ON THE DIFFERENCE EQUATION OF THE POINCARÉ TYPE (Part 3)

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Dedicated to the memory of Professor A.O. Gelfond.

Table of contents

§0. Foreword.
§1. The proof of the Theorem 7 in the case $s = 1$, $q = 0$, where q is defined in Theorem 6 [37].
§2. Begin of the proof of the Theorem 7 in the general case.
§3. On the some linear normed spaces.
§4. End of the proof of the Theorem 7.
§5. Exactness of the obtained results.

§0. Foreword.

Here I complete the presentation of the proof of Theorem 4, which was formulated in the Part 1 ([33]) and proved in [15]; this theorem plays important role in my work ([18] - [38]) and is stated in [39] as Lemma 15. I use the results and notations of the previous parts [33], [37]. I will prove the following Theorem 7, which is more exact than Theorem 4.

Theorem 7. *Let $s \in \mathbb{N} - 1, n \in \mathbb{N}$,*

$$a_i^{\sim} \in \mathbb{C},\ a_i(\nu) \in \mathbb{C},$$

$$a_n(\nu) = 1,\ a_i(\nu) - a_i^{\sim} = O(1/\nu) \tag{1}$$

for $\nu \in \mathbb{N}$ and $i = 0, \dots, n$. Let us consider the following difference equation

$$\sum_{k=0}^{n} a_k(\nu) y(\nu + k) = 0, \tag{2}$$

where $\nu \in \mathbb{N} - 1$. For $m \in \mathbb{N}$ let V_m denotes the linear over $\mathbb{C}$ space of solutions $y = y(\nu)$ of the equation

$$\sum_{k=0}^{n} a_k(\nu)y(\nu+k)=0, \tag{3}$$

where $\nu\in m+\mathbb{N}-1$. *Let the absolute values of all the roots of the characteristical polynomial*

$$T(z)=\sum_{k=0}^{n} a_k^{\sim} z^k \tag{4}$$

are among the numbers $\{\rho_i\colon 1\le i\le 1+s\}$, *such that* $\rho_{s+1}=0$ *and* $\rho_j<\rho_i$ *for* $1\le i<j\le s+1$. *Let* e_i *and* k_i *denote respectively the sum of and the maximum of the multiplicities of those roots, whose absolute value is equal to* ρ_i, *where* $i=1,\ \ldots\ ,\ s+1$, *and let* $k^*=k_{s+1}$. *We suppose that, if* $s>0$, *then*

$$e_i>0 \tag{5}$$

for $i=1\ \ldots\ ,\ s$. *For given* $y=y(\nu)$ *in* $\mathbb{C}^{m-1+\mathbb{N}}$, *let*

$$\omega_{n,y}(\nu)=\max(|y(\nu)|\,,\ldots,|y(\nu+n-1)|).$$

Then there exist $A>0$, $m\in\mathbb{N}$, $\alpha^{\wedge}(\nu)>0$ *with* $\nu\in m+\mathbb{N}-1$ *and the subspaces* $V^{\vee}_{m,1},\ \ldots\ ,\ V^{\vee}_{m,s+1}$ *such that*

$$\lim_{\nu\to\infty}\alpha^{\wedge}(\nu)=0,$$

$$V_m=V^{\vee}_{m,1}\oplus\ \ldots\ ,\oplus V^{\vee}_{m,s+1},\ \dim_{\mathbb{C}}(V^{\vee}_{m,i})=e_i,\ 1\le i\le s+1,$$

and, if $y\in V^{\vee}_{m,\theta}$ *for some* $\theta\in\{1,...,s\}$, *then*

$$\exp(-A(ln(\nu)+\nu^{1-1/k_\theta}))(\rho_\theta)^\nu\omega_n(y)(m)\le\omega_{n,y}(\nu)$$

for $\nu\in m+\mathbb{N}-1$; *moreover, the spaces*

$$V^{\wedge}_{m,j}=V^{\vee}_{m,j}\oplus\ \ldots\ \oplus V^{\vee}_{m,s+1},$$

where $j=1\ \ldots\ ,\ s+1$, *and, if* $s\ge1$ *natural projections* π_j,

$$V^{\wedge}_{m,j}\mapsto V^{\vee}_{m,j},$$

where $j=1\ \ldots\ ,\ s$, *have the folloving properties:*

if $y\in V^{\wedge}_{m,\theta}$ *for some* $\theta\in\{1,...,s\}$, *then*

$$\omega_{n,y}(\nu)\le\exp(A(\ln(\nu)+\nu^{1-1/k_\theta}))(\rho_\theta)^\nu\omega_{n,y}(m), \tag{6}$$

$$(\omega_{n\pi_{theta(y)}}(m)-\alpha(\nu)\omega_{n,y}(m))(\rho_\theta)^\nu\times \tag{7}$$

$$\exp(-A(\ln(\nu)+\nu^{1-1/k_\theta}))\le\omega_{n,y}(\nu),$$

where $\nu\in m+\mathbb{N}-1$;

if

$$k^* > 0, \tag{8}$$

and $y \in V^{\vee}_{m,s+1}\,(= V^{\wedge}_{m,s+1})$, *then*

$$|y(\nu)| \le (A/\nu)^{\nu/k^*}\omega_{n,y}(m),$$

where $\nu \in m + \mathbb{N} - 1$.

Corollary. (cf. Theorem 4). *If*

$$y \in V^{\wedge}_{m,\theta} \diagdown V^{\wedge}_{m,\theta+1}, \tag{9}$$

then

$$\exp(-A(\ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \beta \le \omega_{n,y}(\nu),$$

where $\beta = \beta(y) > 0$ *does not depend from* ν.

Proof. The condition (9) implies the the inequality $\pi_\theta(y) \ne 0_{V^{\vee}_{m,\theta}}$. If $\omega_{n,\pi_\theta(y)}(m) = 0$, then $\pi_\theta(y) = 0^{\vee}_{V_{m,\theta}}$. Consequently, $\omega_{n,\pi_\theta(y)}(m) > 0$ and there exists $\nu_0 \in m + \mathbb{N} - 1$ such that

$$(\omega_{n,\pi_\theta(y)}(m)/2 < (\omega_{n,\pi_\theta(y)}(m) - \alpha(\nu)\omega_{n,y)}(m))$$

for $\nu \in \nu_0 + \mathbb{N} - 1$. Therefore $0 < \omega_{n,y)}(\nu)$ for $\nu \in m + \mathbb{N} - 1$, and, consequently there exists $\beta = \beta(y) > 0$ such that

$$\exp(-A(ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \beta(y) \le \omega_{n,y}(\nu)$$

for $\nu \in m + \mathbb{N} - 1$. ■

So, Theorem 4 is a corollary of the Theorem 7.

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§1. The proof of the Theorem 7 in the case $s = 1,\ k^* = 0$.

Lemma 1. *Let* $a \in \mathbb{N}$, $b \in \mathbb{N} - 1 + a$, $C > 0$ *Then*

$$0 < \sum_{\kappa=a}^{b} \ln(1 + C/\kappa) \le \tag{10}$$

$$\ln(1 + C/a) + b\ln(1 + C/b) - a\ln(1 + C/a) + C\ln((b + C)/(a + C)).$$

Proof. Since the function $\ln(1 + C/x)$ decreases together with increasing of the variable $x \in [1, +\infty)$, it follows that

$$\sum_{\kappa=a}^{b} \ln(1 + C/\kappa) \le \tag{11}$$

$$\ln(1+C/a)+\int\limits_{x=a}^{b}\ln(1+C/x)dx=$$

$$\ln(1+C/a)+b\ln(1+C/b)-a\ln(1+C/a)+\int\limits_{x=a}^{b}(C/(C+x))dx.$$

■

Corollary. *If* $a\in\mathbb{N}$, $b\in\mathbb{N}+a-1$, $b<2a$, $C>0$, *then*

$$\sum_{\kappa=a}^{b}\ln(1+C/\kappa)\leq 3C.$$

Proof. The assertion of the Corollary directly follows from (11).

Lemma 2. *If* $s=1$, $k^*=0$, *then assertion of the Theorem 7 is true.*

Proof. The condition $k^*=0$ implies the inequality

$$a_0^{\sim}=T(0)\neq 0, \tag{12}$$

and the condition $s=1$ implies for p in Theorem 6 of the paper [37] the equality $p=n$. Let

$$A(\nu)=\begin{pmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\\vdots&\vdots&\vdots&\dots&\vdots\\0&0&0&\dots&1\\-a_0(\nu)&-a_1(\nu)&-a_2(\nu)&\dots&-a_{n-1}(\nu)\end{pmatrix}, \tag{13}$$

where $\nu\in\mathbb{N}-1$, and let

$$A^{\sim}=\begin{pmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\\vdots&\vdots&\vdots&\dots&\vdots\\0&0&0&\dots&1\\-a_0^{\sim}&-a_1^{\sim}&-a_2^{\sim}&\dots&-a_{n-1}^{\sim}.\end{pmatrix} \tag{14}$$

Let λ_j, where $j=1,\dots,r$ is the sequence of all the mutually distinct roots of the polynomial (4) and k_j^* is the multiplicity of the root λ_j. Let further $k=\sup k_j^*\colon j=1,\dots,r$. Clearly,

$$\sum_{j=0}^{r}k_j^*=n,$$

$$0<|\lambda_j|\leq\rho_1=\|A\|_{sp}<R=h^{\sim}(A^{\sim})+1,$$

where $j=1,\dots,r$. In view of (2) in [37], there exists $C_1>0$ such that

$$h^{\sim}(A(\nu)-A^{\sim})\leq C_1/(\nu+1). \tag{15}$$

Therefore, according to the Lemma 8 in [37], for any $\varepsilon>0$ the following inequalities holds

$$p^{\sim}_{A^{\sim},\varepsilon}(A(\nu)-A^{\sim})\leq \tag{16}$$

$$(\gamma^*(A^\sim))^2(\max(\varepsilon,1/\varepsilon))^{k-1}h^\sim(A(\nu)-A^\sim)\le$$
$$(\gamma^*(A^\sim))^2(\max(\varepsilon,1/\varepsilon))^{k-1}C_1/(\nu+1),$$

(17)
$$p^\sim_{A^\sim,\varepsilon}(A(\nu))\le\rho_1+sign(k-1)\varepsilon+$$
$$(\gamma^*(A^\sim))^2(\max(\varepsilon,1/\varepsilon))^{k-1}C_1/(\nu+1)\le(\max(\varepsilon,1/\varepsilon))^{k-1}\le$$
$$(\rho_1+(sign(k-1))\varepsilon)\left(1+(\gamma^*(A^\sim))^2(\max(\varepsilon,1/\varepsilon))^{k-1}\frac{C_1}{\rho_1(\nu+1)}\right),$$

(18)
$$\ln(p^\sim_{A^\sim,\varepsilon}(A(\nu)))\le\ln(\rho_1)+\ln(1+(sign(k-1))\varepsilon/\rho_1)+$$
$$\ln\left(1+(\gamma^*(A^\sim))^2(\max(\varepsilon,1/\varepsilon))^{k-1}\frac{C_1}{\rho_1(\nu+1)}\right),$$

where $\nu\in\mathbb{N}-1$. We consider first the case $k>1$. For given $\nu\in\mathbb{N}+1$ we take $d\in\mathbb{N}$ and $a_i\in[1,\nu]\cap\mathbb{N}$, where $i=0,\ \dots\ ,d$, in such a way that

(19)
$$a_0=1,\ a_d=\nu, a_{i-1}<a_i\le 2a_{i-1},$$

where $i=1,\ \dots\ ,d$, and

(20)
$$d\le\frac{\ln(\kappa)}{\ln(2)}+1.$$

According to the Corollary of the Lemma 1 and (19) – (20),

(21)
$$\sum_{\kappa=a_{i-1}}^{a_i-1}\ln(p^\sim_{A^\sim,\varepsilon}(A(\kappa)))\le$$
$$(a_i-a_{i-1})\ln(\rho_1)+(a_i-a_{i-1})\ln(1+\varepsilon/\rho_1)+$$
$$\sum_{\kappa=a_{i-1}}^{a_i-1}\ln\left(1+(\gamma^*(A^\sim))^2(\max(\varepsilon,1/\varepsilon))^{k-1}\frac{C_1}{\rho_1(\kappa+1)}\right)\le$$
$$(a_i-a_{i-1})(\ln(\rho_1)+\ln(1+\varepsilon/\rho_1))+3(\gamma^*(A^\sim))^2(\max(\varepsilon,1/\varepsilon))^{k-1}\frac{C_1}{\rho_1},$$

where $i=1,\ \dots\ ,d$. We take now in (21 $\varepsilon=\varepsilon_i=(a_{i-1})^{-1/k}$. Then we obtain the inequality

$$\sum_{\kappa=a_{i-1}}^{a_i-1}\ln(p^\sim_{A^\sim,\varepsilon}(A(\nu)))\le(a_i-a_{i-1})\ln(\rho_1)+O(2^{(i-1)(1-1/k)}),$$

where $i=1,\ \dots\ ,d$. Therefore

(22)
$$\ln(p^\sim_{A^\sim,\varepsilon_i}\left(\prod_{\kappa=1}^{a_i-a_{i-1}}A(a_i-\kappa)\right)\le$$
$$(a_i-a_{i-1})\ln(\rho_1)+O(2^{(i-1)(1-1/k)},$$

