\sigma_n \in \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_G(X))$ converging to $\overline{\sigma}'$. We have to show that $\overline{\sigma}' = \mathrm{Forg}_G^{-1}\overline{\sigma}$, for some $\overline{\sigma} \in \Gamma_X$.

Since $\mathrm{Inf}_G^{-1}\overline{\sigma}' \in \mathrm{Stab}(\mathrm{D^b}(X))$, applying Inf_G^{-1} to the previous sequence, we have that the sequence $\mathrm{Inf}_G^{-1}\sigma_n'$ converges to $\mathrm{Inf}_G^{-1}\overline{\sigma}'$ in $\mathrm{Stab}(\mathrm{D^b}(X))$. For what we have proved before, we have $\mathrm{Inf}_G^{-1}\sigma_n' = (|G| \cdot \mathrm{id}_{\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})})\sigma_n \in \Gamma_X$ and so $\mathrm{Inf}_G^{-1}\overline{\sigma}' = \overline{\tau} \in \Gamma_X$, because Γ_X is closed.

In particular, σ_n converges to $(1/|G| \cdot \mathrm{id}_{\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})})\overline{\tau}$ in Γ_X. Applying now Forg_G^{-1}, we get that the sequence $\sigma_n' = \mathrm{Forg}_G^{-1}\sigma_n$ converges to $\mathrm{Forg}_G^{-1}\overline{\sigma}$. By uniqueness of limits, $\overline{\sigma}' = \mathrm{Forg}_G^{-1}\overline{\sigma}$. □

The previous results show that $\Gamma_X = (\mathrm{Stab}(\mathrm{D^b}(X)))_G$ embeds as a closed submanifold into $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_G(X))$ by the forgetful functor, as stated in Theorem 1.1.

Remark 2.18. All the previous arguments remain true when dealing with numerical stability conditions, just substituting $(K(X) \otimes \mathbb{C})_G^\vee$ and $(K(X) \otimes \mathbb{C})^\vee$ with $(\mathcal{N}(X) \otimes \mathbb{C})_G$ and $\mathcal{N}(X) \otimes \mathbb{C}$. More precisely, the Euler–Poincaré pairing χ is non-degenerate and G-invariant. Hence it gives a canonical identification $(\mathcal{N}(X) \otimes \mathbb{C})_G^\vee \cong (\mathcal{N}(X) \otimes \mathbb{C})_G$. Moreover, the map Forg_{G*} factorizes in the following way

$$\begin{tikzcd} \mathcal{N}(\mathrm{D}^{\mathrm{b}}_G(X)) \arrow[r, two heads] \arrow[rr, bend right, "\mathrm{Forg}_{G*}"'] & (\mathcal{N}(X) \otimes \mathbb{C})_G \arrow[r, hook] & \mathcal{N}(X) \otimes \mathbb{C}. \end{tikzcd}$$

As a consequence, we also get a submanifold Γ_X in $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ with the properties stated in Theorem 1.1.

We conclude this section with a discussion of the case of Kummer surfaces and weighted projective lines. Enriques surfaces will be treated in detail in the next section.

Example 2.19. (Weighted projective lines) Consider an elliptic curve E with an action of a finite group G. Any $g \in G$ acts as the indentity on the even cohomology of E and, by [5, Thm. 9.1], $\mathrm{Stab}(\mathrm{D^b}(E))$ is the orbit under the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ of the standard stability condition given by slope stability on coherent sheaves. Hence $(\mathrm{Stab}(\mathrm{D^b}(E)))_G = \mathrm{Stab}(\mathrm{D^b}(E))$ and Theorem 1.1 implies that $\mathrm{Stab}(\mathrm{D^b}(E))$ is embedded as a closed submanifold into $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_G(E))$.

Example 5.8 in [12] considers the cases of elliptic curves E and involutions ι such that the categories of coherent sheaves on the stack $[E/\langle\iota\rangle]$ and the ones on a certain weighted projective line C are equivalent. In such a case, conjecturally, $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_{\langle\iota\rangle}(E))$ should be related to the unfolding space of the elliptic singularity corresponding to C ([12]). The embedded closed submanifold $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(E))$ should be the deformation space of the elliptic curve describing the singularity. Related examples are also studied in [22].

Example 2.20. (Kummer surfaces) Let A be an abelian surface and $\mathrm{Km}(A)$ the associated Kummer surface (i.e. the minimal resolution of the quotient $A/\langle\iota\rangle$, where $\iota : A \xrightarrow{\sim} A$ is the involution such that $\iota(a) = -a$). In [18] it was proved that $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ (Example 2.5) is the unique connected component in $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ of maximal dimension equal to $\mathrm{rk}\,\mathrm{NS}(A)+2$. By its very definition, $\iota^* : \mathcal{N}(A) \xrightarrow{\sim} \mathcal{N}(A)$ is the identity. Hence Γ_A is open and closed in $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$, since $(\mathcal{N}(A))_{\langle\iota\rangle} = \mathcal{N}(A)$. Consequently, if non-empty, Γ_A is a connected component.

Take $\sigma_{\omega,\beta} \in \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$, determined by the linear function $Z_{\omega,\beta}$ and the abelian category $\mathcal{A}(\omega,\beta)$, according to Bridgeland's definition in Example 2.5. By the above remarks β and ω are invariant and so $Z_{\omega,\beta}$ and $\mathcal{A}(\omega,\beta)$ are invariant as well. In particular, $\iota^*\sigma_{\omega,\beta} = \sigma_{\omega,\beta}$. Thus $\Gamma_A = \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ and, by Theorem 1.1, $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ is realized as a closed submanifold of $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)))$, the connected component of $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A))$ defined in Example 2.5.

3. Enriques surfaces

In this section we prove Theorem 1.2. In particular, we show that a connected component of the stability manifold of an Enriques surface embeds as a closed submanifold into the stability manifold of the associated K3 surface. In Section 3.1 we prove a derived version of the Torelli Theorem for Enriques surfaces and relate the topology of the connected component just introduced to the description of the group of autoequivalences. Finally, in Section 3.3, we treat the case of generic Enriques surfaces Y where the distinguished connected component of the stability manifold of the Enriques surface is isomorphic to the connected component of the space of stability conditions on the universal cover described by Bridgeland. We also show that the derived category of Y does not contain any spherical objects, concluding with a description of its rigid objects.

3.1. A distinguished connected component.

Let Y be an Enriques surface, $\pi : X \to Y$ its universal cover and $\iota : X \to X$ the fixed-point-free involution such that $Y = X/G$, where G is now the group generated by ι. In this special setting, $\mathbf{Coh}(Y)$ is naturally isomorphic to the abelian category $\mathbf{Coh}_G(X)$. Notice that, via this equivalence, the canonical bundle ω_Y is identified with the G-equivariant sheaf $(\mathcal{O}_X, -\mathrm{id})$. The equivalence $\mathbf{Coh}(Y) \cong \mathbf{Coh}_G(X)$, obviously yields an equivalence $\mathrm{D}^{\mathrm{b}}(Y) \cong \mathrm{D}^{\mathrm{b}}_G(X)$, which will be tacitly meant for the rest of this paper.

It is well-known that the transcendental lattice $T(X)$ admits a primitive embedding $i : T(X) \hookrightarrow \Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ such that $i(T(X))$ is primitive in the sublattice $\Lambda_- := U \oplus U(2) \oplus E_8(-2)$ and no elements of Λ_- with self-intersection -2 are orthogonal to $i(T(X))$. We will denote by $\Lambda_+ := U(2) \oplus E_8(-2)$ the orthogonal complement of Λ_- in Λ (for a definition of the lattices appearing in the previous discussion, see for example [1, Ch. I]).

Consider the connected component $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ described in [6, Thm. 1.1] and Example 2.5. As a consequence of Theorem 1.1, we can now prove the following:

Proposition 3.1. *There exists an open and closed subset* $\Sigma(Y)$ *of* $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$ *which is embedded as a closed submanifold into* $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ *via the functor* Inf_G *and the* *diagram*

$$
\begin{tikzcd}
\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)) \arrow[r, "\mathrm{Forg}_G^{-1}"] \arrow[d] & \Sigma(Y) \arrow[r, "\mathrm{Inf}_G^{-1}"] \arrow[d, "\mathcal{Z}"'] & \Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)) \arrow[d] \\
(\mathcal{N}(X) \otimes \mathbb{C})^{\vee}_G \arrow[r, "\mathrm{Forg}_{G*}^{\vee}"] & (\mathcal{N}(Y) \otimes \mathbb{C})^{\vee} \arrow[r, "\mathrm{Inf}_{G*}^{\vee}"] & (\mathcal{N}(X) \otimes \mathbb{C})^{\vee}_G
\end{tikzcd}
\tag{3.1}
$$

commutes.

Proof. Consider the map $\mathrm{Forg}_G^{-1} : \Gamma_X \to \mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(Y))$. Notice that Γ_X is non-empty. Indeed, we can choose β and ω as in Example 2.5 which are invariant for the action of ι^* and so $\iota^*\sigma_{\omega,\beta} = \sigma_{\omega,\beta}$ as in Example 2.20.

By Theorem 1.1, $\mathrm{Forg}_G^{-1}(\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)))$ is closed. On the other hand, notice that $\mathrm{Inf}_{G*} : (\mathcal{N}(X) \otimes \mathbb{C})_G \to \mathcal{N}(Y) \otimes \mathbb{C}$ is an isomorphism. Indeed $\mathrm{Inf}_{G*} = \pi_*$ and then the result follows, for example, from [28, Lemma 3.1]. Hence Forg_{G*} is an isomorphism as well by Remark 2.18 and

$$\dim(\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X))) = \dim(\mathcal{N}(Y) \otimes \mathbb{C}) = \dim(\mathrm{Forg}_G^{-1}(\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)))).$$

Thus $\mathrm{Forg}_G^{-1}(\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)))$ has the desired property and the diagram in the statement commutes by definition. □

Remark 3.2. (i) It is perhaps worth noticing that a connected component of $\Sigma(Y)$ could be alternatively obtained by repeating the same construction as in [6] for Enriques surfaces. Since we will not need this in the sequel, the easy check is left to the reader.

(ii) It is not difficult to see that the functor $(-) \otimes \omega_Y$ preserves $\Sigma(Y)$.

