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HIGHER TOPOLOGICAL COMPLEXITY AND HOMOTOPY DIMENSION OF CONFIGURATION SPACES ON SPHERES

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Abstract. In the paper "On Higher Analogs of Topological Complexity" Yu. Rudyak introduced the concept of $\mathrm{TC}_n(X)$, the n^{th} topological complexity of a path-connected space X. This concept was developed as a generalization of Farber's notion of Topological Complexity.

In this paper we develop further the properties of the newly discovered concept, relating it to the Lusternik-Schnirelmann category of cartesian powers of X, and to the d_n^X-cup-length. We compute the numerical value of TC_n for products of spheres, closed 1-connected symplectic manifolds (e.g. complex projective spaces), and quaternionic projective spaces. We explore the symmetrized version of the concept ($\mathrm{TC}_n^S(X)$) and introduce a new symmetrization ($\mathrm{TC}_n^\Sigma(X)$) which is a homotopy invariant of X.

We obtain a strong upper bound for $\mathrm{TC}_n^S(X)$ when X is a sphere. This is attained by introducing and studying a new concept: *cellular stratified spaces*. This concept allows us to import techniques from the theory of hyperplane arrangements in order to construct efficient simplicial complexes modelling (up to equivariant homotopy) configuration spaces of distinct points on spheres.

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Contents

1. Introduction 2
2. Preliminaries 7
3. Properties of Higher Topological Complexity 9
4. Symmetric Topological Complexity 14
5. Bounding $\mathfrak{genus}\,(\varepsilon_n)$ for Spheres 21
6. Cellular Stratifications 23
7. The Σ_k-Equivariant Homotopy Model of $C_k(S^n)$ 32
References 41

1. Introduction

The concept of topological complexity "TC" was introduced by Michael Farber, motivated by the most basic problem of robot motion planning; that is, finding the smallest number of continuous instructions for a robot to move from one point to another in a path-connected space.

In greater detail, given a mechanical system S, a *motion planning algorithm* for S is a rule that assigns a (continuous) motion from A to B to each pair (A, B) of positions of S, [La90, LV06]. Let X denote the configuration space of S; thus the positions of S are the points of X, and a motion of S turns out to be a path in X.

Let PX denote the space of all paths in X, that is, the space of all continuous functions $\gamma : [0, 1] \to X$. We will denote by $\pi : PX \to X \times X$ the map associating to any path $\gamma \in PX$ the pair of its initial and final points, $\pi(\gamma) = (\gamma(0), \gamma(1))$. Now, a motion planning algorithm is a map $s : X \times X \to PX$ such that $\pi \circ s = \mathrm{id}_{X\times X}$, i.e. a section of π.

It is easy to see that a *continuous* motion planning algorithm (i.e., a continuous section s) exists only for X contractible. So, it makes sense to express $X \times X$ as a union of subsets, each of which admits a continuous motion planning algorithm (we call such data a *motion planner* on X). To formalize this idea, Michael Farber [Fa03] defined the topological complexity as follows:

Definition. Given a path-connected topological space X, the *topological complexity* of motion planning in X, $\mathrm{TC}(X)$, is the least number k such that the Cartesian product $X \times X$ can be covered by k open subsets

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$$

such that for any $i = 1, 2, \ldots, k$ there exists a continuous motion planning algorithm $s_i : U_i \to PX$, $\pi \circ s_i = \mathrm{id}$ over U_i. If no such k exists, then set $\mathrm{TC}(X) = \infty$.

It is possible and useful to introduce a symmetrized version of topological complexity, the *symmetric topological complexity* $\mathrm{TC}^S(X)$. This invariant appears when we restrict motion planning algorithms to be such that a motion from A to B is the reverse of a motion from B to A, [FG07].

A number of properties of (symmetric) topological complexity can be found in [Fa03, Fa06, Fa08, FG07, FG08, FY04]. The articles [FTY03, GL09] identify the (symmetric) topological complexity of real projective spaces as their immersion (embedding) dimension.

The first of the two main goals of this paper is to make a thorough study of the following natural generalization of Farber's topological complexity. As we have explained, $\mathrm{TC}(X)$ is related to the motion planning problem in which a robot with configuration space X moves from a given state to another state. More generally, we can consider a motion planning problem whose input is not only a pair of initial and final states, but also an additional set of $n-2$ ordered intermediate states. Such a setting arises, for instance, in industrial production processes, where the manufacturing of a given good goes through a series of production steps. The corresponding planning problem leads us to the homotopy invariant $\mathrm{TC}_n(X)$, the n^{th} *topological complexity* of X introduced in [Ru10], and reviewed in Section 2. Of course, the case $n = 2$ recovers Farber's notion, except that our chosen normalization—a trivial fibration has zero Schwarz genus—gives $\mathrm{TC}(X) = \mathrm{TC}_2(X)+1$.

In Section 3, we discuss some elementary properties of TC_n, including methods of calculation of this invariant, e.g. a relation to cup length (Theorem 3.9) and the product formula (Proposition 3.11). As an immediate application, the full determination of the numerical value of $\mathrm{TC}_n(X)$ is given when X is a product of spheres (Corollary 3.12), a closed simply connected symplectic manifold (Corollary 3.15), or a quaternionic projective space (Corollary 3.16).

Although many of our results generalize corresponding results for regular TC, some others have their own flavor. For instance, we show a close connection between higher topological complexity and the Lusternik-Schnirelmann category of cartesian powers of spaces.

Theorem. (Corollary 3.3) *For any path-connected space X,*

$$\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X) \leq \mathrm{cat}(X^n).$$

In fact, for a path-connected topological group G, we prove (Theorem 3.5) that $\mathrm{TC}_n(G) = \mathrm{cat}(G^{n-1})$—this fact can also be thought of as a generalization of a known TC property.

In Section 4 we consider symmetric versions of higher topological complexity. We introduce $\mathrm{TC}^{\Sigma}(X)$, a minor variation of Farber-Grant's symmetric topological complexity $\mathrm{TC}^{S}(X)$—the latter introduced in [FG07]. Although the numerical values of the two invariants differ by at most 1 (Proposition 4.4), unlike the latter, the former concept is a homotopy invariant (it should be noted that this property does not hold for the *monoidal topological complexity* introduced in [IS10, Definition 1.3], where the stasis property is imposed on the motion planning problem—instead of the symmetry condition we have imposed on

TC^Σ). Indeed, we construct the corresponding higher analogues TC^S_n and TC^Σ_n, and prove the homotopy invariance of the latter (Proposition 4.11).

The calculation of the higher *symmetric* topological complexity of a space turns out to be a difficult task, mainly due to what seems to be poor current knowledge of precise homotopy information about braid manifolds. This brings us to the second major goal and most important theoretical achievement of this paper: to set manageable combinatorial grounds for a systematic study of the homotopy properties of configuration spaces on cell complexes. For this purpose, in Section 6 we introduce a new concept: that of a *cellular stratified space*, a generalization of a cell complex where *non-closed cells* can appear. Namely, we allow cells modelled not by a unit disk D^n, but by any subspace of D^n containing $\mathrm{Int}(D^n)$. Such a simple generalization of J. H. C. Whitehead's concept of a CW complex best meets the needs of homotopy theory, and seems to have far-reaching applications.

Our proposal fits well with an active ongoing mutual feedback between topology and combinatorics—of which Kozlov's book [Ko08] is a good example. One of the most fundamental properties on which this relationship rests is the fact that the order complex of the face poset of a regular cell complex X is homeomorphic to X. This allows us to go back and forth between the combinatorial and topological worlds. Our Theorem 6.9 extends this correspondence.

Theorem. (Theorem 6.9) *For a regular totally normal cellular stratification $\mathcal{C}$ on a topological space X, the classifying space* (*order complex*) *of the face poset of* $(X, \mathcal{C})$ *can be embedded in X as a strong deformation retract. Furthermore, when $\mathcal{C}$ is a regular cell complex structure on X, the embedding coincides with the standard homeomorphism.*

Such a result provides us with a systematic method of constructing simplicial complexes modelling, up to homotopy, a broad class of (usually non-compact) spaces. For example, our technique can be used to obtain, in a direct way, the Salvetti complex [Sa87] and its higher versions [BZ92, DS00] for hyperplane arrangements (see Remark 7.7 and Corollary 7.8). More importantly for our TC^S_n-purposes, the fine combinatorial control of the method allows us to deduce an equivariant version of the above theorem.

We demonstrate the power of the new technique by computing, in Section 7, the homotopy dimension of configuration spaces on spheres. Recall that the homotopy dimension of a space X, $\mathrm{hdim}(X)$, is defined as the smallest dimension of a CW complex that is homotopy equivalent to X.

Theorem. (Theorem 5.2 and Remark 5.3) *For positive integers n and k with $n > 1$, the configuration space $B_n(S^k)$ of subsets of cardinality n in the k-dimensional sphere has*

$$\mathrm{hdim}(B_n(S^k)) = (k-1)(n-1)+1. \tag{1.1}$$

It is interesting to note that, as a direct consequence of the calculations in [FZ00], the homotopy dimension of the configuration space $C_n(S^k)$ of n-tuples of distinct points in S^k is also given by the right hand side of (1.1) *provided* $\pi_1(C_n(S^k)) = 0$. Our idea for tackling the case of $B_n(S^k)$ arose from an argument (based on De Concini-Salvetti's n-dimensional analogue of the Salvetti complex) to extend the above situation to the non-simply connected spaces $C_n(S^2)$—i.e., those with $n \geq 3$.

Another notable fact comes from the observation that (1.1) recovers Kallel's strong upper bound for the twisted cohomological dimension of the braid manifolds $B_n(S^k)$ ([Ka08, Theorem 1.1]). Yet our combinatorially-minded methods contrast with the more geometric arguments in [Ka08], where the duality between braid spaces and truncated symmetric products plays a fundamental role.

There is an interesting connection between (1.1) and recent work by Karasev and Volovikov[1]. Corollary 5.10 in [KV10] implies that, for any oriented closed k-dimensional manifold M, the inequality

$$\mathrm{hdim}(B_n(M)) \geq (k-1)(n-1)+1$$

holds when n is prime (a hypothesis in force throughout this paragraph). Note that such a general lower bound is optimal in view of (1.1). But there is one further consequence of the interaction between (1.1) and Karasev-Volovikov's method, which is based on their concept of the *fixed point free genus*, $g_G(X)$, of a G-space X without fixed points (i.e., one where all the stabilizers are proper subgroups of G, see [KV10, Section 3]). We are interested in $G = \Sigma_n$ with its usual (free) action on $C_n(S^k)$ (recalled in the next section), and in the inequalities

$$g_{\mathbb{Z}_n}(C_n(S^k)) \leq \mathfrak{genus}\,(\rho_{n,S^k}) + 1 \leq \mathfrak{genus}\,(\pi_{n,S^k}) + 1. \tag{1.2}$$

Here $\mathfrak{genus}\,(p)$ stands for the (normalized) Schwarz genus of the fibration p (also recalled in the next section), $\mathbb{Z}_n$ stands for the subgroup of cyclic permutations in Σ_n, and $\rho_{n,X}\colon C_n(X) \to C_n(X)/\mathbb{Z}_n$ and $\pi_{n,X}\colon C_n(X) \to B_n(X)$ are projections onto orbit spaces. The

[1]We thank Roman Karasev for bringing this point to our attention, and for clarifying the relevant details.

first inequality in (1.2) is elementary and, in fact, an equality; the second inequality in (1.2) follows from the freeness of the action and the obvious existence of a (non-canonical) $\mathbb{Z}_n$-equivariant map $\Sigma_n \to \mathbb{Z}_n$. The punch line then comes from observing that all three numbers in (1.2) agree. Indeed, while $(k-1)(n-1)+2$ is a lower bound for the left-most term in (1.2) ([KV10, Corollary 5.10]), it is also an upper bound for the right-most term—in view of [Sva66, Theorem 5, page 75] and (1.1).

Our invariant $\mathrm{TC}_n^S(X)$ is given in terms of a slight variation of $\pi_{n,X}$. We use (1.1)—in a similar way to that indicated at the end of the previous paragraph—in order to deduce the following upper bound for $\mathrm{TC}_n^S(S^k)$:

Corollary. (Corollary 5.4) *For integers $k > 0$ and $n > 1$,*

$$\mathrm{TC}_n^S(S^k) \leq \left[(n+2)(k-1)+4\right](n-1)/2k. \tag{1.3}$$

At the end of Section 5 we offer evidence toward the possible optimality of (1.3).

Remark. Farber's TC work was motivated in part by Smale's ideas on the topological complexity of algorithms for finding approximations to the zeroes of a complex polynomial ([Sm87]). In Smale's view, the Schwarz genus of $\pi_{n,\mathbb{R}^m}\colon C_n(\mathbb{R}^m) \to B_n(\mathbb{R}^m)$ plays a fundamental role (for $m = 2$). Of course, a reasonable initial hold on the properties of this Σ_n-cover can be given from a suitably good understanding of the homotopy properties of $B_n(\mathbb{R}^m)$. As done in [Ro08] (or see alternatively the argument suggested in the second half of this remark), such a task can be accomplished by using Fuchs-Vassiliev CW complex structure on $B_n(\mathbb{R}^m)_\infty$, the one-point compactification of $B_n(\mathbb{R}^m)$ ([Fu70, Va87]). But a cleaner approach comes from Björner-Ziegler and De Concini-Salvetti generalization in [BZ92, DS00] of the Salvetti complex, which yields a homotopically sharp CW complex model for $B_n(\mathbb{R}^m)$—Example 7.12 in the final section of the paper reviews the construction, mainly in preparation for the situation for configuration spaces on spheres. Yet, for the reader's amusement, we close this introductory section with the following short argument suggesting a direct way of deducing a homotopically sharp CW complex model for $B_n(\mathbb{R}^m)$: Vassiliev's work gives us a CW complex decomposition in $B_n(\mathbb{R}^m)_\infty$ for which (a) the added point at infinity is the only 0-cell, and (b) all other cells appear in dimensions in between $n+m-1$ and nm. Thus, if the induced cellular stratified space structure on $B_n(\mathbb{R}^m)$ was regular and totally normal, then Theorem 6.9 would immediately yield a simplicial complex of dimension $nm-(n+m-1)=(n-1)(m-1)$ embedded in $B_n(\mathbb{R}^m)$ as a strong deformation retract—an optimal result since, as recalled in (7.1), the homotopy dimension of $B_n(\mathbb{R}^m)$ is known to be $(n-1)(m-1)$. We hope to make the above argument rigorous by proving, in a future work, a more general form of Theorem 6.9.

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2. Preliminaries

We recall Schwarz's (normalized) concept of the genus of a map [Sva66]. This is first defined for fibrations and then extended to arbitrary maps via fibrational substitutes.

2.1. **Definition.** The *Schwarz genus* of a fibration $p : E \to B$ is the least number k such that there is an open covering $U_0, U_1, \ldots, U_k$ of B for which the restriction of p over each $U_i, i=0,1,\ldots,k$ has a continuous section.