(23)
$$\ln(h^{\sim}\left(\prod_{\kappa=1}^{a_i-a_{i-1}} A(a_i-\kappa)\right) \le$$
$$\ln((\gamma^*(A^{\sim}))^2)(\max(\varepsilon,1/\varepsilon))^{k-1}) + (a_i-a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}) \le$$
$$2\ln(\gamma^*(A^{\sim})) + (i-1)(1-1/k)ln(2) + (a_i-a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}) =$$
$$(i-1)(1-1/k)ln(2) + (a_i-a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}),$$
where $i = 1, \ldots, d$. Consequently,

(24)
$$\ln\left(h^{\sim}\left(\prod_{\kappa=1}^{\nu} A(\nu-\kappa)\right)\right) =$$
$$\ln\left(h^{\sim}\left(\left(\prod_{i=1}^{d}\prod_{\kappa=1}^{a_i-a_{i-1}} A(a_i-\kappa)\right)A(0)\right)\right) =$$
$$= \nu\ln(\rho_1) + O((\nu+1)^{(1-1/k)}),$$
where $\nu \in \mathbb{N}-1$. If $k = 1$, then, according to (18),
$$\ln(p^{\sim}_{A^{\sim},1}(A(\nu))) \le \ln(\rho_1) + \ln\left(1 + (\gamma^*(A^{\sim}))^2\frac{C_1}{\rho_1(\nu+1)}\right) \le$$
$$\ln(\rho_1) + (\gamma^*(A^{\sim}))^2\frac{C_1}{\rho_1(\nu+1)},$$

(25)
$$\ln\left(h^{\sim}\left(\prod_{\kappa=1}^{\nu} A(\nu-\kappa)\right)\right) \le$$
$$\ln\left((\gamma^*(A^{\sim}))^2 p^{\sim}_{A^{\sim},1}\left(\prod_{\kappa=1}^{\nu} A(\nu-\kappa)\right)\right) \le$$
$$\nu\ln(\rho_1) + O(\ln(e(\nu+1))).$$
where $\nu \in \mathbb{N}-1$. In view of (24) – (25),

(26)
$$\ln\left(h^{\sim}\left(\prod_{\kappa=1}^{\nu} A(\nu-\kappa)\right)\right) =$$
$$= \nu\ln(\rho_1) + O((\nu+1)^{(1-1/k)}) + O(\ln(e(\nu+1))),$$
where $\nu \in \mathbb{N}-1$.

As in Section 3 of [37], let K denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$ and L denotes a linear normed space over K with norm $p = p(x)$. In the case $L = K^n$ we fix as $p = p(x)$, wehre $x \in K^n$, the maximum of the absolute values of coordinates of the element x in the standard basis, i.e.

(27)
$$p(x) = h(x) = \sup(\{|x_1|, \ldots, |x_n|\}),$$

where
$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

If L is a Banach space with the norm p, then $K-$algebra of all the linear continuous operators acting in L will be denoted by $\mathfrak{M}^\wedge(L)$, and the norm on $\mathfrak{M}^\wedge(L)$, associated with the norm p will be denoted by $p^\sim$. So,

$$p^\sim(A) = sup(\{p(AX) \colon X \in L,\, p(X) \le 1\}).$$

It is well known that the associciated with h norm on $Mat_n(\mathbb{C})$ is defined as follows

$$h^\sim(A) = sup\left(\left\{\sum_{k=1}^{n} |a_{i,j}| \colon i = 1, \dots , n\right\}\right), \tag{28}$$

where $A = (a_{i,k}) \in Mat_n(\mathbb{C})$. The norms h and $h^\sim$ coincide respectiely with with the norms q_∞ and $q_\infty^\sim$ considered in section 6 of the paper [33].

Let $m \in \mathbb{N}$, and let $E_m(L)$ be the set $L^{m-1+\mathbb{N}}$ of all the maps of the set $m - 1 + \mathbb{N}$ into L. The set $E_m(L)$ is a linear space over K, where the muliplication of the elements by the number from K and addition of the elements are defined coordinate-wise. The subspace of $E_m(L)$ composed by all the constant maps is isomorphic to L, and we identify this subspace with L.

As in Section 4 of [37], for any $y \in E_m(\mathbb{C})$ and $n \in \mathbb{N}$ let $Y_{n,y}$ and $Y^{\#}_{n,y}$ denote the elements in the space $E_m(\mathbb{C}^n)$, wich are determined respectively by means the following equalities:

$$Y_{n,y}(\nu) = \begin{pmatrix} y(\nu) \\ \vdots \\ y(\nu + n - 1) \end{pmatrix}, \tag{29}$$

$$Y^{\#}_{n,y}(\nu) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y(\nu) \end{pmatrix}, \tag{30}$$

where $\nu \in m - 1 + \mathbb{N}$. Let is fixed $m \in \mathbb{N}$. If $y = y(\nu)$ is a solution of the equation (3) for $\nu \in \mathbb{N} + m - 1$, then

$$Y_{n,y}(\nu) = \left(\prod_{\kappa=1}^{\nu-m} A(\nu - \kappa)\right) Y_{n,y}(m),$$

where $\nu \in \mathbb{N}$, and, in view of (26),

$$\omega_{n,y}(\nu) = h\left(Y_{n,y}(\nu)\right) \le \tag{31}$$

$$h^\sim\left(\prod_{\kappa=1}^{\nu-m} A(\nu - \kappa)\right) h\left(Y_{n,y}(m)\right) =$$

$$\exp\left(O(1)\left(\nu^{1-1/k} + \ln(\nu)\right)\right)(\rho_1)^\nu\, \omega_{n,y}(m),$$

where $\nu \in \mathbb{N} + m - 1$. So, with $m = 0$ the asserted by the Theorem 7 the upper estimate of the value $\omega_{n,y}(\nu)\,(\rho_\theta)^{-\nu} = \omega_{n,y}(\nu)\,(\rho_1)^{-\nu}$ is obtained in the case $s = 1$, $k^* = 0$, because in this case $\theta = 1$. Let

$$m_1 = [2C_1 h^{\sim}\left((A^{\sim})^{-1}\right)] + 1,$$

and let is fixed $m \in m_1 - 1 + \mathbb{N}$ Then

$$h^{\sim}\left((A^{\sim})^{-1}(A(\nu) - A^{\sim})\right) \le h^{\sim}\left((A^{\sim})^{-1}\right) C_1/(\nu+1) \le$$

$$h^{\sim}\left((A^{\sim})^{-1}\right) C_1/([2C_1 h^{\sim}\left((A^{\sim})^{-1}\right)] + 1) \le 1/2,$$

if $\nu \in m - 1 + \mathbb{N}$, the matrix $E + ((A^{\sim})^{-1}(A(\nu) - A^{\sim}))$ is invertible,

$$h^{\sim}\left(\left(E + ((A^{\sim})^{-1}(A(\nu) - A^{\sim}))\right)^{-1}\right) \le 2,$$

if $\nu \in m - 1 + \mathbb{N}$, there exists the matix

$$(A(\nu))^{-1} =$$

$$\left(E + ((A^{\sim})^{-1}(A(\nu) - A^{\sim}))\right)^{-1}(A^{\sim})^{-1},$$

if $\nu \in m - 1 + \mathbb{N}$, moreover,

$$h^{\sim}\left((A(\nu))^{-1}\right) \le h^{\sim}\left(\left(E + ((A^{\sim})^{-1}(A(\nu) - A^{\sim}))\right)^{-1}\right) h^{\sim}\left((A^{\sim})^{-1}\right) \le$$

$$2h^{\sim}\left((A^{\sim})^{-1}\right)$$

and, finally,

$$h^{\sim}\left((A(\nu))^{-1} - (A^{\sim})^{-1}\right) =$$
$$h^{\sim}\left((A(\nu))^{-1}(A^{\sim} - A(\nu))(A^{\sim})^{-1}\right) \le$$
$$h^{\sim}\left((A(\nu))^{-1}\right) h^{\sim}\left((A^{\sim} - A(\nu))\right) h^{\sim}\left((A^{\sim})^{-1}\right) \le$$
$$2\left(h^{\sim}\left((A^{\sim})^{-1}\right)\right)^2 C_1/(\nu+1),$$

where $\nu \in m - 1 + \mathbb{N}$. Let $C_2 = 2\left(h^{\sim}\left((A^{\sim})^{-1}\right)\right)^2 C_1$. We take $\nu \in m - 1 + \mathbb{N}$. We take $\kappa \in [m, \nu - 1] \cap \mathbb{Z}$ on the role $\nu - \kappa$ in (24) with $\kappa \in [1, \nu - m] \cap \mathbb{Z}$; we take further $(A(\kappa))^{-1}$, where $m \le \kappa \le \nu - 1$, on the role $A(\nu - \kappa)$ in (24), where $1 \le \kappa \le \nu - m$, we take $A_1^{\sim} = (A^{\sim})^{-1}$ on the role $A^{\sim}$, we take C_2 on the role C_1 and, finally, we see that $1/\rho_1$ will play the role $1/\rho_1$. Then we see that

$$\ln\left(h^{\sim}\left(\prod_{\kappa=m}^{\nu-1}(A(\kappa))^{-1}\right)\right) \le \tag{32}$$

$$-\nu\ln(\rho_1) + O\left(\nu^{1-1/k} + ln(\nu)\right),$$

where $\nu \in m - 1 + \mathbb{N} \subset m_1 - 1 + \mathbb{N}$. Since

$$Y_{n,y}(\nu) = \left(\prod_{\kappa=1}^{\nu-m} A(\nu - \kappa)\right) Y_{n,y}(m_1),$$

$$Y_{n,y}(m) = \left(\prod_{\kappa=1}^{\nu-m} A(\nu-\kappa)\right)^{-1} Y_{n,y}(\nu) =$$

$$\left(\prod_{\kappa=m}^{\nu-1} (A(\kappa))^{-1}\right) Y_{n,y}(\nu),$$

it follows that

$$\omega_{n,y}(m) = h\left(Y_{n,y}(m)\right) \leq \tag{33}$$

$$h^{\sim}\left(\prod_{\kappa=m}^{\nu-1} (A(\kappa))^{-1}\right) h\left(Y_{n,y}(\nu)\right) =$$

$$\exp\left(O(1)\left(\nu^{1-1/k} + \ln(\nu)\right)\right)(\rho_1)^{-\nu}\,\omega_{n,y}(\nu),$$

where $\nu \in \mathbb{N}$. So, with $m \in m_1 - 1 + \mathbb{N}$ the asserted by the Theorem 7 the lower estimate of the value $\omega_{n,y}(\nu)\,(\rho_\theta)^{-\nu} = \omega_{n,y}(\nu)\,(\rho_1)^{-\nu}$ is obtained in the case $s = 1$, $k^* = 0$, because in this case $\theta = 1$. ■

Lemma 3. *If $s = 0$, $k^* = n$, (see notations in [33], §0, formulation of the Theorem 4) then assertion of the Theorem 7 is true.*

Proof is given in [37], Theorem 6.

§2. Begin of the proof of the Theorem 7 in the general case.