Example 3.3. A general result of Keum [23, Thm. 2] shows that any Kummer surface is the universal cover of an Enriques surface. More concretely, take two non-isogenous elliptic curves E_1 and E_2 and choose two order-2 points $e_1 \in E_1$ and $e_2 \in E_2$. As remarked in [23, Ex. 3.1], the abelian surface $A := E_1 \times E_2$ has an involution ι defined by

$$\iota : (z_1, z_2) \longmapsto (-z_1 + e_1, z_2 + e_2).$$

Notice that, since $\mathrm{NS}(A)$ is generated by the elliptic curves E_1 and E_2, $\iota^*|_{\mathrm{NS}(A)} = \mathrm{id}_{\mathrm{NS}(A)}$ while ι acts freely on the subgroup of order-2 points of A. Hence the induced involution $\tilde{\iota} : \mathrm{Km}(A) \to \mathrm{Km}(A)$ has no fixed points and the $\mathbb{C}$-linear extension of $\tilde{\iota}$ restricted to the vector space $\mathrm{NS}(A) \otimes \mathbb{C} \subset \mathrm{NS}(\mathrm{Km}(A)) \otimes \mathbb{C}$ is the identity.

If Y is the Enriques surface $\mathrm{Km}(A)/\langle\tilde{\iota}\rangle$, combining Proposition 3.1 and Example 2.20 we obtain a connected component $\Sigma(Y) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(Y))$ and embeddings

$$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(A)) \hookrightarrow \Sigma(Y) \hookrightarrow \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(\mathrm{Km}(A)))$$

of closed submanifolds.

3.2. **The group of autoequivalences.**

As pointed out by Bridgeland in [6] for K3 surfaces, the knowledge of some topological features of a special connected component of the manifold parametrizing stability conditions can give (at least conjecturally) important information about the group of autoequivalences of the derived category.

In this section we want to describe the same strategy for Enriques surfaces. The first step consists in proving a derived version of the Torelli Theorem for Enriques surfaces first stated in [14, 15] (see also [1]). Notice that for K3 surfaces a Derived Torelli Theorem for K3 surfaces is already present in the literature ([16, 17, 19, 32, 33]).

Recall that, for a K3 surface X, the Mukai lattice $\widetilde{H}(X,\mathbb{Z})$ of X is the total cohomology group $H^*(X,\mathbb{Z})$ endowed with the Hodge and lattice structure defined, for example, in [17, Sect. 10.1]. Let σ be a generator of $H^{2,0}(X)$ and ω a Kähler class (e.g. an ample class), then

$$P(X,\sigma,\omega) := \langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1-\omega^2/2, \omega\rangle, \tag{3.2}$$

is a positive four-space in $\widetilde{H}(X,\mathbb{R})$, where $\mathrm{Re}(\sigma)$ and $\mathrm{Im}(\sigma)$ are the real and imaginary parts of σ. This space comes with a natural orientation given by the choice of the basis. We will denote by $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))$ $(\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})))$ the group of (orientation preserving) Hodge isometries of $\widetilde{H}(X,\mathbb{Z})$. The subgroups consisting of the equivariant isometries are denoted by $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G$ and $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))_G$. Obviously, G is naturally a normal subgroup of $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G$.

Proposition 3.4. *Let Y be an Enriques surface and let X and G be as above. There exists a natural morphism of groups*

$$\Pi : \mathrm{Aut}(\mathrm{D^b}(Y)) \longrightarrow \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G/G$$

such that the image of Π contains the index-2 subgroup $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))_G/G$.

Proof. The easy part is defining the morphism Π in the statement. Consider the following set of objects

$$\mathrm{Ker}^{G_\Delta}(\mathrm{D^b}(X)) := \{(\mathcal{G},\lambda)\in \mathrm{D}^{\mathrm{b}}_{G_\Delta}(X\times X) : \Phi_{\mathcal{G}} \in \mathrm{Aut}(\mathrm{D^b}(X))\}$$

and take $\mathrm{Aut}(\mathrm{D^b}(X))_G := \{\Phi \in \mathrm{Aut}(\mathrm{D^b}(X)) : \iota^*\circ\Phi\circ\iota^* \cong \Phi\}$. Notice that G_Δ is the group generated by the involution $\iota\times\iota$.

As pointed out in [34, Sect. 3.3], the functors Forg_G and Inf_G are $2:1$ and fit into the diagram

$$\begin{tikzcd} & \mathrm{Ker}^{G_\Delta}(\mathrm{D^b}(X)) \arrow[dl, "\mathrm{Forg}_G"'] \arrow[dr, "\mathrm{Inf}_G"] & \\ \mathrm{Aut}(\mathrm{D^b}(X))_G & & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_G(X)) = \mathrm{Aut}(\mathrm{D^b}(Y)). \end{tikzcd}$$

Observe that Forg_G and Inf_G can be thought of as group homomorphisms, since $\mathrm{Ker}^{G_\Delta}(\mathrm{D}^{\mathrm{b}}_G(X))$ has a natural group structure. This yields a natural surjective homomorphism

$$\mathrm{Lift} : \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_G(X)) \longrightarrow \mathrm{Aut}(\mathrm{D^b}(X))_G/G.$$

which, composed with the natural map $\mathrm{Aut}(\mathrm{D^b}(X))_G/G \to \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G/G$ (see [32] or [17, Prop. 10.10]), gives the desired morphism Π. Notice that an easy computation shows that the morphism Lift is $2:1$ and the kernel is the group $\langle(-)\otimes\omega_Y\rangle$. In particular we have an isomorphism

$$\mathrm{Aut}(\mathrm{D^b}(Y))/\langle(-)\otimes\omega_Y\rangle \cong \mathrm{Aut}(\mathrm{D^b}(X))_G/G. \tag{3.3}$$

Define j to be the Hodge isometry $j := -\mathrm{id}_{H^2(X,\mathbb{Z})}\oplus \mathrm{id}_{H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})}$ and take the isometry $\psi\in \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G$. Given an equivalence $\Phi\in\mathrm{Aut}(\mathrm{D^b}(X))$, we write Φ^H for the natural Hodge isometry induced on the Mukai lattice ([17, Ch. 10]). The proof is now divided up into a few cases.

Case 1. Suppose that $\psi(0,0,1) = \pm(0,0,1)$.

Under this hypothesis $\pm\psi(1,0,0) = (r,\ell,s) =: v$ with $r=1$. Hence $v = \exp(\mathrm{c}_1(L))$ for some line bundle L on X. If $\psi(0,0,1) = (0,0,1)$ take the equivalence $\Phi = (-)\otimes L^\vee$ (if $\psi(0,0,1) = -(0,0,1)$ just consider $\Phi = (-)\otimes L^\vee[1]$). Since ψ is G-equivariant, $\iota^*v = v$ and thus L and Φ are G-equivariant as well. The composition $\varphi = \Phi^H\circ\psi$ has the property that $\varphi|_{H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})} = \mathrm{id}_{H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})}$.

Since X is the universal cover of Y, there exists an ample line bundle $L'\in\mathrm{Pic}(X)$ such that $\iota^*L'\cong L'$. By [1, Prop. VII.21.1], there are (-2)-curves $C_1,\dots,C_r$ such that

(A.1) for any $i\in\{1,\dots,r\}$, $(C_i,\iota^*C_i)=0$ (by $(-,-)$ we denote the cup-product on $H^2(X,\mathbb{Z})$);

(A.2) the isometry $w = s_{C_1} \circ s_{\iota^* C_1} \circ \cdots \circ s_{C_r} \circ s_{\iota^* C_r}$ is such that $w(L') = \pm L'$.

Here $s_{C_i} \in \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))$ stands for the reflection on cohomology with respect to the vector $(0,[C_i],0)$ (see [1, Sect. VIII.21]). Moreover, by its very definition, $\iota^* \circ w = w \circ \iota^*$. Hence, by the Torelli Theorem for K3 surfaces, there is an isomorphism $f : X \xrightarrow{\sim} X$ with $(j\circ)f_* = w \circ \varphi$ and $\iota^* \circ f = f \circ \iota^*$.

Take now the spherical twists $T_{\mathcal{O}_{C_i}(-1)}$. Since $T^H_{\mathcal{O}_{C_i}(-1)} = s_{C_i}$ and, by [36, Thm. 1.2], $T_{\mathcal{O}_{C_i}(-1)} \circ T_{\mathcal{O}_{\iota^* C_i}(-1)} = T_{\mathcal{O}_{\iota^* C_i}(-1)} \circ T_{\mathcal{O}_{C_i}(-1)}$, the equivalences $\iota^* \circ T_{\mathcal{O}_{C_i}(-1)} \circ T_{\mathcal{O}_{\iota^* C_i}(-1)}$ and $T_{\mathcal{O}_{C_i}(-1)} \circ T_{\mathcal{O}_{\iota^* C_i}(-1)} \circ \iota^*$ are equal. Hence the equivalence

$$\Psi := T_{\mathcal{O}_{C_1}(-1)} \circ T_{\mathcal{O}_{\iota^* C_1}(-1)} \circ \cdots \circ T_{\mathcal{O}_{C_r}(-1)} \circ T_{\mathcal{O}_{\iota^* C_r}(-1)} \circ \Phi^{-1} \circ f_*$$

is such that $\Psi^H = (j\circ)g$ and $\iota^* \circ \Psi = \Psi \circ \iota^*$.

Case 2. Suppose now that $\psi(0,0,1) = (r,\ell,s) =: v$, with $r \neq 0$.

Up to composing with $-\mathrm{id}_{\widetilde{H}(X,\mathbb{Z})}$ (which, on the level of derived categories, just corresponds to the shift [1]), we may assume $r > 0$. Then consider the moduli space $M := M_h(v)$ of stable (with respect to an ample polarization h) sheaves with Mukai vector v. The universal family $\mathcal{E}$ on $M \times X$ induces an equivalence $\Phi_{\mathcal{E}} : \mathrm{D^b}(M) \xrightarrow{\sim} \mathrm{D^b}(X)$.

Claim. *The involution $\iota : X \to X$ and the equivalence $\Phi_{\mathcal{E}}$ induce a fixed-point-free involution $\tilde{\iota}$ on M such that $\mathcal{E}$ is equivariant with respect to the group $\widetilde{G} = \langle \tilde{\iota} \times \iota \rangle$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{D^b}(M) \arrow[r, "\Psi"] \arrow[d, "\Phi_{\mathcal{E}}"'] & \mathrm{D^b}(M) \arrow[d, "\Phi_{\mathcal{E}}"] \\
\mathrm{D^b}(X) \arrow[r, "\iota^*"] & \mathrm{D^b}(X),
\end{tikzcd}$$

where Ψ is the natural composition. For any $x \in M$, $\iota^* \mathcal{E}_x$ is stable and $v(\iota^* \mathcal{E}_x) = v(\mathcal{E}_x)$ (essentially because v is invariant under the action of ι^*). Hence $\Psi(\mathcal{O}_x) = \Phi_{\mathcal{E}}^{-1}(\iota^*(\Phi_{\mathcal{E}}(\mathcal{O}_y))) = \Phi_{\mathcal{E}}^{-1}(\iota^* \mathcal{E}_x) = \mathcal{O}_y$, for some closed point $y \in M$. By [17, Cor. 5.23], $\Psi = ((-)\otimes L) \circ f_*$, for some line bundle $L \in \mathrm{Pic}(X)$ and an isomorphism $f : M \xrightarrow{\sim} X$.

