THE 1-TYPE OF A WALDHAUSEN K-THEORY SPECTRUM

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Abstract. We give a small functorial algebraic model for the 2-stage Postnikov section of the K-theory spectrum of a Waldhausen category and use our presentation to describe the multiplicative structure with respect to biexact functors.

Introduction

Waldhausen's K-theory of a category $\mathbf{C}$ with cofibrations and weak equivalences [Wal85] extends the classical notions of K-theory, such as the K-theory of rings, additive categories and exact categories.

In this paper we give an algebraic model $\mathcal{D}_*\mathbf{C}$ for the 1-type $P_1K\mathbf{C}$ of the K-theory spectrum $K\mathbf{C}$. This model consists of a diagram of groups

$$\begin{tikzcd}
& (\mathcal{D}_0\mathbf{C})^{ab}\otimes(\mathcal{D}_0\mathbf{C})^{ab} \arrow[d, "{\langle\cdot,\cdot\rangle}"'] & & \\
K_1\mathbf{C} \arrow[r, hook] & \mathcal{D}_1\mathbf{C} \arrow[r, "\partial"] & \mathcal{D}_0\mathbf{C} \arrow[ul, dashed, "H"'] \arrow[r, two heads] & K_0\mathbf{C}.
\end{tikzcd}$$

in which the bottom row is exact.

The important features of our model are the following:

- It is *small*, as it has generators given by the objects, weak equivalences and cofiber sequences of the category $\mathbf{C}$.
- It has *minimal nilpotency degree*, since both groups $\mathcal{D}_0\mathbf{C}$ and $\mathcal{D}_1\mathbf{C}$ have nilpotency class 2.
- It encodes the 1-type in a *functorial* way, and the homotopy classes of morphisms $\mathcal{D}_*\mathbf{C}\to\mathcal{D}_*\mathbf{D}$ and $P_1K\mathbf{C}\to P_1K\mathbf{D}$ are in bijection.

From this structure we can recover the homomorphism $\cdot\eta\colon K_0\mathbf{C}\otimes\mathbb{Z}/2\to K_1\mathbf{C}$, which gives the action of the Hopf map in the stable homotopy groups of spheres, in the following way

$$a\cdot\eta \;=\; \langle a,a\rangle,\quad a\in K_0\mathbf{C}.$$

The extra structure given by the quadratic map H is used to describe the behaviour of $\mathcal{D}_*$ with respect to biexact functors between Waldhausen categories

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$\mathbf{C} \times \mathbf{D} \to \mathbf{E}$. In particular the classical homomorphisms

$$\begin{array}{rcl} K_0\mathbf{C} \otimes K_0\mathbf{D} & \longrightarrow & K_0\mathbf{E}, \\ K_0\mathbf{C} \otimes K_1\mathbf{D} & \longrightarrow & K_1\mathbf{E}, \\ K_1\mathbf{C} \otimes K_0\mathbf{D} & \longrightarrow & K_1\mathbf{E}, \end{array}$$

may be obtained from our model $\mathcal{D}_*$.

The object $\mathcal{D}_*\mathbf{C}$ is a *stable quadratic module* in the sense of [Bau91]. This object is defined below by a presentation in terms of generators and relations in the spirit of Nenashev, who gave a model for K_1 of an exact category in [Nen98]. A stable quadratic module is a particular case of a strict symmetric categorical group, or more generally of a commutative monoid in the category of crossed complexes, which were first introduced by Whitehead in [Whi49]. The monoidal structure for crossed complexes was defined in [BH87].

To obtain our presentation of $\mathcal{D}_*$ we introduce the *total crossed complex* ΠX of a bisimplicial set X, and show that there is an Eilenberg–Zilber–Cartier equivalence

$$\begin{tikzcd} \pi\,\mathrm{Diag}(X) \arrow[loop left] \arrow[r, shift left] & \Pi(X) \arrow[l, shift left] \end{tikzcd}$$

generalizing [DP61, Section 2] and [Ton03]. This is then applied to the bisimplicial set given by the nerve of Waldhausen's $wS.$ construction [Wal78].

1. The algebraic 1-type $\mathcal{D}_*\mathbf{C}$ of the K-theory spectrum $K\mathbf{C}$

We begin by defining the algebraic structure which the model $\mathcal{D}_*\mathbf{C}$ will have.

Definition 1.1. A *stable quadratic module* C_* is a diagram of group homomorphisms

$$C_0^{ab} \otimes C_0^{ab} \xrightarrow{\langle\cdot,\cdot\rangle} C_1 \xrightarrow{\partial} C_0$$

such that given $c_i, d_i \in C_i$, $i = 0, 1$,

- $\partial\langle c_0, d_0\rangle = [d_0, c_0]$,
- $\langle \partial(c_1), \partial(d_1)\rangle = [d_1, c_1]$,
- $\langle c_1, d_1\rangle + \langle d_1, c_1\rangle = 0$.