For the proof we use induction on σ, where $\sigma = s$, if $k^* = 0$, and $\sigma = s+1$, if $k^* > 0$, Since for $\sigma = 1$ Theorem 7 is proved in the section 1 (Lemma 2 and Lemma 3) we suppose that

$$\sigma \geq 2 \tag{34}$$

and the assertions of the Theorem 7 are true for $\sigma - 1$. In view of (34), $s \geq 1$, and, in view of (5), $e_1 > 0$. Clearly,

$$\sum_{j=1}^{s+1} e_j = n. \tag{35}$$

If $s = 1$, then, in view of (34) and (35), $e_1 < n$; if $s \geq 2$, then, (5) and (35) imply the inequality $e_1 < n$. Let $p = e_1$, and let $q = n - p$. Since $e_1 < n$, it follows that $p < n$ and $q > 0$. As in (9) of the paper [33], let

$$T_1(z) = \sum_{\alpha=0}^{p} b^{\sim}_{\alpha} z^k,\ T_2(z) = \sum_{\beta=0}^{q} u^{\sim}_{\beta} z^k. \tag{36}$$

Then for the polynomial (4) the following equlity holds

$$T(z) = T_1(z)T_2(z), \tag{37}$$

i.e. the equality (8) of the paper [33] holds, $b^{\sim}_p = u^{\sim}_q = a^{\sim}_n = 1$ and absolute value of each root of the polynomial $T_1(z)$ is greater than the absolute value of each root of the polynomial $T_2(z)$. Therefore with $\xi(x) = 1/(x+1)$ all the *conditions of the Theorem 5 in the paper [33] are fulfilled. Consequently, the following assertion is true:*

Assertion 1. *There exist $m \in \mathbb{N}$, the numbers $b_\alpha(\nu) \in \mathbb{C}$, $u_\beta(\nu) \in \mathbb{C}$, where $\alpha = 0, \dots, p$, $\beta = 0, \dots, q$, $\nu \in \mathbb{N} + m - 1$, such that*

$$b_\alpha(\nu) - b_\alpha^\sim = O(1/\nu), \tag{38}$$

where

$$\alpha = 0, \dots, p,\ b_p(\nu) = 1, b_0(\nu) \neq 0,\ \nu \in \mathbb{N} + m - 1, \tag{39}$$

$$u_\beta(\nu) - u_\beta^\sim = O(1/\nu), \tag{40}$$

where

$$\beta = 0, \dots, q,\ u_q(\nu) = 1,\ \nu \in \mathbb{N} + m - 1, \tag{41}$$

and the connected with the equation (3) the equation

$$\sum_{k=0}^{n} a_k(\nu) y(\nu + k) = 0, \tag{42}$$

where $\nu \in \mathbb{N} - 1 + m$, is equivalent to the equation

$$\sum_{\alpha=0}^{p} b_\alpha(\nu) y(\nu + \alpha) = r(\nu), \tag{43}$$

where $\nu \in \mathbb{N} - 1 + m$ and $r(\nu)$ satisfies to the equation

$$\sum_{\beta=0}^{q} u_\beta(\nu) r(\nu + \beta) = 0 \tag{44}$$

with $\nu \in \mathbb{N} - 1 + m$.

We denote by R_m the space of all the sollutions of the equation (44). The map

$$y(\nu) \to \sum_{\alpha=0}^{p} b_\alpha(\nu) y(\nu + \alpha) = r(\nu) \tag{45}$$

is the homomorphism of the space V_m of solutions of the equation (42) into the space R_m; we denote this homomorphism by ϕ_m.

Let

$$C_3 = \sup\left\{\sum_{\alpha=0}^{p} |b_\alpha(\nu)| \colon \nu \in m - 1 + \mathbb{N}\right\}; \tag{46}$$

$$C_4 = \sup\left\{\sum_{\beta=0}^{q} |u_\beta(\nu)| \colon \nu \in m - 1 + \mathbb{N}\right\};; \tag{47}$$

$$C_5 = \sup\left\{\sum_{k=0}^{n} |a_k(\nu)| \colon \nu \in m-1+\mathbb{N}\right\};; \tag{48}$$

according to (38) – (41) and (1), $1 \le C_k < +\infty < +\infty$ for $k = 1, 23$. In view of (46),

$$|(\phi_m(y)(\nu) \le C_3\omega_{p+1,y}(\nu),$$

$$\omega_{q,\phi_m(y)}(\nu) \le C_3\omega_{n+1,y}(\nu) \le C_3C_5\omega_{n,y}(\nu), \tag{49}$$

where $\nu \in m-1+\mathbb{N}$.

Together with the equation (43) we will consider the corresponding homogeneous equation

$$\sum_{\alpha=0}^{p} b_\alpha(\nu)y(\nu+\alpha) = 0, \tag{50}$$

where $\nu \in \mathbb{N}-1+m$. If $y_1(\nu)$ is a solution of the equation (50), then, clearly,

$$\omega_{p,y_1}(\nu+i) \le C_3^i\omega_{p,y_1}(\nu), \tag{51}$$

where $i \in \mathbb{N}-1$,

$$\omega_{p,y_1}(\nu) \le \omega_{n,y_1}(\nu), \omega_{n,y_1}(\nu) \le C_3^q\omega_{p,y_1}(\nu), \tag{52}$$

where $\nu \in \mathbb{N}-1+m$.

If $r(\nu)$ is a solution of the equation (44), then, clearly,

$$\omega_{q,r}(\nu+i) \le C_4^i\omega_{q,r}(\nu), \tag{53}$$

where $i \in \mathbb{N}-1$,

$$\omega_{q,r}(\nu) \le \omega_{n,r}(\nu), \omega_{n,r}(\nu) \le C_4^p\omega_{q,r}(\nu), \tag{54}$$

where $\nu \in \mathbb{N}-1+m$. Let $C > 0$, $n \in \mathbb{N}$,

$$w_{C,n}(\nu) = \left(\frac{C}{\nu}\right)^{\nu/n}, \tag{55}$$

where $\nu \in \mathbb{N}$; let further $\rho > 0$, for which

$$v_{C,n,\rho}(\nu) = \rho^\nu \exp\left(C\left((\nu)^{1-1/n} + \ln(\nu)\right)\right), \tag{56}$$

where $\nu \in \mathbb{N}$.

According to inductive hypothesis, there exist $A^\vee > 0$, $m \in \mathbb{N}$, $\alpha^\vee(\nu) > 0$ with $\nu \in m+\mathbb{N}-1$ and the subspaces $R^\vee_{m,2}, \ldots, R^\vee_{m,s+1}$ such that

$$\lim_{\nu\to\infty} \alpha^\vee(\nu) = 0,$$

$$R_m = R^\vee_{m,2} \oplus \ldots, \oplus R^\vee_{m,s+1}, \ \dim_{\mathbb{C}}(R^\vee_{m,i}) = e_i, \ 2 \le i \le s+1,$$

if

$$r \in R^\vee_{m,\theta} \tag{57}$$

for some $\theta \in \{2, ..., s\}$, then

$$\exp(-A^{\vee}(\ln(\nu) + \nu^{1-1/k_\theta}))\omega_{q,r}(m)(\rho_\theta)^\nu \leq \omega_{q,r)}(\nu), \tag{58}$$

where $\nu \in \mathbb{N} + m - 1$; moreover, the spaces

$$R^{\wedge}_{m,j} = R^{\vee}_{m,j} \oplus \ldots \oplus R^{\vee}_{m,s+1},$$

where $j = 2, \ldots, s+1$ and natural projections $\pi^{\vee}_j$ of the space $R^{\wedge}_{m,j}$ onto the space $R^{\vee}_{m,j}$, where $j = 2, \ldots, s$ have the folloving properties:

if $s \geq 2$

$$r \in R^{\wedge}_{m,\theta} \tag{59}$$

for some $\theta \in \{2, ..., s\}$, then

$$\omega_{q,r}(\nu) \leq (\rho_\theta)^\nu \exp(A^{\vee}(\ln(\nu) + \nu^{1-1/k_\theta}))\omega_{q,r}(m), \tag{60}$$

$$\omega_{q,r}(\nu) \geq (\omega_{q,\pi^{\vee}_\theta(r)}(m) - \alpha^{\vee}(\nu)\omega_{q,r)}(m)) \times \tag{61}$$

$$(\rho_\theta)^\nu \exp(-A^{\vee}(\ln(\nu) + \nu^{1-1/k_\theta})),$$

where $\nu \in \mathbb{N} + m - 1$; if (8) holds and

$$r \in R^{\vee}_{m,s+1} \, (= R^{\wedge}_{m,s+1}), \tag{62}$$

then

$$|r(\nu)| \leq (A^{\vee}/\nu)^{\nu/k^*}\omega_{q,r}(m), \tag{63}$$

where $\nu \in \mathbb{N} + m - 1$.

Lemma 4. *Let for sufficient big $m \in \mathbb{N}$ the splitting monomorphism*

$$\psi_m \colon R_m \to V_m$$

is specified, having the folloing properties:

a) $\phi_m\psi_m$ is the ientity map $R_m \to R_m$;

b) there exists a constant $C_6 > 0$ such that if (59) holds, then

$$\sup\{\omega_{n,\psi_m(r)}(\nu)/v_{A^{\vee},n,\rho_\theta}(\nu) \colon \nu \in \mathbb{N} - 1 + m\} \leq \tag{64}$$

$$C_6 \sup\{\omega_{q,r}(\nu)/v_{A^{\vee},k_\theta,\rho_\theta}(\nu) \colon \nu \in \mathbb{N} - 1 + m\},$$

and if (62) holds, then

$$\sup\{\omega_{n,\psi_m(r)}(\nu)/w_{A^{\vee},k^*}(\nu) \colon \nu \in \mathbb{N} - 1 + m\} \leq \tag{65}$$

$$C_6 \sup\{\omega_{q,r}(\nu)/w_{A^{\vee},k^*}(\nu) \colon \nu \in \mathbb{N} - 1 + m\},$$

Then assertions of the Theorem 4 are true for σ.

Proof. We denote by I the identity map $V_m \to V_m$. Let further $V^{\vee}_{m,1}$ denotes the space of the solutions of the equation (50). Let us to prove that $V_m = V^{\vee}_{m,1} \oplus \psi_m\left(R_m\right)$, i.e that each element $y \in V_m$ may be uniquely represented in the form

$$y = y_1 + y^{\wedge}, \tag{66}$$

where $y = y_1 \in V^{\vee}_{m,1}$, $y^{\wedge} \in \oplus\psi_m\left(R_m\right)$. Let

$$y^{\wedge} = (\psi_m\phi_m)(y) = (\psi_m(\phi_m(y)), \tag{67}$$

$$y_1^{\wedge} = y - y^{\wedge} = (I - \psi_m\phi_m)(y). \tag{68}$$

In view of (67), $y^{\wedge} \in \psi_m(R_m)$,

$$\phi_m(y^{\wedge}) = \phi_m(\psi_m(\phi_m(y)) = (\phi_m\psi_m)(\phi_m(y)) = \phi_m(y), \tag{69}$$

because $\phi_m(y) \in R_m$ and $\psi_m\phi_m$ is the ientity map $R_m \to R_m$. Therefore we have the equality $\phi_m(y_1) = \phi_m(y) - \phi_m(y^{\wedge}) = 0$, and, consequently, $y_1 \in V^{\vee}_{m,1}$. If $y_2 \in V^{\vee}_{m,1}$, $y_3 \in R_m$, $y_4 = \psi_m(y_3)$ and $y = y_2 + y_4$, then

$$y^{\wedge} = (\psi_m\phi_m)(y) = (\psi_m(\phi_m(y_2)) + \psi_m(\phi_m(y_4)) =$$

$$\psi_m(\phi_m(\psi_m(y_3))) = \psi_m((\phi_m\psi_m)(y_3)) =$$

$$\psi_m(y_3) = y_4,\ y_1^{\wedge} = y - y^{\wedge} = y - y_4 = y_2.$$

So, the existence and the uniqueness of the representation (66) is proved.

Let $V^{\vee}_{m,i} = \psi_m(R^{\vee}_{m,i})$ for $i = 2, \dots, s+1$. Monomorphism ψ_m is an isomorphism of the space

$$R_m = R^{\vee}_{m,2} \oplus \ldots \oplus R^{\vee}_{m,s+1}$$

on the space

$$\psi_m(R_m) = \psi_m(R^{\vee}_{m,2}) \oplus \ldots \oplus \psi_m(R^{\vee}_{m,s+1}) =$$

$$V^{\vee}_{m,2} \oplus \ldots \oplus V^{\vee}_{m,s+1}.$$

Then

$$V_m = V^{\vee}_{m,1} \oplus \psi_m\left(R^{\vee}_m\right) = V_{m,1} \oplus \left(V^{\vee}_{m,2} \oplus \ldots \oplus V^{\vee}_{m,s+1}\right) = \tag{70}$$

$$V^{\vee}_{m,1} \oplus \ldots \oplus V^{\vee}_{m,s+1};$$

moreover $dim_{\mathbb{C}}\left(V^{\vee}_{m,1}\right) = p = e_1$,

$$\dim_{\mathbb{C}}\left(V^{\vee}_{m,i}\right) = \dim_{\mathbb{C}}\left(R^{\vee}_{m,i}\right) = e_i,$$

where $i = 2, \dots, s+1$. The natural projection π_1 of the space $V^{\wedge}_{m,1} = V_m$ onto the space $V^{\vee}_{m,1}$ coincides with $I - \psi_m\phi_m$ and, if $j \in [2, s] \cap \mathbb{N}$,then the natural projection π_j of the space $V^{\wedge}_{m,j}$ onto $V^{\vee}_{m,j}$ coincides with $\psi_m\pi^{\vee}_j$. Therefore $\pi^{\vee}_j = \phi_m\pi_j$. Let $g \in q - 1 + \mathbb{N}$, and let us consider the map

$$Y_{g,r}(m) \to Y_{g,\psi_m(r)}(m),$$

where $r \in R_m$. This map defines a linear isomorphism of $q-$dimesional linear over $\mathbb{C}$ space R_m onto the $q-$dimensinal linear over $\mathbb{C}$ space $V^{\wedge}_{m,2}$. Consequently, there exists two matrices $T_{1,g} \in Mat_g(\mathbb{C})$ and $T_{2,g} \in Mat_g(\mathbb{C})$ such that

$$Y_{g,r}(m) = T_{1,g}Y_{g,\psi_m(r)}(m), Y_{g,\psi_m(r)}(m) = T_{2,g}Y_{g,r}(m).$$

(Clearly, $T_{2,q} = (T_{1,q})^{-1}$, if $g = q$).

