LATTICE REPRESENTATIONS OF HEISENBERG GROUPS

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1 INTRODUCTION

For any positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} := \Big\{ (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\, {}^t\!\lambda \text{ symmetric} \Big\}$$

endowed with the following multiplication law

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') := (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

Here $\mathbb{R}^{(h,g)}$ (resp. $\mathbb{R}^{(h,h)}$) denotes the set of all $h \times g$ (resp. $h \times h$) real matrices.

The Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ is embedded in the symplectic group $Sp(g+h, \mathbb{R})$ via the mapping

$$H_{\mathbb{R}}^{(g,h)} \ni (\lambda, \mu, \kappa) \longmapsto \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in Sp(g+h, \mathbb{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactification of Siegel moduli spaces. In fact, $H_{\mathbb{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g+h, \mathbb{R})$ associated with the rational boundary component F_g (cf. [N] p. 21). For the motivation of the study of this Heisenberg group we refer to [Y4]-[Y8] and [Z]. We also refer to [Y1]- [Y3] for more results on $H_{\mathbb{R}}^{(g,h)}$.

In [C], P. Cartier stated without proof that for $h = 1$, the lattice representation of $H_{\mathbb{R}}^{(g,1)}$ associated to the lattice L is unitarily equivalent to the direct sum of $[L^* : L]^{\frac{1}{2}}$ copies of the Schrödinger representation of $H_{\mathbb{R}}^{(g,1)}$, where L^* is the dual lattice of L with respect to a certain nondegenerate alternating bilinear form. R. Berndt proved the above fact for the case $h = 1$ in his lecture note [B]. In this paper, we give a complete proof of Cartier's theorem for $H_{\mathbb{R}}^{(g,h)}$.

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Main Theorem. Let $\mathcal{M}$ be a positive definite, symmetric half-integral matrix of degree h and L be a self-dual lattice in $\mathbb{C}^{(h,g)}$. Then the lattice representation $\pi_{\mathcal{M}}$ of $H_{\mathbb{R}}^{(g,h)}$ associated with L and $\mathcal{M}$ is unitarily equivalent to the direct sum of $(\det 2\mathcal{M})^g$ copies of the Schrödinger representation of $H_{\mathbb{R}}^{(g,h)}$. For a precise detail, we refer to section 3.

This paper is organized as follows. In section 2, we review the Schrödinger representations of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$. In section 3, we prove the main theorem. In the final section, we provide a relation between lattice representations and theta functions.

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Notations: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. $\mathbb{C}_1^\times$ denotes the multiplicative group consisting of all complex numbers z with $|z|=1$. $Sp(g,\mathbb{R})$ denotes the symplectic group of degree g. H_g denotes the Siegel upper half plane of degree g. The symbol ":=" means that the expression on the right is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A\in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A\in F^{(k,l)}$ and $B\in F^{(k,k)}$, we set $B[A]={}^tABA$. E_k denotes the identity matrix of degree k. For a positive integer n, $\mathrm{Symm}\,(n,K)$ denotes the vector space consisting of all symmetric $n\times n$ matrices with entries in a field K.

2 Schrödinger Representations

First of all, we observe that $H_{\mathbb{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy to see that the inverse of an element $(\lambda,\mu,\kappa)\in H_{\mathbb{R}}^{(g,h)}$ is given by

$$(\lambda,\mu,\kappa)^{-1}=(-\lambda,-\mu,-\kappa+\lambda\,{}^t\mu-\mu\,{}^t\lambda).$$

Now we put

$$[\lambda,\mu,\kappa]:=(0,\mu,\kappa)\circ(\lambda,0,0)=(\lambda,\mu,\kappa-\mu\,{}^t\lambda). \tag{2.1}$$

Then $H_{\mathbb{R}}^{(g,h)}$ may be regarded as a group equipped with the following multiplication

$$[\lambda,\mu,\kappa]\diamond[\lambda_0,\mu_0,\kappa_0]:=[\lambda+\lambda_0,\mu+\mu_0,\kappa+\kappa_0+\lambda\,{}^t\mu_0+\mu_0\,{}^t\lambda]. \tag{2.2}$$

The inverse of $[\lambda,\mu,\kappa]\in H_{\mathbb{R}}^{(g,h)}$ is given by

$$[\lambda,\mu,\kappa]^{-1}=[-\lambda,-\mu,-\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda].$$

We set

$$K := \left\{ [0,\mu,\kappa] \in H_{\mathbb{R}}^{(g,h)} \;\middle|\; \mu \in \mathbb{R}^{(h,g)},\ \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)} \right\}. \tag{2.3}$$

Then K is a commutative normal subgroup of $H_{\mathbb{R}}^{(g,h)}$. Let $\hat{K}$ be the Pontrajagin dual of K, i.e., the commutative group consisting of all unitary characters of K. Then $\hat{K}$ is isomorphic to the additive group $\mathbb{R}^{(h,g)} \times \mathrm{Symm}\,(h,\mathbb{R})$ via

$$< a, \hat{a} > := e^{2\pi i \sigma(\hat{\mu}\,{}^t\mu + \hat{\kappa}\kappa)}, \quad a = [0,\mu,\kappa] \in K,\ \hat{a} = (\hat{\mu},\hat{\kappa}) \in \hat{K}. \tag{2.4}$$

We put

$$S := \left\{ [\lambda,0,0] \in H_{\mathbb{R}}^{(g,h)} \;\middle|\; \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}. \tag{2.5}$$

Then S acts on K as follows:

$$\alpha_\lambda([0,\mu,\kappa]) := [0,\mu,\kappa + \lambda\,{}^t\mu + \mu\,{}^t\lambda], \quad [\lambda,0,0] \in S. \tag{2.6}$$

It is easy to see that the Heisenberg group $\left(H_{\mathbb{R}}^{(g,h)}, \diamond\right)$ is isomorphic to the semidirect product $S \ltimes K$ of S and K whose multiplication is given by

$$(\lambda, a)\cdot(\lambda_0, a_0) := (\lambda + \lambda_0, a + \alpha_\lambda(a_0)), \quad \lambda, \lambda_0 \in S,\ a, a_0 \in K.$$

On the other hand, S acts on $\hat{K}$ by

$$\alpha_\lambda^*(\hat{a}) := (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}), \quad [\lambda,0,0] \in S, \quad a = (\hat{\mu},\hat{\kappa}) \in \hat{K}. \tag{2.7}$$

Then we have the relation $< \alpha_\lambda(a), \hat{a} > = < a, \alpha_\lambda^*(\hat{a}) >$ for all $a \in K$ and $\hat{a} \in \hat{K}$.