Observe now that Ψ^H is not the identity. Indeed, $\Psi^H|_{T(M)} = ((\Phi^H_{\mathcal{E}})^{-1} \circ \iota^*|_{\Phi^H_{\mathcal{E}}(T(M))} \circ \Phi^H_{\mathcal{E}})|_{T(M)}$. By [29, Thm. 1.5], $\Phi^H_{\mathcal{E}}(T(M)) = T(X) \subseteq \Lambda_-$. Hence $\Phi^H_{\mathcal{E}}|_{T(M)} = -\mathrm{id}_{T(M)}$.

Take now the commutative diagram

$$\begin{tikzcd}
\widetilde{H}(M,\mathbb{Z}) \arrow[r, "\Psi^H"] \arrow[d, "\Phi^H_{\mathcal{E}}"'] & \widetilde{H}(M,\mathbb{Z}) \arrow[d, "\Phi^H_{\mathcal{E}}"] \\
\widetilde{H}(X,\mathbb{Z}) \arrow[r, "\iota^*"] \arrow[d, "\psi"'] & \widetilde{H}(X,\mathbb{Z}) \arrow[d, "\psi"] \\
\widetilde{H}(X,\mathbb{Z}) \arrow[r, "\iota^*"] \arrow[d, "\Psi^H_1"'] & \widetilde{H}(X,\mathbb{Z}) \arrow[d, "\Psi^H_1"] \\
\widetilde{H}(X,\mathbb{Z}) \arrow[r, "\iota^*"] & \widetilde{H}(X,\mathbb{Z}),
\end{tikzcd} \tag{3.4}$$

where Ψ^H_1 is the isometry induced on cohomology by an equivalence $\Psi_1 : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$ with the following properties:

(B.1) it is the composition of the tensorization with a(n equivariant) line bundle on X and the composition of spherical twists $T_{\mathcal{O}_{C_i}(-1)} \circ T_{\iota^*\mathcal{O}_{C_i}(-1)}$, where $C_1, \ldots, C_r$ are rational curves on the K3 surface X as in (A.1);

(B.2) there exists an isomorphism $g_1 : M \xrightarrow{\sim} X$ such that $(j\circ)(g_1)_* = \Psi_1^H \circ \psi \circ \Phi_{\mathcal{E}}^H$.

Notice that such a functor exists due to the fact that ψ is equivariant.

Diagram (3.4) can be rewritten on the level of derived categories in the following way

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(M) \arrow[r, "\Psi"] \arrow[d, "(g_1)_*"'] & \mathrm{D}^{\mathrm{b}}(M) \arrow[d, "(g_1)_*"] \\ \mathrm{D}^{\mathrm{b}}(X) \arrow[r, "\iota^*"] & \mathrm{D}^{\mathrm{b}}(X). \end{tikzcd}$$

This proves that $\Psi = f_*$. Moreover, f has no fixed points. Indeed, suppose that for, some closed point $x \in M$,

$$\mathcal{O}_x = \Psi(\mathcal{O}_x) = f^*(\iota^*(f_*(\mathcal{O}_x))).$$

Then $f_*(\mathcal{O}_x) = \iota^*(f_*(\mathcal{O}_x))$. But ι^* does not fix any skyscraper sheaf by definition. This concludes the proof of the claim. $\square$

Now put $\Phi = \Phi_{\mathcal{E}}$ if $r > 0$ and $\Phi = \Phi_{\mathcal{E}[1]}$ if $r < 0$. Since M has a fixed-point-free involution $\tilde{\iota}$ making the equivalence Φ equivariant, $(\Phi_{\mathcal{E}}^{-1})^H \circ \psi$ is as in Case 1. So one repeats again the same argument getting an equivariant isomorphism $f_1 : M \xrightarrow{\sim} X$.

Case 3. Finally assume that $\psi(0,0,1) = (0,\ell,s) =: v$, with $\ell \neq 0$.

Since $\iota^* v = v$, take a G-equivariant line bundle $L' \in \mathrm{Pic}(X)$ such that $(L',\ell) + s \neq 0$. The equivalence $T_{\mathcal{O}_X}$ is G-equivariant. Thus the composition $(T_{\mathcal{O}_X} \circ ((-) \otimes L'))^H \circ \psi$ is as in Case 2 and we proceed as before.

To show that the group $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))_G/G$ is contained in the image of Π as an index-2 subgroup it is enough to show that all the equivalences involved in the above construction are orientation preserving. But for this we can just apply [19, Sect. 5]. $\square$

Remark 3.5. The same proof also shows that two Enriques surfaces Y_1 and Y_2 are derived equivalent if and only if there is a Hodge isometry $\widetilde{H}(X_1,\mathbb{Z}) \cong \widetilde{H}(X_2,\mathbb{Z})$, where X_1 and X_2 are the universal covers of Y_1 and Y_2. Notice that, by [11, Prop. 6.1], $\mathrm{D}^{\mathrm{b}}(Y_1) \cong \mathrm{D}^{\mathrm{b}}(Y_2)$ if and only if $Y_1 \cong Y_2$.

Following [6], given an Enriques surface Y and its universal cover X, define the open subset $\mathcal{P}(X) \subseteq \mathcal{N}(X) \otimes \mathbb{C}$ consisting of those vectors whose real and imaginary parts span a positive definite two plane in $\mathcal{N}(X) \otimes \mathbb{R}$. Denote by $\mathcal{P}^+(X)$ one of the two connected components of $\mathcal{P}(X)$. If $\Delta(X)$ is the set of vectors in $\mathcal{N}(X)$ with self-intersection -2, Bridgeland considers

$$\mathcal{P}_0^+(X) := \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp.$$

Define $\mathcal{P}_0^+(Y) := \mathrm{Forg}_{G*}^\vee(\mathcal{P}_0^+(X))_G$ and take the group $\mathrm{Aut}^0(\mathrm{D}^{\mathrm{b}}(Y))$ ($\mathrm{Aut}^0(\mathrm{D}^{\mathrm{b}}(X))$) of those autoequivalences preserving $\Sigma(Y)$ ($\mathrm{Stab}_{\mathcal{N}}^\dagger(\mathrm{D}^{\mathrm{b}}(X))$) and inducing the identity on cohomology via the morphisms Π.

Corollary 3.6. *The map $\mathcal{Z} : \Sigma(Y) \to \mathcal{N}(Y) \otimes \mathbb{C}$ in Proposition 3.1 defines a covering map onto $\mathcal{P}_0^+(Y)$ such that $\mathrm{Aut}^0(\mathrm{D}^{\mathrm{b}}(Y))/\langle (-) \otimes \omega_Y \rangle$ acts as the group of deck transformations.*

Proof. The fact that $\mathcal{Z}$ is a covering map is an easy consequence of the commutativity of diagram (3.1) in Proposition 3.1.

Let $\Phi \in \mathrm{Aut}^0(\mathrm{D^b}(X))$ be such that $\Phi(\Gamma_X \cap \mathrm{Stab}^\dagger_{\mathcal{N}}(\mathrm{D^b}(X))) = \Gamma_X \cap \mathrm{Stab}^\dagger_{\mathcal{N}}(\mathrm{D^b}(X))$. By definition, $\Phi(\sigma) = \iota^*(\Phi(\sigma)) = \Phi'(\iota^*(\sigma)) = \Phi'(\sigma)$, for some autoequivalence $\Phi' \in \mathrm{Aut}^0(\mathrm{D^b}(X))$ and for all $\sigma \in \Gamma_X \cap \mathrm{Stab}^\dagger_{\mathcal{N}}(\mathrm{D^b}(X))$. Due to [6, Thm. 1.1], this is enough to conclude that $\Phi = \Phi'$ and $\Phi \in \mathrm{Aut}^0(\mathrm{D^b}(X))_G$.

Since $\mathrm{Aut}^0(\mathrm{D^b}(X))_G$ injects into $\mathrm{Aut}(\mathrm{D^b}(X))_G/G$, the commutativity of diagram (3.1) in Proposition 3.1 and the isomorphism (3.3) allow us to conclude that $\mathrm{Aut}^0(\mathrm{D^b}(Y))/\langle(-)\otimes\omega_Y\rangle$ acts as the group of deck transformations. □

Therefore we can state the following conjecture generalizing [6, Conj. 1.2] to the case of Enriques surfaces.

Conjecture 3.7. *The group* $\mathrm{Aut}(\mathrm{D^b}(Y))$ *preserves* $\Sigma(Y)$ *and, moreover,* $\Sigma(Y)$ *is connected and simply connected.*

From the previous conjecture would follow the existence of the short exact sequence

$$1 \longrightarrow \pi_1(\mathcal{P}^+_0(Y)) \longrightarrow \mathrm{Aut}(\mathrm{D^b}(Y))/\langle(-)\otimes\omega_Y\rangle \longrightarrow \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))_G/G \longrightarrow 1.$$

In the forthcoming section we will prove (Proposition 3.10) that $\Sigma(Y)$ is connected when Y is a generic Enriques surface.

3.3. **Spherical objects and generic Enriques surfaces.**

Using the surjectivity of the period map for Enriques and K3 surfaces ([14, 15]) it is quite easy to prove that the universal cover X of an Enriques surface Y which is generic in the moduli space of surfaces of this type has the property

$$\mathrm{Pic}(X) \cong \Lambda_+. \tag{3.5}$$

Remark 3.8. In the generic case, for any $v \in \mathrm{Pic}(X)$, by (3.5), v^2 is divisible by 4. Hence X does not contain rational curves. Since a rational curve in Y lifts to a pair of (disjoint) rational curves in X, Y does not contain rational curves neither.

Let us prove the following easy result.

Lemma 3.9. *Let* Y *be a generic Enriques surface. Then the involution* $\iota : X \to X$ *on the universal cover of* Y *is such that* $\iota^*|_{\mathrm{Pic}(X)} = \mathrm{id}_{\mathrm{Pic}(X)}$ *and* $\iota^*|_{T(X)} = -\mathrm{id}_{T(X)}$.