Therefore there exist $C_{7,g} > 0$ and $C_{8,g} > 0$ such that

$$\omega_{g,r}(m) \le C_{7,g}\omega_{g,\psi_m(r)}(m),\ \omega_{g,\psi_m(r)}(m) \le C_{8,g}\omega_{g,r}(m). \tag{71}$$

Let $y \in V^{\wedge}_{m,\theta}$ for some $\theta \in \{2, ..., s+1\}$. Then, since ψ_m isomorphically maps $R^{\wedge}_{m,\theta}$ onto $V^{\wedge}_{m,\theta}$, it follows that $y = \psi_m(r)$, $r = \phi_m(y)$, where $r \in R^{\wedge}_{m,\theta}$. If $2 \le \theta \le s$, then in view of (60), (64), (49),

$$\sup\{\omega_{n,y}(\nu)/v_{A^\vee,k_\theta,\rho_\theta}(\nu) \colon \nu \in \mathbb{N}-1+m\} \le$$

$$\sup\{\omega_{n,\psi_m(r)}(\nu)/v_{A^\vee,k_\theta,\rho_\theta}(\nu) \colon \nu \in \mathbb{N}-1+m\} \le$$

$$C_6 \sup\{\omega_{q,r}(\nu)/v_{A^\vee,k_\theta,\rho_\theta}(\nu) \colon \nu \in \mathbb{N}-1+m\} \le$$

$$C_6\omega_{q,r}(m) = C_6\omega_{q,\phi_m(y)}(m) \le C_6C_3C_5\omega_{n,y}(m).$$

Therefore

$$\omega_{n,y}(\nu) \le v_{A^\vee,\theta,\rho_\theta}(\nu)C_6C_3C_5\omega_{n,y}(m) = \tag{72}$$

$$\rho^\nu \exp\left(O(1)\left((\nu)^{1-1/k_\theta} + \ln(\nu)\right)\right)\omega_{n,y}(m).$$

If $k^* > 0$ and $\theta = s+1$, then in view of (63), (65), (49),

$$\sup\{\omega_{n,y}(\nu)/w_{A^\vee,k^*}(\nu) \colon \nu \in \mathbb{N}-1+m\} \le$$

$$\sup\{\omega_{n,\psi_m(r)}(\nu)/w_{A^\vee,k^*}(\nu) \colon \nu \in \mathbb{N}-1+m\} \le$$

$$C_6 \sup\{\omega_{q,r}(\nu)/w_{A^\vee,k^*}(\nu) \colon \nu \in \mathbb{N}-1+m\} \le$$

$$C_6\omega_{q,r}(m) = C_6\omega_{q,\phi_m(y)}(m) \le C_6C_3C_5\omega_{n,y}(m).$$

Therefore

$$\omega_{n,y}(\nu) \le w_{A^\vee,k^*}(\nu)C_6C_3C_5\omega_{n,y}(m). \tag{73}$$

If $2 \le \theta \le s$, then $\phi_m(\pi_\theta(y)) = \pi^\vee_\theta(r) \in V^\vee_{m,\theta}$, where $r = \phi_m(y)$, and, in view of (58),

$$\exp(-A^\vee(\ln(\nu) + \nu^{1-1/k_\theta}))\omega_{q,\pi_\theta(r)}(m)(\rho_\theta)^\nu \le \omega_{q,\pi_\theta(r)}(\nu); \tag{74}$$

in view of (71), (54),

$$\omega_{n,y}(m) = \omega_{n,\psi_m(r)}(m) \le C_{8,n}\omega_{n,r}(m) \le C_{8,n}C_4^p\omega_{q,r}(m),$$

$$\omega_{n,\pi_\theta(y)}(m) = \omega_{g,\psi_m(\pi^\vee_\theta(r))}(m) \le C_{8,n}\omega_{n,\pi^\vee_\theta(r)}(m) \le C_{8,n}C_4^p\omega_{q,\pi^\vee_\theta(r)}(m).$$

Therefore, in view of (74), (49),

$$(C_{8,n}C_4^p)^{-1}\omega_{n,\pi_\theta(y)}(m)\exp(-A^\vee(\ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \le$$

$$\omega_{q,\pi_\theta^\vee(r)}(m)\exp(-A^\vee(\ln(\nu)+\nu^{1-1/k_\theta}))(\rho_\theta)^\nu \le$$
$$\omega_{q,\pi_\theta^\vee(r)}(\nu)=\omega_{q,\phi_m(\pi_\theta(y))}(\nu)\le$$
$$C_3C_5\omega_{n,\pi_\theta(y)}(\nu).$$

Therefore

$$\omega_{n,\pi_\theta(y)}(m)\exp(-O(1)(\ln(\nu)+\nu^{1-1/k_\theta}))(\rho_\theta)^\nu \le \omega_{n,\pi_\theta(y)}(\nu). \tag{75}$$

Let $d_\theta=\sum\limits_{j=\theta}^{s+1} e_j$ Clearly, $y-\pi_\theta(y)\in V_{m,1+\theta}$. The map π_θ is a linear map of $d_\theta-$ dimensional linear over $\mathbb{C}$ space $V^\wedge_{m,\theta}$ onto onto the $e_\theta-$ dimensinal linear over $\mathbb{C}$ space $V^\vee_{m,\theta}$. Consequently, there exists $T_{0,\theta}\in Mat_n(\mathbb{C})$ such that

$$Y_{n,\pi_\theta(y)}(m)=T_{0,\theta}Y_{n,y}(m).$$

Therefore there exists $C_{0,\theta}>0$ such that

$$\omega_{n,\pi_\theta(y)}(m)\le C_{0,\theta}\omega_{n,y}(m). \tag{76}$$

Clearly, $y-\pi_\theta(y)\in V_{m,1+\theta}$,

$$\omega_{n,y-\pi_\theta(y)}(m)\le(1+C_{0,\theta})\omega_{n,y}(m). \tag{77}$$

In view of (75), (72), (77)

$$\begin{gathered}\omega_{n,\pi_\theta(y)}(\nu)\ge\omega_{n,\pi_\theta(y)}(\nu)-\omega_{n,y-\pi_\theta(y)}(\nu)\ge\\ \exp(-O(1)(\ln(\nu)+\nu^{1-1/k_\theta}))(\rho_\theta)^\nu\times\\ \left(\omega_{n,\pi_\theta(y)}(m)-\exp(O(1)(\ln(\nu)+\nu^{1-1/k_\theta}+\nu^{1-1/k_{\theta+1}}\left(\frac{\rho_{\theta+1}}{\rho_\theta}\right)^\nu\omega_{n,y}(m)\right),\end{gathered} \tag{78}$$

if $2\le\theta<s$ and $\nu\in m-1+\mathbb{N}$; moreover

$$\exp(O(1)(\ln(\nu)+\nu^{1-1/k_\theta}+\nu^{1-1/k_{\theta+1}}))\left(\frac{\rho_{\theta+1}}{\rho_\theta}\right)^\nu=o(1), \tag{79}$$

when $\nu\to\infty$ and $2\le\theta<s$.

In view of (75), (73), (77)

$$\begin{gathered}\omega_{n,\pi_\theta(y)}(\nu)\ge\omega_{n,\pi_\theta(y)}(\nu)-\omega_{n,y-\pi_\theta(y)}(\nu)\ge\\ \exp(-O(1)(\ln(\nu)+\nu^{1-1/k_\theta}))(\rho_\theta)^\nu\times\\ \left(\omega_{n,\pi_\theta(y)}(m)-\left(\frac{O(1)}{\nu}\right)^{\nu/n}\omega_{n,y}(m)\right),\end{gathered} \tag{80}$$

if $2\le\theta=s$ and $\nu\in m-1+\mathbb{N}$; moreover

$$\left(\frac{\exp(O(1))}{\nu}\right)^{\nu/n}=o(1), \tag{81}$$

when $\nu\to\infty$.

Let $y \in V_m$. We apply (33) in Lemma 2 to $y_1 = \pi_1(y)$. In view of (52), we have

$$(C_3)^{-q}\omega_{n,\pi_1(y)}(m)\exp\left(-O(1)\left(\nu^{1-1/k_1}+\ln(\nu)\right)\right)(\rho_1)^\nu \leq \tag{82}$$

$$\omega_{p,\pi_1(y)}(m)\exp\left(-O(1)\left(\nu^{1-1/k_1}+\ln(\nu)\right)\right) \leq$$

$$(\rho_1)^{-\nu}\,\omega_{p,\pi_1(y)}(\nu) \leq (\rho_1)^{-\nu}\,\omega_{n,\pi_1(y)}(\nu),$$

where $\nu \in m-1+\mathbb{N}$ and $O(1)$ depends only from the equation (3). If (59) holds for $r = \phi_m(y)$, then in view of (64), (60), (49),

$$\sup\{\omega_{n,\psi_m(\phi_m(y))}(\nu)/v_{A^\vee,k_\theta,\rho_\theta}(\nu)\colon \nu \in \mathbb{N}-1+m\} \leq$$

$$C_6\sup\{\phi_m(y)(\nu)/v_{A^\vee,k_\theta,\rho_\theta}(\nu)\colon \nu \in \mathbb{N}+m-1\} \leq C_6\omega_{q,\phi_m(y)}(m) \leq$$

$$C_6C_3C_5\omega_{n,y}(m);$$

and therefore

$$\omega_{n,\psi_m(\phi_m(y))}(\nu) \leq C_6C_3C_5\omega_{n,y}(m)v_{A^\vee,k_\theta,\rho_\theta}(\nu) = \tag{83}$$

$$C_6C_3C_5\omega_{n,y}(m)(\rho_1)^\nu v_{A^\vee,k_\theta,\rho_\theta}(\nu)/(\rho_1)^\nu,$$

where $\nu \in \mathbb{N}+m-1$. In view of (82) and (83),

$$\omega_{n,y}(\nu) = \omega_{n,\pi_1(y)+\psi_m(\phi_m(y))}(\nu) \geq \tag{84}$$

$$\omega_{n,\pi_1(y)}(\nu) - \omega_{n,\psi_m(\phi_m(y))}(\nu) \geq$$

$$(C_3)^{-q}\omega_{n,\pi_1(y)}(m)\exp\left(-O(1)\left(\nu^{1-1/k_1}+\ln(\nu)\right)\right)(\rho_1)^\nu -$$

$$C_6C_3C_5\omega_{n,y}(m)(\rho_1)^\nu v_{A^\vee,k_\theta,\rho_\theta}(\nu)/(\rho_1)^\nu =$$

$$\left(\omega_{p,\pi_1(y)}(m)\exp\left(-O(1)\left(\nu^{1-1/k_1}+\ln(\nu)\right)\right)(\rho_1)^\nu -\right.$$

$$\left.\omega_{n,y}(m)\left(\exp\left(O(1)\left(\nu^{1-1/k_1}+\ln(\nu)\right)\right)\frac{v_{A^\vee,k_\theta,\rho_\theta}(\nu)}{(\rho_1)^\nu}\right)\right),$$

where $\nu \in \mathbb{N}+m-1$ and

$$\lim_{\nu\to\infty}\left(\exp\left(O(1)\left(\nu^{1-1/k_1}+\ln(\nu)\right)\right)\frac{v_{A^\vee,k_\theta,\rho_\theta}(\nu)}{(\rho_1)^\nu}\right) = 0. \tag{85}$$

If for $r = \phi_m(y)$ (62) holds, then in view of (65), (63), (49),

$$\sup\{\omega_{n,\psi_m(\phi_m(y))}(\nu)/w_{A^\vee,k^*}(\nu)\colon \nu \in \mathbb{N}-1+m\} \leq$$

$$C_6\sup\{\phi_m(y)(\nu)/w_{A^\vee,k^*}(\nu)\colon \nu \in \mathbb{N}+m-1\} \leq C_6\omega_{q,\phi_m(y)}(m) \leq$$

$$C_6C_3C_5\omega_{n,y}(m);$$

and therefore

$$\omega_{n,\psi_m(\phi_m(y))}(\nu) \leq C_6C_3C_5\omega_{n,y}(m)w_{A^\vee,k^*}(\nu), \tag{86}$$

where $\nu \in \mathbb{N}+m-1$. In view of (82) and (86),

$$\omega_{n,y}(\nu) = \omega_{n,\pi_1(y)+\psi_m(\phi_m(y))}(\nu) \geq \tag{87}$$

$$\omega_{n,\pi_1(y)}(\nu) - \omega_{n,\psi_m(\phi_m(y))}(\nu) \geq$$
$$(C_3)^{-q}\omega_{n,\pi_1(y)}(m) \exp\left(-O(1)\left(\nu^{1-1/k_1} + \ln(\nu)\right)\right)(\rho_1)^\nu -$$
$$C_6C_3C_5\omega_{n,y}(m)w_{A^\vee,k^*}(\nu) =$$
$$\omega_{p,\pi_1(y)}(m) \exp\left(-O(1)\left(\nu^{1-1/k_1} + \ln(\nu)\right)\right)(\rho_1)^\nu -$$
$$\omega_{n,y}(m)\left(\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu+1)\right)\right)\frac{w_{A^\vee,k^*}(\nu)}{(\rho_1)^\nu}\right),$$

where $\nu \in \mathbb{N} + m - 1$, and

$$\lim_{\nu\to\infty}\left(\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu+1)\right)\right)w_{A^\vee,k^*}(\nu)/(\rho_1)^\nu\right) = 0. \tag{88}$$