2.2. **Remark.** Under the above conditions, Schwarz's original definition in [Sva66] endows B with a genus equal to $k+1$, that is, 1 greater than our genus. We have chosen genus k for a covering with $k+1$ open sets to simplify our formulae and to comply with the most common definition of $\operatorname{cat}(X)$.

2.3. **Definition.** A *fibrational substitute* of a map f is a fibration $\widehat{f}$ such that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & E \arrow[d, "\widehat{f}"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

where h is a homotopy equivalence.

2.4. **Definition.** The *Schwarz genus* of a map f, denoted by $\mathfrak{genus}\,(f)$, is defined to be the Schwarz genus of its fibrational substitute. We agree to set $\mathfrak{genus}\,(f) = -1$ for $f : X \to Y$ with $X = \varnothing = Y$.

2.5. **Remark.** The Schwarz genus of a map is well defined since, for a path-connected Y, every map $f : X \to Y$ has a fibrational substitute unique up to fiber homotopy equivalence, [Do63, Se51].

We shall use the following proposition, [Sva66, Prop. 22, page 84], in the proof of Proposition 3.11.

2.6. **Proposition.** *Let $f : X \to Y$ and $f' : X' \to Y'$ be two maps between paracompact Hausdorff spaces, and let $f \times f' : X \times X' \to Y \times Y'$ be the product map. Then $\mathfrak{genus}\,(f \times f') \le \mathfrak{genus}\,(f) + \mathfrak{genus}\,(f')$.*

2.7. **Definition.** Let X be a path-connected space. The n^{th} *topological complexity* of X, denoted by $\mathrm{TC}_n(X)$, is the Schwarz genus of the fibration

$$e_n^X = e_n : X^{J_n} \to X^n, \quad e_n(\gamma) = (\gamma(1_1), \dots, \gamma(1_n)) \tag{2.1}$$

where J_n is the wedge of n closed intervals $[0, 1]$ (each with $0 \in [0, 1]$ as the base point), and 1_i stands for 1 in the i^{th} interval.

As explained in [Ru10], the n^{th} topological complexity is directly related to robot motion planning, in which the robot passes through n points for $n \ge 2$. Farber's TC is just $\mathrm{TC}_2 + 1$.

Throughout the paper, we denote by $d_n^X = d_n : X \to X^n$ the diagonal map. Also, frequently we denote $d_n(X) \subset X^n$ by Δ. We can define TC_n equivalently as follows, see [Ru10, Remark 3.2.3].

2.8. **Proposition.** *The n^{th} topological complexity $\mathrm{TC}_n(X)$ equals the Schwarz genus of the diagonal map $d_n : X \to X^n$. Indeed, e_n is a fibrational substitute of d_n.*

We close this section with some auxiliary notation relevant for the construction of symmetric versions of higher topological complexity.

As indicated in the introduction, $C_n(X)$ stands for the configuration space of n ordered distinct points in a space X—with the topology it inherits from X^n. The symmetric group Σ_n acts on $e_n^{-1}(C_n(X))$ and $C_n(X)$ by permuting paths in the former case, and by permuting coordinates in the latter. These actions are free and the restricted fibration $e_n\colon e_n^{-1}(C_n(X)) \to C_n(X)$ is equivariant.

There is a resulting fibration $\varepsilon_n^X = \varepsilon_n\colon Y_n(X) \to B_n(X)$ at the level of orbit spaces, where $Y_n(X) = e_n^{-1}(C_n(X))/\Sigma_n$ and $B_n(X) = C_n(X)/\Sigma_n$ (the latter stands for the "braid" configuration space of n unordered distinct points in X). Note that $\mathfrak{genus}\,(\varepsilon_n^X)$ gives a measure of the topological complexity of the motion planning problem on X when not only a pair of end positions are relevant as input, but where $n-2$ intermediate stages are also to be attained through the course of the motion. This concept will be connected in Section 4 to symmetrized forms of TC_n, while Section 5 is devoted to exploring $\mathfrak{genus}\,(\varepsilon_n^X)$ for X a sphere.

Note that the commutative diagram (where horizontal arrows are canonical projections)

$$\begin{tikzcd} e_n^{-1}(C_n(X)) \arrow[r] \arrow[d, "e_n"'] & Y_n(X) \arrow[d, "\varepsilon_n"] \\ C_n(X) \arrow[r] & B_n(X) \end{tikzcd} \tag{2.2}$$

is a pull-back square. Thus, the homotopy fiber of ε_n is $(\Omega X)^{n-1}$, just as for e_n ([Ru10, Remark 3.2.3]).

3. Properties of Higher Topological Complexity

The Schwarz genus of a fibration over X does not exceed $\mathrm{cat}(X)$. Thus,

$$\mathrm{TC}_n(X) \leq \mathrm{cat}(X^n) \leq n\,\mathrm{cat}(X) \leq n\dim(X). \tag{3.1}$$

On the other hand, the inequality $\mathrm{cat}(X) \leq \mathrm{TC}_2(X)$ is well known, see for instance [Fa08, FG08, FY04]. Our first results follow from extrapolating those ideas.

3.1. **Proposition.** *For any path-connected space X,*

$$\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X).$$

Proof. Let $\mathrm{TC}_n(X) = k$ and choose a covering $B_0 \cup B_1 \cup \cdots \cup B_k = X^n$ such that there is a continuous section s_i for e_n^X over B_i for $i = 0, \ldots, k$. Then, let $p : X^n \to X$ be the projection onto the first factor, choose $x_1 \in X$, and put $A_i = p^{-1}(x_1) \cap B_i$.

Note that $\{A_i\}_{i=0}^k$ is an open cover for $p^{-1}(x_1)$. Since the latter is homeomorphic to X^{n-1}, it suffices to show that each A_i is contractible within $p^{-1}(x_1)$.

For a point $(x_1, x_2, \ldots, x_n) \in A_i$ consider the n paths $\gamma_1, \ldots, \gamma_n$ in X, where γ_j is the restriction of $s_i(x_1, x_2, \ldots, x_n)$ to the j-th wedge summand of J_n. [From now on we will express this situation by saying that $s_i(x_1, x_2, \ldots, x_n)$ is the multipath $\{\gamma_j\}_{j=1}^n$]. So $\gamma_j(1) = x_j$ and $\gamma_j(0) = x_0$ for some $x_0 \in X$. Then, the constant path δ_1 at x_1, and the paths δ_j ($j = 2, \ldots, n$) formed by using γ_j^{-1} the first half of the time and γ_1 the second half are the components of a path $\delta = (\delta_1, \ldots, \delta_n)$ in $p^{-1}(x_1)$ from $\delta(0) = (x_1, x_2, \ldots, x_n)$ to $\delta(1) = (x_1, x_1, \ldots, x_1)$. The continuity of s_i implies that δ depends continuously on $(x_1, x_2, \ldots, x_n)$, so that we have actually constructed a contraction A_i to $(x_1, x_1, \ldots, x_1)$ in $p^{-1}(x_1)$. Thus, $\mathrm{cat}(X^{n-1}) \le k = \mathrm{TC}_n(X)$. □

3.2. **Remark.** Using the fact that $\mathrm{cat}(X^n) \ge n$ when X is not contractible ([CLOT03, Theorem 1.47]), we see that Proposition 3.1 recovers [Ru10, Proposition 3.5].

3.3. **Corollary.** *For any path-connected space X,*

$$\mathrm{cat}(X^{n-1}) \le \mathrm{TC}_n(X) \le \mathrm{cat}(X^n).$$

Proof. This follows directly from inequality (3.1) and the previous proposition. □

Our next goal is to give a complete characterization of $\mathrm{TC}_n(X)$ in terms of $\mathrm{cat}(X^{n-1})$ for $X = G$ a path-connected topological group.

3.4. **Proposition.** *For any path-connected topological group G,*

$$\mathrm{TC}_n(G) \le \mathrm{cat}(G^{n-1}).$$

Proof. Let ϵ denote the neutral element of G. Let $k = \mathrm{cat}(G^{n-1})$ and choose a covering $A_0 \cup \cdots \cup A_k = G^{n-1}$ where A_i is open and contractible to $(\epsilon, \ldots, \epsilon) = \epsilon^{(n-1)}$ in G^{n-1} for all $i = 0, \ldots, k$.

Take $i \in \{0, \ldots, k\}$ and put $B_i = \{(g, ga_2, \ldots, ga_n) \,|\, (a_2, \ldots, a_n) \in A_i,\ g \in G\}$, which is open in G^n. We claim that e_n^G admits a (continuous) section over each B_i. Indeed, the contractibility of A_i yields a path γ_a in G^{n-1} joining $\epsilon^{(n-1)}$ to $a = (a_2, \ldots, a_n) \in A_i \subset G^{n-1}$, and γ_a depends continuously on a. For any $g \in G$, consider the obvious path $g\gamma_a$ in G^n joining $(g, \ldots, g) = g\epsilon^{(n)} \in G^n$ to $(g, ga_2, \ldots, ga_n)$, where the first component is the identity path at g. Then, we get a section

$$s_i : B_i \to G^{J_n}$$

where, at the j^{th} interval of J_n, $s_i(g, ga_2, \ldots, ga_n)$ is the j^{th} coordinate of $g\gamma_a$.

The proof will be complete once we check that $B_0 \cup \cdots \cup B_k = G^n$. Take $(b_1, \ldots, b_n) \in G^n$ and put $g = b_1$ and $a_i = g^{-1}b_i$. Then there exists j such that $(a_2, \ldots, a_n) \in A_j$. So, $(b_1, \ldots, b_n) \in B_j$. □

3.5. **Theorem.** *For any path-connected topological group G,*

$$\mathrm{TC}_n(G) = \mathrm{cat}(G^{n-1}).$$

Proof. This follows directly from the last two propositions. □

Alternatively we can look at the growth of TC_n in terms of the difference of any two consecutive values of n.

3.6. **Corollary.** *For any path-connected topological group G,*

$$\mathrm{TC}_n(G) - \mathrm{TC}_{n-1}(G) \leq \mathrm{cat}(G).$$

Proof. This follows from Theorem 3.5:

$$\begin{aligned} &\mathrm{TC}_n(G) - \mathrm{TC}_{n-1}(G) = \mathrm{cat}(G^{n-1}) - \mathrm{cat}(G^{n-2}) \\ &\leq \mathrm{cat}(G^{n-2}) + \mathrm{cat}(G) - \mathrm{cat}(G^{n-2}) = \mathrm{cat}(G). \end{aligned}$$

□

Unlike the case of groups, higher topological complexities of a general path-connected space X do not seem to be completely determined by $\mathrm{cat}(X)$. Nonetheless, we can directly obtain the following bound on the difference of consecutive higher topological complexities of X.

3.7. **Corollary.** *For $n \geq 3$ and any path-connected space X,*

$$\mathrm{TC}_n(X) - \mathrm{TC}_{n-1}(X) \leq \mathrm{cat}(X^2).$$

Proof. We have the sequence of inequalities

$$\begin{aligned} \mathrm{cat}(X) \leq \mathrm{TC}_2(X) \leq \mathrm{cat}(X^2) \leq \mathrm{TC}_3(X) \leq \cdots \leq \mathrm{cat}(X^{n-1}) \\ \leq \mathrm{TC}_n(X) \leq \mathrm{cat}(X^n) \leq \cdots . \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{TC}_n(X) - \mathrm{TC}_{n-1}(X) &\leq \mathrm{cat}(X^n) - \mathrm{cat}(X^{n-2}) \\ &\leq \mathrm{cat}(X^2) + \mathrm{cat}(X^{n-2}) - \mathrm{cat}(X^{n-2}) \\ &\leq \mathrm{cat}(X^2). \end{aligned}$$

□

This is quite coarse, however, since $\mathrm{TC}_n(X)$ grows as n [Ru10], not as $2n$.

Next we work out the higher analogue of the usual cup-length lower bound for TC. In the following definition (modified cup-length of a space X) we consider cohomology with local coefficients.

3.8. **Definition.** Given a space X and a natural number n, define the *d_n^X-cup-length*, denoted by $\mathrm{cl}(X, n)$, to be the largest m with the following property: There exist cohomology classes $u_i \in H^*(X^n; A_i)$ such that $d_n^* u_i = 0$ for $i = 1, \ldots, m$ and

$$u_1 \smile \cdots \smile u_m \neq 0 \in H^*(X^n; A_1 \otimes \cdots \otimes A_m).$$

The following theorem, which follows directly from [Sva66, Theorem 4], gives a lower bound for TC_n in terms of $\mathrm{cl}(X,n)$.

3.9. **Theorem.** *For any path-connected space X we have the inequality $\mathrm{cl}(X,n) \le \mathrm{TC}_n(X)$.*

We will also need the following bound on $\mathrm{cl}(X \times S^k, n)$ in terms of $\mathrm{cl}(X,n)$.

3.10. **Theorem.** *For any path-connected space X and natural numbers n and k we have $\mathrm{cl}(X \times S^k, n) \ge \mathrm{cl}(X,n) + n - 1$. This inequality can be improved to $\mathrm{cl}(X \times S^k, n) \ge \mathrm{cl}(X,n) + n$ provided k is even and $H^*(X)$ is torsion-free.*

Proof. Let v be a generator of $H^k(S^k) = \mathbb{Z}$. Let $p_i : (S^k)^n \to S^k$ be the projection onto the i^{th} factor and put $v_i = p_i^* v$ for $i = 1, \dots, n$.

Assume that $\mathrm{cl}(X,n) = m$ and take $u_1, \dots, u_m$ such that $d_n^* u_j = 0$ for $j = 1, \dots, m$ and $u_1 \smile \cdots \smile u_m \ne 0$.

To prove the first assertion note that $d_n^*(v_i - v_1) = 0$ for $i > 1$, while the basis element $v_2 \smile \cdots \smile v_n \in H^*\big((S^k)^n\big)$ appears in the reduced expansion (using distributivity) of $(v_2 - v_1) \smile \cdots \smile (v_n - v_1)$. Hence,

$$u_1 \smile \cdots \smile u_m \smile (v_2 - v_1) \smile \cdots \smile (v_n - v_1) \ne 0.$$

Thus $\mathrm{cl}(X \times S^k, n) \ge \mathrm{cl}(X,n) + n - 1$.

Assume now that k is even and that $H^*(X)$ is torsion-free. The element $v_1 + v_2 + \cdots + v_{n-1} - (n-1)v_n$ lies in the kernel of d_n^* and has cup n^{th} power equal to a non-zero multiple of $v_1 \smile v_2 \smile \cdots \smile v_n$. Hence,

$$u_1 \smile \cdots \smile u_m \smile (v_1 + v_2 + \cdots + v_{n-1} - (n-1)v_n)^n \ne 0.$$

Thus $\mathrm{cl}(X \times S^k, n) \ge \mathrm{cl}(X,n) + n$. $\square$

In [Fa03] M. Farber showed that $\mathrm{TC}(X \times Y) \le \mathrm{TC}(X) + \mathrm{TC}(Y) - 1$, that is, $\mathrm{TC}_2(X \times Y) \le \mathrm{TC}_2(X) + \mathrm{TC}_2(Y)$. This result is generalized in the following proposition.