We have two types of S-orbits in $\hat{K}$.

TYPE I. Let $\hat{\kappa} \in \mathrm{Symm}\,(h,\mathbb{R})$ with $\hat{\kappa} \neq 0$. The S-orbit of $\hat{a}(\hat{\kappa}) := (0,\hat{\kappa}) \in \hat{K}$ is given by

$$\hat{\mathcal{O}}_{\hat{\kappa}} := \left\{ (2\hat{\kappa}\lambda, \hat{\kappa}) \in \hat{K} \;\middle|\; \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}. \tag{2.8}$$

TYPE II. Let $\hat{y} \in \mathbb{R}^{(h,g)}$. The S-orbit $\hat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}) := (\hat{y}, 0)$ is given by

$$\hat{\mathcal{O}}_{\hat{y}} := \{ (\hat{y}, 0) \} = \hat{a}(\hat{y}). \tag{2.9}$$

We have

$$\hat{K} = \left(\bigcup_{\hat{\kappa} \in \mathrm{Symm}(h,\mathbb{R})} \hat{\mathcal{O}}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y} \in \mathbb{R}^{(h,g)}} \hat{\mathcal{O}}_{\hat{y}} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa}) = (0,\hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}. \tag{2.10}$$

And the stabilizer $S_{\hat y}$ of S at $\hat a(\hat y)=(\hat y,0)$ is given by

$$S_{\hat y}=\left\{\,[\lambda,0,0]\,\Big|\;\lambda\in\mathbb{R}^{(h,g)}\,\right\}=S\;\cong\;\mathbb{R}^{(h,g)}. \tag{2.11}$$

From now on, we set $G:=H_{\mathbb{R}}^{(g,h)}$ for brevity. K is a closed, commutative normal subgroup of G. Since $(\lambda,\mu,\kappa)=(0,\mu,\kappa+\mu\,{}^t\lambda)\circ(\lambda,0,0)$ for $(\lambda,\mu,\kappa)\in G$, the homogeneous space $X:=K\backslash G$ is identified with $\mathbb{R}^{(h,g)}$ via

$$Kg=K\circ(\lambda,0,0)\longmapsto\lambda,\quad g=(\lambda,\mu,\kappa)\in G.$$

We observe that G acts on X by

$$(Kg)\cdot g_0:=K\,(\lambda+\lambda_0,0,0)=\lambda+\lambda_0, \tag{2.12}$$

where $g=(\lambda,\mu,\kappa)\in G$ and $g_0=(\lambda_0,\mu_0,\kappa_0)\in G$.

If $g=(\lambda,\mu,\kappa)\in G$, we have

$$k_g=(0,\mu,\kappa+\mu\,{}^t\lambda),\quad s_g=(\lambda,0,0) \tag{2.13}$$

in the Mackey decomposition of $g=k_g\circ s_g$ (cf. [M]). Thus if $g_0=(\lambda_0,\mu_0,\kappa_0)\in G$, then we have

$$s_g\circ g_0=(\lambda,0,0)\circ(\lambda_0,\mu_0,\kappa_0)=(\lambda+\lambda_0,\mu_0,\kappa_0+\lambda\,{}^t\mu_0) \tag{2.14}$$

and so

$$k_{s_g\circ g_0}=(0,\mu_0,\kappa_0+\mu_0\,{}^t\lambda_0+\lambda\,{}^t\mu_0+\mu_0\,{}^t\lambda). \tag{2.15}$$

For a real symmetric matrix $c={}^tc\in\mathbb{R}^{(h,h)}$ with $c\neq 0$, we consider the one-dimensional unitary representation σ_c of K defined by

$$\sigma_c\left((0,\mu,\kappa)\right):=e^{2\pi i\sigma(c\kappa)}\,I,\quad(0,\mu,\kappa)\in K, \tag{2.16}$$

where I denotes the identity mapping. Then the induced representation $U(\sigma_c):=\mathrm{Ind}_K^G\,\sigma_c$ of G induced from σ_c is realized in the Hilbert space $\mathcal{H}_{\sigma_c}=L^2(X,d\dot g,\mathbb{C})\cong L^2\left(\mathbb{R}^{(h,g)},d\xi\right)$ as follows. If $g_0=(\lambda_0,\mu_0,\kappa_0)\in G$ and $x=Kg\in X$ with $g=(\lambda,\mu,\kappa)\in G$, we have

$$\left(U_{g_0}(\sigma_c)f\right)(x)=\sigma_c\left(k_{s_g\circ g_0}\right)\left(f(xg_0)\right),\quad f\in\mathcal{H}_{\sigma_c}. \tag{2.17}$$

It follows from (2.15) that

$$\left(U_{g_0}(\sigma_c)f\right)(\lambda)=e^{2\pi i\sigma\{c(\kappa_0+\mu_0\,{}^t\lambda_0+2\lambda\,{}^t\mu_0)\}}\,f(\lambda+\lambda_0). \tag{2.18}$$

Here we identified $x=Kg$ (resp. $xg_0=Kgg_0$) with λ (resp. $\lambda+\lambda_0$). The induced representation $U(\sigma_c)$ is called the *Schrödinger representation* of G associated with σ_c. Thus $U(\sigma_c)$ is a monomial representation.