We apply (31) in Lemma 2 to $y_1 = \pi_1(y)$. Then, in view of (52) , we have

$$(C_3)^{-q}\omega_{n,\pi_1(y)}(\nu) \leq \omega_{p,\pi_1(y)}(\nu) \leq$$
$$\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu)\right)\right)(\rho_1)^\nu\,\omega_{p,\pi_1(y)}(m) \leq$$
$$\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu)\right)\right)(\rho_1)^\nu\,\omega_{n,\pi_1(y)}(m) \leq$$
$$\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu)\right)\right)(\rho_1)^\nu\,\omega_{p,\pi_1(y)}(m) \leq$$
$$\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu)\right)\right)(\rho_1)^\nu\,\omega_{n,y-\psi_m(\phi_m(y))}(m) \leq$$
$$\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu)\right)\right)(\rho_1)^\nu\,\omega_{n,y}(m) +$$
$$\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu)\right)\right)(\rho_1)^\nu\,\omega_{n,\psi_m(\phi_m(y))}(m).$$

If (59) holds for $r = \phi_m(y)$, then, in view of (83),

$$\omega_{n,\psi_m(\phi_m(y))}(m) \leq C_6C_3C_5\omega_{n,y}(m)v_{A^\vee,k_\theta,\rho_\theta}(m).$$

If (62) holds for $r = \phi_m(y)$, then, in view of (86),

$$\omega_{n,\psi_m(\phi_m(y))}(m) \leq C_6C_3C_5\omega_{n,y}(m)w_{A^\vee,k^*}(m).$$

Therefore

$$\omega_{n,\pi_1(y)}(\nu) \leq \exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu+1)\right)\right)(\rho_1)^\nu\,\omega_{n,y}(m).$$

In view of (83),(86)

$$\omega_{n,(y)}(\nu) = \omega_{n,\pi_1(y)+\psi_m(\phi_m(y))}(\nu) \leq \tag{89}$$
$$\exp\left(O(1)\left(\nu^{1-1/k_1} + \ln(\nu+1)\right)\right)(\rho_1)^\nu\,\omega_{n,y}(m),$$

where $\nu \in \mathbb{N} + m - 1$. The assertions of the Theorem 7 for σ are contained in (75), (78), (79), (80), (81), (82), (84), (85), (87), (88), (89). ■

So, we must point to the spliting monomorphism ψ_m of the Lemma 4.

§3. On some linear normed spaces.

Let K be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Let further L be a linear normed space over K with norm $p(x)$. In the case $L = K^n$ we fix as $p(x)$, wehre $x \in K^n$, the maximum of the absolute values of coordinates of x in the standard basis, i.e.

$$p(x) = h(x) = \sup(\{|x_1|, \ldots, |x_n|\}), \tag{90}$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

If L is a Banach space with the norm p, then $K-$algebra of all the linear continuous operators acting in L will be denoted by $\mathfrak{M}^\wedge(L)$, and the norm on $\mathfrak{M}^\wedge(L)$, associated with the norm p will be denoted by $p^\sim$. So,

$$p^\sim(A) = sup(\{p(AX) \colon X \in L, \, p(X) \le 1\}).$$

It is well known that the associciated with h norm on $Mat_n(\mathbb{C})$ is defined as follows

$$h^\sim(A) = sup\left(\left\{\sum_{k=1}^n |a_{i,j}| \colon i = 1, \ldots, n\right\}\right), \tag{91}$$

where $A = (a_{i,k}) \in Mat_n(\mathbb{C})$. The norms h and $h^\sim$ coincide respectiely with the norms q_∞ and $q_\infty^\sim$ considered in [33], section 6. Let $m \in \mathbb{N} - 1$, and we denote by $E_m(L)$ be the set $L^{m-1+\mathbb{N}}$ of all the maps of the set $m - 1 + \mathbb{N}$ into L. The set $E_m(L)$ is a linear space over K, where the muliplication of the elements by the number from K and addition of the elements is defined coordinate-wise. The subspace of $E_m(L)$ composed by all the constant maps is isomorphic to L, and we identify this subspace with L.

We denote by $\mathfrak{M}^\vee(L)$ the space of all the $K-$linear maps of the space L in L. If $\phi \in \mathfrak{M}^\vee(L)$ and $\psi \in \mathfrak{M}^\vee(L)$, then $\phi \circ \psi$ denotes the composition of operators ϕ and ψ, so that $(\phi \circ \psi)f = \phi((\psi f))$ for each $f \in L$. For $x \in E_m(L)$ let

$$p_{m,\infty}(x) = \sup(\{p(x(\nu)) \colon \nu \in m - 1 + \mathbb{N}\}.$$

Let further

$$E_{m,\infty}(L) = \{x \in E_m(L) \colon p_{m,\infty}(x) \ne \infty\},$$

$$E_{m,0}(L) = \{x \in E_m(L) \colon \lim_{\nu \to \infty} p(x(\nu)) = 0\},$$

$$E_m^{\to}(L) = L + E_{m,0}(L).$$

Clearly, the space $E_m^{\to}(L)$ consists of all the $y \in E_m(L)$, for which there exists

$$\lim(y) = \lim_{\nu \to \infty}(y(\nu)).$$

Let $m \in \mathbb{N} - 1$, $\mu \in m - 1 + \mathbb{N}$ and ler $r_{m,\mu}$ be the operator of restriction of the elements $y \in E_m(L)$ on te set $m - 1 + \mathbb{N}$. Clearly, the map $r_{m,\mu}$ is an epimorphism of the space $E_m(L)$ onto the space $E_\mu(L)$. If L is a K-algebra, then $E_m(L)$ is a K-algebra, where the muliplication and addition of the elements is defined coordinate-wise; so, in this case $r_{m,\mu}$ is an epimorphism of K-algebra $E_m(L)$ onto K-algebra $E_\mu(L)$. If L be an algebra with unity, let L^* denotes the group of all its invertible elements. Then

$$(L^*)^{m-1+\mathbb{N}} \subset L^{m-1+\mathbb{N}};$$

we denote below $(L^*)^{m-1+\mathbb{N}}$ by $E_m(L^*)$. Clearly,

$$E_m(L^*) = (E_m(L))^*.$$

Let $L = \mathbb{C}^n$, $y \in E_m(L)$, and let $y_i(\nu)$ denotes the i-th coordinate of the element $y(\nu)$, where $i = 1, \ldots, n$, $\nu \in m-1+\mathbb{N}$; then the space $(E_m(\mathbb{C}))^n$ contains an element $\omega(y)$, which has $y_i(\nu)$ as the value of its i-th coordinate at the point $\nu \in m-1+\mathbb{N}$. So we obtain the natural isomorphism ω of the algebra $E_m(\mathbb{C}^n)$ onto $(E_m(\mathbb{C}))^n$. This map ω induces an isomorphism of the algebra $E_m(Mat_n(\mathbb{C}))$ onto $Mat_n(E_m(\mathbb{C}))$. If L is a $K-algebra$, then each element $a \in E_m(L)$ determines an acting on the space $E_m(L)$ K-linear operator $\mu_a \in \mathfrak{M}^\vee(E_m(L))$, which turns any $y \in E_m(L)$ into $\mu_a y = ay$. On the space $E_m(L)$ acts also K-linear operator $\bigtriangledown \in \mathfrak{M}^\vee(L)$, which turns any element $y \in E_m(L)$ in the $\bigtriangledown y \in E_m(L)$ such that

$$(\bigtriangledown y)(\nu) = y(\nu+1)$$

for any $\nu \in m-1+\mathbb{N}$. Let us consider the subring $\mathfrak{A}_m(L)$ of the ring $\mathfrak{M}^\vee(\mathfrak{L})$ generated by the operator $\bigtriangledown$ and by all the operators μ_a, where $a \in E_m(L)$. Clearly,

$$\mu_a \circ \bigtriangledown^r \circ \mu_b \circ \bigtriangledown^s = \mu_{a\bigtriangledown^r b_k} \circ \bigtriangledown^{r+s}, \tag{92}$$

where $\{r, s\} \subset \mathbb{N}-1$, $\{a, b\} \subset E_m(L)$. For each $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$ are uniquelly defined the number $\deg(\alpha)$ and representation of α in the form

$$\alpha = \sum_{k=0}^{\deg(\alpha)} \mu_{a_k} \circ \bigtriangledown^k, \tag{93}$$

where $a_k \in E_m(L)$ for $k = 0, \ldots, \deg(\alpha)$ and $a_{\deg(\alpha)} \neq 0_{E_m(L)}$. Clearly, (93) may be rewritten in the form

$$\alpha = \sum_{k=0}^{\infty} \mu_{a_k} \circ \bigtriangledown^k, \tag{94}$$

where $a_k = 0_{E_m(L)}$ for $k \in \deg(\alpha)+\mathbb{N}$. It follows from (92) that $\mathfrak{A}_m(L)$ is a graduated algebra, and if

$$\beta = \sum_{r=0}^{p} \mu_{b_r} \circ \bigtriangledown^r \in \mathfrak{A}_m(L), \tag{95}$$

$$\gamma = \sum_{s=0}^{q} \mu_{c_s} \circ \bigtriangledown^s \in \mathfrak{A}_m(L), \tag{96}$$

then

$$\beta\gamma = \sum_{k=0}^{p+q} \sum_{\substack{\leq r \leq p \\ 0 \leq s \leq q \\ r+s=k}} \mu_{b_r \triangledown^r c_s} \circ \triangledown^{r+s}; \tag{97}$$

clearly, $\deg(\beta\gamma) = \deg(\beta)+\deg(\gamma)$, if $b_p(\nu)^r c_q(\nu+p)$ is different from 0 at least for one $\nu \in m-1+\mathbb{N}$. Let $\mathfrak{A}_m^{\rightarrow}(L)$ be the ring generated by the operator $\triangledown$ and by all the operators μ_a, where $a \in E_m^{\rightarrow}(L)$. Since $\triangledown a \in E_m^{\rightarrow}(L)$, if $a \in E_m^{\rightarrow}(L)$, it follows, in view of (92), that $\mathfrak{A}_m^{\rightarrow}(L)$ is a graduated subalgebra $\mathfrak{A}_m^{\rightarrow}(L)$ of the algebra $\mathfrak{A}_m(L)$, each $\alpha \in \mathfrak{A}_m(L) \diagdown \{0_{\mathfrak{A}_m(L)}\}$ admits a representation in the form (33) with $a_k \in E_m^{\rightarrow}(L)$ for $k = 0, \ldots, \deg(\alpha)$ and $a_{\deg(\alpha)} \neq 0_{E_m(L)}$; to each such α corresponds the limit operator

$$\lim(\alpha) = \sum_{k=0}^{\deg(\alpha)} \mu_{\lim(a_k)} \circ \triangledown^k, \tag{98}$$

and polynomial

$$P(\alpha, z) = \sum_{k=0}^{\deg(\alpha)} \lim(a_k) z^k \in L[z]. \tag{99}$$

If $\alpha = 0_{\mathfrak{A}_m(L)}$, then we put

$$\lim(\alpha) = 0_{\mathfrak{A}_m(L)},\ P(\alpha, z) = 0_{L[z]}.$$

The equality (92) shows that the map

$$\alpha \to P(\alpha, z) \tag{100}$$

is an epimorphism of the algebra $\mathfrak{A}_m^{\rightarrow}(L)$ on on the algebra $L[z]$ (if the algebra L is noncommutative, then we can treat $L[z]$ as a semigroup ring of the semigroup $(\mathbb{N}-1, +)$ over the algebra L). We note that, if $\alpha \in \mathfrak{A}_m(\mathbb{C})$, then, clearly, $Ker(\alpha)$ coincides with the linear space of all the solutions of the equation (3); moreover if $\alpha \in \mathfrak{A}_m^{\rightarrow}(\mathbb{C})$, then the corresponding to α equation (3) is an equation of the Poincar'e type and $P(\alpha, z)$ is its characterictical polynomial. Let L be an algebra with unity. The set of all $\alpha \in \mathfrak{A}_m(L) \diagdown \{0_{\mathfrak{A}_m(L)}\}$, which have the representation (93) with $a_{\deg(\alpha)} \in E_m(L^*)$ will be denoted further by $\mathfrak{A}_m(L)^{\circ}$. The set of all the the elements $\alpha \in \mathfrak{A}_m(L) \diagdown \{0_{\mathfrak{A}_m(L)}\}$, which have the representation (93) with $a_{\deg(\alpha)} = 1_{E_m(L)}$ will be denoted further by $\mathfrak{A}_m(L)^{\vee}$. The set of all the $\alpha \in \mathfrak{A}_m(L) \diagdown \{0_{\mathfrak{A}_m(L)}\}$, which have the representation (93) with $a_0 \in E_m(L^*)$, will be denoted further by $\mathfrak{A}_m(L)^{\wedge}$. Let further

$$\mathfrak{A}_m(L)^{\circ\wedge} = \mathfrak{A}_m(L)^{\circ} \bigcap \mathfrak{A}_m(L)^{\wedge},\ \mathfrak{A}_m(L)^{\vee\wedge} = \mathfrak{A}_m(L)^{\vee} \bigcap \mathfrak{A}_m(L)^{\wedge},$$

$$\mathfrak{A}_m(L)^{\circ\rightarrow} = \mathfrak{A}_m(L)^{\circ} \bigcap \mathfrak{A}_m(L)^{\rightarrow},\ \mathfrak{A}_m(L)^{\vee\rightarrow} = \mathfrak{A}_m(L)^{\vee} \bigcap \mathfrak{A}_m(L)^{\rightarrow},$$

$$\mathfrak{A}_m(L)^{\circ\wedge\rightarrow} = \mathfrak{A}_m(L)^{\circ\wedge} \bigcap \mathfrak{A}_m(L)^{\rightarrow},\ \mathfrak{A}_m(L)^{\vee\wedge\rightarrow} = \mathfrak{A}_m(L)^{\vee\wedge} \bigcap \mathfrak{A}_m(L)^{\rightarrow},$$