3.11. **Proposition.** *For path-connected paracompact Hausdorff spaces X and Y,*

$$\mathrm{TC}_n(X \times Y) \le \mathrm{TC}_n(X) + \mathrm{TC}_n(Y).$$

Proof. The natural homeomorphisms

$$(X \times Y)^n \to X^n \times Y^n,$$
$$((x_1, y_1), \dots, (x_n, y_n)) \mapsto (x_1, \dots, x_n, y_1, \dots, y_n), \quad x_i \in X, \quad y_j \in Y$$

and

$$(X \times Y)^{J_n} \to X^{J_n} \times Y^{J_n},$$

$$(\varphi : J_n \to X \times Y) \mapsto ((p_X \varphi : J_n \to X), (p_Y \varphi : J_n \to Y))$$

can be included into a commutative diagram

$$\begin{array}{ccc} (X \times Y)^{J_n} & \longrightarrow & X^{J_n} \times Y^{J_n} \\ {\scriptstyle e_n^{X\times Y}}\big\downarrow & & \big\downarrow{\scriptstyle e_n^X \times e_n^Y} \\ (X \times Y)^n & \longrightarrow & X^n \times Y^n \end{array}$$

Now, by Proposition 2.6, we have

$$\begin{aligned} \mathrm{TC}_n(X \times Y) &= \mathfrak{genus}\,(e_n^{X\times Y}) = \mathfrak{genus}\,(e_n^X \times e_n^Y) \\ &\le \mathfrak{genus}\,(e_n^X) + \mathfrak{genus}\,(e_n^Y) = \mathrm{TC}_n(X) + \mathrm{TC}_n(Y). \end{aligned}$$

□

Next, we apply the previous results in order to compute the higher topological complexity of concrete families of spaces.

3.12. **Corollary.** *$\mathrm{TC}_n(S^{k_1} \times S^{k_2} \times \cdots \times S^{k_m}) = m(n-1) + l$ where l is the number of even dimensional spheres.*

Proof. Note that $\mathrm{TC}_n(S^k) = \mathrm{cl}(S^k, n)$ for all k, [Ru10, Section 4]. Then the inequality $\mathrm{cl}(S^{k_1} \times \cdots \times S^{k_m}, n) \ge m(n-1) + l$ follows from Theorem 3.10 by induction, so $\mathrm{TC}_n(S^{k_1} \times \cdots \times S^{k_m}) \ge m(n-1)+l$ by Theorem 3.9. The opposite estimate follows from Proposition 3.11. □

The calculation of the n^{th} topological complexity of the k-dimensional torus $T^k = (S^1)^k$, partially solved for $k = 2$ in [Ru10, Proposition 5.1], can now be completed using either Corollary 3.12 or Theorem 3.5.

3.13. **Corollary.** *$\mathrm{TC}_n(T^k) = k(n-1)$.*

3.14. **Theorem.** *Let X be a CW complex of finite type, and R a principal ideal domain. Take $u \in H^d(X; R)$ with $d > 0$, d even, and assume that the n-fold iterated self R-tensor product $u^m \otimes \cdots \otimes u^m \in (H^{md}(X; R))^{\otimes n}$ is an element of infinite additive order. Then $\mathrm{TC}_n(X) \ge mn$.*

Proof. Let $p_i : X^n \to X$ be the projection onto the i^{th} factor and put $u_i = p_i^* u \in H^d(X^n; R)$. In view of Theorem 3.9, the required inequality follows from

$$v := (u_2 - u_1)^{2m}(u_3 - u_1)^m \cdots (u_n - u_1)^m \neq 0. \tag{3.2}$$

In order to check (3.2), note that v comes from the tensor product—which injects into the cohomology of the Cartesian product by the Künneth Theorem (this is where the finiteness hypotheses are used). So, calculations can actually be performed in the former R-module. Now, assuming that $\dim(X) \leq dm + 1$, we have

$$\begin{aligned} v &= (u_2 - u_1)^{2m}(u_3 - u_1)^m \cdots (u_n - u_1)^m \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m (u_3 - u_1)^m \cdots (u_n - u_1)^m \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m u_3^m (u_4 - u_1)^m \cdots (u_n - u_1)^m \\ &= \cdots \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m \cdots u_n^m, \end{aligned}$$

which is non-zero by hypothesis.

For $\dim(X)$ arbitrary, consider the skeletal inclusion $j : X^{(dm+1)} \to X$ and note that $v \neq 0$ since $j^* v \neq 0$. □

3.15. **Corollary.** *For every closed simply connected symplectic manifold M^{2m} we have $\mathrm{TC}_n(M) = nm$.*

Proof. This follows from Theorem 3.14 (taking u to be the cohomology class given by the symplectic 2-form on M—note that the hypothesis on $u^m \otimes \cdots \otimes u^m$ holds since coefficients are taken over the reals), the inequality $\mathrm{cat}(M^{2m}) \leq m$ (which follows from [Sva66, Theorem 5, page 75]), and inequality (3.1). (This argument even yields $\mathrm{cat}(M^{2m}) = m$, a fact that is well known to experts.) □

Of course, Corollary 3.15 applies to complex projective spaces. In the quaternionic case essentially the same proof gives:

3.16. **Corollary.** *The quaternionic projective space of real dimension $4m$, $\mathbb{HP}^m$, has $\mathrm{TC}_n(\mathbb{HP}^m) = nm$.* □

In the next section we introduce a couple of symmetric versions of TC_n. One of them, $\mathrm{TC}_n^{\Sigma}(X)$, has the advantage of actually being a homotopy invariant of X. The other, $\mathrm{TC}_n^{S}(X)$, gives the natural generalization of Farber-Grant's symmetric topological complexity $\mathrm{TC}_2^{S}(X)$. Section 5 represents our contribution toward computing $\mathrm{TC}_n^{S}(X)$ when X is a sphere.

4. Symmetric Topological Complexity

Farber and Grant studied in [FG07] a symmetrized version, TC^S, of topological complexity. Here we start by proposing a homotopically more convenient version TC^{Σ}. Although the invariant TC^{Σ} is inspired by [FG08, Definition 2], those authors did not develop this approach.

Consider the involutions $\tau : X \times X \to X \times X$ and $\overline{\tau} : PX \to PX$ defined by $\tau(x, y) = (y, x)$ and $\overline{\tau}(\gamma)(t) = \gamma(1 - t)$, for $(x, y) \in X \times X$ and $\gamma \in PX$.

4.1. **Definition.** A subset A in $X \times X$ is *symmetric* if $\tau A = A$.

4.2. **Definition.** A function $s : A \to PX$ is *equivariant* if $\overline{\tau}(s(a)) = s(\tau(a))$ for $a \in A$, where A is a symmetric subset of $X \times X$.

4.3. **Definition.** $\mathrm{TC}^{\Sigma}(X)$ is the least number k such that $X \times X = A_0 \cup A_1 \cup \cdots \cup A_k$ where each A_i is open, symmetric, and has a continuous equivariant section $s_i : A_i \to PX$ of the map e_2 in (2.1).

Before proving (in Proposition 4.11 below) that $\mathrm{TC}^{\Sigma}(X)$ is a homotopy invariant of X, we show that Definition 4.3 differs by at most 1 from Farber-Grant's symmetric topological complexity. In our terms the Farber-Grant definition amounts to setting

$$\mathrm{TC}^{S}(X) = 2 + \mathfrak{genus}\,(\varepsilon_2)$$

where ε_2 is the map on the right hand side of (2.2). However, in accordance with the normalization discussed in the introduction (implicit in Definition 4.3), we should compare Definition 4.3 with

$$\mathrm{TC}^{S}_2(X) = 1 + \mathfrak{genus}\,(\varepsilon_2). \tag{4.1}$$

4.4. **Proposition.** *For each ENR X we have*

$$\mathrm{TC}^{S}_2(X) - 1 \leq \mathrm{TC}^{\Sigma}(X) \leq \mathrm{TC}^{S}_2(X).$$

4.5. **Remark.** We will prove a more general version of Proposition 4.4 (Theorem 4.12 below). The proof of the extended situation is considerably more elaborated as it requires an involved use of the theory of *equivariant* euclidean neighborhood retracts. For the sake of clarity, we offer first the easy argument proving Proposition 4.4—which will also serve as a warm up for the proof of Theorem 4.12.

Proof. To prove the first inequality, take an open covering $X \times X = A_0 \cup \cdots \cup A_k$ where each A_i is symmetric and has an equivariant section of e_2.

The $\mathbb{Z}/2$-action τ on $X \times X$ yields the orbit map $\pi : X \times X \to (X \times X)/\tau$. Then each $\pi(A_i - \Delta)$ is open and has a section of ε_2, cf. [FG07, Lemma 8], and thus $\mathfrak{genus}\,(\varepsilon_2) \leq \mathrm{TC}^{\Sigma}(X)$.

For the second inequality, take $B_0, \dots, B_l$, with $B_0 \cup \dots \cup B_l = \pi(X \times X - \Delta)$ where each B_i is open and has a section of ε_2. Then each $\pi^{-1}(B_i)$ is symmetric, open in $X \times X$, and admits an equivariant section of e_2. Further, since X is an ENR, there is a symmetric open neighborhood of Δ supporting an equivariant section of e_2 (see the proof of [FG07, Corollary 9]). Consequently $\mathrm{TC}^{\Sigma}(X) \leq 1 + \mathfrak{genus}\,(\varepsilon_2)$. □

As the following examples show, Proposition 4.4 is optimal in the sense that the two bounds given in this result can actually be attained.

4.6. **Example.** For X contractible we have $\mathrm{TC}_2(X) = \mathrm{TC}^{\Sigma}(X) = 0$ while $\mathrm{TC}_2^S(X) = 1$. Indeed, take a point $x_0 \in X$ and a contraction $H : X \times I \to X$, with $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $x \in X$. Given $a, b \in X$, take the path $s(a, b) : I \to X$ such that $s(t) = H(a, 2t)$ for $0 \leq t \leq 1/2$ and $s(t) = H(b, 2 - 2t)$. Then s is an equivariant section for e_2^X and, in view of the general inequality

$$\mathrm{TC}_2(X) \leq \mathrm{TC}^{\Sigma}(X),$$

this gives $\mathrm{TC}_2(X) = \mathrm{TC}^{\Sigma}(X) = 0$. The same argument, but now using (4.1), gives $\mathrm{TC}_2^S(X) = 1$ (see [FG08, Example 7]).

4.7. **Example.** Farber and Grant proved in [FG07, Corollary 18] that $\mathrm{TC}_2^S(S^k) = 2$ for any k. On the other hand, Farber proved in [Fa03] that $\mathrm{TC}_2(S^k) = 1$ for k odd, while $\mathrm{TC}_2(S^k) = 2$ for k even. Here we observe that

$$\mathrm{TC}_2(S^k) = \mathrm{TC}^{\Sigma}(S^k) = \mathrm{TC}_2^S(S^k) \text{ for even } k, \tag{4.2}$$

for $2 = \mathrm{TC}_2(S^k) \leq \mathrm{TC}^{\Sigma}(S^k) \leq \mathrm{TC}_2^S(S^k) = 2$. For an odd k the construction from [Fa08, Example 4.8] gives us an open symmetric partition $S^k \times S^k = A_0 \cup A_1$ with continuous sections of e_2 over each A_i. However, one of these sections is not equivariant, which prevents us from deducing $\mathrm{TC}^{\Sigma}(S^k) = 1$.

The construction of TC^{Σ} can be generalized in a straightforward way. Recall that for a given n, the symmetric group Σ_n acts on X^n as well as on X^{J_n} by permuting coordinates and paths, respectively. Further, the fibration e_n in (2.1) is Σ_n-equivariant.

4.8. **Definition.** A subset A in X^n is *symmetric* if $\sigma A = A$ for all $\sigma \in \Sigma_n$.

4.9. **Definition.** For a symmetric $A \subset X^n$, a function $s : A \to X^{J_n}$ is *equivariant* if $\sigma(s(a)) = s(\sigma(a))$ for all $a \in A$ and $\sigma \in \Sigma_n$.

Definition 4.3 can now be extended to:

4.10. **Definition.** $\mathrm{TC}_n^{\Sigma}(X)$ is the least number k such that $X^n = A_0 \cup A_1 \cup \cdots \cup A_k$ where each A_i is open, symmetric and has a continuous equivariant section $s_i : A_i \to X^{J_n}$ for e_n. So, $\mathrm{TC}^{\Sigma}(X) = \mathrm{TC}_2^{\Sigma}(X)$.

4.11. **Proposition.** *$\mathrm{TC}_n^{\Sigma}(X)$ is a homotopy invariant of X.*

Proof. It suffices to prove that, given $f : Y \to X$ and $g : X \to Y$ with $gf \simeq 1_Y$, we have $\mathrm{TC}_n^{\Sigma}(X) \geq \mathrm{TC}_n^{\Sigma}(Y)$ for all n. Let $H : 1_Y \simeq gf$ be a homotopy $H : Y \times [0,1] \to Y$ such that $H(y,0) = y$ and $H(y,1) = gf(y)$.

Let A be an open symmetric subset of X^n, and let $s : A \to X^{J_n}$ be an equivariant section of e_n^X over A. Given $a = (a_1, \ldots, a_n) \in A$, let $s_i(a)$ denote the restriction of $s(a) \in X^{J_n}$ to the i^{th} wedge summand of J_n (this is a path in X joining x_0 and a_i for some $x_0 \in X$ that depends continuously on a). Note that the symmetry of s gives

$$s_i(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = s_{\sigma(i)}(a_1, \ldots, a_n), \quad \text{for} \quad \sigma \in \Sigma_n. \tag{4.3}$$

Take $B := (f^n)^{-1}(A)$ and consider the map $s' : B \to Y^{J_n}$ which, at a given $b \in B$ with $f^n(b) = a$, has $s'_i(b) := (g \circ s_i(a)) \cdot \gamma_i$ as its restriction to the i^{th} wedge summand of J_n, where γ_i is the path in Y given by

$$\gamma_i(t) = H(b_i, 1 - t).$$

Then, s' is an equivariant (in view of (4.3)) continuous section of e_n^Y over B.

Now, if $X = A_0 \cup \cdots \cup A_k$ where each A_j ($j = 0, \ldots, k$) is open, symmetric, and admits a continuous equivariant section of e_n^X, then $Y = B_0 \cup \cdots \cup B_k$ where each B_j—defined as above using A_j—is open, symmetric, and admits a continuous equivariant section of e_n^Y. Hence, $\mathrm{TC}_n^{\Sigma}(X) \geq \mathrm{TC}_n^{\Sigma}(Y)$. ☐

The following assertion is our higher analogue of Proposition 4.4.