Now we denote by $\mathcal{H}^{\sigma_c}$ the Hilbert space consisting of all functions $\phi:G\longrightarrow\mathbb{C}$ which satisfy the following conditions:

(1) $\phi(g)$ is measurable with respect to dg.
(2) $\phi\left((0,\mu,\kappa)\circ g)\right) = e^{2\pi i\sigma(c\kappa)}\phi(g)$ for all $g\in G$.
(3) $\parallel \phi \parallel^2 := \int_X |\phi(g)|^2\, d\dot{g} < \infty,\quad \dot{g} = Kg,$

where dg (resp. $d\dot{g}$) is a G-invariant measure on G (resp. $X = K\backslash G$). The inner product $(\ ,\ )$ on $\mathcal{H}^{\sigma_c}$ is given by

$$(\phi_1,\phi_2) := \int_G \phi_1(g)\,\overline{\phi_2(g)}\,dg,\quad \phi_1,\ \phi_2\in\mathcal{H}^{\sigma_c}.$$

We observe that the mapping $\Phi_c : \mathcal{H}_{\sigma_c} \longrightarrow \mathcal{H}^{\sigma_c}$ defined by

$$(\Phi_c(f))\,(g) := e^{2\pi i\sigma\{c(\kappa+\mu\,{}^t\lambda)\}}\, f(\lambda),\quad f\in\mathcal{H}_{\sigma_c},\ g=(\lambda,\mu,\kappa)\in G \tag{2.19}$$

is an isomorphism of Hilbert spaces. The inverse $\Psi_c : \mathcal{H}^{\sigma_c} \longrightarrow \mathcal{H}_{\sigma_c}$ of Φ_c is given by

$$(\Psi_c(\phi))\,(\lambda) := \phi((\lambda,0,0)),\quad \phi\in\mathcal{H}^{\sigma_c},\ \lambda\in\mathbb{R}^{(h,g)}. \tag{2.20}$$

The Schrödinger representation $U(\sigma_c)$ of G on $\mathcal{H}^{\sigma_c}$ is given by

$$(U_{g_0}(\sigma_c)\phi)\,(g) = e^{2\pi i\sigma\{c(\kappa_0+\mu_0\,{}^t\lambda_0+\lambda\,{}^t\mu_0-\lambda_0\,{}^t\mu)\}}\,\phi\left((\lambda_0,0,0)\circ g\right), \tag{2.21}$$

where $g_0=(\lambda_0,\mu_0,\kappa_0),\ g=(\lambda,\mu,\kappa)\in G$ and $\phi\in\mathcal{H}^{\sigma_c}$. (2.21) can be expressed as follows.

$$(\,U_{g_0}(\sigma_c)\phi\,)\,(g) = e^{2\pi i\sigma\{c(\kappa_0+\kappa+\mu_0\,{}^t\lambda_0+\mu\,{}^t\lambda+2\lambda\,{}^t\mu_0)\}}\,\phi((\lambda_0+\lambda,0,0)). \tag{2.22}$$

Theorem 2.1. Let c be a positive symmetric half-integral matrix of degree h. Then the Schrödinger representation $U(\sigma_c)$ of G is irreducible.

Proof. The proof can be found in [Y1], Theorem 3. □

3 Proof of Main Theorem

Let $L := \mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)}$ be the lattice in the vector space $V\cong\mathbb{C}^{(h,g)}$. Let B be an alternating bilinear form on V such that $B(L,L)\subset\mathbb{Z}$, that is, $\mathbb{Z}$-valued on $L\times L$. The dual L_B^* of L with respect to B is defined by

$$L_B^* := \{\, v\in V \mid B(v,l)\in\mathbb{Z}\ for\ all\ l\in L\,\}.$$

Then $L\subset L_B^*$. If B is nondegenerate, L_B^* is also a lattice in V, called the *dual lattice* of L. In case B is nondegenerate, there exist a $\mathbb{Z}$-basis $\{\,\xi_{11},\xi_{12},\cdots,\xi_{hg},\eta_{11},\eta_{12},\cdots,\eta_{hg}\,\}$ of L and a set $\{\,e_{11},e_{12},\cdots,e_{hg}\,\}$ of positive integers with $e_{11}|e_{12},\ e_{12}|e_{13},\cdots,e_{h,g-1}|e_{hg}$ such that

$$\begin{pmatrix} B(\xi_{ka},\xi_{lb}) & B(\xi_{ka},\eta_{lb}) \\ B(\eta_{ka},\xi_{lb}) & B(\eta_{ka},\eta_{lb} \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix},$$

where $1\le k,l\le h,\ 1\le a,b\le g$ and $e:=\operatorname{diag}(e_{11},e_{12},\cdots,e_{hg})$ is the diagonal matrix of degree hg with entries $e_{11},e_{12},\cdots,e_{hg}$. It is well known that $[L_B^*:L] = (\det e)^2 = (e_{11}e_{12}\cdots e_{hg})^2$ (cf. [I] p. 72). The number $\det e$ is called the *Pfaffian* of B.

Now we consider the following subgroups of G:

$$\Gamma_L := \left\{ (\lambda,\mu,\kappa) \in G \,|\, (\lambda,\mu) \in L,\ \kappa \in \mathbb{R}^{(h,h)} \right\} \tag{3.1}$$

and

$$\Gamma_{L_B^*} := \left\{ (\lambda,\mu,\kappa) \in G \,|\, (\lambda,\mu) \in L_B^*,\ \kappa \in \mathbb{R}^{(h,h)} \right\}. \tag{3.2}$$

Then both Γ_L and $\Gamma_{L_B^*}$ are the normal subgroups of G. We put

$$\mathcal{Z}_0 := \left\{ (0,0,\kappa) \in G \,|\, \kappa = {}^t\kappa \in \mathbb{Z}^{(h,h)} \text{ integral} \right\}. \tag{3.3}$$