Clearly, $\mathfrak{A}_m(L)^\circ$. consists of epimorphisms of the space $E_m(L)$ onto $E_m(L)$. The above map $r_{m,\mu}$ induces epimorphism $r^{\triangleright}_{m,\mu}$ of the algebra $\mathfrak{A}_m(L)$ on the algebra $\mathfrak{A}_\mu(L)$ defined as follows:

if $\alpha \in \mathfrak{A}_m(L)$,

$$\alpha = \sum_{k=0}^{n} \mu_{a_k} \circ \bigtriangledown^k, \tag{101}$$

then

$$r^{\triangleright}_{m,\mu}(\alpha) = \sum_{k=0}^{n} \mu_{r_{m,\mu}(a_k)} \circ \bigtriangledown^k, \tag{102}$$

where the operator $\bigtriangledown$ in (101) acts in $E_m(L)$ and the operator $\bigtriangledown$ in (102) acts in $E_\mu(L)$. Clearly, $r^{\triangleright}_{m,\mu}$ surjectively maps

$\mathfrak{A}_m(L)^\circ$ onto $\mathfrak{A}_\mu(L)^\circ$, $\mathfrak{A}_m(L)^\vee$ onto $\mathfrak{A}_\mu(L)^\vee$,
$\mathfrak{A}_m(L)^\wedge$ onto $\mathfrak{A}_m u(L)^\wedge$, $\mathfrak{A}_m(L)^{\circ\wedge}$ onto $\mathfrak{A}_\mu(L)^{\circ\wedge}$,
$\mathfrak{A}_m(L)^{\vee\wedge}$ onto $\mathfrak{A}_\mu(L)^{\vee\wedge}$, $\mathfrak{A}_m(L)^{\circ\to}$ onto $\mathfrak{A}_\mu(L)^{\circ\to}$,
$\mathfrak{A}_m(L)^{\vee\to}$ onto $\mathfrak{A}_\mu(L)^{\vee\to}$, $\mathfrak{A}_m(L)^{\circ\wedge\to}$ onto $\mathfrak{A}_\mu(L)^{\circ\wedge\to}$,
$\mathfrak{A}_m(L)^{\vee\wedge\to}$ onto $\mathfrak{A}_\mu(L)^{\vee\wedge\to}$. Since the diagram

$$\begin{array}{ccc} E_m(L) & \xrightarrow{r_{m,\mu}} & E_\mu(L) \\ \alpha\Big\downarrow & & \Big\downarrow r^{\triangleright}_{m,\mu}(\alpha) \\ E_m(L) & \xrightarrow[r_{m,\mu}]{} & E_\mu(L) \end{array}$$

is commuative and therefore

$$r_{m,\mu}\alpha = r^{\triangleright}_{m,\mu}(\alpha) r_{m,\mu}, \tag{103}$$

it follows that $r_{m,\mu}$ surjectively maps $Ker(r_{m,\mu}\alpha)$ onto

$$Ker(r^{\triangleright}_{m,\mu}(\alpha)) \supset r_{m,\mu}Ker(\alpha).$$

Lemma 5. *If $\mu \in m-1+\mathbb{N}$ and $\alpha \in \mathfrak{A}_m(L)^\wedge$, then the operator α bijectively maps $Ker(r_{m,\mu})$ onto $Ker(r_{m,\mu})$.*

Proof. Proof is given in (37), Lemma 3.

Corollary 1. *Let $\mu \in m-1+\mathbb{N}$ and let $\alpha \in \mathfrak{A}_m(L)^\wedge$. If*

$$g \in E_m(L),\ x \in E_\mu(L),\ m \le \mu,\ \alpha \in \mathfrak{A}_m(L)^\wedge,$$

$$r_{m,\mu}(g) = (r^{\triangleright}_{m,\mu}(\alpha))(x),$$

then there exists a unique $y \in E_m(L)$ such that

$$\alpha(y) = g,\ r_{m,\mu}(y) = x;$$

Proof. Proof is given in (37), Corollary 1 to the Lemma 3.

Corollary 2. *Let $\mu \in m-1+\mathbb{N}$ and $\alpha \in \mathfrak{A}_m(L)^\wedge$. Then and $r_{m,\mu}$ bijectively maps $Ker(\alpha)$ onto $Ker(r^{\triangleright}_{m,\mu}(\alpha)) = r_{m,\mu}(Ker(\alpha))$.*

Proof. Proof is given in (37), Corollary 2 to the Lemma 3.

If for the equation (2) are fulfilled the conditions (1) then

$$a_k = (a_k(0),\, a_k(1),\, \dots\, ,\, a_k(\nu),\, \dots) \in E_0^{\rightarrow}(\mathbb{C}), \tag{104}$$

where $k = 0, \dots, n$. Moreover, $a_n = 1_{E_0(\mathbb{C})}$, for α in (101) $Ker(\alpha)$ coincides with the linear over $\mathbb{C}$ space of all the solutions of the equation (2), polynomial (4) is equal to the polynomial $P(\alpha, z) = P(r^{\triangleright}_{0,m}(\alpha), z)$, where $m \in \mathbb{N}-1$, and the set $Ker(r^{\triangleright}_{0,m}(\alpha))$ coincides with linear over $\mathbb{C}$ space V_m of all the solutions of the equation (3).

Let $\mathfrak{v}$ be the element in $E_{0,0}$, for which

$$\mathfrak{v}(\nu) = \frac{1}{\nu+1},$$

where $\nu \in \mathbb{N}-1$. Clearly, $r_{0,m}(\mathfrak{v})E_{m,\infty}(\mathbb{C}) \subset E_{m,0}(\mathbb{C})$ for any $m \in \mathbb{N}-1$. Let

$$E^{\succ}_{m,0}(L) = r_{0,m}(\mathfrak{v})E_{m,\infty}(L),\; E^{\succ}_{m}(L) = L + E^{\succ}_{m,0}(L).$$

Let us consider the ring $\mathfrak{A}^{\succ}_m(L)$ generated by the operator $\bigtriangledown$ and by all the operators μ_a, where $a \in E_m(L)^{\succ}$. The Assertion 1 may be reformulated as follows:

Lemma 6. *Let $\alpha \in \mathfrak{A}^{\succ}_0(\mathbb{C})\bigcap\mathfrak{A}^{\vee}_0(\mathbb{C})$, and $P(\alpha, z)$ coincides with the polynomial $T(z)$ in (4) and (37).*

Then there exist $m \in \mathbb{N}$ and representation of the oprator $r^{\triangleright}_{0,m}(\alpha)$ in the form

$$r^{\triangleright}_{0,m}(\alpha) = \eta\beta \tag{105}$$

such that

$$\eta \in \mathfrak{A}^{\succ}_m(\mathbb{C})\bigcap\mathfrak{A}^{\vee}_m(\mathbb{C}),\; deg(\eta) = q \tag{106}$$

$$\beta \in \mathfrak{A}^{\succ}_m(\mathbb{C})\bigcap\mathfrak{A}^{\vee\wedge}_m(\mathbb{C}),\; \deg(\beta) = p = n - q, \tag{107}$$

and the polynomials $P(\beta, z)$, $P(\eta, z)$ coincide respectively with the polynomials $T_1(z)$, $T_2(z)$ in (36) .

Let L be a Banach space with the norm p. Let $a = a(\nu)$, where $\nu \in \mathbb{N}-1$, be an arbitrary sequence of positive real numbers. Let $E_{m,a}(L) = aE_{m,\infty}(L)$. Then $E_{m,a}(L)$ consists of all such $y \in E_m(L)$, for which

$$p_{m,a}(y) = p_{m,\infty}\left(a^{-1}y\right) = \tag{108}$$

$$\sup\left\{p\left((a(\nu))^{-1}y(\nu)\right) : \nu \in \mathbb{N}+m-1\right\} < +\infty.$$

Then $p_{m,a}$ is a norm on $E_{m,a}(L)$ and $E_{m,a}(L)$ is a Banah space relatively the norm $p_{m,a}$. Let $\lambda > 0$ and let $\mathfrak{a}_{m,\lambda}$ be a sequence, for which

$$\mathfrak{a}_{m,\lambda}(\nu) = \lambda^{\nu},$$

where $\nu \in \mathbb{N}+m-1$. Then

$$E_{m,\mathfrak{a}_{m,1}}(L) = E_{m,\infty}(L),\; p_{m,\mathfrak{a}_{m,1}} = p_{m,\infty},$$

the map, which turns any element $z \in E_{0,\mathfrak{a}_{m,1}}(L)$ *into* $y \in E_m(L)$ *such that* $y(\nu) = a(\nu)z(\nu - m)$, *where* $\nu \in \mathbb{N} + m - 1$, *is a isometry of the space* $E_{0,\mathfrak{a}_{m,1}}(L) = E_{0,\infty}(L)$ *on the space* $E_{m,a}(L)$. *Clearly,* $p_{m,\mathfrak{a}_{m,1}}$ *and* $p_{m,\infty}$ *coincide on the space* $E_{m,\mathfrak{a}_{m,1}}(L) = E_{m,\infty}(L)$. *Clearly,*

$$\{y \in E_m(L) \colon p_{m,a}(y) \le 1\} = \{at \colon p_{m,\infty}(t) \le 1\}. \tag{109}$$

Let $C > 0$, $n \in \mathbb{N}$. *Let* $w_{C,n}$ *denotes the element in* $E_{0,0}(\mathbb{C})$, *for which*

$$w_{C,n}(\nu) = \left(\frac{C}{\nu+1}\right)^{\nu/n}, \tag{110}$$

where $\nu \in \mathbb{N} - 1$; *let further* $\rho > 0$ *and* $v_{C,n,\rho}$ *denotes the element in* $E_0(\mathbb{C})$, *for which*

$$v_{C,n,\rho}(\nu) = \rho^\nu \exp\left(C(\nu+1)^{1-1/n} + \ln(\nu+1)\right), \tag{111}$$

where $\nu \in \mathbb{N} - 1$. *Let* C *is bigger than all the absolute values of* $O(1)$ *in (31), (33) and* $A^\vee$ *in (58), (63). In view of (31), the sequence*

$$Y_{n,y} = Y_{n,y}(\nu),$$

where $\nu \in \mathbb{N} + m - 1$, *contains in* $E_{m,v_{C,n,\rho_1}}(\mathbb{C}^n)$, *and*

$$h_{m,v_{C,n,\rho_1}}(Y_{n,y}) \le h(Y_{n,y}(m)).$$

In view of (60), if $s \ge 2$, $r \in R^\wedge_{m,\theta} \backslash R^\wedge_{m,\theta+1}$ *for some* $\theta \in \{2, ..., s\}$, *then*

$$Y_{q,r} = Y_{q,r}(\nu),$$

where $\nu \in \mathbb{N} + m - 1$, *contains in* $E_{m,v_{C,k_\theta,\rho_\theta}}(\mathbb{C}^q)$, *and*

$$h_{m,v_{C,k_\theta,\rho_\theta}}(Y_{q,r}) \le h(Y_{q,r}(m)) = \omega_{q,r}(m)).$$