4.12. **Theorem.** *Assume X is an ENR, and let ε_k be the map on the right hand side of (2.2), then*

$$\mathfrak{genus}\,(\varepsilon_n) \leq \mathrm{TC}_n^{\Sigma}(X) \leq \mathfrak{genus}\,(\varepsilon_n) + \cdots + \mathfrak{genus}\,(\varepsilon_2) + n - 1. \tag{4.4}$$

4.13. **Remark.** The first inequality in (4.4) follows just as in the proof of Proposition 4.4: If e_n admits an equivariant section over $A \subset X^n$, then ε_n admits a section over $\pi(A \cap C_n(X))$ where $\pi : X^n \to X^n/\Sigma_n$ stands for the canonical projection. Our efforts will therefore focus on the second inequality in (4.4), whose proof requires some preparation.

4.14. **Definition.** A topological space X with an action of a compact Lie group G is called a euclidean neighborhood G-retract (or G-ENR for short) if X can be G-equivariantly embedded as a G-equivariant retract of a G-symmetric neighborhood in an orthogonal representation of G.

In what follows we will make implicit use of the following fact: if a G-ENR X is G-equivariantly embedded in a given orthogonal representation $\mathbb{R}^N$ of G, then there exists a G-symmetric neighborhood U of X in $\mathbb{R}^N$ and a G-equivariant retraction $U \to X$. As suggested at the end of the introduction in [J76], such a property follows by using the equivariant version of the Tietze Theorem (Tietze-Gleason Theorem, [Br72, Gl50]) in the non-equivariant argument in [Do95, Proposition and Definition IV.8.5].

The following theorem is a weak version of [J76, Theorem 2.1][2].

4.15. **Theorem** (Jaworowski)**.** *Let L be a finite group acting on an ENR Z. Then Z is an L-ENR if for every subgroup G of L, the fixed point set Z^G is an ENR.*

There is a Σ_n-equivariant filtration

$$\Delta = D^1(X) \subset \cdots \subset D^{n-1}(X) \subset D^n(X) = X^n.$$

Here $D^i(X)$ is the closed set consisting of the n-tuples $(x_1, x_2, \ldots, x_n)$ such that the *set* $\{x_1, x_2, \ldots, x_n\}$ has cardinality at most i (compare with the filtration considered at the end of Section 1 in [Ka08]). For instance, $D^{n-1}(X)$ is the fat diagonal in X^n (denoted by $\Delta_n(X)$ in Corollary 7.2 in the final section of the paper).

Set $D^0(X) = \emptyset$, and for $1 \leq i \leq n$ let C^i stand for the difference $D^i(X) - D^{i-1}(X)$, the subspace of n-tuples $(x_i, x_2, \ldots, x_n)$ such that the set $\{x_1, x_2, \ldots, x_n\}$ has cardinality i. Note that $C^n = C_n(X)$ but, for $i < n$, each partition $\mathcal{P} = \{P_1, \ldots, P_i\}$ of $\{1, 2, \ldots, n\}$ into i nonempty sets determines a closed subspace $C^i_{\mathcal{P}} \subset C^i$ formed by those tuples $(x_1, \ldots, x_n)$ in C^i satisfying $x_r = x_s$ whenever r and s lie in a same part P_j.

Note that C^i is the disjoint union of the $C^i_{\mathcal{P}}$, each of which maps homeomorphically, under a suitable coordinate projection, onto $C_i(X)$. [For instance, for $n = 3$ the three closed subspaces partitioning C^2 are determined, respectively, by the three requirements $x_1 = x_2$, $x_1 = x_3$, and

[2] Although Jaworowski's theorem was originally set in terms of a combination of the concepts of ANR's and ENR's, for our formulation the reader should keep in mind the (elementary in view of the Tietze Theorem) fact that any ENR is an ANR.

$x_2 = x_3$; in the latter case, the required projection can be chosen to be $(x_1, x_2, x_3) \mapsto (x_1, x_2)$.] In particular we have a continuous (surjective) map $\pi_i \colon C^i \to C_i(X)$.

Let P^i denote the subspace of $e_n^{-1}(C^i)$ consisting of those multipaths $\alpha = \{\alpha_i\}_{i=1}^n$ satisfying $\alpha_k = \alpha_\ell$ whenever $\alpha_k(1_k) = \alpha_\ell(1_\ell)$. Proceeding as above, we get a continuous surjection $\Pi_i \colon P^i \to e_i^{-1}(C_i(X))$ in such a way that in the commutative diagram

$$\begin{array}{ccccccc} X^{J_n} & \longleftarrow & P^i & \xrightarrow{\Pi_i} & e_i^{-1}(C_i(X)) & \longrightarrow & Y_i(X) \\ \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle e_i} & & \downarrow{\scriptstyle \varepsilon_i} \\ X^n & \longleftarrow & C^i & \xrightarrow{\pi_i} & C_i(X) & \longrightarrow & B_i(X) \end{array} \tag{4.5}$$

the second and third squares are pullbacks (the left-most horizontal maps are inclusions but they do not determine a pull-back square).

Our last ingredient in preparation to (4.4) is given by taking an arbitrary open subset W of $B_i(X)$. We let $A = \pi_i^{-1}(W')$ where W' stands for the inverse image of W under the cover map $C_i(X) \to B_i(X)$. Clearly W' is Σ_i-symmetric, and A is Σ_n-symmetric. This set-up will be in force in the following two auxiliary results—the basis of our proof of the second inequality in (4.4):

4.16. **Lemma.** *The space A is a Σ_n-ENR.*

Proof. Note first that every $C^i_{\mathcal{P}}$ is an ENR, because it is homeomorphic to $C_i(X)$ which, in turn, is an open subset of the ENR X^i. Now, every $g \in \Sigma_n$ yields a homeomorphism from any given $C^i_{\mathcal{P}}$ onto some $C^i_{\mathcal{P}'}$. It is easy to see that, in case $\mathcal{P} = \mathcal{P}'$ and if there is some point $x \in C^i_{\mathcal{P}}$ fixed by g, then in fact $g \cdot y = y$ for any $y \in C^i_{\mathcal{P}}$, i.e. $(C^i_{\mathcal{P}})^g = C^i_{\mathcal{P}}$. Hence, for any subgroup G of Σ_n, the set $(C^i_{\mathcal{P}})^G$ is either empty or the whole $C^i_{\mathcal{P}}$, and therefore an ENR. In particular, $(C^i)^G$ is an ENR since C^i is the disjoint union of the various $C^i_{\mathcal{P}}$'s, while A^G is an ENR since A is open in C^i. Thus, by Theorem 4.15, A is a Σ_n-ENR, as asserted. □

4.17. **Lemma.** *Assume $s : A \to P^i$ is a Σ_n-equivariant section of the second vertical map in (4.5). Then there is a Σ_n-symmetric neighborhood U of A in X^n that admits a Σ_n-equivariant section $\sigma \colon U \to X^{J_n}$ of the first vertical map in (4.5).*

Proof. Start by noticing that, as a consequence of Theorem 4.15, X^n is a Σ_n-ENR; indeed, for any subgroup G of Σ_n, the fixed point set of G on X^n is an intersection of hyperplanes $x_i = x_j$ in X^n, and therefore it is homeomorphic to X^m, $m \leq n$, and hence is an ENR. Thus, we can take Σ_n-equivariant embeddings $A \to X^n \to \mathbb{R}^N$, and a Σ_n-equivariant retraction $r'\colon O \to A$ of a Σ_n-symmetric neighborhood O of A in $\mathbb{R}^N$, where $\mathbb{R}^N$ is an orthogonal representation of Σ_n.

Put $V = O \cap X^n$. Then V is a Σ_n-symmetric neighborhood of A in X^n, and $r = r'_{|V} : V \to A$ is a Σ_n-equivariant retraction. Note that V is an open Σ_n-symmetric subset of the Σ_n-ENR X^n, and so V is a Σ_n-ENR in itself. So we can choose an open Σ_n-symmetric neighborhood Y of V in $\mathbb{R}^N$, and a Σ_n-equivariant retraction $\rho : Y \to V$. Let $U \subset V$ consist of all points $v \in V$ such that the segment from v to $ir(v)$ lies in Y (cf. [Do95, Corollary IV.8.7])—here i stands for the inclusion $A \hookrightarrow V$. Clearly U is a neighborhood of A in V, and hence in X^n. Furthermore, the composite $ir_{|U}$ and the inclusion $U \hookrightarrow V$ are homotopic via the homotopy

$$\Phi : U \times I \to V, \quad \Phi(u,t) = \rho\left(t \cdot u + (1-t) \cdot ir(u)\right).$$

Note that U is Σ_n-symmetric and Φ is Σ_n-equivariant, since the Σ_n-action on $\mathbb{R}^N$ is orthogonal and so maps lines to lines.

We use the homotopy Φ (and the Σ_n-equivariant section $s : A \to P^i$ given by hypothesis) in order to construct a Σ_n-equivariant section $\sigma\colon U \to X^{J_n}$ of the first vertical map in (4.5). For $x \in U$, consider the path $\beta\colon I \to V$, $\beta(t) = \Phi(x,t)$, starting at $y = \beta(0) = r(x) \in A$ and ending at x. Since $V \subset X^n$, we set $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and $\beta = (\beta_1, \dots, \beta_n)$, so each β_i is a path in X from y_i to x_i. Further, $s(y)$ gives a multipath $\{\alpha_i\}_{i=1}^n$ with $\alpha_i(1) = y_i$ and $\alpha_i(0) = \alpha_j(0)$ for all $1 \le i,j \le n$. Then the multipath $\{\alpha_i \cdot \beta_i\}_{i=1}^n$ determines an element $\sigma(x) \in X^{J_n}$ with $e_n(\sigma(x)) = x$. This defines the required Σ_n-equivariant section over U. □

Note that the two pull-back squares in (4.5) imply that the hypothesis in Lemma 4.17 holds whenever W is chosen to admit a section of the fourth vertical map in (4.5). Thus we get:

Proof of Theorem 4.12. In view of Lemmas 4.16 and 4.17 we can choose $1+\mathfrak{genus}\,(\varepsilon_i)$ Σ_n-equivariant local sections for e_n whose domains cover C^i, and thus a total of

$$\sum_{i=2}^{n}(1+\mathfrak{genus}\,(\varepsilon_i))+1 = \mathfrak{genus}\,(\varepsilon_n)+\cdots+\mathfrak{genus}\,(\varepsilon_2)+n \tag{4.6}$$

Σ_n-equivariant local sections for e_n whose domains cover X^n, where the last "+1" in (4.6) accounts for the obvious equivariant section on the diagonal $D^1(X)$. The theorem follows. □

A comparison of Proposition 4.4 and Theorem 4.12 suggests the following generalization of (4.1):

4.18. **Definition.** For $n \geq 2$ set

$$\mathrm{TC}^S_n(X) = \mathfrak{genus}\,(\varepsilon_n) + \cdots + \mathfrak{genus}\,(\varepsilon_2) + n - 1.$$

This variation of the one proposed in [Ru10] will be explored, for X a sphere, in the next section.

5. Bounding $\mathfrak{genus}\,(\varepsilon_n)$ for Spheres

The following result, a consequence of [Sva66, Theorem 5, page 75], is the basis for this section's goal.

5.1. **Proposition.** *If X is an $(s-1)$-connected space and $B_n(X)$ has the homotopy type of a k-dimensional CW space, then $\mathfrak{genus}\,(\varepsilon_n) \leq k/s$.*

The following paragraph illustrates our strategy to settle a (potentially optimal) upper bound for $\mathrm{TC}^S_n(S^k)$.

As recalled in Example 4.7, the equality

$$\mathrm{TC}^S_2(S^k) = 2 \tag{5.1}$$

holds for any k. Farber and Grant prove that $\mathrm{TC}^S_2(S^k)$ is no greater than 2 by actually producing a symmetric motion planner with two local rules. Their construction makes use of a well-known explicit Σ_2-equivariant deformation retraction

$$C_2(S^k) \to S^k \tag{5.2}$$

that implies a homotopy equivalence $B_2(S^k) \simeq \mathbb{RP}^k$. But note that Proposition 5.1 gives an alternative direct way to deduce the inequality $\mathrm{TC}^S_2(S^k) \leq 2$, for $B_2(S^k)$ is homotopy equivalent to a CW space of dimension k.

We next give the main ingredient for extending the previous argument in order to obtain a strong upper bound for $\mathrm{TC}^S_n(S^k)$.

5.2. **Theorem.** *Put $d(k,n) = (k-1)(n-1)+1$. For $n \geq 2$ and $k \geq 1$, $B_n(S^k)$ has the homotopy type of a CW complex of dimension $d(k,n)$.*

Note that the case $k = 1$ in Theorem 5.2 is well known as

$$B_n(S^1) \text{ has the homotopy type of } S^1 \tag{5.3}$$

(cf. [Ka08, Proposition 2.5]). Our proof of Theorem 5.2, given in the final section of the paper (after a key section where our main new combinatorial ingredient is introduced), can be thought of as a rather elaborate generalization of the case $n = 2$—given by (5.2). Namely, we construct an explicit $d(k,n)$-dimensional simplicial complex embedded in $C_n(S^k)$ as a strong Σ_n-equivariant deformation retract.

5.3. **Remark.** The analogue of Theorem 5.2 for the ordered configuration space $C_k(S^n)$ follows from Theorem 7.17 at the end of the paper. Further, the calculations in [FZ00] imply that $C_k(S^n)$ cannot have the homotopy type of a cell complex of dimension less than $d(k,n)$. So the result is optimal in the ordered configuration case. The corresponding optimality of Theorem 5.2—i.e., the claim in (1.1)—follows since the homotopy dimension of a CW space is not less than the homotopy dimension of any of its covering spaces. It is interesting to compare with the situation in (7.1).

An immediate consequence of Proposition 5.1 and Theorem 5.2 is:

5.4. **Corollary.** *For $X = S^k$ and $i \geq 2$, $\mathfrak{genus}\,(\varepsilon_i) \leq i - 1 - (i-2)/k$. In particular* $\mathrm{TC}^S_n(S^k) \leq \left[(n+2)(k-1)+4\right](n-1)/2k$ *for $n \geq 2$.*

Corollary 5.4 is optimal for $n = 2$, in view of (5.1). The next result gives further evidence toward the optimality of Corollary 5.4.

5.5. **Proposition.** *For $X = S^1$ and $i \geq 1$, $\mathfrak{genus}\,(\varepsilon_{2i}) = 1$. Consequently, for $n \geq 2$, $2\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor \leq \mathrm{TC}^S_n(S^1) \leq 2(n-1)$, where $\lfloor x \rfloor$ stands for the integral part of the real number x.*

Proof. We just observed that $\mathfrak{genus}\,(\varepsilon_2) = 1$ (over any sphere), so we assume $i \geq 2$. Further, in view of Corollary 5.4 it suffices to prove that ε_{2i} has no global section, or equivalently (cf. [FG07, Lemma 7]), that $e_{2i} : e_{2i}^{-1}(C_{2i}(S^1)) \to C_{2i}(S^1)$ has no Σ_{2i}-equivariant section.