Here $[x, y] = -x - y + x + y$ is the commutator of two elements $x, y \in K$ in any group K, and K^{ab} is the abelianization of K. We will also write $\langle\cdot,\cdot\rangle_{C_*}$ and ∂_{C_*} for the structure homomorphisms of C_*. Note that the axioms imply that C_0 and C_1 are groups of nilpotency degree 2.

Stable quadratic modules were introduced in [Bau91, Definition IV.C.1]. Notice, however, that we adopt the opposite convention for the homomorphism $\langle\cdot,\cdot\rangle$.

As usual, one can define stable quadratic modules in terms of generators and relations in degrees zero and one.

We assume the reader has certain familiarity with Waldhausen categories and related concepts. We refer to [Wei] for the basics, see also [Wal85].

Definition 1.2. Let $\mathbf{C}$ be a Waldhausen category with distinguished zero object $*$. Cofibrations and weak equivalences are denoted by $\rightarrowtail$ and $\xrightarrow{\sim}$, respectively. A generic cofiber sequence is denoted by

$$A \rightarrowtail B \twoheadrightarrow B/A.$$

We define $\mathcal{D}_*\mathbf{C}$ as the stable quadratic module generated in dimension zero by the symbols

- $[A]$ for any object in $\mathbf{C}$,

and in dimension one by

- $[A \xrightarrow{\sim} A']$ for any weak equivalence,
- $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence,

such that the following relations hold.

(1) $\partial[A \xrightarrow{\sim} A'] = -[A'] + [A]$.
(2) $\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A]$.
(3) $[*] = 0$.
(4) $[A \xrightarrow{1_A} A] = 0$.
(5) $[A \xrightarrow{1_A} A \twoheadrightarrow *] = 0$, $[* \rightarrowtail A \xrightarrow{1_A} A] = 0$.
(6) For any pair of composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$,

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B].$$

(7) For any commutative diagram in $\mathbf{C}$ as follows

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[r, two heads] \arrow[d, "\sim"] & B/A \arrow[d, "\sim"] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & B'/A' \end{tikzcd}$$

we have

$$\begin{array}{rcl} [A \xrightarrow{\sim} A'] + [B/A \xrightarrow{\sim} B'/A'] & & \\ +\langle [A], -[B'/A'] + [B/A] \rangle & = & -[A' \rightarrowtail B' \twoheadrightarrow B'/A'] \\ & & +[B \xrightarrow{\sim} B'] \\ & & +[A \rightarrowtail B \twoheadrightarrow B/A]. \end{array}$$

(8) For any commutative diagram consisting of four obvious cofiber sequences in $\mathbf{C}$ as follows

$$\begin{tikzcd} & & C/B \\ & B/A \arrow[r, tail] & C/A \arrow[u, two heads] \\ A \arrow[r, tail] & B \arrow[r, tail] \arrow[u, two heads] & C \arrow[u, two heads] \end{tikzcd}$$

we have

$$\begin{array}{rcl} [B \rightarrowtail C \twoheadrightarrow C/B] & & \\ +[A \rightarrowtail B \twoheadrightarrow B/A] & = & [A \rightarrowtail C \twoheadrightarrow C/A] \\ & & +[B/A \rightarrowtail C/A \twoheadrightarrow C/B] \\ & & +\langle [A], -[C/A] + [C/B] + [B/A] \rangle. \end{array}$$

(9) For any pair of objects A, B in $\mathbf{C}$

$$\langle A, B \rangle = -[A \overset{i_1}{\rightarrowtail} A \vee B \overset{p_2}{\twoheadrightarrow} B] + [B \overset{i_2}{\rightarrowtail} A \vee B \overset{p_1}{\twoheadrightarrow} A].$$

Here

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \vee B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

are the inclusions and projections of a coproduct in $\mathbf{C}$.

Remark 1.3. Notice that relation (7) implies that if (9) holds for a particular choice of the coproduct $A \vee B$ then it holds for any other choice, since two different coproducts are canonically isomorphic by an isomorphism which preserves the inclusions and the projections of the factors.

Definition 1.4. A *morphism* $f\colon C_* \to D_*$ in the category **squad** of stable quadratic modules is given by group homomorphisms $f_i\colon C_i \to D_i$, $i = 0, 1$, such that

- $\partial_{D_*} f_1(c_1) = f_0 \partial_{C_*}(c_1)$,
- $\langle f_0(c_0), f_0(d_0)\rangle_{D_*} = f_1\langle c_0, d_0\rangle_{C_*}$.

The *homotopy groups* of C_* are

$$\pi_1 C_* = \operatorname{Ker}\partial \text{ and } \pi_0 C_* = \operatorname{Coker}\partial.$$

A *weak equivalence* in **squad** is a morphism which induces isomorphisms in homotopy groups. The *homotopy category*

$$\operatorname{Ho}\mathbf{squad}$$

is obtained from **squad** by inverting weak equivalences.