Proof. By [15, Thm. 5.1], for the K3 surface X there exists an isometry $\psi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda$ such that the following diagram commutes

$$\begin{tikzcd} H^2(X,\mathbb{Z}) \arrow[r, "\iota^*"] \arrow[d, "\psi"'] & H^2(X,\mathbb{Z}) \arrow[d, "\psi"] \\ \Lambda \arrow[r, "\theta"] & \Lambda. \end{tikzcd} \tag{3.6}$$

Here the isometry θ is $-\mathrm{id}$ on the first copy of U in Λ, exchanges the other two copies of U and is the identity on $E_8(-1)^{\oplus 2}$. By the genericity of Y, $\mathrm{Pic}(X)$ is mapped isometrically by ψ onto the invariant part Λ_+ of Λ and this concludes the proof. □

We can now say more about the connected component $\Sigma(Y)$ defined in the previous section.

Proposition 3.10. *Let* Y *be a generic Enriques surface. Then* $\mathrm{Stab}^\dagger_{\mathcal{N}}(\mathrm{D^b}(X)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ *is isomorphic to* $\Sigma(Y)$*, which is then connected.*

Proof. By Lemma 3.9, $\mathcal{P}_0^+(Y) = \mathcal{P}_0^+(X)_G = \mathcal{P}_0^+(X)$. Hence Γ_X is an open and closed subset in the connected component $\mathrm{Stab}^\dagger_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ of the same dimension. $\square$

In this section we will also be interested in characterizing special objects in $\mathrm{D}^{\mathrm{b}}(Y)$. In particular, recall the following definition:

Definition 3.11. Let Z be a smooth projective variety with canonical bundle ω_Z.
(i) An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(Z)$ is *rigid* if $(\mathcal{E},\mathcal{E})^1 = 0$ and $\mathcal{E} \cong \mathcal{E} \otimes \omega_Z$;
(ii) An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(Z)$ is *spherical* if $(\mathcal{E},\mathcal{E})^i = 1$ for $i \in \{0,2\}$ and otherwise zero and $\mathcal{E} \cong \mathcal{E} \otimes \omega_Z$.

For a variety Z we denote by $\mathrm{Sph}(Z)$ and $\mathrm{Rig}(Z)$ the sets of spherical and rigid objects in $\mathrm{D}^{\mathrm{b}}(Z)$ respectively.

Lemma 3.12. *Let $\pi : X \to Y$ be as before. If $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(Y)$ is rigid, then there exists $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$ rigid and such that $\mathcal{E} = \pi_*(\mathcal{F})$.*

Proof. Since $\mathcal{E}$ is rigid, $\mathcal{E} \cong \mathcal{E} \otimes \omega_Y$. By [17, Lemma 7.16], there exists $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$ with $\mathcal{E} = \pi_*\mathcal{F}$. The easy remark that $\pi^*\pi_*\mathcal{E} = \mathcal{E} \oplus \iota^*\mathcal{E}$ yields the following list of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}^i_{\mathrm{D}^{\mathrm{b}}(Y)}(\mathcal{E},\mathcal{E}) &\cong \mathrm{Hom}^i_{\mathrm{D}^{\mathrm{b}}(Y)}(\pi_*\mathcal{F},\pi_*\mathcal{F}) \\
&\cong \mathrm{Hom}^i_{\mathrm{D}^{\mathrm{b}}(X)}(\pi^*\pi_*\mathcal{F},\mathcal{F}) \\
&\cong \mathrm{Hom}^i_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{F}\oplus\iota^*\mathcal{F},\mathcal{F}) \\
&\cong \mathrm{Hom}^i_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{F},\mathcal{F}) \oplus \mathrm{Hom}^i_{\mathrm{D}^{\mathrm{b}}(X)}(\iota^*\mathcal{F},\mathcal{F}).
\end{aligned}$$

Therefore if $\mathcal{E}$ is rigid, $\mathcal{F}$ is rigid as well. $\square$

Lemma 3.13. *Consider in $\mathrm{D}^{\mathrm{b}}(Y)$ a triangle*

$$\mathcal{A} \longrightarrow \mathcal{E} \longrightarrow \mathcal{B} \longrightarrow \mathcal{A}[1]$$

such that $(\mathcal{A},\mathcal{B})^r = (\mathcal{B},\mathcal{B})^s = 0$, for $r \le 0$ and $s < 0$. Assume moreover that $\mathcal{E} \cong \mathcal{E} \otimes \omega_Y$ and $\mathrm{Hom}^{\le 0}_{\mathrm{D}^{\mathrm{b}}(Y)}(\mathcal{A},\mathcal{B}\otimes\omega_Y) = 0$. Then $\mathcal{A} \cong \mathcal{A} \otimes \omega_Y$, $\mathcal{B} \cong \mathcal{B} \otimes \omega_Y$ and $(\mathcal{A},\mathcal{A})^1 + (\mathcal{B},\mathcal{B})^1 \le (\mathcal{E},\mathcal{E})^1$.

Proof. Let $f : \mathcal{E} \to \mathcal{E} \otimes \omega_Y$ be the isomorphism in the hypotheses. Since $\omega_Y^{\otimes 2} = \mathcal{O}_Y$, the isomorphisms $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y)}(\mathcal{A},\mathcal{B}\otimes\omega_Y) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y)}(\mathcal{A}\otimes\omega_Y,\mathcal{B}) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y)}(\mathcal{A},\mathcal{B}) = 0$ imply the existence of morphisms f_1, f_2, g_1 and g_2 making the following diagram commutative:

(3.7)
$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "f_1"'] \arrow[dd, bend right=60, "h_1"'] & \mathcal{E} \arrow[r] \arrow[d, "f"'] \arrow[dd, bend left=50, "\mathrm{id}"] & \mathcal{B} \arrow[d, "f_2"'] \arrow[dd, bend left=60, "h_2"] \\
\mathcal{A}\otimes\omega_Y \arrow[r] \arrow[d, "g_1"'] & \mathcal{E}\otimes\omega_Y \arrow[r] \arrow[d, "f^{-1}"'] & \mathcal{B}\otimes\omega_Y \arrow[d, "g_2"'] \\
\mathcal{A} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{B}.
\end{tikzcd}$$

Since $\mathrm{Hom}^{-1}_{\mathrm{D}^{\mathrm{b}}(Y)}(\mathcal{A},\mathcal{B}) = 0$, the morphisms h_1 and h_2 are uniquely determined and so they must be the identity. Repeating the same argument, starting from the triangle $\mathcal{A}\otimes\omega_Y \longrightarrow \mathcal{E}\otimes\omega_Y \longrightarrow \mathcal{B}\otimes\omega_Y \longrightarrow \mathcal{A}\otimes\omega_Y[1]$, one immediately concludes that f_1 and f_2 are isomorphisms.

The fact that $(\mathcal{A},\mathcal{A})^1 + (\mathcal{B},\mathcal{B})^1 \le (\mathcal{E},\mathcal{E})^1$ is now obtained repeating the proof of [18, Lemma 2.7], simply using Serre duality and the isomorphisms f_1 and f_2 previously defined. $\square$

In the specific case of Enriques surfaces we will need a third class of objects in the derived category.

Definition 3.14. An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(Y)$ is *quasi-spherical* if $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{F} \otimes \omega_Y$ where $\pi^*\mathcal{F}$ is a G-invariant spherical object.

We now complete the proof of the last statement in Theorem 1.2.

Proposition 3.15. *Let Y be a generic Enriques surface. Then* $\mathrm{Sph}(Y)$ *is empty, while* $\mathrm{Rig}(Y)$ *consists of objects which are extensions of quasi-spherical objects.*

Proof. Let $\mathcal{E} \in \mathrm{D^b}(Y)$ be spherical. Due to Lemma 3.12, there exists $\mathcal{G} \in \mathrm{Rig}(X)$ such that $\mathcal{E} = \pi_*\mathcal{G}$. Then, by the argument in the proof of Lemma 3.12,

$$\mathrm{Hom}^i_{\mathrm{D^b}(Y)}(\mathcal{E},\mathcal{E}) \cong \mathrm{Hom}^i_{\mathrm{D^b}(X)}(\mathcal{G},\mathcal{G}) \oplus \mathrm{Hom}^i_{\mathrm{D^b}(X)}(\iota^*\mathcal{G},\mathcal{G}).$$

This implies that $\chi(\mathcal{E},\mathcal{E}) = \chi(\mathcal{G},\mathcal{G}) + \chi(\iota^*\mathcal{G},\mathcal{G}) = 2\chi(\mathcal{G},\mathcal{G}) = 4$. The penultimate equality is due to the fact that, being Y generic, ι^* acts as the identity on $H^0(X,\mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X,\mathbb{Z})$ (Lemma 3.9). In particular, $\mathcal{E}$ is not a (-2)-class.

Suppose now that $\mathcal{E} \in \mathbf{Coh}(Y)$ is rigid and consider the short exact sequence

$$0 \longrightarrow \mathcal{E}_{\mathrm{tor}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{E}_{\mathrm{tor}}$ is the torsion part of $\mathcal{E}$ and $\mathcal{F}$ is torsion free. Since, clearly,

$$\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{E}_{\mathrm{tor}},\mathcal{F}) = \mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{E}_{\mathrm{tor}},\mathcal{F}\otimes\omega_Y) = 0,$$

we can now apply Lemma 3.13, concluding that $\mathcal{E}_{\mathrm{tor}}$ and $\mathcal{F}$ are both rigid objects.

Fix an ample polarization ℓ on Y and consider the HN-filtration of $\mathcal{F}$ and the exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0,$$

where $\mathcal{F}_2$ is the first μ_ℓ-semistable factor in the filtration. Since ω_Y is a torsion class, $\mu_\ell^-(\mathcal{F}_1) > \mu_\ell(\mathcal{F}_2) = \mu_\ell(\mathcal{F}_2\otimes\omega_Y)$ (here $\mu_\ell^-(\mathcal{F}_1)$ is the slope of the last μ_ℓ-semistable factor in the HN-filtration of $\mathcal{F}_1$). Therefore, $\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{F}_1,\mathcal{F}_2) = \mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{F}_1,\mathcal{F}_2\otimes\omega_Y) = 0$ and Lemma 3.13 applies allowing us to conclude that $\mathcal{F}_1$ and $\mathcal{F}_2$ are rigid as well. Thus, by induction on the length of the HN-filtration, we can assume that $\mathcal{F}$ is μ_ℓ-semistable.