It is easy to show that

$$\Gamma_{L_B^*} = \left\{ g \in G \,|\, g\gamma g^{-1}\gamma^{-1} \in \mathcal{Z}_0 \text{ for all } \gamma \in \Gamma_L \right\}.$$

We define

$$Y_L := \left\{ \varphi \in \operatorname{Hom}(\Gamma_L, \mathbb{C}_1^\times) \,|\, \varphi \text{ is trivial on } \mathcal{Z}_0 \right\} \tag{3.4}$$

and

$$Y_{L,S} := \left\{ \varphi \in Y_L \,|\, \varphi(\kappa) = e^{2\pi i\sigma(S\kappa)} \text{ for all } \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)} \right\} \tag{3.5}$$

for each symmetric real matrix S of degree h. We observe that if S is not half-integral, then $Y_L = \emptyset$ and so $Y_{L,S} = \emptyset$. It is clear that if S is symmetric half-integral, then $Y_{L,S}$ is not empty.
Thus we have

$$Y_L = \cup_{\mathcal{M}}\, Y_{L,\mathcal{M}}, \tag{3.6}$$

where $\mathcal{M}$ runs through the set of all symmetric half-integral matrices of degree h.

Lemma 3.1. Let $\mathcal{M}$ be a symmetric half-integral matrix of degree h with $\mathcal{M} \neq 0$. Then any element φ of $Y_{L,\mathcal{M}}$ is of the form $\varphi_{\mathcal{M},q}$. Here $\varphi_{\mathcal{M},q}$ is the character of Γ_L defined by

$$\varphi_{\mathcal{M},q}((l,\kappa)) := e^{2\pi i\sigma(\mathcal{M}\kappa)} \cdot e^{\pi i q(l)}, \quad (l,\kappa) \in \Gamma_L, \tag{3.7}$$

where $q : L \longrightarrow \mathbb{R}/2\mathbb{Z} \cong [0,2)$ is a function on L satisfying the following condition:

$$q(l_0 + l_1) \equiv q(l_0) + q(l_1) - 2\sigma\{\mathcal{M}(\lambda_0\, {}^t\mu_1 - \mu_0\, {}^t\lambda_1)\} \pmod 2 \tag{3.8}$$

for all $l_0 = (\lambda_0,\mu_0) \in L$ and $l_1 = (\lambda_1,\mu_1) \in L$.

Proof. (3.8) follows immediately from the fact that $\varphi_{\mathcal{M},q}$ is a character of Γ_L. It is obvious that any element of $Y_{L,\mathcal{M}}$ is of the form $\varphi_{\mathcal{M},q}$. □

Lemma 3.2. An element of $Y_{L,0}$ is of the form $\varphi_{k,l}\,(k,l\in\mathbb{R}^{(h,g)})$. Here $\varphi_{k,l}$ is the character of Γ_L defined by

$$\varphi_{k,l}(\gamma) := e^{2\pi i\sigma(k\,{}^t\lambda + l\,{}^t\mu)},\quad \gamma=(\lambda,\mu,\kappa)\in\Gamma_L. \tag{3.9}$$

Proof. It is easy to prove it and so we omit the proof. □

Lemma 3.3. Let $\mathcal{M}$ be a nonsingular symmetric half-integral matrix of degree h. Let $\varphi_{\mathcal{M},q_1}$ and $\varphi_{\mathcal{M},q_2}$ be the characters of Γ_L defined by (3.7). The character φ of Γ_L defined by $\varphi := \varphi_{\mathcal{M},q_1}\cdot\varphi_{\mathcal{M},q_2}^{-1}$ is an element of $Y_{L,0}$.

Proof. It follows from the fact that there exists an element $g=(\mathcal{M}^{-1}\lambda,\mathcal{M}^{-1}\mu,0)\in G$ with $(\lambda,\mu)\in V$ such that

$$\varphi_{\mathcal{M},q_1}(\gamma) = \varphi_{\mathcal{M},q_2}(g\gamma g^{-1})\quad\text{for all }\gamma\in\Gamma_L.$$

□

For a unitary character $\varphi_{\mathcal{M},q}$ of Γ_L defined by (3.7), we let

$$\pi_{\mathcal{M},q} := \ \mathrm{Ind}_{\Gamma_L}^G\,\varphi_{\mathcal{M},q} \tag{3.10}$$

be the representation of G induced from $\varphi_{\mathcal{M},q}$. Let $\mathcal{H}_{\mathcal{M},q}$ be the Hilbert space consisting of all measurable functions $\phi : G\longrightarrow\mathbb{C}$ satisfying

(L1) $\phi(\gamma g)=\varphi_{\mathcal{M},q}(\gamma)\,\phi(g)$ for all $\gamma\in\Gamma_L$ and $g\in G$.
(L2) $\|\phi\|_{\mathcal{M},q}^2 := \int_{\Gamma_L\backslash G}|\phi(\bar g)|^2\,d\bar g<\infty,\quad \bar g=\Gamma_L g.$

The induced representation $\pi_{\mathcal{M},q}$ is realized in $\mathcal{H}_{\mathcal{M},q}$ as follows:

$$\Big(\pi_{\mathcal{M},q}(g_0)\phi\Big)(g) := \phi(gg_0),\quad g_0,g\in G,\ \phi\in\mathcal{H}_{\mathcal{M},q}. \tag{3.11}$$

$\pi_{\mathcal{M},q}$ is called the *lattice representation* of G associated with the lattice L.

Main Theorem. Let $\mathcal{M}$ be a positive definite, symmetric half integral matrix of degree h. Let $\varphi_{\mathcal{M}}$ be the character of Γ_L defined by $\varphi_{\mathcal{M}}((\lambda,\mu,\kappa)) := e^{2\pi i\sigma(\mathcal{M}\kappa)}$ for all $(\lambda,\mu,\kappa)\in\Gamma_L$. Then the lattice representation

$$\pi_{\mathcal{M}} := \ \mathrm{Ind}_{\Gamma_L}^G\,\varphi_{\mathcal{M}}$$

induced from the character $\varphi_{\mathcal{M}}$ is unitarily equivalent to the direct sum

$$\bigoplus U(\sigma_{\mathcal{M}}) := \bigoplus \mathrm{Ind}_K^G\,\sigma_{\mathcal{M}}\quad (\,(\det 2\mathcal{M}\,)^g\text{-copies}\,)$$

of the Schrödinger representation $\mathrm{Ind}_K^G\,\sigma_{\mathcal{M}}$.