Let **WCat** be the category of Waldhausen categories as above and exact functors. The construction $\mathcal{D}_*$ defines a functor

$$\mathcal{D}_*\colon \mathbf{WCat} \longrightarrow \mathbf{squad}.$$

For an exact functor $F\colon \mathbf{C} \to \mathbf{D}$ the stable quadratic module morphism $\mathcal{D}_*F\colon \mathcal{D}_*\mathbf{C} \to \mathcal{D}_*\mathbf{D}$ is given on generators by

$$\begin{aligned}
(\mathcal{D}_*F)([A]) &= [F(A)],\\
(\mathcal{D}_*F)([A \xrightarrow{\sim} A']) &= [F(A) \xrightarrow{\sim} F(A')],\\
(\mathcal{D}_*F)([A \rightarrowtail B \twoheadrightarrow B/A]) &= [F(A) \rightarrowtail F(B) \twoheadrightarrow F(B/A)].
\end{aligned}$$

Let $\operatorname{Ho}\mathbf{Spec}_0$ be the homotopy category of connective spectra. In Lemma 4.18 below we define a functor

$$\lambda_0\colon \operatorname{Ho}\mathbf{Spec}_0 \longrightarrow \operatorname{Ho}\mathbf{squad}$$

together with natural isomorphisms

$$\pi_i\lambda_0 X \cong \pi_i X, \quad i = 0, 1.$$

This functor induces an equivalence of categories

$$\lambda_0\colon \operatorname{Ho}\mathbf{Spec}_0^1 \xrightarrow{\sim} \operatorname{Ho}\mathbf{squad},$$

were $\operatorname{Ho}\mathbf{Spec}_0^1$ is the homotopy category of spectra with trivial homotopy groups in dimensions other than 0 and 1.

The naive algebraic model for the 1-type of the algebraic K-theory spectrum $K\mathbf{C}$ of a Waldhausen category $\mathbf{C}$ would be $\lambda_0 K\mathbf{C}$. However this stable quadratic module is much bigger than $\mathcal{D}_*\mathbf{C}$ and it is not directly defined in terms of the basic structure of the Waldhausen category $\mathbf{C}$. This makes meaningful the following theorem, which is the main result of this paper.

Theorem 1.5. *Let* $\mathbf{C}$ *be a Waldhausen category. There is a natural isomorphism in* Ho **squad**

$$\mathcal{D}_*\mathbf{C} \xrightarrow{\cong} \lambda_0 K\mathbf{C}.$$

This theorem shows that the model $\mathcal{D}_*\mathbf{C}$ satisfies the functoriality properties claimed in the introduction. It also shows that the exact sequence of the introduction is available. The theorem will be proved in section four.

From a local point of view the 1-type of a connective spectrum is determined up to non-natural isomorphism by the first k-invariant. We now establish the connection between this k-invariant and the algebraic model $\mathcal{D}_*\mathbf{C}$.

Definition 1.6. The *k-invariant* of a stable quadratic module C_* is the homomorphism

$$k\colon \pi_0 C_* \otimes \mathbb{Z}/2 \longrightarrow \pi_1 C_*, \quad k(x\otimes 1) = \langle x, x\rangle.$$

Given a connective spectrum X the k-invariant of $\lambda_0 X$ coincides with the action of the Hopf map $0 \neq \eta \in \pi_1^s \cong \mathbb{Z}/2$ in the stable stem of the sphere.

$$\begin{tikzcd}
\pi_0 X \otimes \pi_1^s \arrow[r] \arrow[d, "\cong"'] & \pi_1 X \arrow[d, "\cong"] \\
\pi_0\lambda_0 X \otimes \mathbb{Z}/2 \arrow[r, "k"] & \pi_1\lambda_0 X
\end{tikzcd}$$

See Lemma 4.18 below. We recall that the action of η coincides with the first Postnikov invariant of X. This is used to derive the following corollary of Theorem 1.5.

Corollary 1.7. *The first Postnikov invariant of the spectrum* $K\mathbf{C}$

$$\cdot\eta\colon K_0\mathbf{C}\otimes\mathbb{Z}/2 \longrightarrow K_1\mathbf{C}$$

is defined by

$$[A]\cdot\eta = [\tau_{A,A}\colon A\vee A \xrightarrow{\cong} A\vee A],$$

where $\tau_{A,A}$ *is the automorphism which exchanges the factors of a coproduct* $A\vee A$ *in* $\mathbf{C}$.

Proof. The corollary follows from the commutativity of the following diagram

$$\begin{tikzcd}
A \arrow[r, tail, "i_1"] \arrow[d, equal] & A\vee A \arrow[r, two heads, "p_2"] \arrow[d, "\cong"', "\tau_{A,A}"] & A \arrow[d, equal] \\
A \arrow[r, tail, "i_2"] & A\vee A \arrow[r, two heads, "p_1"] & A
\end{tikzcd}$$

and relations (4), (7) and (9) in Definition 1.2. □

The next corollary can be easily obtained from the previous one by using again the relations defining $\mathcal{D}_*$ and matrix arguments as for example in the proof of [Ran85, Proposition 2.1 (iv)].