Take now a μ_ℓ-stable factor $\mathcal{E}$ of $\mathcal{F}$ with a morphism $\mathcal{F} \to \mathcal{E}$. Then there exist $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F}_2$ is extension of $\mathcal{E}$, they fit in the short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0,$$

and $\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{F}_1,\mathcal{F}_2) = \mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{F}_1,\mathcal{E}) = 0$. Obviously $\mathcal{E}\otimes\omega_Y$ is again μ_ℓ-stable and $\mu_\ell(\mathcal{E}) = \mu_\ell(\mathcal{E}\otimes\omega_Y)$. Suppose that $\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{F}_1,\mathcal{E}\otimes\omega_Y) \neq 0$. Then, as before, there exist $\mathcal{F}_3$, $\mathcal{F}_4$ and a short exact sequence

$$0 \longrightarrow \mathcal{F}_3 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_4 \longrightarrow 0,$$

where $\mathcal{F}_4$ is extension of $\mathcal{E}\otimes\omega_Y$ and $\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{F}_3,\mathcal{E}\otimes\omega_Y) = 0$. If $\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{F}_3,\mathcal{E}) \neq 0$, we continue filtering until we get a short exact sequence

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}_2 \longrightarrow 0, \tag{3.8}$$

with $\mathcal{G}_2$ extension of $\mathcal{E}$ and $\mathcal{E}\otimes\omega_Y$ and $\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{G}_1,\mathcal{E}) = \mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{G}_1,\mathcal{E}\otimes\omega_Y) = 0$. Notice that the process clearly terminates, since JH-filtrations are finite.

By construction, $\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{G}_1,\mathcal{G}_2) = \mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{G}_1,\mathcal{G}_2\otimes\omega_Y) = 0$ and so we are in good shape to apply Lemma 3.13 to (3.8) concluding that $\mathcal{G}_1$ and $\mathcal{G}_2$ are both rigid and extension of $\mathcal{E}$ and $\mathcal{E}\otimes\omega_Y$.

Since $\chi(\mathcal{E},\mathcal{E}) = \chi(\mathcal{E},\mathcal{E}\otimes\omega_Y) = \chi(\mathcal{E}\otimes\omega_Y,\mathcal{E}\otimes\omega_Y)$, the following holds true

$$0 < \chi(\mathcal{F},\mathcal{F}) = k^2\chi(\mathcal{E},\mathcal{E}),$$

where k is the number of μ_ℓ-stable factors of $\mathcal{F}$.

Since $\mathcal{E}$ is stable and $\mu_\ell(\mathcal{E}) = \mu_\ell(\mathcal{E}\otimes\omega_Y)$, if $\mathrm{Hom}^2_{\mathrm{D^b}(Y)}(\mathcal{E},\mathcal{E}) \neq 0$, then $\mathcal{E} \cong \mathcal{E}\otimes\omega_Y$. Hence $\chi(\mathcal{E},\mathcal{E}) = 2 - (\mathcal{E},\mathcal{E})^1 > 0$ implies $(\mathcal{E},\mathcal{E})^1 = 0$ (notice that $(\mathcal{E},\mathcal{E})^1$ is even in this case). In particular $\mathcal{E}$ should be spherical, which is impossible.

Suppose now that $\mathrm{Hom}^2_{\mathrm{D^b}(Y)}(\mathcal{E},\mathcal{E})=0$. Obviously, $\chi(\mathcal{E},\mathcal{E})=1-(\mathcal{E},\mathcal{E})^1>0$ and so $(\mathcal{E},\mathcal{E})^1=0$. Thus $\mathcal{F}\cong\mathcal{E}^{\oplus m}\oplus(\mathcal{E}\otimes\omega_Y)^{\oplus m}$, for some positive integer m. Let us show that $\mathcal{E}$ is locally free. Indeed, it is enough to apply Lemma 3.13 to the triangle

$$\mathcal{T}[-1]\longrightarrow\mathcal{F}\longrightarrow\mathcal{F}^{\vee\vee}\longrightarrow\mathcal{T},$$

where $\mathcal{T}$ is a torsion sheaf supported on points. Since

$$\mathrm{Hom}^{\leq 0}_{\mathrm{D^b}(Y)}(\mathcal{T}[-1],\mathcal{F}^{\vee\vee})=\mathrm{Hom}^{\leq 0}_{\mathrm{D^b}(Y)}(\mathcal{T}[-1],\mathcal{F}^{\vee\vee}\otimes\omega_Y)=0,$$

Lemma 3.13 implies once more that $\mathcal{T}$ should be rigid which is a contradiction unless $\mathcal{T}=0$. Hence $\mathcal{F}$ is locally free. Observe that $\pi^*\mathcal{E}$ is spherical. Indeed, by adjunction, $\mathrm{Hom}^i_{\mathrm{D^b}(X)}(\pi^*\mathcal{E},\pi^*\mathcal{E})=\mathrm{Hom}^i_{\mathrm{D^b}(Y)}(\mathcal{E},\pi_*\pi^*\mathcal{E})=\mathrm{Hom}^i_{\mathrm{D^b}(Y)}(\mathcal{E},\mathcal{E}\oplus\mathcal{E}\otimes\omega_Y)$ and thus, by Serre duality,

$$\mathrm{Hom}^i_{\mathrm{D^b}(X)}(\pi^*\mathcal{E},\pi^*\mathcal{E})\cong\begin{cases}\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{E},\mathcal{E}) & \text{if } i\in\{0,2\}\\ 0 & \text{otherwise.}\end{cases}$$

The fact that $\pi^*\mathcal{E}$ is G-invariant is now easy.

Consider a rigid complex $\mathcal{E}\in\mathrm{Rig}(Y)$ and let N be the maximal integer such that $\mathcal{H}^N(\mathcal{E})\neq 0$. Hence we have the following triangle

$$\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow\mathcal{H}^N(\mathcal{E})[-N]$$

with $\mathrm{Hom}^{\leq 0}_{\mathrm{D^b}(Y)}(\mathcal{F},\mathcal{H}^N(\mathcal{E})[-N])=\mathrm{Hom}^{\leq 0}_{\mathrm{D^b}(Y)}(\mathcal{F},\mathcal{H}^N(\mathcal{E})\otimes\omega_Y[-N])=0$. Lemma 3.13 implies that $\mathcal{H}^N(\mathcal{E})$ is rigid and hence locally free. The same happens for the other cohomology sheaves, proceeding by induction on $\mathcal{F}$. $\square$

Remark 3.16. The case of generic Enriques surfaces is interesting from many viewpoints. Namely they provides examples of smooth surfaces with no spherical objects but plenty of rigid objects. This marks a deep difference with the case of the closest relatives of Enriques surfaces: K3 surfaces. Indeed, in that case, spherical objects are always present (at least in the untwisted case). As was proved in [18], the only way to reduce drastically the number of rigid and spherical objects is to pass to twisted or generic analytic K3 surfaces.

4. Local Calabi–Yau varieties

We consider some further situations in which the techniques of Section 2.2 can be applied. In Section 4.1 we compare the spaces of stability conditions on projective spaces $\mathbb{P}^N$and those on their canonical bundles $|\omega_{\mathbb{P}^N}|$. In this case the relation is slightly weaker than before and we obtain only a map between particular open subsets. For $N=1$, where the two stability manifolds can be completely described, the open subset of $\mathrm{Stab}(\mathbb{P}^1)$ in question corresponds to the "non-degenerate" stability conditions, while the one of $\mathrm{Stab}(|\omega_{\mathbb{P}^2}|)$ to a fundamental domain with respect to the action of the group of autoequivalences.

In Section 4.2 we examine spaces of stability conditions for Kleinian singularities (with particular attention for the case of A_2-singularities) and compare them with the spaces of stability conditions of the corresponding quivers. We then conclude by studying the case of local K3 surfaces (analytically) embedded into projective ones. Here the relation is even weaker: only few stability conditions on projective K3 surfaces induce stability conditions on the local ones. This can be intuitively understood by thinking of the space of stability conditions on the canonical bundle over the projective line as a sort of "limit" of the stability manifold for projective K3 surfaces.

4.1. Stability conditions on canonical bundles on projective spaces.

We start by recalling briefly the notion of mutation of exceptional objects in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N)$. Let $(\mathcal{E}, \mathcal{F})$ a strong exceptional pair. We define the objects $\mathcal{L}_{\mathcal{E}}\mathcal{F}$ and $\mathcal{R}_{\mathcal{F}}\mathcal{E}$ (which we call *left mutation* and *right mutation* respectively) by means of the distinguished triangles

$$\mathcal{L}_{\mathcal{E}}\mathcal{F} \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N)}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F},$$

$$\mathcal{E} \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N)}(\mathcal{E}, \mathcal{F})^* \otimes \mathcal{F} \to \mathcal{R}_{\mathcal{F}}\mathcal{E}.$$

A *mutation* of a strong exceptional collection $E = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$ is defined as a mutation of a pair of adjacent objects in E and it will be denoted by

$$\begin{aligned} \mathcal{R}_i E =& \{\mathcal{E}_0, \ldots, \mathcal{E}_{i-1}, \mathcal{E}_{i+1}, \mathcal{R}_{\mathcal{E}_{i+1}}\mathcal{E}_i, \mathcal{E}_{i+2}, \ldots, \mathcal{E}_n\}, \\ \mathcal{L}_i E =& \{\mathcal{E}_0, \ldots, \mathcal{E}_{i-1}, \mathcal{L}_{\mathcal{E}_i}\mathcal{E}_{i+1}, \mathcal{E}_i, \mathcal{E}_{i+2}, \ldots, \mathcal{E}_n\}, \end{aligned}$$

for $i \in \{0, \ldots, n-1\}$. By a result of Bondal ([4, Assertion 9.2]), applying a mutation to a complete strong exceptional collection consisting of sheaves on $\mathbb{P}^N$ we get a complete strong exceptional collection consisting of sheaves as well. An *iterated mutation* of a strong exceptional collection is the iterated application of a finite number of mutations.