Proof. We first recall that the induced representation $\pi_{\mathcal{M}}$ is realized in the Hilbert space $\mathcal{H}_{\mathcal{M}}$ consisting of all measurable functions $\phi : G \longrightarrow \mathbb{C}$ satisfying the conditions

$$\phi((\lambda_0,\mu_0,\kappa_0)\circ g)=e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,\phi(g),\quad (\lambda_0,\mu_0,\kappa_0)\in\Gamma_L,\ g\in G \tag{3.13}$$

and

$$\|\phi\|^2_{\pi,\mathcal{M}}:=\int_{\Gamma_L\backslash G}|\phi(\bar g)|^2\,d\bar g<\infty,\quad \bar g=\Gamma_L\circ g. \tag{3.14}$$

Now we write

$$g_0=[\lambda_0,\mu_0,\kappa_0]\in\Gamma_L\quad and\quad g=[\lambda,\mu,\kappa]\in G.$$

For $\phi\in\mathcal{H}_{\mathcal{M}}$, we have

$$\phi(g_0\diamond g)=\phi([\lambda_0+\lambda,\mu_0+\mu,\kappa_0+\kappa+\lambda_0\,{}^t\mu+\mu\,{}^t\lambda_0]). \tag{3.15}$$

On the other hand, we get

$$\begin{aligned}\phi(g_0\diamond g)&=\phi((\lambda_0,\mu_0,\kappa_0-\mu_0\,{}^t\lambda_0)\circ g)\\&=e^{2\pi i\sigma\{\mathcal{M}(\kappa_0-\mu_0\,{}^t\lambda_0)\}}\,\phi(g)\\&=e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,\phi(g)\quad(\text{ because }\sigma(\mathcal{M}\mu_0\,{}^t\lambda_0)\in\mathbb{Z}\,)\end{aligned}$$

Thus putting $\kappa':=\kappa_0+\lambda_0\,{}^t\mu+\mu\,{}^t\lambda_0$, we get

$$\phi([\lambda_0+\lambda,\mu_0+\mu,\kappa+\kappa'])=e^{2\pi i\sigma(\mathcal{M}\kappa')}\cdot e^{-4\pi i\sigma(\mathcal{M}\lambda_0\,{}^t\mu)}\,\phi([\lambda,\mu,\kappa]). \tag{3.16}$$

Putting $\lambda_0=\kappa_0=0$ in (3.16), we have

$$\phi([\lambda,\mu+\mu_0,\kappa])=\phi([\lambda,\mu,\kappa])\ \text{ for all }\mu_0\in\mathbb{Z}^{(h,g)}\text{ and }[\lambda,\mu,\kappa]\in G. \tag{3.17}$$

Therefore if we fix λ and κ, ϕ is periodic in μ with respect to the lattice $\mathbb{Z}^{(h,g)}$ in $\mathbb{R}^{(h,g)}$. We note that

$$\phi([\lambda,\mu,\kappa])=\phi([0,0,\kappa]\diamond[\lambda,\mu,0])=e^{2\pi i\sigma(\mathcal{M}\kappa)}\,\phi([\lambda,\mu,0])$$

for $[\lambda,\mu,\kappa]\in G$. Hence ϕ admits a Fourier expansion in μ :

$$\phi([\lambda,\mu,\kappa])=e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}}c_N(\lambda)\,e^{2\pi i\sigma(N\,{}^t\mu)}. \tag{3.18}$$

If $\lambda_0\in\mathbb{Z}^{(h,g)}$, then we have

$$\begin{aligned}\phi([\lambda+\lambda_0,\mu,\kappa])&=e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}}c_N(\lambda+\lambda_0)\,e^{2\pi i\sigma(N\,{}^t\mu)}\\&=e^{-4\pi i\sigma(\mathcal{M}\lambda_0\,{}^t\mu)}\,\phi([\lambda,\mu,\kappa])\qquad(\text{ by }(3.16)\,)\\&=e^{-4\pi i\sigma(\mathcal{M}\lambda_0\,{}^t\mu)}\,e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}}c_N(\lambda)\,e^{2\pi i\sigma(N\,{}^t\mu)},\\&=e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}}c_N(\lambda)\,e^{2\pi i\sigma\{(N-2\mathcal{M}\lambda_0)\,{}^t\mu\}}.\quad(\text{ by }(3.18)\,)\end{aligned}$$

So we get

$$\begin{aligned}&\sum_{N\in\mathbb{Z}^{(h,g)}} c_N(\lambda+\lambda_0)\, e^{2\pi i\sigma(N\,{}^t\mu)}\\ =&\sum_{N\in\mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i\sigma\{(N-2\mathcal{M}\lambda_0)\,{}^t\mu\}}\\ =&\sum_{N\in\mathbb{Z}^{(h,g)}} c_{N+2\mathcal{M}\lambda_0}(\lambda)\, e^{2\pi i\sigma(N\,{}^t\mu)}.\end{aligned}$$

Hence we get

$$c_N(\lambda+\lambda_0)=c_{N+2\mathcal{M}\lambda_0}(\lambda)\quad\text{for all } \lambda_0\in\mathbb{Z}^{(h,g)} \text{ and } \lambda\in\mathbb{R}^{(h,g)}. \tag{3.19}$$

Consequently, it is enough to know only the coefficients $c_\alpha(\lambda)$ for the representatives α in $\mathbb{Z}^{(h,g)}$ modulo $2\mathcal{M}$. It is obvious that the number of all such representatives α's is $(\det 2\mathcal{M})^g$. We denote by $\mathcal{J}$ a complete system of such representatives α's in $\mathbb{Z}^{(h,g)}$ modulo $2\mathcal{M}$.