Corollary 1.8. *Let* $\mathbf{A}$ *be a Waldhausen category which is additive. Then the first Postnikov invariant of the spectrum* $K\mathbf{A}$

$$\cdot\eta\colon K_0\mathbf{A}\otimes\mathbb{Z}/2 \longrightarrow K_1\mathbf{A}$$

is defined by

$$[A]\cdot\eta = [-1_A\colon A \xrightarrow{\cong} A].$$

2. The multiplicative properties of $\mathcal{D}_*$

In order to describe the multiplicative properties of $\mathcal{D}_*\mathbf{C}$ with respect to biexact functors we would need a symmetric monoidal structure on **squad** which models the smash product of spectra. Unfortunately such a monoidal structure does not exist and we need to enrich $\mathcal{D}_*\mathbf{C}$ with an extra structure map H,

$$\begin{tikzcd} (\mathcal{D}_0\mathbf{C})^{ab}\otimes(\mathcal{D}_0\mathbf{C})^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"'] & \\ \mathcal{D}_1\mathbf{C} \arrow[r, "\partial"] & \mathcal{D}_0\mathbf{C}, \arrow[ul, bend right, "H"'] \end{tikzcd}$$

so that the diagrams

$$\begin{aligned} \mathcal{D}_0^{sg}\mathbf{C} &= \left(\mathcal{D}_0\mathbf{C} \underset{\langle\partial,\partial\rangle}{\overset{H}{\rightleftarrows}} (\mathcal{D}_0\mathbf{C})^{ab}\otimes(\mathcal{D}_0\mathbf{C})^{ab} \right), \\ \mathcal{D}_1^{sg}\mathbf{C} &= \left(\mathcal{D}_1\mathbf{C} \underset{\langle\cdot,\cdot\rangle}{\overset{H\partial}{\rightleftarrows}} (\mathcal{D}_0\mathbf{C})^{ab}\otimes(\mathcal{D}_0\mathbf{C})^{ab} \right), \end{aligned}$$

are square groups in the sense of [BP99].

Definition 2.1. A *square group* M is a diagram

$$M_e \underset{P}{\overset{H}{\rightleftarrows}} M_{ee}$$

where M_e is a group, M_{ee} is an abelian group, P is a homomorphism, H is a quadratic map, i.e. the symbol

$$(x|y)_H = H(x+y) - H(y) - H(x), \quad x, y \in M_e,$$

is bilinear, and the following identities hold, $a \in M_{ee}$,

$$\begin{aligned} (P(a)|x)_H &= 0, \\ (x|P(a))_H &= 0, \\ P(x|y)_H &= [x,y], \\ PHP(a) &= P(a)+P(a). \end{aligned}$$

Note that $(\cdot|\cdot)_H$ induces a homomorphism

$$(\cdot|\cdot)_H\colon \operatorname{Coker} P \otimes \operatorname{Coker} P \longrightarrow M_{ee}. \tag{2.2}$$

Moreover

$$T = HP - 1\colon M_{ee} \longrightarrow M_{ee}$$

is an involution, i.e. a homomorphism with $T^2 = 1$, and

$$\Delta\colon \operatorname{Coker} P \longrightarrow X_{ee}\colon x \mapsto (x|x)_H - H(x) + TH(x)$$

defines a homomorphism.

A *morphism* $f\colon M \to N$ in the category of square groups is given by group homomorphisms $f_e\colon M_e \to N_e$, $f_{ee}\colon M_{ee} \to N_{ee}$ commuting with H and P.

A *quadratic pair module* $f\colon M \to N$ is a square group morphism such that $M_{ee} = N_{ee}$ and f_{ee} is the identity.

Morphisms in the category **qpm** of quadratic pair modules are defined again by homomorphisms commuting with all operators.

A stable quadratic module C_* is termed 0-*free* if $C_0 = \langle E\rangle_{nil}$ is a free group of nilpotency class 2. Here E is the basis.

Lemma 2.3. *Let C_* be a 0-free stable quadratic module with $C_0 = \langle E\rangle_{nil}$ and let $H\colon C_0 \to \mathbb{Z}[E]\otimes\mathbb{Z}[E]$ be the unique quadratic map such that $H(e) = 0$ for any $e \in E$ and $(x|y)_H = y\otimes x$ for $x, y \in C_0$. Then*

$$\begin{aligned} C_0^{sg} &= \left(C_0 \underset{\langle\partial,\partial\rangle}{\overset{H}{\rightleftarrows}} C_0^{ab}\otimes C_0^{ab} \right), \\ C_1^{sg} &= \left(C_1 \underset{\langle\cdot,\cdot\rangle}{\overset{H\partial}{\rightleftarrows}} C_0^{ab}\otimes C_0^{ab} \right), \end{aligned}$$

are square groups. Moreover, the homomorphism $\partial\colon C_1 \to C_0$ defines a quadratic pair module

$$C_1^{sg} \longrightarrow C_0^{sg}.$$

The square group C_0^{sg} in this lemma will also be denoted by $\mathbb{Z}_{nil}[E]$ or just $\mathbb{Z}_{nil}$ if E is a singleton, as in [BJP05].