An open subset of the stability manifold of $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N)$ was studied in [24]. Let $E = \{\mathcal{E}_0, \ldots, \mathcal{E}_N\}$ be a strong complete exceptional collection on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N)$ consisting of sheaves. Define $\Sigma_E(\mathbb{P}^N)$ as the union of the open subsets Θ_E (see Example 2.6) over all iterated mutations F of E. It is proved in [25, Cor. 3.20] that $\Sigma_E(\mathbb{P}^N) \subseteq \mathrm{Stab}(\mathbb{P}^N)$ is an open and connected $(N+1)$-dimensional submanifold. For $N = 1$ all strong complete exceptional collections are obtained as iterated mutations of

$$O := \{\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\}$$

and $\Sigma_O(\mathbb{P}^1)$ is equal to the full stability manifold $\mathrm{Stab}(\mathbb{P}^1)$ ([25, 30]).

Let X be the total space of the canonical bundle $V := \omega_{\mathbb{P}^N} \xrightarrow{\pi} \mathbb{P}^N$ and let $i : \mathbb{P}^N \hookrightarrow X$ denote the zero-section and C its image. Denote by $\mathrm{Stab}(X)$ the stability manifold of $\mathrm{D}^{\mathrm{b}}_0(X) := \mathrm{D}^{\mathrm{b}}_C(\mathbf{Coh}(X))$, the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$ whose objects have cohomology sheaves supported on C.

We want to compare the open subset $\Sigma_E(\mathbb{P}^N)$ with the space of stability conditions on X. Consider the exact faithful functor $i_* : \mathrm{D}^{\mathrm{b}}(\mathbb{P}^N) \to \mathrm{D}^{\mathrm{b}}_0(X)$. By dévissage, the essential image of i_* generates $\mathrm{D}^{\mathrm{b}}_0(X)$. Take the abelian category

$$\mathbf{Q}_0^E := \langle \mathcal{E}_0[N], \mathcal{E}_1[N-1] \ldots, \mathcal{E}_N \rangle \tag{4.1}$$

and consider the open set $U_0^E \subseteq \Theta_E$ consisting of stability conditions which have, up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, $\mathbf{Q}_0^E$ as heart. It is easy to see that $\mathbf{Q}_0^E$ is i_*-admissible. Then, by Proposition 2.12 (or Theorem 2.14), we have an open connected subset $\widetilde{U}_0^E$ of $\mathrm{Stab}(X)$ such that $i_*^{-1} : \widetilde{U}_0^E \xrightarrow{\sim} U_0^E$. Denote by Γ^E the open subset of $\mathrm{Stab}(X)$ defined as the union of the open subsets $\widetilde{U}_0^F$, for F an iterated mutation of E. By [25, Cor. 3.20] it is connected. By Lemma 2.8, we have the following result.

Proposition 4.1. *The morphism $i_*^{-1} : \Gamma^E \to \Sigma_E(\mathbb{P}^N)$ is an open embedding,*

Remark 4.2. Notice that the image of the morphism i_*^{-1} consists of those stability conditions in $\Sigma_E(\mathbb{P}^N)$ whose heart, up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, is *faithful*, i.e. its bounded derived category is equivalent to $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N)$. More precisely, up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, the heart of a stability condition in the image of i_*^{-1} is equivalent to the abelian category of finitely generated modules over the algebra $\mathrm{End}\,(\bigoplus \mathcal{G}_i)$, for $G = \{\mathcal{G}_0, \ldots, \mathcal{G}_N\}$ an iterated mutation of E.

Remark 4.3. The abelian categories $\Phi(\langle i_*\mathbf{Q}_0^E\rangle)$, for a Fourier-Mukai autoequivalence Φ of $\mathrm{D}_0^{\mathrm{b}}(X)$, are called in [7] *quivery subcategories.* If $\{\mathcal{S}_0,\dots,\mathcal{S}_N\}$ are the minimal objects of a quivery subcategory, then there is a canonical cyclic ordering in which

$$\mathrm{Hom}^k_{\mathrm{D}_0^{\mathrm{b}}(X)}(\mathcal{S}_i,\mathcal{S}_j)=0,\ \text{unless } 0\le k\le N \text{ and } i-j\equiv k \pmod N.$$

An ordered quivery subcategory is a quivery subcategory in which an ordering of its minimal objects is fixed and is compatible with the canonical cyclic ordering. Bridgeland proved [7, Thm. 4.11] that there is an action of the braid group B_{N+1} (i.e. the group generated by elements τ_i, indexed by the cyclic group $\mathbb{Z}/(N+1)\mathbb{Z}$, together with a single element r, subject to the relations $r\tau_i r^{-1}=\tau_{i+1}$, $r^{N+1}=0$ and, if $N\ge 2$, $\tau_i\tau_{i+1}\tau_i=\tau_{i+1}\tau_i\tau_{i+1}$ and for $j-i\ne\pm1$, $\tau_i\tau_j=\tau_j\tau_i$[1]) on the set of ordered quivery subcategories of $\mathrm{D}_0^{\mathrm{b}}(X)$ which essentially corresponds to tilting at minimal objects:

$$\begin{aligned} r\{\mathcal{S}_0,\dots,\mathcal{S}_N\} &:= \{\mathcal{S}_N,\mathcal{S}_0,\dots,\mathcal{S}_{N-1}\}\\ \tau_i\{\mathcal{S}_0,\dots,\mathcal{S}_N\} &:= \{\mathcal{S}_0,\dots,\mathcal{S}_{i-2},\mathcal{S}_i[-1],T_{\mathcal{S}_i}\mathcal{S}_{i-1},\mathcal{S}_{i+1},\dots,\mathcal{S}_N\},\end{aligned}$$

with $i\in\{1,\dots,N\}$. For the lower dimensional cases $N\in\{1,2\}$ this action is free (see [7, Thm. 5.6] for $N=2$, whose proof works also for $N=1$).

Consider the spherical twists $T_{i_*\mathcal{E}_k}$, for $\mathcal{E}_k\in E$, and take the subgroup G_E of $\mathrm{Aut}(\mathrm{D}_0^{\mathrm{b}}(X))$ generated by these functors, tensorizations by line bundles, automorphisms of X, and shifts. When $N=1$, [21, Thm. 1.3] or, in a more intrinsic way, Theorem 4.5, [8, Thm. 1.1] and the classification of the spherical objects in $\mathrm{D}_0^{\mathrm{b}}(X)$ (see [21]) yields $G_E=\mathrm{Aut}(\mathrm{D}_0^{\mathrm{b}}(X))$. (Notice that, for $N=1$, all autoequivalences of $\mathrm{D}_0^{\mathrm{b}}(X)$ are of Fourier-Mukai type, by [20, Appendix A].)

Define $\Sigma_E(\omega_{\mathbb{P}^N})$ as the open subset of $\mathrm{Stab}(X)$ consisting of stability conditions which are in Γ^E, modulo the action of the subgroup G_E. Using [7, Prop. 4.10] and [25, Cor. 3.20], it follows that $\Sigma_E(\omega_{\mathbb{P}^N})$ is connected. A further topological study of $\Sigma_E(\omega_{\mathbb{P}^N})$, using the description of $\Sigma_E(\mathbb{P}^N)$, is contained in [27].

Remark 4.4. Note that the unique open subset $\Sigma(\omega_{\mathbb{P}^2}) := \Sigma_{\{\mathcal{O}_{\mathbb{P}^2},\mathcal{O}_{\mathbb{P}^2}(1),\mathcal{O}_{\mathbb{P}^2}(2)\}}(\omega_{\mathbb{P}^2})$ is a little larger then the space $\mathrm{Stab}^0(|\omega_{\mathbb{P}^2}|)$ of [9, Prop. 2.4]. Essentially we include the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ and more degenerate stability conditions.

Unfortunately, for $N>1$, Proposition 4.1 cannot be improved. Thus we can just prove that there is an open subset of $\Sigma_E(\mathbb{P}^N)$, consisting of "non-degenerate" stability conditions, which is isomorphic to a fundamental domain in $\Sigma_E(\omega_{\mathbb{P}^N})$ with respect to the action of the group G_E.

This picture is clearer for the case $N=1$, when both the two spaces cover the full stability manifolds. Indeed, we have the following result, which is a particular case of a more general theorem due to Ishii, Uehara and Ueda [20]. Another proof of it may be found in [31]. Here we present a shorter proof which first appeared in [26]. The proof of simply-connectedness may be useful in more general situations (see [24, Cor. 4.12] and [27]). Moreover, it does not rely on the structure of the autoequivalences group, but only on Remark 4.3.

Theorem 4.5. $\Sigma(\omega_{\mathbb{P}^1}) := \Sigma_O(\omega_{\mathbb{P}^1}) = \mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$. *In particular,* $\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$ *is connected and its open subset* $\Gamma := \Gamma^O$ *is a fundamental domain for the action of the autoequivalences group. Moreover,* $\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$ *is simply-connected.*

Proof. We start by proving that $\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$ is connected. Say $\sigma\in\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$. For our convenience we take the abelian category $\mathcal{P}([0,1))$ as heart of the stability condition σ instead of $\mathcal{P}((0,1])$. This is no problem since, up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, these two categories are the same.

By [18, Prop. 2.9] we know that every stable factor of a spherical object must be spherical. It follows, since $K(|\omega_{\mathbb{P}^1}|)\cong\mathbb{Z}^2$, that there are at least two σ-stable spherical objects (with independent

[1]Notice that the condition $N\ge 2$ is missing in [7]. Indeed, if $N=1$, $\tau_0\tau_1\tau_0\ne\tau_1\tau_0\tau_1$.

classes in K-theory). Now, [21, Prop. 1.6] says that any spherical object is the image, up to shifts, of $\mathcal{O}_C$ via an autoequivalence $\Phi \in G := G_{\{\mathcal{O}_C, \mathcal{O}_C(1)\}}$. Since G preserves the connected component $\Sigma(|\omega_{\mathbb{P}^1}|)$, we may as well assume that $\mathcal{O}_C$ is a stable spherical object. Write $\mathcal{S}$ for another spherical stable object. Since $\mathcal{O}_C$ and $\mathcal{S}$ are σ-stable we may assume that, acting with $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, $\phi_\sigma(\mathcal{O}_C) = 0$ and that, up to shift, $\mathcal{S} \in \mathcal{P}([0,1))$ and $\mathrm{Hom}^i_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{O}_C, \mathcal{S}) \neq 0$ only if $i \in \{0,1\}$.