The projection $(S^1)^{2i} \to (S^1)^2$ onto the first two coordinates, and the inclusion $J_2 \hookrightarrow J_{2i}$ into the first two wedge summands yield the commutative diagram

$$\begin{array}{ccc} (S^1)^{J_{2i}} & \longrightarrow & (S^1)^{J_2} \\ {\scriptstyle e_{2i}}\big\downarrow & & \big\downarrow{\scriptstyle e_2} \\ (S^1)^{2i} & \longrightarrow & (S^1)^2. \end{array} \tag{5.4}$$

Consider the embedding $c\colon C_2(S^1) \to C_{2i}(S^1)$ defined by $c(x_1,x_2) = (x_1, x_2, x_3, \ldots, x_{2i})$ where the x_j with $j \geq 3$ are defined as follows. Regard x_k as a cyclic coordinate on $S^1 = [0, 2\pi]/(0 \sim 2\pi)$ and put

$$x_j = \begin{cases} x_1 + \frac{j-2}{i} \cdot \theta, & \text{for } \ j = 3, \ldots, i+1; \\ x_1 - \frac{j-i-1}{i} \cdot (2\pi - \theta), & \text{for } \ j = i+2, \ldots, 2i; \end{cases}$$

where θ is the positive angle from x_1 to x_2, $0 < \theta < 2\pi$. Then (5.4) leads to the commutative diagram

$$\begin{tikzcd}
 & e_{2i}^{-1}(C_{2i}(S^1)) \arrow[r, "a"] \arrow[d, "e_{2i}"'] & e_2^{-1}(C_2(S^1)) \arrow[d, "e_2"] \\
C_2(S^1) \arrow[r, "c"] & C_{2i}(S^1) \arrow[r, "b"] & C_2(S^1)
\end{tikzcd}$$

where $bc = \mathrm{id}$. The conclusion of the proof is now immediate: if $s : C_{2i}(S^1) \to e_{2i}^{-1}(C_{2i}(S^1))$ were a Σ_{2i}-equivariant section of e_{2i}, then the composite asc would be a Σ_2-equivariant section of e_2, contradicting the case $i = 1$ of the corollary. $\square$

It might be reasonable to expect that the conclusion of Proposition 5.5 extends to

$$\mathfrak{genus}\,(\varepsilon_j) = 1 \text{ for any } j \geq 2 \tag{5.5}$$

(implying that $\mathrm{TC}_n^S(S^1) = 2(n-1)$ for any $n \geq 2$). A straightforward modification of the previous proof shows that, in order to establish (5.5), it would suffice to consider the case where j is prime.

6. Cellular Stratifications

There have been several attempts at weakening the conditions for CW-complexes in order to handle spaces with a more general type of decomposition into cells. Schürmann's book [Sc03] is a good example of such a situation. This section contains our proposal for such a goal—the main technical tool in preparation for the proof of Theorem 5.2. We introduce the notion of cellular stratified spaces and describe their basic properties. Our idea is to allow cells that are not necessarily "closed" so that open manifolds such as complements of hyperplane arrangements and configuration spaces can be handled, up to homotopy, as finite CW complexes.

6.1. **Definition.** Let X be a topological space, and n a non-negative integer. An *n-cell structure* on a subspace $e \subset X$ is a pair (D, φ) consisting of a subspace D of the n-disk D^n with $\mathrm{Int}(D^n) \subset D$, and a continuous map $\varphi\colon D \to X$, satisfying the following conditions:

(1) $\varphi(D) = \overline{e}$;
(2) the restriction $\varphi|_{\mathrm{Int}(D^n)} : \mathrm{Int}(D^n) \to e$ is a homeomorphism;
(3) the pair (D, φ) is maximal among pairs satisfying the above two conditions.

When the meaning is clear from the context, we refer to an n-cell structure (D, φ) on e just by e, in which case we also say that e is a cell of dimension n. The map φ is called the *characteristic map* of e, and D is called the *domain* for e.

When a subspace A of X contains a cell $e \subset X$ with structure (D, φ), we will think of e as a cell of A with domain $D_A := \varphi^{-1}(\overline{e} \cap A)$ and characteristic map $\varphi_{|D_A} \colon D_A \to \overline{e} \cap A = \overline{e}_A$.

6.2. **Definition.** Let X be a topological space. A *cellular stratification* $\mathcal{C}$ on X is a filtration $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ by subspaces of X satisfying the following conditions:

(i) $X = \bigcup_{n \geq 0} X_n$;

(ii) for $n \geq 0$, the set $X_n - X_{n-1}$ (where we put $X_{-1} = \varnothing$) decomposes as a topological disjoint union,
$$X_n - X_{n-1} = \coprod_{\lambda \in \Lambda_n} e_\lambda,$$
where each e_λ has an n-cell structure $(D_\lambda, \varphi_\lambda)$.

(iii) (closure-finiteness) for each n-cell e_λ, $\partial e_\lambda := \overline{e_\lambda} - e_\lambda$ is covered by finitely many cells of dimension less than n;

(iv) (weak topology) X has the weak topology determined by the closures $\overline{e_\lambda}$ for all $\lambda \in \Lambda_n$ and all $n \geq 0$.

We remark that when we say that $\mathcal{C}$ is a cellular stratification on X, we refer not only to the filtration $\{X_n\}$ but also to the fixed set of (domains and) characteristic maps of cells. We refer to X_n as the n^{th} *skeleton* of $\mathcal{C}$ and denote it by $\mathrm{sk}_n(\mathcal{C})$, or by $\mathrm{sk}_n(X)$ if the cellular stratification $\mathcal{C}$ is clear from the context.

A *cellular stratified space* is a pair $(X, \mathcal{C})$ where $\mathcal{C}$ is a cellular stratification of X. As usual, we abbreviate $(X, \mathcal{C})$ to X if there is no danger of confusion.

A subspace $A \subset X$ which is the union of some cells of a cellular stratification $\mathcal{C}$ on X is called a *cellular stratified subspace* of $(X, \mathcal{C})$ provided $(A, \mathcal{C}_{|A})$ becomes a cellular stratified space, where $\mathcal{C}_{|A}$ is the filtration $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$ with $A_n = A \cap X_n$ (and where domains and characteristic maps of cells in A are taken as indicated at the end of Definition 6.1).

We usually impose further conditions on cellular stratified spaces.

6.3. **Definition.** Let $X = (X, \mathcal{C})$ be a cellular stratified space.

a. X is said to be *finite* if the set of cells (of all dimensions) is finite. X is said to be of *finite type* if for every $n \geq 0$ the set of cell of dimension n is finite.

b. An n-cell e_λ of X is said to be *regular* if its characteristic map is a homeomorphism onto $\overline{e_\lambda}$. Furthermore, X is called *regular* if all its cells are regular.

c. An n-cell e_λ of X is said to be *closed* if $D_\lambda = D^n$, and X is usually called a *CW complex* if all its cells are closed.

d. We call X *normal* if, for each n-cell e_λ, ∂e_λ is a union of cells of dimension less than n.

e. A pair of cells (e_λ, e_μ) of X is said to be *strongly normal* provided $e_\mu \subset \overline{e_\lambda}$ and there exists an embedding $b_{\mu,\lambda} : D_\mu \to D_\lambda$ with $\varphi_\mu = \varphi_\lambda \circ b_{\mu,\lambda}$ (note that there is a unique such embedding $b_{\mu,\lambda}$ when e_λ is regular). We call X *strongly normal* if it is normal and all pairs of cells (e_λ, e_μ) with $e_\mu \subset \partial e_\lambda$ are strongly normal.

f. We call X *totally normal* if it is normal and the following two conditions hold for each n and each n-cell e_λ:
   - there exists a structure of regular CW complex on S^{n-1} (which depends on the cell e_λ) containing $\partial D_\lambda := D_\lambda - \mathrm{Int}(D^n)$ as a stratified subspace;
   - for any cell e in ∂D_λ there exists a cell e_μ contained in ∂e_λ having the same domain as e and such that the characteristic map $\varphi\colon D \to \overline{e}$ of e—a homeomorphism—lifts φ_μ through φ_λ, that is, yields a commutative diagram

$$\begin{tikzcd} \overline{e} \arrow[r, "\varphi_{\lambda|\overline{e}}"] & \overline{e_\mu} \\ D = D_\mu \arrow[u, "\varphi"] \arrow[ur, "\varphi_\mu"'] & \end{tikzcd}$$

g. Given a stratified subspace $A = (A, \mathcal{C}_{|A})$ of X, we say that the pair (X, A) is *relatively* regular (respectively normal, strongly normal, totally normal), if $X - A$ is a cellular stratified subspace of X which is regular (respectively normal, strongly normal, totally normal).

6.4. **Remark.** Conditions (iii) and (iv) in Definition 6.2 hold for free in the case of a finite cellular stratified space. Consequently, any subspace A of X which is the union of finitely many cells in a cellular stratification $\mathcal{C}$ of X is automatically a cellular stratified subspace with the filtration $\mathcal{C}_{|A}$.

Each of the concepts in Definitions 6.1–6.3 can be illustrated by taking a difference $X - A$, for a suitably chosen subcomplex A of a CW complex X (Examples 7.13 and 7.14 in the next section describe particularly amenable situations).

For future reference, we record a few obvious properties.

6.5. **Lemma.** *We have the following implications:*

(1) *a totally normal cellular stratified space is strongly normal;*
(2) *a regular normal cellular stratified space is strongly normal;*
(3) *a regular CW complex is totally normal. More generally,*
(4) *a normal regular cellular stratified subspace of a CW complex is totally normal.*

It is a standard technique in combinatorial algebraic topology to translate geometric properties of a regular CW complex into combinatorial properties of its face poset. The goal of Definitions 6.1–6.3 is to isolate the critical features that allow us to extend, via the following definition, the above fruitful interaction to the case of cellular stratified spaces.

6.6. **Definition.** Let $(X, \mathcal{C})$ be a regular cellular stratified space.

(1) The *face poset* of $(X, \mathcal{C})$ is the set $F(X, \mathcal{C}) = \{e \mid e \text{ is a cell in } \mathcal{C}\}$ with partial order defined by $e' \leq e$ whenever $e' \subset \overline{e}$.
(2) The *barycentric subdivision* of $(X, \mathcal{C})$, denoted by $\mathrm{Sd}(X, \mathcal{C})$, is the geometric realization of $\Delta F(X, \mathcal{C})$—the order complex of $F(X, \mathcal{C})$.

6.7. **Remark.** Given a poset P, the order complex ΔP of P is defined to be the ordered simplicial complex consisting of finite totally ordered subsets of P. When P is regarded as a small category, it is well known that the geometric realization of ΔP (denoted by $|\Delta P|$) coincides with the classifying space BP of P. Thus

$$\mathrm{Sd}(X, \mathcal{C}) = BF(X, \mathcal{C}) = |\Delta F(X, \mathcal{C})|.$$

When X or $\mathcal{C}$ is obvious from the context, we use the shorthand $F(\mathcal{C})$ or $F(X)$ instead of $F(X, \mathcal{C})$, and $\mathrm{Sd}(\mathcal{C})$ or $\mathrm{Sd}(X)$ instead of $\mathrm{Sd}(X, \mathcal{C})$.

6.8. **Remark.** The obvious generalization of Definition 6.6 does not quite describe the "right" objects when $\mathcal{C}$ is non-regular. The current statements are enough for the purposes of this paper. The non-regular case will be dealt with elsewhere.

It is well known (see for instance [BLSWZ99, Proposition 4.7.8]) that, if $\mathcal{C}$ is a regular CW complex structure on a space X, then $\mathrm{Sd}(\mathcal{C})$ is homeomorphic to X. As a CW complex, $\mathrm{Sd}(\mathcal{C})$ gives the usual barycentric subdivision of $\mathcal{C}$. However $\mathrm{Sd}(\mathcal{C})$ is usually much thinner than X

when $\mathcal{C}$ is just a regular cellular stratification on X. For instance, if X is an open disk with exactly one n-cell and characteristic map $\varphi = \mathrm{id}\colon \mathrm{Int}(D^n) \to \mathrm{Int}(D^n)$, then $\mathrm{Sd}(\mathcal{C})$ is a single point—which is homotopy equivalent but not homeomorphic to X. In fact, the next result, whose proof is the central goal of this section, asserts that, under suitable conditions, no homotopy property is lost when X is replaced by the combinatorial model $\mathrm{Sd}(\mathcal{C})$.

6.9. **Theorem.** *For a regular totally normal cellular stratification $\mathcal{C}$ on X, the barycentric subdivision $\mathrm{Sd}(\mathcal{C})$ can be embedded in X as a strong deformation retract. When $\mathcal{C}$ is a regular CW complex structure, the embedding is a homeomorphism. Furthermore, when X is equipped with a cellular action of a group G, both the embedding and the deformation retraction can be taken to be G-equivariant.*

In preparation for the proof of Theorem 6.9, we make explicit the construction of the stated embedding (which is a straightforward extension of the situation for regular CW complexes).

6.10. **Lemma.** *Let $\mathcal{C}$ be a strongly normal cellular stratification on X. Then for each non-negative integer k and each non-degenerate k-chain*

$$\mathbf{e} : e_{\lambda_0} < \cdots < e_{\lambda_k} \tag{6.1}$$

in $F(\mathcal{C})$—that is, for each k-simplex in $\Delta F(\mathcal{C})$—there exist embeddings

$$d_{\mathbf{e}} : \Delta^k \to D_{\lambda_k} \quad \textit{and} \quad i_{\mathbf{e}}\colon \Delta^k \to \overline{e_{\lambda_k}} \subset X \tag{6.2}$$

satisfying the following conditions:

1. *$i_{\mathbf{e}} = \varphi_{\lambda_k} \circ d_{\mathbf{e}}$, where $\varphi_{\lambda_k}\colon D_{\lambda_k} \to \overline{e_{\lambda_k}} \subset X$ stands for the characteristic map of e_{λ_k}.*
2. *For $0 \le j \le k$ let $\mathbf{e}_j$ denote the $(k-1)$-subchain $e_{\lambda_0} < \cdots < \widehat{e_{\lambda_j}} < \cdots < e_{\lambda_k}$ of (6.1), where the cell e_{λ_j} has been removed. Then the restriction of $i_{\mathbf{e}}$ to the face of Δ^k opposite to the vertex of $\Delta F(\mathcal{C})$ corresponding to e_{λ_j} coincides with $i_{\mathbf{e}_j}$.*

Consequently the embeddings $\{i_{\mathbf{e}}\}$ fit together assembling an embedding $i\colon \mathrm{Sd}(\mathcal{C}) \hookrightarrow X$.