The stable quadratic module $\mathcal{D}_*\mathbf{C}$ defined in the previous section is 0-free. The basis of $\mathcal{D}_0\mathbf{C}$ is the set of objects in $\mathbf{C}$, excluding the zero object $*$. In particular $\mathcal{D}_0^{sg}\mathbf{C}$ and $\mathcal{D}_1^{sg}\mathbf{C}$ above are square groups and $\mathcal{D}_*\mathbf{C}$ is endowed with the structure of a quadratic pair module. Moreover, the morphisms induced by exact functors are compatible with H, so that $\mathcal{D}_*$ lifts to a functor

$$\mathcal{D}_*\colon \mathbf{WCat} \longrightarrow \mathbf{qpm}.$$

The category of square groups is a symmetric monoidal category with the tensor product $\circledcirc$ defined in [BJP05] that we now recall.

Definition 2.4. The *tensor product* $M \circledcirc N$ of square groups M, N is defined as follows. The group $(M\circledcirc N)_e$ is generated by the symbols $x \circledcirc y$, $a\bar{\otimes}b$ for $x \in M_e$, $y \in N_e$, $a \in M_{ee}$ and $b \in N_{ee}$, subject to the following relations

(1) the symbol $a\bar{\otimes}b$ is bilinear and central,
(2) $x\circledcirc(y_1+y_2) = x\circledcirc y_1 + x\circledcirc y_2 + H(x)\bar{\otimes}(y_2|y_1)_H$,
(3) the symbol $x\circledcirc y$ is left linear, $(x_1+x_2)\circledcirc y = x_1\circledcirc y + x_2\circledcirc y$,
(4) $P(a)\circledcirc y = a\bar{\otimes}(y|y)_H$.
(5) $T(a)\bar{\otimes}T(b) = -a\bar{\otimes}b$,
(6) $x\circledcirc P(b) = \Delta(x)\bar{\otimes}b$.

The abelian group $(M\circledcirc N)_{ee}$ is defined as the tensor product $M_{ee}\otimes N_{ee}$. The homomorphism

$$P\colon (M\circledcirc N)_{ee} \longrightarrow (M\circledcirc N)_e$$

is $P(a\otimes b) = a\bar{\otimes}b$, and

$$H\colon (M\circledcirc N)_{e} \longrightarrow (M\circledcirc N)_{ee}$$

is the unique quadratic map satisfying

$$\begin{aligned} H(x\circledcirc y) &= \Delta(x)\otimes H(y) + H(x)\otimes(y|y)_H, \\ H(a\bar{\otimes}b) &= a\otimes b - T(a)\otimes T(b), \\ (a\bar{\otimes}b|-)_H &= 0, \\ (-|a\bar{\otimes}b)_H &= 0, \\ (a\circledcirc b|c\circledcirc d)_H &= (a|c)_H\otimes(b|d)_H. \end{aligned}$$

The unit for the tensor product is the square group $\mathbb{Z}_{nil}$.

Theorem 2.5. *Let $\mathbf{C}\times\mathbf{D}\to\mathbf{E}\colon (A,B)\mapsto A\wedge B$ be a biexact functor between Waldhausen categories. Then there are morphisms of square groups*

$$\varphi^{ij}\colon \mathcal{D}_i^{sg}\mathbf{C}\circledcirc\mathcal{D}_j^{sg}\mathbf{D}\to\mathcal{D}_{i+j}^{sg}\mathbf{E},$$

for i, j, $i+j\in\{0,1\}$, defined by

$$\begin{array}{rcl}
\varphi_e^{00}([A]\circledcirc[C]) &=& [A\wedge C],\\
\varphi_e^{01}([A]\circledcirc[C\stackrel{\sim}{\to}C']) &=& [A\wedge C\stackrel{\sim}{\to}A\wedge C'],\\
\varphi_e^{01}([A]\circledcirc[C\rightarrowtail D\twoheadrightarrow D/C]) &=& [A\wedge C\rightarrowtail A\wedge D\twoheadrightarrow A\wedge(D/C)],\\
\varphi_e^{10}([A\stackrel{\sim}{\to}A']\circledcirc[C]) &=& [A\wedge C\stackrel{\sim}{\to}A'\wedge C],\\
\varphi_e^{10}([A\rightarrowtail B\twoheadrightarrow B/A]\circledcirc[C]) &=& [A\wedge C\rightarrowtail B\wedge C\twoheadrightarrow (B/A)\wedge C],\\
\varphi_{ee}^{ij}([A]\otimes[A']\otimes[C]\otimes[C']) &=& [A\wedge C]\otimes[A'\wedge C'].
\end{array}$$