Suppose we know that $\mathcal{S} \cong \mathcal{O}_C(a)$, for some integer $a > 0$. To show that $\mathcal{O}_C(-1)[1] \in \mathcal{P}([0,1))$, consider the triangle

$$\mathcal{O}_C^{\oplus a+1} \xrightarrow{\psi} \mathcal{O}_C(a) \longrightarrow \mathcal{O}_C(-1)^{\oplus a}[1], \tag{4.2}$$

induced by the analogous one in $\mathrm{D}^\mathrm{b}(\mathbb{P}^1)$. We claim that ψ is an injection. For suppose not. Then, one can filter $\ker(\psi)$ as $\mathcal{G}' \longrightarrow \ker(\psi) \longrightarrow \mathcal{G}$, with $\mathcal{G}' \in \mathcal{P}(0,1)$ and $\mathcal{G} \in \mathcal{P}(0)$. The inclusion $\mathcal{G}' \hookrightarrow \mathcal{O}_C^{\oplus a+1}$ forces $\mathcal{G}' = 0$ and this means that $\ker(\psi)$ is semistable of phase 0. As $\mathcal{O}_C$ is stable, $\ker(\psi) \cong \mathcal{O}_C^{\oplus k}$, for some $k > 0$. This and the triangle above give the conclusion that $\mathrm{Hom}_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{O}_C, \mathcal{O}_C(-1)^{\oplus a}) \neq 0$, which is absurd. Therefore, $\mathcal{O}_C(-1)^{\oplus a}[1] \in \mathcal{P}([0,1))$, so that $\mathcal{O}_C(-1)[1] \in \mathcal{P}([0,1))$. Since $\langle \mathcal{O}_C(-1)[1], \mathcal{O}_C \rangle$ is the heart of a t-structure, and we have proved that $\langle \mathcal{O}_C(-1)[1], \mathcal{O}_C \rangle \subset \mathcal{P}([0,1))$, Lemma 2.3 in [25] yields $\langle \mathcal{O}_C(-1)[1], \mathcal{O}_C \rangle = \mathcal{P}([0,1))$. So, what we have shown is that, up to component preserving autoequivalences and the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, $\sigma \in \Sigma(|\omega_{\mathbb{P}^1}|)$.

If $\mathcal{S} \cong \mathcal{O}_C(b)[1]$, for some integer $b < 0$, we reason in a similar way by considering the triangle

$$\mathcal{O}_C^{\oplus(-b-1)} \longrightarrow \mathcal{O}_C(b)[1] \longrightarrow \mathcal{O}_C(-1)^{\oplus(-b)}[1].$$

Therefore it remains to prove that either $\mathcal{S} = \mathcal{O}_C(d)$, for some integer $d > 0$, or $\mathcal{S} = \mathcal{O}_C(d)[1]$, for some $d < 0$. By [21, Cor. 3.10],

$$\bigoplus_q \mathcal{H}^q(\mathcal{S}) \cong \mathcal{O}_C(a)^{\oplus r} \oplus \mathcal{O}_C(a+1)^{\oplus s}, \tag{4.3}$$

for some $a \in \mathbb{Z}$, $s \geq 0, r > 0$. So let us analyze all different possibilities.

Case $a \geq 2$ and $a \leq -3$. If $a \geq 2$, an easy check using the spectral sequences

$$\mathrm{Hom}^p_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{O}_C, \mathcal{H}^q(\mathcal{S})) \Longrightarrow \mathrm{Hom}^{p+q}_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{O}_C, \mathcal{S}) \tag{4.4}$$

$$\mathrm{Hom}^p_{\mathrm{D}^\mathrm{b}(C)}(\mathcal{O}_C, \mathcal{O}_C \otimes \wedge^q \mathcal{N}_C) \Longrightarrow \mathrm{Hom}^{p+q}_{\mathrm{D}^\mathrm{b}_0(X)}(i_*\mathcal{O}_C, i_*\mathcal{O}_C), \tag{4.5}$$

where $\mathcal{N}_C$ is the normal bundle of C, shows that $\mathcal{S} \cong \mathcal{O}_C(a)$. The case $a \leq -3$ is dealt with similarly, using the spectral sequence $\mathrm{Hom}^p_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{H}^{-q}(\mathcal{S}), \mathcal{O}_C) \Longrightarrow \mathrm{Hom}^{p+q}_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{S}, \mathcal{O}_C)$.

Case $a = 1, -2$. Both cases are completely similar, so we explicitly deal just with $a = 1$. Using (4.4), (4.5) and the fact that $\ker(d_2^{0,2}) = \mathrm{Hom}_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{O}_C, \mathcal{H}^2(\mathcal{S}))$ in (4.4), it is easy to see that $\mathcal{H}^q(\mathcal{S}) \neq 0$ only if $q \in \{0,1\}$. Again, analyzing these spectral sequences one sees that $\mathcal{H}^1(\mathcal{S})$ cannot contain $\mathcal{O}_C(2)$, so $\mathcal{H}^1(\mathcal{S}) = \mathcal{O}_C(1)^{\oplus r_1}$ for some $r_1 \geq 0$. The exact triangle

$$\mathcal{O}_C(1)^{\oplus r_0} \oplus \mathcal{O}_C(2)^{\oplus s} \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_C(1)^{\oplus r_1}[-1]$$

together with the vanishing $\mathrm{Hom}^2_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{O}_C(1), \mathcal{O}_C(2)) = 0$ imply that $\mathcal{O}_C(2)^{\oplus s}$ is a direct factor of $\mathcal{S}$. But as $\mathcal{S}$ is stable, this means that $\mathcal{S} = \mathcal{O}_C(2)$, $s = 1$ and $r = 0$, which contradicts our choice that $r > 0$. So, one must have $s = 0$. The long exact sequence arising from the application of the functor $\mathrm{Hom}_{\mathrm{D}^\mathrm{b}_0(X)}(\mathcal{S}[1], -)$ to the triangle

$$\mathcal{O}_C(1)^{\oplus r_0} \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_C(1)^{\oplus r_1}[-1]$$

shows that $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_0(X)}(\mathcal{S}[1],\mathcal{O}_C(1)^{\oplus r_0}) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_0(X)}(\mathcal{S}[1],\mathcal{S})$. It follows that the second map of the triangle above must be zero, for otherwise we get the contradiction that $\mathrm{Hom}^{-1}_{\mathrm{D}^{\mathrm{b}}_0(X)}(\mathcal{S},\mathcal{S})\neq 0$. Using the fact that the cohomology of $\mathcal{S}$ is concentrated in degrees 0 and 1, we get that $r_0=1$ and $\mathcal{S}=\mathcal{O}_C(1)$.

Case $a=0,-1$. Consider first $a=0$, i.e. $\bigoplus_q \mathcal{H}^q(\mathcal{S})\cong \mathcal{O}_C^{\oplus r}\oplus\mathcal{O}_C(1)^{\oplus s}$ and define $l(\mathcal{S}):=r+s$. If $l(\mathcal{S})=1$, one is done as above, so assume $l(\mathcal{S})>1$. By Lemma 4.2 in [21], we have that $l(T_{\mathcal{O}_C}(\mathcal{S}))<l(\mathcal{S})$. Also, $T_{\mathcal{O}_C}(\mathcal{O}_C)=\mathcal{O}_C[-1]$. We see therefore that $\mathcal{O}_C$ is stable for the stability condition $\tau=T_{\mathcal{O}_C}(\sigma)$ and $\bigoplus_q\mathcal{H}^q(\mathcal{S}')\cong\mathcal{O}_C(a')^{\oplus r'}\oplus\mathcal{O}_C(a'+1)^{\oplus s'}$, with $\mathcal{S}'=T_{\mathcal{O}_C}(\mathcal{S})$ and $r'+s'<l(\mathcal{S})$.

If $a'\notin\{0,-1\}$, then we conclude applying the previous cases. If $a'=0$, then proceed further repeating the same procedure and considering $\mathcal{S}'':=T_{O_C}(\mathcal{S}')$ with $l(\mathcal{S}'')<l(\mathcal{S}')$. If $\bigoplus_q\mathcal{H}^q(\mathcal{S}')\cong\mathcal{O}_C(-1)^{\oplus r'}\oplus\mathcal{O}_C^{\oplus s'}$, take $\Psi:=(T_{\mathcal{O}_C(-1)}(-)\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(2)[-1])$. Then, $l(\Psi(\mathcal{S}'))=l(T_{\mathcal{O}_C(-1)}(\mathcal{S}'))<l(\mathcal{S}')$, and $\Psi(\mathcal{O}_C)=\mathcal{O}_C$. Repeating the argument a finite number of times, we reduce either to the case $a\notin\{0,-1\}$ or to the case $l(\mathcal{S})=1$. In both cases we conclude that either $\mathcal{S}=\mathcal{O}_C(d)$, for some integer $d>0$, or $\mathcal{S}=\mathcal{O}_C(d)[1]$, for some $d<0$.

It remains to prove that $\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$ is simply-connected. Consider the open subset $V_0:=\widetilde{U}_0^O$, with $O:=\{\mathcal{O}_{\mathbb{P}^1},\mathcal{O}_{\mathbb{P}^1}(1)\}$. Then the natural map from V_0 to

$$\mathcal{C}:=\left\{(m_0,m_1,\phi_0,\phi_1)\in\mathbb{R}^4\,:\,m_i>0,\ \phi_1-1<\phi_0<\phi_1+1\right\},$$

sending $\sigma\in V_0$ to $(|Z(\mathcal{O}_C[1])|,|Z(\mathcal{O}_C(1))|,\phi(\mathcal{O}_C[1]),\phi(\mathcal{O}_C(1)))$ is an homeomorphism. Hence V_0 is contractible and is the union of the three contractible regions

$$\begin{aligned}V_{0,1}&:=\{x\in\mathcal{C}\,:\,\phi_0<\phi_1\}\cong\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})\\ V_{0,2}&:=\{x\in\mathcal{C}\,:\,\phi_0>\phi_1\}\cong\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})\\ V_{0,3}&:=\{x\in\mathcal{C}\,:\,\phi_0=\phi_1\}\cong\mathbb{C}\times\mathbb{R}_{>0},\end{aligned}$$

where $V_{0,2}=rV_{0,1}$ and r was defined in Remark 4.3. By the same remark and by the previous part of the proof, we know that

$$\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)=\bigcup_{l\in B_2} lV_0,$$

where lV_0 is the set of stability conditions whose heart is, up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, the quivery subcategory $l\langle\mathcal{O}_C[1],\mathcal{O}_C(1)\rangle$. Notice that $lV_0\cap V_0\neq\emptyset$ if and only if $l\in\{\tau_0^m,\tau_1^m,r\}$ for some integer $m\in\mathbb{Z}$. Moreover in such a case, for all $m\in\mathbb{Z}$,

$$\begin{aligned}\tau_1^mV_0\cap V_0&=V_{0,1}\\ \tau_0^mV_0\cap V_0&=V_{0,2}.\end{aligned}$$

In particular, observe that $lV_{0,3}\cap V_{0,3}=\emptyset$, a fact which will be used implicitly later in the proof. Thus we have the following easy consequence of Seifert–Van Kampen theorem: [$*$] Let V be the open subset of $\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$ consisting of stability conditions whose heart is, up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, a fixed quivery subcategory of $\mathrm{D}^{\mathrm{b}}_0(|\omega_{\mathbb{P}^1}|)$ and let $h\in L:=\{\tau_0,\tau_1,r\}$. Then $V\cup hV$ and $V\cup h^{-1}V$ are connected and simply-connected.