Proof. We construct embeddings $i_{\mathbf{e}}$ by induction on k. The vertices of $\mathrm{Sd}(\mathcal{C})$ are in one-to-one correspondence with cells in $\mathcal{C}$. For each cell e_λ, set $v_\lambda = \varphi_\lambda(0)$. This defines an obvious embedding $i_0 : \mathrm{sk}_0\mathrm{Sd}(\mathcal{C}) \to X$, where $\mathrm{sk}_\ell\mathrm{Sd}(\mathcal{C})$ stands for the ℓ^{th} skeleton of $\mathrm{Sd}(\mathcal{C})$.

Suppose we have constructed embeddings as required for each non-degenerate j-chain with $j < k$. For a non-degenerate k-chain (6.1), we next define embeddings (6.2) satisfying the above properties. By hypothesis there is an embedding $b_{\lambda_{k-1},\lambda_k}: D_{\lambda_{k-1}} \to \partial D_{\lambda_k}$ with $\varphi_{\lambda_{k-1}} = \varphi_{\lambda_k} \circ b_{\lambda_{k-1},\lambda_k}$. By the inductive assumption, we have an embedding $i_{\mathbf{e}'}: \Delta^{k-1} \to \overline{e_{\lambda_{k-1}}} \subset X$ corresponding to the $(k-1)$-chain $\mathbf{e}': e_{\lambda_0} < \cdots < e_{\lambda_{k-1}}$, and an embedding $d_{\mathbf{e}'}: \Delta^{k-1} \to D_{\lambda_{k-1}}$ with $i_{\mathbf{e}'} = \varphi_{\lambda_{k-1}} \circ d_{\mathbf{e}'}$. Extending the composite $b_{\lambda_{k-1},\lambda_k} \circ d_{\mathbf{e}'}$ to the joins yields the first map in the composite of embeddings

$$\Delta^k = \Delta^{k-1} * e_{\lambda_k} \longrightarrow b_{\lambda_{k-1},\lambda_k}(d_{\mathbf{e}'}(\Delta^{k-1})) * 0 \longrightarrow D_{\lambda_k}.$$

This works as the first embedding in (6.2), while the second embedding is forced from (1). Condition (2) is obvious from the construction. $\square$

At the end of the section we deduce Theorem 6.9 from a key special case, where attention is focused on the case of a disk. In turn, our proof of the special case uses standard techniques in simplicial topology (Lemmas 6.12 and 6.13 below) based on the following concept.

6.11. **Definition.** Let K be a cellular stratified space. For $x \in K$, the *open star* around x in K, $\mathrm{St}(x; K)$, is the union of those cells whose closure contains x. For a subset $A \subset K$, define

$$\mathrm{St}(A; K) = \bigcup_{x \in A} \mathrm{St}(x; K).$$

When K is a simplicial complex and A is a subcomplex, $\mathrm{St}(A; K)$ is called the *regular neighborhood* of A in K.

6.12. **Lemma.** *Let K be a regular CW complex. For any stratified subspace L of K, the image of the regular neighborhood $\mathrm{St}(\mathrm{Sd}(L); \mathrm{Sd}(\overline{L}))$ of $\mathrm{Sd}(L)$ in $\mathrm{Sd}(\overline{L})$ under the embedding (actually a homeomorphism) $i: \mathrm{Sd}(K) \hookrightarrow K$ in Lemma 6.10 contains L.*

Proof. For a point $x \in L$, there exists a cell e in L with $x \in e$. Under the barycentric subdivision of $\overline{L}$, e is triangulated, namely there exists a non-degenerate n-chain $\mathbf{e}: e_0 < e_1 < \cdots < e_n = e$ of cells in $\overline{L}$ such that $x \in i_{\mathbf{e}}(\mathrm{Int}(\Delta^n))$ and $v(e) \in \overline{i_{\mathbf{e}}(\mathrm{Int}(\Delta^n))}$ where $v(e)$ is the vertex in $\mathrm{Sd}(L)$ corresponding to e. By definition of St, we have

$$i_{\mathbf{e}}(\mathrm{Int}(\Delta^n)) \subset i(\mathrm{St}(v(e); \mathrm{Sd}(\overline{L}))) \subset i(\mathrm{St}(\mathrm{Sd}(L); \mathrm{Sd}(\overline{L}))),$$

so that $L \subset i(\mathrm{St}(\mathrm{Sd}(L); \mathrm{Sd}(\overline{L})))$. $\square$

It follows from the construction of the barycentric subdivision that, under the conditions of the lemma, $\mathrm{Sd}(L)$ is a full subcomplex of $\mathrm{Sd}(\overline{L})$, that is, for any collection of vertices $v_0, \ldots, v_k$ in $\mathrm{Sd}(L)$ which forms a simplex σ in $\mathrm{Sd}(\overline{L})$, the simplex σ belongs to $\mathrm{Sd}(L)$. The important property of such a situation is, in general, that a full subcomplex A of a simplicial complex K is a simplicial strong deformation retract of its regular neighborhood $\mathrm{St}(A;K)$. Indeed, as shown in [ES52, Lemma II.9.3] (see alternatively the case $K' = \varnothing$ in the proof of Lemma 6.13 below), there is a simplicial homotopy rel A between the identity on $\mathrm{St}(A;K)$ and the composite

$$\mathrm{St}(A;K) \xrightarrow{r_A} A \overset{\iota_A}{\hookrightarrow} \mathrm{St}(A;K). \tag{6.3}$$

Here r_A is the retraction given by

$$r_A(x) = \frac{1}{\sum_{v\in A\cap\sigma} t(v)} \sum_{v\in A\cap\sigma} t(v)v$$

whenever $x = \sum_{v\in\sigma} t(v)v$ belongs to a simplex σ. Further, by a simplicial homotopy H on $\mathrm{St}(A;K)$ we mean one for which, whenever a point x lies in a cell e of $\mathrm{St}(A;K)$, the curve $H(x,s)$ stays in e for $s<1$. The above basic property can be extended as follows:

6.13. **Lemma.** *Let A be a full subcomplex of a finite simplicial complex K, and let K' be a subcomplex of K. Then any simplicial homotopy H' rel $A' := A\cap K'$ between the identity on $\mathrm{St}(A';K')$ and $\iota_{A'}\circ r_{A'}$ can be extended to a simplicial homotopy rel A between the identity on $\mathrm{St}(A;K)$ and (6.3).*

Proof. We regard K as a subcomplex of a large simplex S, and let $V(B)$ denote the set of vertices of a subcomplex B of S. Then every point $x\in|K|$ can be expressed as a formal (barycentric) sum

$$x = \sum_{v\in V(K)} a_v v$$

with $\sum_{v\in V(K)} a_v = 1$ and $a_v \geq 0$.

Let

$$H'\colon \mathrm{St}(A';K')\times[0,1]\to \mathrm{St}(A';K')$$

be a simplicial homotopy rel A' between the identity on $\mathrm{St}(A';K')$ and $\iota_{A'}\circ r_{A'}$. Consider the homotopy

$$H\colon \mathrm{St}(A;K)\times[0,1]\to \mathrm{St}(A;K) \tag{6.4}$$

defined by

$$\begin{aligned} H(x,s) \;=\; & \frac{\alpha+(1-s)\beta}{(1-s)+s\,(\alpha+\gamma)}\, H'\!\left(\sum_i \frac{a_i}{\alpha+\beta}u_i' + \sum_j \frac{b_j}{\alpha+\beta}v_j', s\right) \\ & + \sum_k \frac{c_k}{(1-s)+s\,(\alpha+\gamma)}u_k + \sum_\ell \frac{(1-s)d_\ell}{(1-s)+s(\alpha+\gamma)}v_\ell, \end{aligned}$$

where $s\in[0,1]$, $\alpha=\sum_i a_i$, $\beta=\sum_j b_j$, $\gamma=\sum_k c_k$, and $x\in\mathrm{St}(A;K)$ has the form $x=\sum_i a_iu_i'+\sum_j b_jv_j'+\sum_k c_ku_k+\sum_\ell d_\ell v_\ell$ with

- $u'_i \in V(A')$,
- $v'_j \in V(K') - V(A')$,
- $u_k \in V(A) - V(A')$,
- $v_\ell \in V(K) - (V(K') \cup V(A))$.

Then

$$\begin{aligned}
H(x,0) &= (\alpha+\beta)\left(\sum_i \frac{a_i}{\alpha+\beta}u'_i + \sum_j \frac{b_j}{\alpha+\beta}v'_j\right) + \sum_k c_k u_k + \sum_\ell d_\ell v_\ell \\
&= \sum_i a_i u'_i + \sum_j b_j v'_j + \sum_k c_k u_k + \sum_\ell d_\ell v_\ell \\
&= x, \\
H(x,1) &= \frac{\alpha}{\alpha+\gamma}\, r_{A'}\left(\sum_i \frac{a_i}{\alpha+\beta}u'_i + \sum_j \frac{b_j}{\alpha+\beta}v'_j\right) + \sum_k \frac{c_k}{\alpha+\gamma}u_k \\
&= \frac{\alpha}{\alpha+\gamma}\sum_i \left(\frac{\frac{a_i}{\alpha+\beta}}{\frac{\alpha}{\alpha+\beta}}\right) u'_i + \sum_k \frac{c_k}{\alpha+\gamma}u_k \\
&= \sum_i \frac{a_i}{\alpha+\gamma}u'_i + \sum_k \frac{c_k}{\alpha+\gamma}u_k \\
&= r_A(x).
\end{aligned}$$

Further, when $x \in K'$, we have $c_k = d_\ell = 0$ and $x = \sum_i a_i u'_i + \sum_j b_j v'_j$. Since $\alpha+\beta = 1$, we then have

$$\begin{aligned}
H(x,s) &= \frac{\alpha+(1-s)\beta}{(1-s)+s\alpha}\, H'\left(\sum_i a_i u'_i + \sum_j b_j v'_j, s\right) \\
&= \frac{1-s\beta}{1-s(1-\alpha)}\, H'(x,s) \quad = \quad H'(x,s).
\end{aligned}$$

Lastly, when $x \in A$, we have $b_j = d_\ell = 0$ and $x = \sum_i a_i u'_i + \sum_k c_k u_k$. Since $\alpha+\gamma = 1$, we then have

$$\begin{aligned}
H(x,s) &= \alpha\, H'\left(\sum_i \frac{a_i}{\alpha}u'_i, s\right) + \sum_k c_k u_k \\
&= \sum_i a_i u'_i + \sum_k c_k u_k \quad = \quad x.
\end{aligned}$$

Thus H is the required homotopy. $\square$

6.14. **Remark.** We need to make a slight adaptation of Lemma 6.13 for its proper use in the proof of Theorem 6.9. Namely, under the previous conditions, assume that M is a cellular stratified subspace of $\mathrm{St}(A; K)$ containing A, and that M' is a cellular stratified subspace of

$\mathrm{St}(A';K')\cap M$ containing A'. Then any simplicial homotopy rel A' between the identity on M' and the composite

$$M' \hookrightarrow \mathrm{St}(A';K') \xrightarrow{r_{A'}} A' \hookrightarrow M'$$

can be extended to a simplicial homotopy rel A between the identity on M and the composite

$$M \hookrightarrow \mathrm{St}(A;K) \xrightarrow{r_A} A \hookrightarrow M.$$

The point here is that, since the homotopy (6.4) is defined on a cell-by-cell basis, the construction in the proof of Lemma 6.13 applies to prove the above variation of Lemma 6.13.

We are now ready to deduce a key special case of Theorem 6.9. Suppose $\mathcal{C}$ is a regular CW complex structure on S^{n-1}, and $L\subset S^{n-1}$ is a stratified subspace. Set $K=L\cup\mathrm{Int}(D^n)$ with the obvious stratification obtained from that in L by adding $\mathrm{Int}(D^n)$ as an n-cell (this is a stratified subspace of the regular CW complex decomposition on D^n coming from $\mathcal{C}$ by adding $\mathrm{Int}(D^n)$ as an n-cell).

6.15. **Corollary.** *Under the above considerations, any simplicial homotopy relative to $i(\mathrm{Sd}(L))$ between the identity on L and the composite*

$$L \hookrightarrow i(\mathrm{St}(\mathrm{Sd}(L);\mathrm{Sd}(\overline{L}))) \xrightarrow{r_{i(\mathrm{Sd}(L))}} i(\mathrm{Sd}(L)) \hookrightarrow L$$

can be extended to a simplicial homotopy relative to $i(\mathrm{Sd}(K))$ between the identity on K and the composite

$$K \hookrightarrow i(\mathrm{St}(\mathrm{Sd}(K);\mathrm{Sd}(\overline{K}))) \xrightarrow{r_{i(\mathrm{Sd}(K))}} i(\mathrm{Sd}(K)) \hookrightarrow K.$$

Proof. We have observed that $\mathrm{Sd}(K)$ is a full subcomplex of $\mathrm{Sd}(\overline{K})$. Moreover, in view of Lemma 6.12, K can be considered as a cellular stratified subspace of the regular neighborhood of $\mathrm{Sd}(K)$ in $\mathrm{Sd}(\overline{K})$. Likewise, L can be considered as a cellular stratified subspace of the regular neighborhood $\mathrm{St}(\mathrm{Sd}(L);\mathrm{Sd}(\overline{L}))\cap K$. Therefore Lemma 6.13 (as adapted in Remark 6.14) can be applied and the result follows. $\square$

Proof of Theorem 6.9. For each k-cell e_λ of X, let $\mathcal{C}_{\lambda,k}$ be a fixed regular CW complex structure on S^{k-1} as in the definition of total normality. We construct, by induction on k, strong deformation retractions

$$H_k\colon \mathrm{sk}_k X\times[0,1]\to \mathrm{sk}_k X \tag{6.5}$$

of $\mathrm{sk}_k X$ onto $i(\mathrm{Sd}(\mathrm{sk}_k\mathcal{C}))$ with the following property: For each k-cell e_λ, the restriction $H_k|_{\overline{e_\lambda}\times[0,1]}$ lands in $\overline{e_\lambda}$ giving, in terms of the corresponding characteristic map $\varphi_\lambda\colon D_\lambda \to \overline{e_\lambda}$ (which is a homeomorphism), a simplicial homotopy rel $i(\mathrm{Sd}(D_\lambda))$ between the identity on D_λ and the composite

$$D_\lambda \hookrightarrow i(\mathrm{St}(\mathrm{Sd}(D_\lambda); \mathrm{Sd}(\overline{D_\lambda}))) \xrightarrow{r_{i(\mathrm{Sd}(D_\lambda))}} i(\mathrm{Sd}(D_\lambda)) \hookrightarrow D_\lambda \tag{6.6}$$

—here $\mathrm{Sd}(D_\lambda)$ is taken with respect to the obvious stratified structure coming from $\mathcal{C}_{\lambda,k|\partial D_\lambda}$ by adding the k-cell $\mathrm{Int}(D^k)$.