In view of (63), if $k^* > 0$, $r \in R^\wedge_{m,s+1}$, *then*

$$Y_{q,r} = Y_{q,r}(\nu),$$

where $\nu \in \mathbb{N} + m - 1$, *contains in* $E_{m,w_{C,k^*}}(\mathbb{C}^q)$, *and*

$$h_{m,w_{C,k^*}}(Y_{q,r}) \le h(Y_{q,r}(m)) = \omega_{q,r}(m)).$$

Lemma 7. *The operaor* $\bigtriangledown$ *turns the space* $E_{m,a}(L)$ *into* $E_{m,a}(L)$ *if and only if, when*

$$h_{m,a}(\bigtriangledown a) = \tag{112}$$

$$\sup\{a(\nu+1)/a(\nu) \colon \nu \in \mathbb{N} + m - 1\} < +\infty.$$

If (112) holds then $\bigtriangledown$ *is continuous on* $E_{m,a}(L)$ *and*

$$p^\sim_{m,a}(\bigtriangledown) = h_{m,a}(\bigtriangledown a) \tag{113}$$

Proof. Let $f \in L \diagdown \{0\}$, $g = (p(f))^{-1} f$,

$$y_0(\nu) = g,\ y_1 = ay_0.$$

Then $p_{m,a}(y_1) = 1$,

$$p\left((a(\nu))^{-1}(\bigtriangledown y_1)(\nu)\right) = a(\nu+1)/a(\nu),$$

where $\nu \in \mathbb{N} + m - 1$; if $\bigtriangledown$ turns $E_{m,a}(L)$ into $E_{m,a}(L)$ then

$$h_{m,a}(\bigtriangledown a) = \sup\{a(\nu+1)/a(\nu) \colon \nu \in \mathbb{N} + m - 1\} = \tag{114}$$

$$\sup\left\{p\left((a(\nu))^{-1}(\bigtriangledown y_1)(\nu)\right) : \nu \in \mathbb{N} + m - 1\right\} = p_{m,a}(\bigtriangledown y_1) < +\infty,$$

and therefore (112) holds. If (112) holds, $y \in E_{m,a}(L)$ and $p_{m,a}(y) \leq 1$, then

$$p\left((a(\nu))^{-1}(\bigtriangledown y)(\nu)\right) = \tag{115}$$

$$p\left((a(\nu))^{-1}a(\nu+1)(a(\nu+1))^{-1}y(\nu+1)\right) =$$
$$(a(\nu))^{-1}a(\nu+1)p\left((a(\nu+1))^{-1}y(\nu+1)\right) \leq$$
$$h_{m,a}(\bigtriangledown a)p_{m,a}(y).$$

Therefore $\bigtriangledown$ is continuous on $E_{m,a}(L)$ and

$$p^{\sim}_{m,a}(\bigtriangledown) \leq h_{m,a}(\bigtriangledown a)p_{m,a}(y) \tag{116}$$

The equality (113) follows from (116) and from (114). ■

Each $A \in E_m(\mathfrak{M}^\vee(L))$ defines a linear operator $A^\vee \in \mathfrak{M}^\vee(E_m(L))$, such that

$$(A^\vee y)(\nu) = (A(\nu))(y(\nu)),$$

where $\nu \in m - 1 + \mathbb{N}$, $y \in E_m(L)$. The operator $A^\vee$ is invertible if and only if, when the operator $A(\nu)$ is invertible for any $\nu \in m - 1 + \mathbb{N}$; in this case the map

$$\nu \mapsto (A(\nu))^{-1},$$

where $\nu \in m - 1 + \mathbb{N}$, will be denoted by A^{-1}. So,

$$A^{-1}(\nu) = (A(\nu))^{-1},$$

where $\nu \in m - 1 + \mathbb{N}$. Clearly, $(A^\vee)^{-1} = (A^{-1})^\vee$. If

$$A \in E_{m,\infty}(\mathfrak{M}(L)), \tag{117}$$

then, clearly,

$$(p^\sim)_{m,\infty}(A) = \sup\{p^\sim(A(\nu)) \colon \nu \in m - 1 + \mathbb{N}\} = \tag{118}$$

$$\sup\{\sup\{p(A(\nu)y(\nu)) \colon : y(\nu) \in L,\, p(y(\nu)) \leq 1\} \colon \nu \in m - 1 + \mathbb{N}\} =$$
$$\sup\{p_{m,\infty}(A^\vee(y)) \colon y \in E_{m,\infty}(L), p_{m,\infty}(y) \leq 1\} =$$
$$(p_{m,\infty})^\sim(A^\vee).$$

In particular, if

$$a \in E_{m,\infty}(K) \tag{119}$$

and we take on the role of A in (117) the we take the sequence of the (left) multiplications by $a(\nu)$ of the elements in L, then the role of $A^\vee$ will play the above operator μ_a, and

$$p_{m,\infty}(a) = (p_{m,\infty})^\sim(\mu_a). \tag{120}$$

If $a = a(\nu)$ and $d = d(\nu)$, where $\nu \in \mathbb{N} - 1$, are two sequences of te positive real mumbers and

$$A \in E_{m,a/d}(\mathfrak{M}(L)), \tag{121}$$

then $A = \mu_{a/d}B$, where $B \in E_{m,\infty}(\mathfrak{M}(L))$, the operator $A^\vee$ maps the space $E_{m,d}(L)$ into the space $E_{m,a}(L)$, and

$$\begin{gathered}(p^\sim)_{m,a/d}(A) = (p^\sim)_{m,\infty}(B) = (p_{m,\infty})^s im(B^\vee) = \\ \sup\{p_{m,\infty}(B^\vee(y)) \colon y \in E_{m,\infty}(L), p_{m,\infty}(y) \le 1\} = \\ \sup\{p_{m,a}(B^\vee(\mu_a(y))) \colon y \in E_{m,\infty}(L), p_{m,d}(\mu_d y) \le 1\} = \\ \sup\{p_{m,a}(B^\vee(\mu_{a/d}(z))) \colon z \in E_{m,d}(L), p_{m,d}(z) \le 1\} = \\ \sup\{p_{m,a}((\mu_{a/d}B)^\vee(z)) \colon z \in E_{m,d}(L), p_{m,d}(z) \le 1\} = \\ \sup\{p_{m,a}(A^\vee(z)) \colon z \in E_{m,d}(L), p_{m,d}(z) \le 1\};\end{gathered} \tag{122}$$

in particular, if $a = d$ the it follows from (122) that $A^\vee$ maps $E_{m,a}(L)$ into $E_{m,a}(L)$, and

$$\begin{gathered}(p_{m,a})^\sim(A^\vee) = \\ \sup\{p_{m,a}(A^\vee(z)) \colon z \in E_{m,a}(L), p_{m,a}(z) \le 1\} = \\ (p^\sim)_{m,a/a}(A) = (p^\sim)_{m,\infty}(A).\end{gathered} \tag{123}$$

Lemma 8. *Let $m \in \mathbb{N} - 1$ and α from (93) belongs to $\mathfrak{A}^{\rightarrow}_m(\mathbb{C})$. Let further $a = a(\nu)$, where $\nu \in \mathbb{N} - 1$, be an arbitrary sequence of positive real numbers, $\lambda > 0$ and*

$$h_{m,a}(\bigtriangledown a) = \sup\{a(\nu+1)/a(\nu) \colon \nu \in \mathbb{N} + m - 1\} \le \lambda. \tag{124}$$

Then α is bounded linear operator on the space $E_{m,a}(\mathbb{C})$, and

$$(p_{m,a})^\sim(\alpha) \le \sum_{k=0}^{\deg(\alpha)} p_{m,\infty}(a_k)\lambda^k \tag{125}$$

Proof. In view of (124) and Lemma 7, (113) holds. The asserion of the Lemma now follows from (120). ■

Corollary. *If α from (93) belongs to $\mathfrak{A}^{\rightarrow}_m(\mathbb{C})$, then α is bounded linear operator on the space $E_{m,\mathfrak{a}_{m,\lambda}}(\mathbb{C})$, and (125) holds.*

Proof. In fact,

$$\mathfrak{a}_{m,\lambda}(\nu+1)/\mathfrak{a}_{m,\lambda}(\nu) = \lambda$$

So, all the conditions of the Lemma 8 are fulfilled. ■

Clearly, if $a_1(\nu)$ and $a_2(\nu)$, where $\nu \in \mathbb{N}-1$, are two arbitrary sequences of positive real numbers, and

(126) $$A_i \in E_{m,a_i}(\mathfrak{M}(L)),$$

where $i = 1, 2$ then $A_1A_2 \in E_{m,a_1a_2}(\mathfrak{M}(L))$, and

$$(p^{\sim})_{m,a_1a_2}(A_1A_2) = (p^{\sim})_{m,\infty}\left((a_1a_2)^{-1}A_1A_2\right) =$$
$$(p^{\sim})_{m,\infty}\left(\left((a_1)^{-1}A_1\right)\left((a_2)^{-1}A_2\right)\right) \leq$$
$$(p^{\sim})_{m,\infty}\left((a_1)^{-1}A_1\right)(p^{\sim})_{m,\infty}\left((a_2)^{-1}A_2\right) =$$
$$(p^{\sim})_{m,a_1}(A_1)(p^{\sim})_{m,a_2}(A_2).$$

Clearly,

(127) $$(\nabla A^{\vee})^k = \left(\prod_{\kappa} = 1^k(\nabla^{\kappa}A\right)^{\vee}\nabla^k,$$

where $k \in \mathbb{N}-1$, $A \in E_m(\mathfrak{M}(L))$

Let L is a Banach space over th field K and $A \in E_{m,\infty}(\mathfrak{M}^{\wedge}(L))$. Let further there exists $A^{-1} \in E_{m,\infty}(\mathfrak{M}^{\wedge}(L))$, $\lambda > 0$ and

(128) $$(p^{\sim})_{m,1}(A^{-1}) = \rho < 1/\lambda.$$

Let further $a = a(\nu)$, where $\nu \in \mathbb{N}-1$, be an arbitrary sequence of positive real numbers, and (refeq:cde1) holds. As we see above, (see text after (122) the operator $(A^{-1})^{\vee}$ maps $E_{m,a}(L)$ into $E_{m,a}(L)$, and, in view of (123) and (128),

(129) $$(p_{m,a})^{\sim}((A^{-1})^{\vee}) = (p^{\sim})_{m,\infty}(A^{-1}) = \rho < 1/\lambda.$$

According to the (129), (113) and (124)

(130) $$(p_{m,a})^{\sim}(\nabla \circ (A^{-1})^{\vee}) \leq \rho\lambda < 1,$$

$\mathfrak{M}(E_{m,a}(L))$ contains the operator

$$-\sum k = 0^{\infty}(\nabla \circ (A^{-1})^{\vee})^k =$$
$$(\nabla \circ (A^{-1})^{\vee} - 1_{\mathfrak{M}(E_{m,a}(L))})^{-1} = A^{v}ee(\nabla - A^{\vee})^{-1},$$

and

(131) $$(p_{m,a})^{\sim}\left(A^{\vee}(\nabla - A^{\vee})^{-1}\right) \leq 1/(1-\rho\lambda).$$

Therefore, $\mathfrak{M}(E_{m,a}(L))$ contains the linear operator

(132) $$-(A^{-1})^{\vee}\sum_{k=0}^{\infty}(\nabla \circ (A^{-1})^{\vee})^k = (\nabla - A^{\vee})^{-1},$$

and in view of (132) and (129),

(133) $$(p_{m,a})^{\sim}((\nabla - A^{\vee})^{-1}) \leq \rho/(1-\rho\lambda).$$

According to (127), the equality (132) may be rewritten in the form

(134) $$(\nabla - A^{\vee})^{-1} = -(A^{-1})^{\vee}\sum_{k=0}^{\infty}\left(\prod_{\kappa=1}^{k}(\nabla^{\kappa}(A^{-1}))^{\vee}\right) \circ \nabla^k.$$

§4. End of the proof of the Theorem 7.