Then we have

$$\begin{aligned}&\phi([\lambda,\mu,\kappa])\\ =\;& e^{2\pi i\sigma(\mathcal{M}\kappa)}\Big\{\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}}\\ &\quad+\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\beta+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\beta+2\mathcal{M}N)\,{}^t\mu\}}\\ &\quad\cdot\\ &\quad\cdot\\ &\quad+\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\gamma+2\mathcal{M}N}(\lambda)\, e^{2\pi i\{(\gamma+2\mathcal{M}N)\,{}^t\mu\}}\Big\},\end{aligned}$$

where $\{\,\alpha,\beta,\cdots,\gamma\,\}$ denotes the complete system $\mathcal{J}$.

For each $\alpha\in\mathcal{J}$, we denote by $\mathcal{H}_{\mathcal{M},\alpha}$ the Hilbert space consisting of Fourier expansions

$$e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{\,(\alpha+2\mathcal{M}N)\,{}^t\mu\}},\quad (\lambda,\mu,\kappa)\in G,$$

where $c_N(\lambda)$ denotes the coefficients of the Fourier expansion (3.18) of $\phi\in\mathcal{H}_\mathcal{M}$ and ϕ runs over the set $\{\,\phi\in\pi_\mathcal{M}\,\}$. It is easy to see that $\mathcal{H}_{\mathcal{M},\alpha}$ is invariant under $\pi_\mathcal{M}$. We denote the restriction of $\pi_\mathcal{M}$ to $\mathcal{H}_{\mathcal{M},\alpha}$ by $\pi_{\mathcal{M},\alpha}$. Then we have

$$\pi_\mathcal{M}=\bigoplus_{\alpha\in\mathcal{J}}\pi_{\mathcal{M},\alpha}. \tag{3.20}$$

Let $\phi_\alpha\in\pi_{\mathcal{M},\alpha}$. Then for $[\lambda,\mu,\kappa]\in G$, we get

$$\phi_\alpha([\lambda,\mu,\kappa])=e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}}. \tag{3.21}$$

We put

$$I_\lambda := \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{(h\times g)\text{-times}} \subset \left\{ [\lambda,0,0] \mid \lambda\in\mathbb{R}^{(h,g)} \right\}$$

and

$$I_\mu := \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{(h\times g)\text{-times}} \subset \left\{ [0,\mu,0] \mid \mu\in\mathbb{R}^{(h,g)} \right\}.$$

Then we obtain

$$\int_{I_\mu} \phi_\alpha([\lambda,\mu,\kappa])\, e^{-2\pi i\sigma(\alpha\,{}^t\mu)}\, d\mu = e^{2\pi i\sigma(\mathcal{M}\kappa)}\, c_\alpha(\lambda), \quad \alpha\in\mathcal{J}. \tag{3.22}$$

Since $\Gamma_L\backslash G \cong I_\lambda\times I_\mu$, we get

$$\begin{aligned}
\|\phi_\alpha\|^2_{\pi,\mathcal{M},\alpha} : &= \|\phi_\alpha\|^2_{\pi,\mathcal{M}} = \int_{\Gamma_L\backslash G} |\phi_\alpha(\bar g)|^2\, d\bar g \\
&= \int_{I_\lambda}\int_{I_\mu} |\phi_\alpha(\bar g)|^2\, d\lambda d\mu \\
&= \int_{I_\lambda\times I_\mu} \left| \sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}} \right|^2 d\lambda d\mu \\
&= \int_{I_\lambda} \sum_{N\in\mathbb{Z}^{(h,g)}} |c_{\alpha+2\mathcal{M}N}(\lambda)|^2\, d\lambda \\
&= \int_{I_\lambda} \sum_{N\in\mathbb{Z}^{(h,g)}} |c_\alpha(\lambda+N)|^2\, d\lambda \qquad (\text{by } (3.19)) \\
&= \int_{\mathbb{R}^{(h,g)}} |c_\alpha(\lambda)|^2\, d\lambda.
\end{aligned}$$

Since $\phi_\alpha\in\pi_{\mathcal{M},\alpha},\ \|\phi_\alpha\|_{\pi,\mathcal{M},\alpha}<\infty$ and so $c_\alpha(\lambda)\in L^2\left(\mathbb{R}^{(h,g)},d\xi\right)$ for all $\alpha\in\mathcal{J}$.

For each $\alpha\in\mathcal{J}$, we define the mapping $\vartheta_{\mathcal{M},\alpha}$ on $L^2\left(\mathbb{R}^{(h,g)},d\xi\right)$ by

$$(\vartheta_{\mathcal{M},\alpha}f)([\lambda,\mu,\kappa]) := e^{2\pi i\sigma(\mathcal{M}\kappa)} \sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+N)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}}, \tag{3.23}$$

where $f\in L^2\left(\mathbb{R}^{(h,g)},d\xi\right)$ and $[\lambda,\mu,\kappa]\in G$.

Lemma 3.4. For each $\alpha\in\mathcal{J}$, the image of $L^2\left(\mathbb{R}^{(h,g)},d\xi\right)$ under $\vartheta_{\mathcal{M},\alpha}$ is contained in $\mathcal{H}_{\mathcal{M},\alpha}$. Moreover, the mapping $\vartheta_{\mathcal{M},\alpha}$ is a one-to-one unitary operator of $L^2\left(\mathbb{R}^{(h,g)},d\xi\right)$ onto $\mathcal{H}_{\mathcal{M},\alpha}$ preserving the norms. In other words, the mapping

$$\vartheta_{\mathcal{M},\alpha} : L^2\left(\mathbb{R}^{(h,g)},d\xi\right) \longrightarrow \mathcal{H}_{\mathcal{M},\alpha}$$

is an isometry.