Now, fix a point $\sigma_0\in V_{0,3}$. Take a continuous loop $\alpha:[0,1]\to\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$ with base point σ_0. Assume that there exists $t_0\in[0,1)$ such that $\alpha([0,t_0))\subseteq V_0$ and $\alpha(t_0)\notin V_0$. Then there exist $i\in\{0,1\}$ and $m\in\mathbb{Z}$ such that $\alpha(t_0)\in V_1:=\tau_i^mV_0$. Continuing further, there exists $t_1\in((t_0,1])$ such that $\alpha([t_0,t_1))\subseteq V_1$ and $\alpha(t_1)\notin V_1$, and as before one has $j\in\{0,1\}$ and $n\in\mathbb{Z}$ such that $\alpha(t_1)\in V_2:=\tau_j^nV_1$. By compactness, and since $\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$ is a manifold, we may assume that

$V_k = V_0$, after a finite number of steps. Hence there exists $l \in F(L)$ such that $lV_0 = V_0$, where $F(L)$ is the free group generated by the set L. By Remark 4.3, the action of B_2 on the set of ordered quivery subcategories is free. Hence, up to multiplying by r, the class of l in B_2 is equal to the identity. This means that, up to contracting/adding pieces of the form hh^{-1} or $h^{-1}h$, with $h \in L$, by [$*$], we can assume

$$l = (k_1 s_1^{\pm 1} k_1^{-1}) \dots (k_a s_a^{\pm 1} k_a^{-1}),$$

with $k_1, \dots, k_a \in F(L)$ and $s_1, \dots, s_a \in R(L)$, where $R(L) := \left\{ r\tau_1 r\tau_0^{-1},\, r^2 \right\}$. But then, again by [$*$], the loop α can be split as a composition of contractible loops, i.e. it is contractible. Hence $\mathrm{Stab}(|\omega_{\mathbb{P}^1}|)$ is simply-connected. □

It seems reasonable that, in the case $N = 2$, by adding all "geometric" stability conditions constructed along the lines of Example 2.5, we may be able to describe an actual connected component of both $\mathrm{Stab}(\mathbb{P}^2)$ and $\mathrm{Stab}(|\omega_{\mathbb{P}^2}|)$. In such a case, we would have again a fundamental domain in (a connected component of) $\mathrm{Stab}(|\omega_{\mathbb{P}^2}|)$ isomorphic to an open "non-degenerate" subset of (a connected component of) $\mathrm{Stab}(\mathbb{P}^2)$.

Remark 4.6. (i) A statement analogous to Theorem 4.5 in the context of the local 3-dimensional flop $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$ is a particular case of [38, Thm. 1.1] and [39, Thm. 7.2].

(ii) We would like to mention that the case of $\mathrm{D}^{\mathrm{b}}_0(X)$ fits in with the equivariant examples considered in Section 2.3. Indeed, as pointed out by Tony Pantev, the category $\mathrm{D}^{\mathrm{b}}_0(\mathbf{Coh}(X))$ of coherent sheaves on X supported on the zero-section is equivalent to the derived category of $\omega^{\vee}_{\mathbb{P}^N}$-equivariant coherent sheaves $\mathrm{D}^{\mathrm{b}}_{\omega^{\vee}_{\mathbb{P}^N}}(\mathbf{Coh}(\mathbb{P}^N))$, with $\omega^{\vee}_{\mathbb{P}^N}$ acting trivially on $\mathbb{P}^N$. The proof will appear in [27]. Notice that the forgetful functor π_* that we consider in this case does not satisfy condition (Ind). Hence it does not induce stability conditions as in Section 2.2.

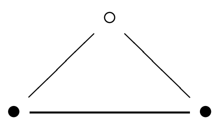
4.2. **Further examples.**

Kleinian singularities. Let $\pi : X \to \mathbb{C}^2/G$ be the minimal resolution of singularities, where G is a finite subgroup of $\mathrm{Sl}_2(\mathbb{C})$, and $\mathbf{D}$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$ consisting of those objects $\mathcal{E}$ such that $\pi_*\mathcal{E} = 0$. Stability conditions on $\mathbf{D}$ have been studied by Bridgeland in [8] (see also [37] for the special case of A_n-singularities). More specifically, he describes a connected component Σ which is a covering space of some period domain (see [8, Thm. 1.1] for more details).

As mentioned in [8], stability conditions on the category $\mathbf{D}$ should be related to those on the derived category of the path algebra of the Dynkin quiver associated to the group G. We can use our techniques to make this assertion precise. For simplicity we will explicitly treat only the case of A_2-singularity although the approach works in general.

Consider the Dynkin quiver $A_2 : \bullet \to \bullet$ and the derived category $\mathrm{D}^{\mathrm{b}}(A_2)$ of its path algebra. Notice that, up to shifts, the category $\mathrm{D}^{\mathrm{b}}(A_2)$ contains just three exceptional objects, two of which $\mathcal{S}_0[1]$ and $\mathcal{S}_1$ are the simples corresponding to the vertexes of A_2 and the third $\mathcal{S}_2$ is the unique indecomposable extension of $\mathcal{S}_0[1]$ and $\mathcal{S}_1$. Stability conditions on $\mathrm{D}^{\mathrm{b}}(A_2)$ were described in [25].

Via the McKay correspondence, the category $\mathrm{D}^{\mathrm{b}}(X)$ is equivalent to $\mathrm{D}^{\mathrm{b}}(\widehat{A}_2)$, the derived category of the preprojective algebra associated to the affine Dynkin diagram $\widehat{A}_2$:



By [8], the natural faithful functor $\mathrm{D}^{\mathrm{b}}(A_2) \longrightarrow \mathrm{D}^{\mathrm{b}}(\widehat{A}_2)$ factorizes as $\mathrm{D}^{\mathrm{b}}(A_2) \xrightarrow{i} \mathbf{D} \hookrightarrow \mathrm{D}^{\mathrm{b}}(\widehat{A}_2)$. Hence, the procedure described in Section 2.2 may be applied to relate $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(A_2))$ and $\mathrm{Stab}(\mathbf{D})$.

Consider $S_0 := \{\mathcal{S}_0, \mathcal{S}_1\}$, $S_1 := \{\mathcal{S}_1, \mathcal{S}_2\}$ and $S_2 := \{\mathcal{S}_2[-1], \mathcal{S}_0\}$ and the corresponding abelian categories $\mathbf{Q}_0^{S_i}$ defined as in (4.1). As in the previous section, it is easy to see that $\mathbf{Q}_0^{S_i}$ is i-admissible. Take the open subset $U^{S_i} \subseteq \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(A_2))$ consisting of stability conditions whose heart is $\mathbf{Q}_0^{S_i}$ (up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$) and let $\Gamma = U^{S_0} \cup U^{S_1} \cup U^{S_2}$.

By Proposition 2.12, there exists an open connected subset $\widetilde{\Gamma} \subset \mathrm{Stab}(\mathbf{D})$ such that $i^{-1} : \widetilde{\Gamma} \to \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(A_2))$ is an open embedding whose image is Γ. By [8, Thm. 1.1], $\widetilde{\Gamma}$ is a fundamental domain for the action of the group $\mathrm{Br}(\mathbf{D}) \subset \mathrm{Aut}(\mathbf{D})$ preserving the connected component Σ.

Rational curves in K3 surfaces. Let X be a smooth projective K3 surface containing a smooth rational curve C. Since the formal neighbourhood of C in X is isomorphic to the formal neighbourhood of $\mathbb{P}^1$ in $\omega_{\mathbb{P}^1}$, the category $\mathrm{D}_0^{\mathrm{b}}(|\omega_{\mathbb{P}^1}|)$ can be seen as a fully faithful subcategory of $\mathrm{D}^{\mathrm{b}}(X)$. The embedding functor $i : \mathrm{D}_0^{\mathrm{b}}(|\omega_{\mathbb{P}^1}|) \to \mathrm{D}^{\mathrm{b}}(X)$ can be used to induce stability conditions from $\mathrm{D}^{\mathrm{b}}(X)$ to $\mathrm{D}_0^{\mathrm{b}}(|\omega_{\mathbb{P}^1}|)$.

Using the exact sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, it is easy to see that the sheaves $\mathcal{O}_C(k)$, $k \in \mathbb{Z}$, are not all stable in any stability condition σ in the open subset $U(X)$ described in [6, Sect. 10]. By [6, Thm. 12.1], there exists a stability condition σ in the boundary $\partial U(X)_{(C_k)}$ of $U(X)$ of type (C_k) (i.e. in σ the objects $\mathcal{O}_C(k)[1]$ and $\mathcal{O}_C(k+1)$ are both stable of the same phase). Hence σ induces a stability condition on $\mathrm{Stab}(\mathrm{D}_0^{\mathrm{b}}(|\omega_{\mathbb{P}^1}|))$ via i^{-1} (in other words, $\sigma \in \mathrm{Dom}(i^{-1})$). Since all stability conditions in $\mathrm{Stab}_{\mathcal{N}}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X))$ are obtained from the stability conditions in the closure of $U(X)$ by applying autoequivalences, it is easy to see that $\mathrm{Dom}(i^{-1}) \cap \mathrm{Stab}_{\mathcal{N}}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X))$ concides with

$$\{\Phi(\sigma) \in \mathrm{Stab}_{\mathcal{N}}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X)) : \sigma \in \partial U(X)_{(C_k)} \text{ and } \Phi \in \langle T_{\mathcal{O}_C(m)} : m \in \mathbb{Z} \rangle\}.$$

Intuitively, letting the ample class ω in Example 2.5 go to infinity produces a degeneration of Bridgeland's stability conditions which, by acting with the group $\langle T_{\mathcal{O}_C(m)} : m \in \mathbb{Z} \rangle$, yields a surjection onto $\mathrm{Stab}(\mathrm{D}_0^{\mathrm{b}}(|\omega_{\mathbb{P}^1}|))$.

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