Let, in accordance with (50), (105),

$$\nabla^p + \sum_{k=0}^{p-1} \mu_{b_k} \circ \nabla^k, \tag{135}$$

where $b_k \in E_m^{\succ}\mathbb{C} \subset E_m^{\rightarrow}\mathbb{C}$ for $k = 0, \dots, p-1$, and $b_k(\nu)$ are coefficients of the equation (50) for $k = 0, \dots, p-1$ and $\nu \in m-1+\mathbb{N}$. In view of (38),

$$b_k^{\sim} = \lim(b_k), \tag{136}$$

where $k = 0, \dots, p-1$,

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -b_0 & -b_1 & -b_2 & \dots & -b_{p-1} \end{pmatrix},$$

$B = \omega^{-1}(B_1)$, where ω is the above isomorphism of the algebra $E_m(Mat_p(\mathbb{C}))$ onto $Mat_p(E_m(\mathbb{C}))$, and let

$$B^{\sim} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -b_0^{\sim} & -b_1^{\sim} & -b_2^{\sim} & \dots & -b_{p-1}^{\sim}. \end{pmatrix}$$

We use now the Lemmata 8 and 9 in [37] and we take on the role of the matrix A in these Lemmata the matrix $B^{\sim}$. In view of (36), $b_0^{\sim} = T_1(0) \neq 0$. Therefore $(B^{\sim})^{-1}$ exists. We take on the role of ε and ε_1 in the specified Lemma 8 and Lemma 9 of the paper [37] the number

$$\varepsilon_0 = (\rho_1 - \rho_2)/4,$$

and we take on $\mathbb{C}^p$ the norm $p_{B^{\sim},\varepsilon_0}$. In view of (110),

$$w_{C,n}(\nu+1)/w_{C,n}(\nu) \le \left(\frac{C}{\nu+1}\right)^{1/n}, \tag{137}$$

where $\nu \in \mathbb{N}$. In view of (111),

$$v_{C,n,\rho}(\nu+1)/v_{C,n,\rho}(\nu) \le \rho \exp\left((2C)(\nu)^{-1/n}\right), \tag{138}$$

where $\nu \in \mathbb{N}$. Making use the equality $\lim(B) = B^{\sim}$, (137), (138) and the inequalities $\rho_\theta \le \rho 2$, $k_\theta \le n$ if $2 \le \theta \le s+1$, the above m may be replaced by means the map $r^{\#}_{m,\mu}$ on some bigger m, such that

$$p^{\sim}(B(\nu) - B^{\sim}) \le \varepsilon_0, \tag{139}$$

if $\nu \in \mathbb{N} - 1 + m$,

(140) $$w_{C,n}(\nu+1)/w_{C,n}(\nu) \leq \rho_2 + \varepsilon_0,$$

if $\nu \in \mathbb{N} - 1 + m$,

(141) $$v_{C,k_\theta,\rho_\theta}(\nu+1)/v_{C,n,\rho_2}(\nu) \leq \rho_\theta + \varepsilon_0,$$

if $2 \leq \theta \leq s$ and $\nu \in \mathbb{N} - 1 + m$,

(142) $$C/(\nu+1) < 1,$$

if $\nu \in \mathbb{N} - 1 + m$. Then, in view of (142 and the inequalities $\rho_\theta \leq \rho 2$, $k_\theta \leq n$ if $2 \leq \theta \leq s+1$, the following inequalities hold:

(143) $$\frac{w_{C,k^*}(\nu+1)}{w_{C,k^*}(\nu)} \leq \frac{w_{C,n}(\nu+1)}{w_{C,n}(\nu)} \leq \rho_2 + \varepsilon_0,$$

if $k^* > 0$ and $\nu \in \mathbb{N} - 1 + m$,

(144) $$\frac{v_{C,k_\theta,\rho_\theta}(\nu+1)}{v_{C,n,\rho_2}(\nu)} \leq \rho_\theta + \varepsilon_0 \leq \rho_2 + \varepsilon_0,$$

if $2 \leq \theta \leq s$ and $\nu \in \mathbb{N}-1+m$. In view of the inequality (139) and according to the Lemma 9 in [37], the matrix $(B(\nu))^{-1}$ exists, if $\nu \in \mathbb{N} - 1 + m$, and

(145) $$p^{\sim}(B(\nu)^{-1}) \leq (\rho_1 - 2\varepsilon_0)^{-1},$$

where $\nu \in \mathbb{N} - 1 + m$. Cosequently, there exists $B^{-1} \in E_{m,\infty}$ and

(146) $$(p^{\sim})_{m,\infty}\left(B^{-1}\right) \leq (\rho_1 - 2\varepsilon_0)^{-1}.$$

Making use of (129) - (134) with

(147) $$\rho = (\rho_1 - 2\varepsilon_0)^{-1},\ \lambda = \rho_2 + \varepsilon_0,\ a = r^{\#}_{1,m} v_{C,k_\theta,\rho_\theta},$$

if $2 \leq \theta \leq s$, and

(148) $$\rho = (\rho_1 - 2\varepsilon_0)^{-1},\ \lambda = \rho_2 + \varepsilon_0,\ a = w_{C,k^*},$$

if $k^* > 0$, $\theta = s+1$, we see that on $E_{m,a}(\mathbb{C}^p)$ acts a linear operator

(149) $$-(B^{-1})^{\vee}\sum_{k=0}^{\infty}(\bigtriangledown \circ (B^{-1})^{\vee})^k = (\bigtriangledown - B^{\vee})^{-1},$$

and

(150) $$(p_{m,a})^{\sim}(\bigtriangledown - B^{\vee})^{-1}) \leq \rho/(1-\rho\lambda) = 1/\varepsilon_0.$$

For any $y \in E_m(\mathbb{C})$ and $n \in \mathbb{N}$ let $Y_{n,y}$ and $Y^{\#}_{n,y}$ denote the elements in the space $E_m(\mathbb{C}^n)$, wich are determined respectively by means the following equalities:

(151) $$Y_{n,y}(\nu) = \begin{pmatrix} y(\nu) \\ \vdots \\ y(\nu+n-1) \end{pmatrix},$$

$$Y^{\#}_{n,y}(\nu) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y(\nu) \end{pmatrix}, \tag{152}$$

where $\nu \in m-1+\mathbb{N}$. Clearly, if (124) holds, then

$$h_{m,a}(y) = h_{m,a}(Y^{\#}_{n,y}) \leq h_{m,a}(Y_{n,y}) \leq \tag{153}$$

$$\max(1, \lambda^{n-1})h_{m,a}(y),$$

where $y \in E_{m,a}(\mathbb{C})$; terefore the maps

$$y \to Y^{\#}_{n,y}, \tag{154}$$

$$y \to Y_{n,y}, \tag{155}$$

where

$$y \in E_{m,a}(\mathbb{C})$$

are isomorphisms of the Banach space $E_{m,a}(\mathbb{C})$ onto closed subspaces of the Banach space $E_{m,a}(\mathbb{C}^n)$ and the functions

$$y \to h_{m,a}(Y^{\#}_{n,y})$$

and

$$y \to h_{m,a}(Y_{n,y}),$$

where $y \in E_{m,a}(\mathbb{C})$, are the equivalent to $h_{m,a}(y)$ norms on the space $E_{m,a}(\mathbb{C})$. For any $X \in \mathbb{C}^p$ let $\pi(X)$ denotes the first coordiate of the column X, and we denote by π the map of $\mathbb{C}^p$ on $\mathbb{C}$, which turns each $X \in \mathbb{C}^p$ into $\pi(X)$. The equation (43) is equivalent to the equation

$$((\nabla - B^{\vee})U)(\nu) = Y^{\#}_{p,r}(\nu),\ \nu \in m-1+\mathbb{N}: \tag{156}$$

the column $U(\nu)$, $\nu \in m-1+\mathbb{N}$ is a solution of the equation (156) if and only if, when $y(\nu) = (\pi(U))(\nu)$ with $\nu \in m-1+\mathbb{N}$ is a solution of the equation (43). Let a is defined by means (147) – (148), and let ξ is the map of $E_{m,a}(\mathbb{C})$ into $E_m(\mathbb{C})$ such that

$$\xi(z)(\nu) = \pi\left((\nabla - B^{\vee})^{-1}Y^{\#}_{p,z}\right)(\nu), \tag{157}$$

for all the $z \in E_{m,a}(\mathbb{C})$ and $\nu \in m-1+\mathbb{N}$. In view of (147) – (149), (153) and (157), if $y = \xi(z)$, for some $z \in E_{m,a}(\mathbb{C})$ then

$$h_{m,a}(y) = h_{m,a}(\xi(z)) = h_{m,a}\left(\pi\left((\nabla - B^{\vee})^{-1}Y^{\#}_{p,z}\right)\right). \tag{158}$$

According to the Lemma 8 in the paper [37], and in view of (150),

$$h_{m,a}\left(\pi\left((\nabla - B^{\vee})^{-1}Y^{\#}_{p,z}\right)\right) \leq \tag{159}$$

$$\gamma^*(B^{\sim})\left(\max(1, \varepsilon_0)\right)^{p-1} p_{m,a}\left((\nabla - B^{\vee})^{-1}Y^{\#}_{p,z}\right) \leq$$

$$\gamma^*(B^\sim)\left(\max(1,\varepsilon_0)\right)^{p-1}(p_{m,a})^\sim\left((\nabla-B^\vee)^{-1}\right)p_{m,a}\left(Y^\#_{p,z}\right)\le$$
$$\gamma^*(B^\sim)\left(\max(1,\varepsilon_0)\right)^{p-1}(\varepsilon_0)^{-1}p_{m,a}(z)\le$$
$$(\gamma^*(B^\sim))^2\left(\max\left((\varepsilon_0)^{-1},\varepsilon_0\right)\right)^{p-1}(\varepsilon_0)^{-1}h_{m,a}(z).$$

In view of (158), (159), (153) , (147), (148),

(160) $$h_{m,a}(Y_{n,y})\le\max\left(1,(\rho_2+\varepsilon_0)^{n-1}\right)h_{m,a}(y)\le$$

$$\max\left(1,(\rho_2+\varepsilon_0)^{n-1}\right)(\gamma^*(B^\sim))^2\left(\max\left((\varepsilon_0)^{-1},\varepsilon_0\right)\right)^{p-1}(\varepsilon_0)^{-1}h_{m,a}(z).$$

We see that for ξ are fulfilled all the conditions reqired in the Lemma 4 for the splitting monomorphism ψ_m. Therefore, in view of (105), the role of the splitting monomorphism ψ_m of the Lemma 4 may play the restriction of the above homomorphism ξ on $Ker(\eta)$. ■

§5. Exactness of the obtained results.

The simple example

(161) $$y(\nu)=\nu^\alpha$$

shows that the results of Theorem 7 are exact. if $k_\theta=1$. We shall show that, if $k_\theta=n>1$, then the results of Theorem 7 are exact. We take $\gamma\in\mathbb{R}\diagdown\{0\}$, put

(162) $$y_1(x)=\exp(\gamma x^{1/n}),$$

where $x>0$, and define $\alpha(x)$ from the equation

(163) $$(\triangle^n(y_1))(x)=(\gamma/|\gamma|)^n\alpha(x)y_1(x).$$

Let us consider the function

(164) $$f(t)=f_x(t)=y_1(x+t)/y_1(x),$$

where $t\in[0,n]$. Since $(x+t)^{1/n}-x^{1/n}\le x^{1/n-1}$, if $x>0$, $t\in[0,n]$, it follows that

(165) $$f_x(t)=1+O\left(x^{1/n-1}\right),$$

if $x>0$, $t\in[0,n]$. Clearly,

$$\alpha(x)=(\gamma/|\gamma|)^n(\triangle^n(f_x))\Big|_{t=0}.$$

We take $t_0=0$, and, making use of the Lagrange theorem, define t_k for $k=1,\ldots,n$ from the equalities

(166) $$\left(\triangle^{n+1-k}\left((f_x)^{(k-1)}\right)\right)\Big|_{t=t_{k-1}}=$$

$$\left(\triangle\left(\triangle^{n-k}\left((f_x))^{(k-1)}\right)\right)\right)\Big|_{t=t_{k-1}}=$$

$$\left(\triangle^{n-k}\left((f_x)^{(k)}\right)\right)\Big|_{t=t_k},$$

where $0 \leq t_{k-1} < t_k < t_{k-1}+1 \leq n$ for $k = 1, \ldots, n$. Then

(167) $$\alpha(x) = (\gamma/|\gamma|)^n(\triangle^n(f_x))\Big|_{t=0} =$$

$$(\gamma/|\gamma|)^n f_x^(n)(t_n) =$$
$$(\gamma/|\gamma|)^n(\gamma x^{-1/n})^n(1+O(1/x)(1+O(x^{1/n-1})) =$$
$$x^{-1}|(\gamma|/n)^n + O(x^{1/n-2}),$$

when $x \to \infty$. So, the Poincaré equation

$$(\triangle^n(y))(\nu) = (\gamma/|\gamma|)^n\alpha(\nu)y(\nu),$$

with $\nu \in \mathbb{N}$, and $\alpha(\nu) = O(1/\nu)$ has a solution

$$y = y_1(\nu)\exp(\gamma\nu^{1/n})$$

Therefore in the case $k_\theta = n > 1$ the results of Theorem 7 are exact.

The following result is proved in [38]:

Theorem . ([38], Theorem 2) *Let is given the equation*

(168) $$y(\nu+2) \ - \ 2y(\nu+1) \ + \ y(\nu) = a(\nu)y(\nu),$$

and are fulfilled all the conditions

(169) $$0 < a(\nu) < 1,$$

(170) $$a(\nu+1) < a(\nu),$$

(171) $$a(\nu) = \alpha^2/(4\nu) + \beta\nu^{-3/2} + O(\nu^{-2}),$$

where $\nu \in \mathbb{N}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

Then the equation (168) have a solutions $y_1(\nu)$ and $y_2(\nu)$ such that

$$y_1(\nu) = \exp(\alpha\nu^{1/2} + O(ln(\nu)),\ y_2(\nu) = \exp(-\alpha\nu^{1/2} + O(ln(\nu)).$$

Remark. *Taking in (171) $\beta = 0$, we see that estimates of the Theorem 7 are exact for equations of second order with coefficients contained in the set of rational functions of ν.*

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