When $k = 0$, there is nothing to prove, since $\mathrm{Sd}(\mathrm{sk}_0(X)) = \mathrm{sk}_0 X$. Assuming we have constructed the required H_{k-1}, we next extend it to all k-cells. Using Corollary 6.15, we obtain, for each k-cell e_λ, a simplicial homotopy $H_\lambda : D_\lambda \times [0,1] \to D_\lambda$ satisfying the two conditions:

(a) H_λ is a homotopy rel $i(\mathrm{Sd}(D_\lambda))$ between the identity on D_λ and (6.6).

(b) H_λ extends the homotopy $H_{k-1,\lambda}$ given by the composition

$$\partial D_\lambda \times [0,1] \xrightarrow{(\varphi_\lambda|_{\partial D_\lambda})\times[0,1]} \partial e_\lambda \times [0,1] \xrightarrow{H_{k-1}|_{\partial e_\lambda \times [0,1]}} \partial e_\lambda \xrightarrow{(\varphi_\lambda|_{\partial D_\lambda})^{-1}} \partial D_\lambda$$

(the fact that the middle map lands in ∂e_λ follows from the inductive construction).

By the regularity hypothesis, H_{k-1} and the various H_λ fit together to produce the new required homotopy (6.5), completing the inductive step in the construction of the strong deformation retraction of X onto $\mathrm{Sd}(X)$.

Since our construction is done on a cell-by-cell basis, if X is equipped with a cellular action of a group G, we obtain a G-equivariant homotopy. □

7. The Σ_k-Equivariant Homotopy Model of $C_k(S^n)$

This final section is devoted to the proof of Theorem 5.2: by using the method in [BZ92, DS00], we construct a Σ_k-equivariant cellular homotopy model of $C_k(S^n)$ based on the stratification of euclidean spaces induced by the braid arrangement.

The following is an immediate consequence of Theorem 6.9.

7.1. **Corollary.** *Let (X, A) be a relatively regular and totally normal pair of cellular stratified spaces. Then $B(F(X) - F(A))$ can be embedded in $X - A$ as a strong deformation retract. If in addition a group G acts cellularly on the stratified pair (X, A), then the strong deformation retraction can be taken to be G-equivariant.*

Our main interest lies on configuration spaces, for which the following special case of Corollary 7.1 is fundamental.

7.2. **Corollary.** *Let X be a topological space and let $\mathcal{C}$ be a cellular stratification on X^k under which the fat diagonal*

$$\Delta_k(X) = \{(x_1, \dots, x_k) \in X^k \mid x_i = x_j \text{ for some pair } i, j \text{ with } i \neq j\}$$

is a stratified subspace. Assume further that the pair $(X^k, \Delta_k(X))$ is relatively regular and totally normal. Let $C_\Delta(\mathcal{C})$ be the induced cellular stratification on the ordered configuration space $C_k(X) = X^k - \Delta_k(X)$. Then $\mathrm{Sd}(C_\Delta(\mathcal{C}))$ is contained in $C_k(X)$ as a strong deformation retract. Furthermore, if $C_\Delta(\mathcal{C})$ is compatible with the Σ_k-action, then $\mathrm{Sd}(C_\Delta(\mathcal{C}))$ is a strong Σ_k-equivariant deformation retract of $C_k(X)$.

The cellular structure $\mathcal{C}$ we use when X is a sphere is motivated by the braid arrangement. Recall that a *hyperplane arrangement* is a finite set of hyperplanes in a finite dimensional real affine space. An especially important arrangement is the *rank $k-1$ braid arrangement* $\mathcal{A}_{k-1}$ formed by the set of all hyperplanes $x_i - x_j = 0$, $1 \leq i < j \leq k$, in $\mathbb{R}^k$.

We start with the stratification of $\mathbb{R}^k \otimes \mathbb{R}^n$ by a general hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^k$. The construction, introduced by Björner and Ziegler in [BZ92], requires the concept of higher dimensional sign vectors.

7.3. **Definition.** Consider the set $S_n = \{0, \pm e_1, \cdots, \pm e_n\}$. The n-dimensional *sign vector* is the function $\mathrm{sign}_n : \mathbb{R}^n \to S_n$ given on a tuple $\mathbf{x} = (x_1, \dots, x_n)$ by

$$\mathrm{sign}_n(\mathbf{x}) = \begin{cases} \mathrm{sign}(x_n)e_n & \text{if } \mathrm{sign}(x_n) \neq 0, \\ \mathrm{sign}(x_{n-1})e_{n-1} & \text{if } \mathrm{sign}(x_{n-1}) \neq 0 = \mathrm{sign}(x_n), \\ \quad \dots & \\ \mathrm{sign}(x_1)e_1 & \text{if } \mathrm{sign}(x_1) \neq 0 = \mathrm{sign}(x_i),\ 2 \leq i \leq n, \\ 0 & \text{if } \mathbf{x} = 0. \end{cases}$$

7.4. **Definition.** Let $\mathcal{A} = \{H_1, \dots, H_q\}$ be a hyperplane arrangement in a real vector space V given by affine 1-forms $\ell_1, \cdots, \ell_q \colon V \to \mathbb{R}$. These q forms yield an affine map $L : V \to \mathbb{R}^q$ with coordinates $\ell_i, i = 1, \dots, q$. We tensor L by $\mathbb{R}^n$ and get the map

$$L \otimes \mathbb{R}^n : V \otimes \mathbb{R}^n \to \mathbb{R}^q \otimes \mathbb{R}^n = (\mathbb{R}^n)^q.$$

Consider the composite

$$\mathrm{sign}_{\mathcal{A} \otimes \mathbb{R}^n} := (\mathrm{sign}_n)^q \circ (L \otimes \mathbb{R}^n) : V \otimes \mathbb{R}^n \longrightarrow S_n^q.$$

The stratification of $V \otimes \mathbb{R}^n$ given by this map,

$$V \otimes \mathbb{R}^n \;=\; \coprod_{\mathbf{s}\in \mathrm{Im}\left(\mathrm{sign}_{\mathcal{A}\otimes\mathbb{R}^n}\right)} (\mathrm{sign}_{\mathcal{A}\otimes\mathbb{R}^n})^{-1}(\mathbf{s}),$$

is called the n^{th} Björner-Ziegler stratification associated with $\mathcal{A}$ and is denoted by $\mathcal{C}^n_{\mathcal{A}}$.

This stratification is obtained by cutting $V \otimes \mathbb{R}^n$ into convex regions by hyperplanes. Hyperplanes are also cut by other hyperplanes, intersections of two hyperplanes are cut by other hyperplanes, and so on. Each cell in the resulting stratification is an open convex polyhedron bounded by lower dimensional open convex polyhedra. (Here the term "open convex polyhedron" means a subspace of a euclidean space defined by linear equations and linear strict inequalities. See Ziegler's book [Zi95] for more details.) This leads to the cellular stratification (in the sense of Definition 6.2) whose n^{th} skeleton consists of the union of those cells (the open convex polyhedra just described) having dimension at most n. It will be safe to abuse notation and write $\mathcal{C}^n_{\mathcal{A}}$ to refer either to the n^{th} Björner-Ziegler stratification of $\mathcal{A}$, or to its associated cellular stratification.

Thus we obtain the following fundamental fact:

7.5. **Lemma.** *$\mathcal{C}^n_{\mathcal{A}}$ is a finite, regular, and totally normal cellular stratification of $V \otimes \mathbb{R}^n$.*

$\mathcal{C}^n_{\mathcal{A}}$ is designed to include $\bigcup_i H_i \otimes \mathbb{R}^n$ as a stratified subspace. Thus it also includes the complement

$$M(\mathcal{A}\otimes\mathbb{R}^n) := V \otimes \mathbb{R}^n - \bigcup_i H_i \otimes \mathbb{R}^n$$

as a stratified subspace.

7.6. **Definition.** The induced stratification on $M(\mathcal{A}\otimes\mathbb{R}^n)$ is denoted by $\mathcal{C}^{n,\text{comp}}_{\mathcal{A}}$. The classifying space (order complex) of the face poset of $\mathcal{C}^{n,\text{comp}}_{\mathcal{A}}$, i.e. $\mathrm{Sd}(\mathcal{C}^{n,\text{comp}}_{\mathcal{A}})$, is called the n^{th} *order Salvetti complex of $\mathcal{A}$* and is denoted by $\mathrm{Sal}^{(n)}(\mathcal{A})$.

7.7. **Remark.** When $n = 2$, we obtain the classical Salvetti complex constructed by Salvetti in [Sa87]. Note, however, the term Salvetti complex is used for a CW complex $\mathrm{Sal}(\mathcal{A})$ whose face poset $F(\mathrm{Sal}(\mathcal{A}))$ is isomorphic to that of $\mathcal{C}^{2,\text{comp}}_{\mathcal{A}}$. In other words, our $\mathrm{Sal}^{(2)}(\mathcal{A})$ is the barycentric subdivision of the standard Salvetti complex.

As a corollary to Theorem 6.9, we obtain the following result, which first appeared in the paper [BZ92] by Björner and Ziegler.

7.8. **Corollary.** *Let $\mathcal{A} = \{H_1, \dots, H_q\}$ be a real hyperplane arrangement in a real vector space V. Then $\mathrm{Sal}^{(n)}(\mathcal{A})$ can be embedded into the complement $M(\mathcal{A} \otimes \mathbb{R}^n)$ as a strong deformation retract.*

7.9. **Remark.** The moral is, then, that the general form of Theorem 6.9 provides us with a unified framework for working with complements of hyperplane arrangements as well as with configuration spaces (yet, as suggested in the remark at the end of the introduction, possibilities seem much wider). In fact, our proof of Theorem 5.2 at the end of the section takes advantage, through Corollary 7.11 below, of the natural connection (recalled in Example 7.12 below) between hyperplane arrangements and configuration spaces of euclidean spaces.

A detailed analysis of the stratification $\mathcal{C}^n_{\mathcal{A}}$, including a proof of Corollary 7.8 can be found in the paper [DS00] by De Concini and Salvetti. In particular, their Theorem 1.4.7.(v) determines the dimension of $\mathrm{Sal}^{(n)}(\mathcal{A})$ as follows:

7.10. **Proposition.** *Let $\mathcal{A}$ be a real central essential arrangement[3] in a real vector space of dimension d. Then we have*

$$\dim(\mathrm{Sal}^{(n)}(\mathcal{A})) = d(n-1).$$

Although $\mathcal{A}_{k-1}$ is not essential, we can apply Proposition 7.10 to the essential arrangement $\mathcal{A}'_{k-1}$ given as the restriction of $\mathcal{A}_{k-1}$ to the hyperplane V_k determined by the 1-form $x_1 + \cdots + x_k = 0$. Note that the restriction process does not loose any combinatorial information since the linear inclusion $V_k \hookrightarrow \mathbb{R}^k$ induces an inclusion of cellular stratified spaces

$$(V_k \otimes \mathbb{R}^n, \mathcal{C}^n_{\mathcal{A}'_{k-1}}) \hookrightarrow (\mathbb{R}^k \otimes \mathbb{R}^n, \mathcal{C}^n_{\mathcal{A}_{k-1}})$$

for which the restricted map

$$\left(M(\mathcal{A}'_{k-1} \otimes \mathbb{R}^n), \mathcal{C}^{n,\mathrm{comp}}_{\mathcal{A}'_{k-1}}\right) \hookrightarrow \left(M(\mathcal{A}_{k-1} \otimes \mathbb{R}^n), \mathcal{C}^{n,\mathrm{comp}}_{\mathcal{A}_{k-1}}\right)$$

renders an isomorphism of face posets. Consequently $\mathrm{Sal}^{(n)}(\mathcal{A}_{k-1})$ is simplicially isomorphic to $\mathrm{Sal}^{(n)}(\mathcal{A}'_{k-1})$, and we get:

7.11. **Corollary.** $\dim(\mathrm{Sal}^{(n)}(\mathcal{A}_{k-1})) = (k-1)(n-1)$.

We now have all the ingredients for a proof of Theorem 5.2, but before we start assembling all the pieces, we illustrate the basic building block by recalling, in the following example, the well-known Salvetti complex approach to configuration spaces of points in a euclidean space.

[3] A hyperplane arrangement is called central if the hyperplanes are linear subspaces. A central hyperplane arrangement is called essential if the normal vectors to the hyperplanes span the ambient vector space.

7.12. **Example.** A straightforward check gives that the configuration space $C_k(\mathbb{R}^n)$ agrees with the complement $M(\mathcal{A}_{k-1} \otimes \mathbb{R}^n)$. Corollary 7.8 then claims that the n^{th} Salvetti complex $\mathrm{Sal}^{(n)}(\mathcal{A}_{k-1})$ sits inside $C_k(\mathbb{R}^n)$ as a strong deformation retract. Furthermore, the Σ_k-action on $\mathbb{R}^k \otimes \mathbb{R}^n$ is cellular (with respect to $\mathcal{C}^n_{\mathcal{A}_{k-1}}$) and closed on the fat diagonal (the latter being identified with

$$\bigcup_{1 \leq i < j \leq k} H_{i,j} \otimes \mathbb{R}^n,$$

where $H_{i,j}$ stands for the hyperplane $x_i - x_j = 0$ in $\mathbb{R}^k$). Therefore the final assertion in Corollary 7.2 applies giving a corresponding simplicial complex contained in the unordered configuration space $B_k(\mathbb{R}^n)$ as a strong deformation retract. The important observation here is that Corollary 7.11 implies that both models above are dimensionally optimal since, as explained in Remark 5.3 (for spheres rather than euclidean spaces), well-known cohomological calculations of $C_k(\mathbb{R}^n)$ yield in fact

$$\mathrm{hdim}(C_k(\mathbb{R}^n)) = \mathrm{hdim}(B_k(\mathbb{R}^n)) = (k-1)(n-1). \tag{7.1}$$

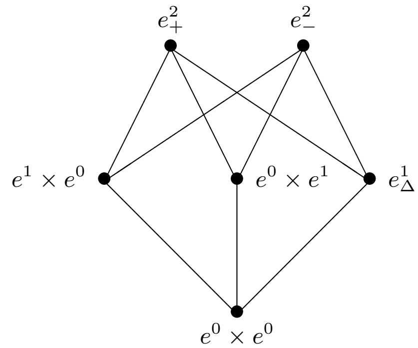
The remainder of the paper can be thought of as adapting the considerations in the previous example to the case of configurations spaces on spheres. For starters, the next two examples work in full detail the situation for configuration spaces of two different points on the circle and the 2-sphere.