Proof. We already showed that $\vartheta_{\mathcal{M},\alpha}$ preserves the norms. First we observe that if $(\lambda_0,\mu_0,\kappa_0)\in\Gamma_L$ and $g=[\lambda,\mu,\kappa]\in G$,

$$\begin{aligned}(\lambda_0,\mu_0,\kappa_0)\circ g &= [\lambda_0,\mu_0,\kappa_0+\mu_0\,{}^t\lambda_0]\diamond[\lambda,\mu,\kappa]\\ &= [\lambda_0+\lambda,\mu_0+\mu,\kappa+\kappa_0+\mu_0\,{}^t\lambda_0+\lambda_0\,{}^t\mu+\mu\,{}^t\lambda_0].\end{aligned}$$

Thus we get

$$\begin{aligned}&(\vartheta_{\mathcal{M},\alpha}f)((\lambda_0,\mu_0,\kappa_0)\circ g)\\ &= e^{2\pi i\sigma\{\mathcal{M}(\kappa+\kappa_0+\mu_0\,{}^t\lambda_0+\lambda_0\,{}^t\mu+\mu\,{}^t\lambda_0)\}}\sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+\lambda_0+N)\,e^{2\pi i\{(\alpha+2\mathcal{M}N)\,{}^t(\mu_0+\mu)\}}\\ &= e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\cdot e^{2\pi i\sigma(\mathcal{M}\kappa)}\cdot e^{2\pi i\sigma(\alpha\,{}^t\mu_0)}\sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+N)\,e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}}\\ &= e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,(\vartheta_{\mathcal{M},\alpha}f)(g).\end{aligned}$$

Here in the above equalities we used the facts that $2\sigma(\mathcal{M}N\,{}^t\mu_0)\in\mathbb{Z}$ and $\alpha\,{}^t\mu_0\in\mathbb{Z}$. It is easy to show that

$$\int_{\Gamma_L\backslash G}|\vartheta_{\mathcal{M},\alpha}f(\bar{g})|^2\,d\bar{g}=\int_{\mathbb{R}^{(h,g)}}|f(\lambda)|^2\,d\lambda=\|f\|_2^2<\infty.$$

This completes the proof of lemma 3.4. //

Finally it is easy to show that for each $\alpha\in\mathcal{J}$, the mapping $\vartheta_{\mathcal{M},\alpha}$ intertwines the Schrödinger representation $\big(U(\sigma_{\mathcal{M}}),L^2(\mathbb{R}^{(h,g)},d\xi)\big)$ and the representation $(\pi_{\mathcal{M},\alpha},\mathcal{H}_{\mathcal{M},\alpha})$. Therefore, by lemma 3.4, for each $\alpha\in\mathcal{J}$, $\pi_{\mathcal{M},\alpha}$ is unitarily equivalent to $U(\sigma_{\mathcal{M}})$ and so $\pi_{\mathcal{M},\alpha}$ is an irreducible unitary representation of G. According to (3.20), the induced representation $\pi_{\mathcal{M}}$ is unitarily equivalent to

$$\bigoplus U(\sigma_{\mathcal{M}})\quad((\det 2\mathcal{M})^g\text{-copies}).$$

This completes the proof of Main Theorem. □

4 Relation of Lattice Representations to Theta Functions

In this section, we state the connection between the lattice representations and theta functions. As before, we write $V=\mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)}\cong\mathbb{C}^{(h,g)},\ L=\mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)}$ and $\mathcal{M}$ is a positive symmetric half-integral matrix of degree h. The function $q_{\mathcal{M}}:L\longrightarrow\mathbb{R}/2\mathbb{Z}=[0,2)$ defined by

$$q_{\mathcal{M}}((\xi,\eta)):=2\,\sigma(\mathcal{M}\xi\,{}^t\eta),\quad(\xi,\eta)\in L \tag{4.1}$$

satisfies the condition (3.8). We let $\varphi_{\mathcal{M},q_{\mathcal{M}}}:\Gamma_L\longrightarrow\mathbb{C}_1^\times$ be the character of Γ_L defined by

$$\varphi_{\mathcal{M},q_{\mathcal{M}}}((l,\kappa))=e^{2\pi i\,\sigma(\mathcal{M}\kappa)}\,e^{\pi i\,q_{\mathcal{M}}(l)},\quad(l,\kappa)\in\Gamma_L.$$

We denote by $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ the Hilbert space consisting of measurable functions $\phi:G\longrightarrow\mathbb{C}$ which satisfy the conditions (4.2) and (4.3):

$$\phi((l,\kappa)\circ g) = \varphi_{\mathcal{M},q_{\mathcal{M}}}((l,\kappa))\,\phi(g) \quad \text{for all } (l,\kappa)\in\Gamma_L \text{ and } g\in G. \tag{4.2}$$

$$\int_{\Gamma_L\backslash G} \|\phi(\dot g)\|^2\, d\dot g < \infty, \quad \dot g = \Gamma_L\circ g. \tag{4.3}$$

Then the lattice representation

$$\pi_{\mathcal{M},q_{\mathcal{M}}} := \mathrm{Ind}_{\Gamma_L}^{G}\,\varphi_{\mathcal{M},q_{\mathcal{M}}}$$

of G induced from the character $\varphi_{\mathcal{M},q_{\mathcal{M}}}$ is realized in $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ as

$$(\,\pi_{\mathcal{M},q_{\mathcal{M}}}(g_0)\,\phi\,)\,(g) = \phi(gg_0), \quad g_0, g\in G,\ \phi\in\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}.$$

Let $\mathbf{H}_{\mathcal{M},q_{\mathcal{M}}}$ be the vector space consisting of measurable functions $F: V\longrightarrow\mathbb{C}$ satisfying the conditions (4.4) and (4.5).

$$F(\lambda+\xi,\mu+\eta) = e^{2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,F(\lambda,\mu) \tag{4.4}$$

for all $(\lambda,\mu)\in V$ and $(\xi,\eta)\in L$.