such that the following diagram of square groups commutes

$$\begin{tikzcd}
& \mathcal{D}_1^{sg}\mathbf{C}\circledcirc\mathcal{D}_1^{sg}\mathbf{D} \arrow[dl, "\partial\circledcirc 1"'] \arrow[dr, "1\circledcirc\partial"] & \\
\mathcal{D}_0^{sg}\mathbf{C}\circledcirc\mathcal{D}_1^{sg}\mathbf{D} \arrow[dr, "\varphi^{01}"] \arrow[dddr, bend right, "1\circledcirc\partial"'] & & \mathcal{D}_1^{sg}\mathbf{C}\circledcirc\mathcal{D}_0^{sg}\mathbf{D} \arrow[dl, "\varphi^{10}"'] \arrow[dddl, bend left, "\partial\circledcirc 1"] \\
& \mathcal{D}_1^{sg}\mathbf{E} \arrow[d, "\partial"] & \\
& \mathcal{D}_0^{sg}\mathbf{E} & \\
& \mathcal{D}_0^{sg}\mathbf{C}\circledcirc\mathcal{D}_0^{sg}\mathbf{D} \arrow[u, "\varphi^{00}"'] &
\end{tikzcd}$$

Now given a biexact functor $\mathbf{C}\times\mathbf{D}\to\mathbf{E}\colon (A,B)\mapsto A\wedge B$ we recover the classical homomorphisms

$$\begin{array}{rcl}
\bar\varphi^{00}\colon K_0\mathbf{C}\otimes K_0\mathbf{D} &\longrightarrow& K_0\mathbf{E},\\
\bar\varphi^{01}\colon K_0\mathbf{C}\otimes K_1\mathbf{D} &\longrightarrow& K_1\mathbf{E},\\
\bar\varphi^{10}\colon K_1\mathbf{C}\otimes K_0\mathbf{D} &\longrightarrow& K_1\mathbf{E},
\end{array}$$

from φ^{ij} in Theorem 2.5 as follows. Given i, j, $i+j\in\{0,1\}$,

$$\bar\varphi^{ij}(a\otimes b) \quad=\quad \varphi_e^{ij}(a\circledcirc b). \tag{2.6}$$

Here we use the natural exact sequence

$$K_1\mathbf{C}\hookrightarrow\mathcal{D}_1\mathbf{C}\stackrel{\partial}{\longrightarrow}\mathcal{D}_0\mathbf{C}\twoheadrightarrow K_0\mathbf{C}$$

available for any Waldhausen category $\mathbf{C}$ to identify $K_1\mathbf{C}$ with its image in $\mathcal{D}_1\mathbf{C}$, and we use the same notation for an element in $\mathcal{D}_0\mathbf{C}$ and for its image in $K_0\mathbf{C}$. One can use the relations defining the tensor product $\circledcirc$ of square groups to check that the homomorphisms $\bar\varphi^{ij}$ are well defined by the formula (2.6) above.

In the proof of Theorem 2.5 we use a technical lemma about square groups, which measures the failure of $\circledcirc$ to preserve certain coproducts.

Let $M \cdot E$ be the E-fold coproduct of a square group M, for any indexing set E. We know from [BJP05] that $\mathbb{Z}_{nil}[E] = \mathbb{Z}_{nil} \cdot E$, so we have canonical morphisms $M \circledcirc \iota_x \colon M \to M \circledcirc \mathbb{Z}_{nil}[E]$ for $x \in E$. However, the natural comparison morphism

$$\iota = (M \circledcirc \iota_x)_{x \in E} \colon M \cdot E \longrightarrow M \circledcirc \mathbb{Z}_{nil}[E]$$

is not an isomorphism.

Consider the homomorphisms

$$\begin{array}{rcl} \overline{\Delta} \colon \mathbb{Z}[E] \otimes \mathbb{Z}[E] & \twoheadrightarrow & \mathbb{Z}[E], \\ H \colon \wedge^2 \mathbb{Z}[E] & \to & \operatorname{Ker} \overline{\Delta}, \\ q \colon \operatorname{Ker} \overline{\Delta} & \twoheadrightarrow & \wedge^2 \mathbb{Z}[E], \end{array}$$

where $\overline{\Delta}(e \otimes e) = e$ for $e \in E$ and $\overline{\Delta}(e \otimes e') = 0$ if $e \neq e' \in E$, with $H(x \wedge y) = y \otimes x - x \otimes y$ and $q(x \otimes y) = x \wedge y$. For any abelian group A we consider

$$(A \otimes \wedge^2 \mathbb{Z}[E])^{\otimes} = \left(A \otimes \wedge^2 \mathbb{Z}[E] \underset{A \otimes q}{\overset{A \otimes H}{\rightleftarrows}} A \otimes \operatorname{Ker} \overline{\Delta} \right).$$

This is isomorphic to the square group defined in [BJP05, Section 1]. The construction is obviously functorial in A.

Lemma 2.7. *For any square group M and any set E there is a pushout diagram in the category of square groups*

$$\begin{tikzcd} (\operatorname{Coker} P \otimes \operatorname{Coker} P \otimes \wedge^2 \mathbb{Z}[E])^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "push"] & M \cdot E \arrow[d, "\iota"] \\ (M_{ee} \otimes \wedge^2 \mathbb{Z}[E])^{\otimes} \arrow[r] & M \circledcirc \mathbb{Z}_{nil}[E] \end{tikzcd}$$

which is natural in M and E.