7.13. **Example.** Let $S^1 = e^0 \cup e^1$ be the minimal CW complex decomposition, and consider the corresponding product decomposition in $S^1 \times S^1$. Since this does not contain the diagonal $\Delta_2(S^1)$ as a stratified subspace, we subdivide $e^1 \times e^1$ along the diagonal. The resulting cellular stratification $\mathcal{B}_{1,2}$ on $S^1 \times S^1$ is

$$S^1 \times S^1 \;=\; e^0 \times e^0 \;\cup\; e^0 \times e^1 \;\cup\; e^1 \times e^0 \;\cup\; e^1_{\Delta} \;\cup\; e^2_{+} \;\cup\; e^2_{-},$$

with Hasse diagram[4]

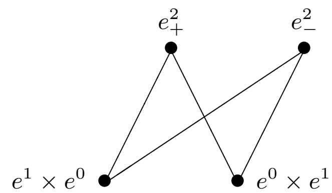
[4]Recall that the Hasse diagram of a poset is the graph whose vertices are elements of the poset, where two vertices x, y are connected by an edge if $x < y$ and there is no element z with $x < z < y$. Elements are ordered from bottom to top, starting with minimal ones.



Although $\mathcal{B}_{1,2}$ is not regular (but it is strongly normal), the pair $(S^1 \times S^1, \Delta_2(S^1))$ is relatively regular and totally normal. Further,

$$F(C_\Delta(\mathcal{B}_{1,2})) = F(S^1 \times S^1) - F(\Delta_2(S^1))$$

is the subposet of $F(S^1 \times S^1)$ obtained by removing the cells $e^0 \times e^0$ and e^1_Δ. The corresponding Hasse diagram is obtained by removing the corresponding vertices together with those edges having these vertices as one of their ends:



This is the Hasse diagram of the minimal Σ_2-equivariant CW complex decomposition of S^1, i.e. $S^1 = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_-$, so that

$$\mathrm{Sd}(C_\Delta(\mathcal{B}_{1,2})) = BF(C_\Delta(\mathcal{B}_{1,2})) \cong_{\Sigma_2} S^1.$$

In view of Corollary 7.2, this gives a direct combinatorial explanation of the case $n = 2$ in (5.3).

The situation for the 2-sphere is quite more involved:

7.14. **Example.** Consider the product decomposition

$$S^2 \times S^2 \ = \ e^0 \times e^0 \ \cup \ e^0 \times e^2 \ \cup \ e^2 \times e^0 \ \cup \ e^2 \times e^2$$

coming from the minimal CW complex decomposition $S^2 = e^0 \cup e^2$. This time the diagonal in $e^2 \times e^2$ does not divide $e^2 \times e^2$ into pieces; instead the required subdivision arises from a direct comparison with the situation of the complexification of a real hyperplane arrangement. In terms of the identification $e^2 \times e^2 \cong \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{C} \times \mathbb{C}$, the diagonal corresponds to the complexification of the braid arrangement $\mathcal{A}_1$. The associated Björner-Ziegler stratification on $\mathbb{C}^2$ is given by

$$\begin{aligned}
\mathbb{C}^2 \;=\; & \left\{ (z_1,z_2)\in\mathbb{C}^2 \;\mid\; z_1=z_2 \right\}\\
& \cup \left\{ (z_1,z_2)\in\mathbb{C}^2 \;\mid\; \operatorname{Im}(z_1-z_2)=0,\ \operatorname{Re}(z_1-z_2)>0 \right\}\\
& \cup \left\{ (z_1,z_2)\in\mathbb{C}^2 \;\mid\; \operatorname{Im}(z_1-z_2)=0,\ \operatorname{Re}(z_1-z_2)<0 \right\}\\
& \cup \left\{ (z_1,z_2)\in\mathbb{C}^2 \;\mid\; \operatorname{Im}(z_1-z_2)>0 \right\}\\
& \cup \left\{ (z_1,z_2)\in\mathbb{C}^2 \;\mid\; \operatorname{Im}(z_1-z_2)<0 \right\}
\end{aligned}$$

which yields a subdivision

$$e^2\times e^2 = e^2_\Delta \;\cup\; e^3_+ \;\cup\; e^3_- \;\cup\; e^4_+ \;\cup\; e^4_-$$

and a corresponding cellular stratification $\mathcal{B}_{2,2}$ of $S^2\times S^2$. Let us check that the pair $(S^2\times S^2,\Delta_2(S^2))$ is relatively regular and normal[5] with respect to $C_\Delta(\mathcal{B}_{2,2})$. The explicit form of the stratification is

$$C_2(S^2)=S^2\times S^2-\Delta_2(S^2)=e^0\times e^2 \;\cup\; e^2\times e^0 \;\cup\; e^3_+ \;\cup\; e^3_- \;\cup\; e^4_+ \;\cup\; e^4_-,$$

and we have

$$\begin{aligned}
\partial e^3_+ &= \partial e^3_- = e^0\times e^2\cup e^2\times e^0\\
\partial e^4_+ &= \partial e^4_- = e^0\times e^2\cup e^2\times e^0\cup e^3_+\cup e^3_-
\end{aligned}$$

so all cells are normal. The 2-cells $e^0\times e^2$ and $e^2\times e^0$ are regular, since they are at the bottom (their boundaries are empty). A characteristic map for e^3_+ can be constructed as follows. Let

$$\varphi_{2,2} : I^2\times I^2 \longrightarrow S^2\times S^2$$

be the characteristic map of the 4-cell in $S^2\times S^2$. Let

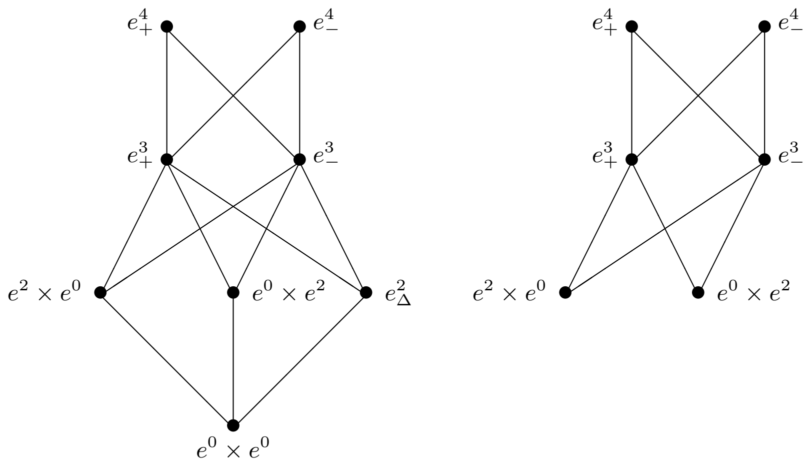
$$D_{3,+}=\varphi_{2,2}^{-1}(e^0\times e^2\cup e^2\times e^0\cup e^3_+)$$

and $\varphi_{3,+} : D_{3,+}\longrightarrow C_2(S^2)$ be the restriction of $\varphi_{2,2}$. We have

$$\begin{aligned}
\partial D_{3,+} \;=\; & \{(1,x,y,x)\in I^4 \mid 1>y>0,\ 1>x>0\}\\
& \cup\{(x,y,0,y)\in I^4 \mid 1>x>0,\ 1>y>0\}.
\end{aligned}$$

These two components are mapped homeomorphically onto $e^0\times e^2$ and $e^2\times e^0$, respectively. Thus e^3_+ is regular. Analogous considerations prove that other cells are regular too.
Now, the Hasse diagrams of $F(\mathcal{B}_{2,2})$ and $F(C_\Delta(\mathcal{B}_{2,2}))$ are respectively given by

[5] Total normality is then a consequence of Lemma 6.5(4).



Note that $F(C_\Delta(\mathcal{B}_{2,2}))$ is isomorphic to the face poset of the minimal Σ_2-equivariant regular CW complex decomposition of S^2, so that

$$\mathrm{Sd}(C_\Delta(\mathcal{B}_{2,2})) = BF(C_\Delta(\mathcal{B}_{2,2})) \cong_{\Sigma_2} S^2.$$

Once again, in view of Corollary 7.2, this gives our combinatorial explanation for the case $k = 2$ in (5.2).

The above examples suggest to use the braid arrangement to subdivide the product CW complex decomposition of $(S^n)^k$ into a cellular stratification $\mathcal{B}_{n,k}$. We are now ready to describe the cellular structure $\mathcal{B}_{n,k}$.

7.15. **Definition.** Consider the minimal CW complex decomposition $S^n = e^0 \cup e^n$. Each cell e in the corresponding product decomposition of the k-fold cartesian product $S^n \times \cdots \times S^n$ is naturally homeomorphic to the m-fold cartesian product $e^n \times \cdots \times e^n$ for some $m = m(e)$, $0 \leq m \leq k$. In terms of the identifications

$$e \cong \underbrace{e^n \times \cdots \times e^n}_{m} \cong \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m} = \mathbb{R}^m \otimes \mathbb{R}^n,$$

the braid arrangement $\mathcal{A}_{m-1}$ in $\mathbb{R}^m$ induces a cellular stratification on e. The resulting cellular stratification in the k-fold cartesian product $S^n \times \cdots \times S^n$ is denoted by $\mathcal{B}_{n,k}$.

Note that $\mathcal{B}_{n,k}$ is compatible with the Σ_k-action. Furthermore:

7.16. **Lemma.** *Under the stratification $\mathcal{B}_{n,k}$, the fat diagonal $\Delta_k(S^n)$ is a stratified subspace of $(S^n)^k$, and $((S^n)^k, \Delta_k(S^n))$ is a relatively regular and totally normal cellular stratified space.*

Proof. It remains to check the assertion about regularity and total normality. In fact, as noted in Example 7.14, Lemma 6.5(4) reduces our work to showing that $C_{\Delta}(\mathcal{B}_{n,k})$ is normal and regular. We proceed as in Example 7.14.

A cell in $(S^n)^k$ is a product of e^0's and e^n's. If it contains more than one e^0's, the cell belongs to $\Delta_k(S^n)$ and does not contribute anything to $C_{\Delta}(\mathcal{B}_{n,k})$. Thus, instead of $(S^n)^k$, we start with the following stratified subspace of $(S^n)^k$:

$$X_{n,k} = (e^n)^k \cup \bigcup_{\ell} \left((e^n)^{\ell-1} \times e^0 \times (e^n)^{k-\ell}\right).$$

Our stratified space $(C_k(S^n), C_{\Delta}(\mathcal{B}_{n,k}))$ is obtained from $X_{n,k}$ by subdividing each cell via $\mathcal{A}_{k-2}$ or $\mathcal{A}_{k-1}$, and then removing cells in the fat diagonal. Let us denote the Björner-Ziegler cellular stratifications on cells in $X_{n,k}$ by

$$\begin{aligned}(e^n)^{\ell-1} \times e^0 \times (e^n)^{k-\ell} &= \bigcup_{\mu \in M_{k-1,\ell}} e_{k-1,\ell,\mu} \\ (e^n)^k &= \bigcup_{\lambda \in \Lambda_k} e_{k,\lambda}.\end{aligned}$$

Cells of the form $e_{k-1,\ell,\mu}$ are normal, since the boundary of $(e^n)^{\ell-1} \times e^0 \times (e^n)^{k-\ell}$ is empty and the Björner-Ziegler stratification is normal. For a cell of the form $e_{k,\lambda}$, the boundary $\partial e_{k,\lambda}$ is a union of subspaces of cells $(e^n)^{\ell-1} \times e^0 \times (e^n)^{k-\ell}$ and subspaces of the cell $(e^n)^k$ corresponding to cells in $(\mathbb{R}^k \otimes \mathbb{R}^n, \mathcal{C}^n_{\mathcal{A}_{k-1}})$. Since the defining inequalities of $e_{k,\lambda}$ descend to those of cells of the form $e_{k-1,\ell,\mu}$ when we replace the ℓ-th coordinate by the point in e^0, the first components are unions of cells of the form $e_{k-1,\ell,\mu}$. Thus cells of the form $e_{k,\lambda}$ are also normal.

The regularity of cells of the form $e_{k-1,\ell,\mu}$ follows from the regularity of the Björner-Ziegler stratification. In order to prove the regularity of cells of the form $e_{k,\lambda}$, we need to specify characteristic maps. As we have done in Example 7.14, characteristic maps are defined by the restriction of the characteristic map of the nk-cell in $(S^n)^k$. More precisely, let

$$\varphi_{n,k} : I^{nk} = (I^n)^k \longrightarrow (S^n)^k$$

be the characteristic map of $(e^n)^k$. Let $\overline{e_{k,\lambda}}$ be the closure of $e_{k,\lambda}$ in $C_k(S^n)$ (not in $X_{n,k}$) and define

$$D_{k,\lambda} = \varphi_{n,k}^{-1}(\overline{e_{k,\lambda}}).$$

Then the restriction of $\varphi_{n,k}$ gives us a regular characteristic map

$$\varphi_{k,\lambda} : D_{k,\lambda} \longrightarrow C_k(S^n).$$

□

Corollary 7.2 implies that $\mathrm{Sd}(C_\Delta(\mathcal{B}_{n,k}))$ can be embedded in $C_k(S^n)$ as a strong Σ_k-equivariant deformation retract. The dimension of $\mathrm{Sd}(C_\Delta(\mathcal{B}_{n,k}))$ can be computed by comparing with the face poset of the Salvetti complex for the braid arrangement. The following explicit description implies Theorem 5.2.

7.17. **Theorem.** $\mathrm{Sd}(C_\Delta(\mathcal{B}_{n,k}))$ *is a simplicial complex of dimension* $d(n,k)$ *embedded in* $C_k(S^n)$ *as a strong* Σ_k*-equivariant deformation retract.*

Proof. We complete the only remaining task (namely, counting the dimension of $\mathrm{Sd}(C_\Delta(\mathcal{B}_{n,k}))$) by using the strategy in Lemma 7.16: we focus on the two kinds of cells

$$\underbrace{e^n \times \cdots \times e^n \times e^0 \times e^n \times \cdots \times e^n}_{k} \quad \text{and} \quad \underbrace{e^n \times \cdots \times e^n}_{k}.$$

The braid arrangement $\mathcal{A}_{k-2}$ gives rise to a cellular stratification of the cells of the first type. The top cells in the resulting stratification (which are also cells of $C_\Delta(\mathcal{B}_{n,k})$) are evidently in dimension $n(k-1)$. On the other hand, Corollary 7.11 implies that the minimal dimension of cells in $C_\Delta(\mathcal{B}_{n,k})$ coming from the various $\mathcal{A}_{k-2}$ stratifications is

$$n(k-1) - \dim(\mathrm{Sal}^{(n)}(\mathcal{A}_{k-2})) = n(k-1) - (n-1)(k-2) = n+k-2.$$

Likewise, the cells of $C_\Delta(\mathcal{B}_{n,k})$ coming from the $\mathcal{A}_{k-1}$ stratification on $(e^n)^k$ are in dimensions in between nk and

$$nk - \dim(\mathrm{Sal}^{(n)}(\mathcal{A}_{k-1})) = nk - (n-1)(k-1) = n+k-1.$$

Thus the rank of $F(C_\Delta(\mathcal{B}_{n,k}))$, i.e. the dimension of $\mathrm{Sd}(C_\Delta(\mathcal{B}_{n,k}))$, is given by

$$nk - n - k + 2 = (n-1)(k-1) + 1 = d(n,k).$$

□

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