$$\int_{L\backslash V}\|F(\dot v)\|^2\,d\dot v = \int_{I_\lambda\times I_\mu}\|F(\lambda,\mu)\|^2\,d\lambda d\mu < \infty. \tag{4.5}$$

Given $\phi\in\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ and a fixed element $\Omega\in H_g$, we put

$$E_\phi(\lambda,\mu) := \phi((\lambda,\mu,0)), \quad \lambda,\mu\in\mathbb{R}^{(h,g)}, \tag{4.6}$$

$$F_\phi(\lambda,\mu) := \phi([\lambda,\mu,0]), \quad \lambda,\mu\in\mathbb{R}^{(h,g)}, \tag{4.7}$$

$$F_{\Omega,\phi}(\lambda,\mu) := e^{-2\pi i\,\sigma(\mathcal{M}\lambda\Omega\,{}^t\lambda)}\,F_\phi(\lambda,\mu), \quad \lambda,\mu\in\mathbb{R}^{(h,g)}. \tag{4.8}$$

In addition, we put for $W=\lambda\Omega+\mu\in\mathbb{C}^{(h,g)}$,

$$\vartheta_{\Omega,\phi}(W) := \vartheta_{\Omega,\phi}(\lambda\Omega+\mu) := F_{\Omega,\phi}(\lambda,\mu). \tag{4.9}$$

We observe that E_ϕ, F_ϕ, $F_{\Omega,\phi}$ are functions defined on V and $\vartheta_{\Omega,\phi}$ is a function defined on $\mathbb{C}^{(h,g)}$.

Proposition 4.1. If $\phi\in\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$, $(\xi,\eta)\in L$ and $(\lambda,\mu)\in V$, then we have the formulas

$$E_\phi(\lambda+\xi,\mu+\eta) = e^{2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,E_\phi(\lambda,\mu). \tag{4.10}$$

$$F_\phi(\lambda+\xi,\mu+\eta) = e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu). \tag{4.11}$$

$$F_{\Omega,\phi}(\lambda+\xi,\mu+\eta) = e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu). \tag{4.12}$$

If $W=\lambda\Omega+\eta\in\mathbb{C}^{(h,g)}$, then we have

$$\vartheta_{\Omega,\phi}(W+\xi\Omega+\eta) = e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi)\}}\,\vartheta_{\Omega,\phi}(W). \tag{4.13}$$

Moreover, F_ϕ is an element of $\mathbf{H}_{\mathcal{M},q_{\mathcal{M}}}$.

Proof. We note that

$$(\lambda+\xi,\mu+\eta,0)=(\xi,\eta,-\xi\,{}^t\mu+\eta\,{}^t\lambda)\circ(\lambda,\mu,0).$$

Thus we have

$$\begin{aligned}
E_\phi(\lambda+\xi,\mu+\eta)&=\phi((\lambda+\xi,\mu+\eta,0))\\
&=\phi((\xi,\eta,-\xi\,{}^t\mu+\eta\,{}^t\lambda)\circ(\lambda,\mu,0))\\
&=e^{2\pi i\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,\phi((\lambda,\mu,0))\\
&=e^{2\pi i\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,E_\phi(\lambda,\mu).
\end{aligned}$$

This proves the formula (4.10). We observe that

$$[\lambda+\xi,\mu+\eta,0]=(\xi,\eta,-\xi\,{}^t\mu-\mu\,{}^t\xi-\eta\,{}^t\xi)\circ[\lambda,\mu,0].$$

Thus we have

$$\begin{aligned}
F_\phi(\lambda+\xi,\mu+\eta)&=\phi([\lambda+\xi,\mu+\eta,0])\\
&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\mu+\mu\,{}^t\xi+\eta\,{}^t\xi)\}}\\
&\quad\times e^{2\pi i\,\sigma(\mathcal{M}\xi\,{}^t\eta)}\,\phi([\lambda,\mu,0])\\
&=e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,\phi([\lambda,\mu,0])\\
&=e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu).
\end{aligned}$$

This proves the formula (4.11). According to (4.11), we have

$$\begin{aligned}
F_{\Omega,\phi}(\lambda+\xi,\mu+\eta)&=e^{-2\pi i\,\sigma\{\mathcal{M}(\lambda+\xi)\Omega\,{}^t(\lambda+\xi)\}}\,F_\phi(\lambda+\xi,\mu+\eta)\\
&=e^{-2\pi i\,\sigma\{\mathcal{M}(\lambda+\xi)\Omega\,{}^t(\lambda+\xi)\}}\\
&\quad\times e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu)\\
&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\\
&\quad\times e^{-2\pi i\,\sigma(\mathcal{M}\lambda\Omega\,{}^t\lambda)}\,F_\phi(\lambda,\mu)\\
&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu).
\end{aligned}$$

This proves the formula (4.12). The formula (4.13) follows immediately from the formula (4.12). Indeed, if $W=\lambda\Omega+\mu$ with $\lambda,\mu\in\mathbb{R}^{(h,g)}$, we have

$$\begin{aligned}
\vartheta_{\Omega,\phi}(W+\xi\Omega+\eta)&=F_{\Omega,\phi}(\lambda+\xi,\mu+\eta)\\
&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2(\lambda\Omega+\mu)\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu)\\
&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi)\}}\,\vartheta_{\Omega,\phi}(W).
\end{aligned}$$

□

REMARK 4.2. The function $\vartheta_{\Omega,\phi}(W)$ is a theta function of level $2\mathcal{M}$ with respect to Ω if $\vartheta_{\Omega,\phi}$ is holomorphic. For any $\phi\in\mathcal{H}_{\mathcal{M},q_\mathcal{M}}$, the function $\vartheta_{\Omega,\phi}$ satisfies the well known transformation law of a theta function. In this sense, the lattice representation $(\pi_{\mathcal{M},q_\mathcal{M}},\mathcal{H}_{\mathcal{M},q_\mathcal{M}})$ is closely related to theta functions.

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