Proof. One can check inductively by using [BJP05, Proposition 5] and [BJP05, Section 5.6 (6)] that there is a map of central extensions of square groups in the sense of [BJP05, Section 5.5] as follows.

$$\begin{tikzcd} (\operatorname{Coker} P \otimes \operatorname{Coker} P \otimes \wedge^2 \mathbb{Z}[E])^{\otimes} \arrow[r, hook, "\mu"] \arrow[d] & M \cdot E \arrow[r, two heads] \arrow[d, "\iota"'] & \prod_E M \arrow[d, equal] \\ (M_{ee} \otimes \wedge^2 \mathbb{Z}[E])^{\otimes} \arrow[r, hook, "\nu"] & M \circledcirc \mathbb{Z}_{nil}[E] \arrow[r, two heads] & \prod_E M \end{tikzcd}$$

Here the left-hand morphism is induced by (2.2). The morphism μ is completely determined by the homomorphism

$$\mu_{ee} \colon \operatorname{Coker} P \otimes \operatorname{Coker} P \otimes \operatorname{Ker} \overline{\Delta} \longrightarrow (M \cdot E)_{ee}$$

defined by $\mu_{ee}(a \otimes b \otimes x \otimes y) = P(\iota_x(a)|\iota_y(b))_H$ for $a, b \in \operatorname{Coker} P$ and $x \neq y \in E$. Similarly ν is determined by the homomorphism

$$\nu_{ee} \colon M_{ee} \otimes \operatorname{Ker} \overline{\Delta} \longrightarrow M_{ee} \otimes \mathbb{Z}[E] \otimes \mathbb{Z}[E]$$

induced by the inclusion $\operatorname{Ker} \overline{\Delta} \subset \mathbb{Z}[E] \otimes \mathbb{Z}[E]$. It is straightforward to check that the square on the left is the desired pushout. $\square$

Proof of Theorem 2.5. It is obvious that φ_{ee}^{ij} is a well-defined abelian group homomorphism in all cases. The square group morphism φ^{00} is well-defined as a consequence of [BJP05, Proposition 34].

Let E be the set of objects of $\mathbf{D}$, excluding $*$, so that $\mathbb{Z}_{nil}[E] = \mathcal{D}_0^{sg}\mathbf{D}$, and let $M = \mathcal{D}_1^{sg}\mathbf{C}$. To see that φ^{10} is well defined by the formulas in the statement we note that it is just the morphism determined, using Lemma 2.7, by the square group morphisms

$$(M_{ee} \otimes \wedge^2\mathbb{Z}[E])^{\otimes} \xrightarrow{\xi} \mathcal{D}_1^{sg}\mathbf{E} \xleftarrow{\zeta} M \cdot E$$

defined as follows. The square group morphism ξ is completely determined by

$$\xi_{ee} = \varphi_{ee}^{10} \colon (\mathcal{D}_0\mathbf{C})^{ab} \otimes (\mathcal{D}_0\mathbf{C})^{ab} \otimes \operatorname{Ker}\overline{\Delta} \longrightarrow (\mathcal{D}_0\mathbf{E})^{ab} \otimes (\mathcal{D}_0\mathbf{E})^{ab}.$$

For each $A \in E$, the component $\zeta \circ \iota_A : \mathcal{D}_1^{sg}\mathbf{C} \to \mathcal{D}_1^{sg}\mathbf{C} \cdot E \to \mathcal{D}_1^{sg}\mathbf{E}$ is the unique square group morphism such that

$$\begin{array}{ccccc}
\mathcal{D}_1^{sg}\mathbf{C} & \xrightarrow{\quad\iota_A\quad} & \mathcal{D}_1^{sg}\mathbf{C}\cdot E & \xrightarrow{\quad\zeta\quad} & \mathcal{D}_1^{sg}\mathbf{E} \\
{\scriptstyle\partial}\downarrow & & & & \downarrow{\scriptstyle\partial} \\
\mathcal{D}_0^{sg}\mathbf{C} & \xrightarrow{\quad\iota_A\quad} & \mathcal{D}_0^{sg}\mathbf{C}\cdot E & \xrightarrow{\quad\varphi^{00}\quad} & \mathcal{D}_1^{sg}\mathbf{E}
\end{array}$$

coincides with the morphism of quadratic pair modules

$$\mathcal{D}_*(\,\cdot\, \wedge A) \colon \mathcal{D}_*\mathbf{C} \to \mathcal{D}_*\mathbf{E}$$

induced by the exact functor $\cdot \wedge A \colon \mathbf{D} \to \mathbf{E}$.

By using this alternative definition of φ^{10} in terms of Lemma 2.7 it is also immediate that the lower right cell in the diagram of the statement is commutative.

To see that φ^{01} is well-defined and that the lower left cell of the diagram in the statement commutes one proceeds similarly, using the fact that $\circledcirc$ is symmetric.

Now we just need to check that the upper cell is commutative. For this it is enough to show that the following equalities hold

$$\begin{aligned}
\varphi^{01}((\partial[A \xrightarrow{\sim} A']) \circledcirc [C \xrightarrow{\sim} C']) &= \varphi^{10}([A \xrightarrow{\sim} A'] \circledcirc (\partial[C \xrightarrow{\sim} C'])), \\
\varphi^{01}((\partial[A \xrightarrow{\sim} A']) \circledcirc [C \rightarrowtail D \twoheadrightarrow D/C]) &= \varphi^{10}([A \xrightarrow{\sim} A'] \circledcirc (\partial[C \rightarrowtail D \twoheadrightarrow D/C])), \\
\varphi^{01}((\partial[A \rightarrowtail B \twoheadrightarrow B/A]) \circledcirc [C \xrightarrow{\sim} C']) &= \varphi^{10}([A \rightarrowtail B \twoheadrightarrow B/A] \circledcirc (\partial[C \xrightarrow{\sim} C'])), \\
\varphi^{01}((\partial[A \rightarrowtail B \twoheadrightarrow B/A]) \circledcirc [C \rightarrowtail D \twoheadrightarrow D/C]) &= \varphi^{10}([A \rightarrowtail B \twoheadrightarrow B/A] \circledcirc (\partial[C \rightarrowtail D \twoheadrightarrow D/C])).
\end{aligned}$$

This is a tedious but straightforward task which makes use of the laws of stable quadratic modules and the tensor product of square groups, the elementary properties of a biexact functor, and the relations (1), (2), (6), (7) and (8) in Definition 1.2. $\square$

3. Natural transformations and induced homotopies on $\mathcal{D}_*$

In this section we define induced homotopies along the functor $\mathcal{D}_*$ from section 1.

Definition 3.1. Two morphisms $f, g \colon C_* \to D_*$ are *homotopic* $f \simeq g$ if there exists a function $\alpha \colon C_0 \to D_1$ satisfying

- $\alpha(c_0 + d_0) = \alpha(c_0) + \alpha(d_0) + \langle f_0(d_0), -f_0(c_0) + g_0(c_0)\rangle_{D_*}$,
- $g_0(c_0) = f_0(c_0) + \partial_{D_*}\alpha(c_0)$,
- $g_1(c_1) = f_1(c_1) + \alpha\partial_{C_*}(c_1)$.

Such a function is called a *homotopy.*

The category **squad** of stable quadratic modules is a 2-category with 2-morphisms given by homotopies; it is indeed a category enriched over groupoids. See [BM05, Proposition 7.2] for details. In addition the qutient of the full subcategory $\mathbf{squad}_f \subset \mathbf{squad}$ given by 0-free objects by the homotopy relation is equivalent to the homotopy category of stable quadratic modules

$$\mathbf{squad}_f/\simeq \stackrel{\sim}{\longrightarrow} \operatorname{Ho}\mathbf{squad},$$

compare [BM05, Proposition 7.7].

The category **WCat** of Waldhausen categories and exact functors is also a 2-category, where a 2-morphism $\varepsilon\colon F \Rightarrow G$ between two exact functors $F, G\colon \mathbf{C} \to \mathbf{D}$ is a natural transformation ε given by weak equivalences $\varepsilon(A)\colon F(A) \xrightarrow{\sim} G(A)$ in $\mathbf{D}$ for any object A in $\mathbf{C}$.

Theorem 3.2. *The construction $\mathcal{D}_*\colon \mathbf{WCat} \to \mathbf{squad}$ defines a 2-functor.*

Proof. The homotopy $\mathcal{D}_*\varepsilon\colon \mathcal{D}_0\mathbf{C} \to \mathcal{D}_1\mathbf{D}$ induced by a 2-morphism $\varepsilon\colon F \Rightarrow G$ in **WCat** is determined by the formula

$$(\mathcal{D}_*\varepsilon)([A]) = [\varepsilon(A)\colon F(A) \xrightarrow{\sim} G(A)].$$

We leave the details to the reader. □

Remark 3.3. The homotopies defined in Theorem 3.2 are constructed by using just one kind of degree 1 generators of $\mathcal{D}_*$, namely those given by weak equivalences. In case we have a cofiber sequence $F \rightarrowtail G \twoheadrightarrow H$ of exact functors $F, G, H\colon \mathbf{C} \to \mathbf{D}$ one can define a homotopy using the other class of degree 1 generators, given by cofiber sequences, to give a direct proof of the additivity theorem [Wal85, Proposition 1.3.2 (4)] for the algebraic model of the 1-type $\mathcal{D}_*$.

4. Proof of Theorem 1.5

In this section we use the notions of crossed module and crossed complex in the category of groups and in the category of groupoids. There are different but equivalent ways of presenting these objects, and the functors between them, depending on a series of conventions such as using left or right actions, choice of basepoint of an n-simplex, etc. In this paper we adopt the conventions which are compatible with [Ton03]. As examples of a crossed complex we can mention the fundamental crossed complex $\pi_{CW}Y$ of a CW-complex Y and the fundamental crossed complex πX of a simplicial set X; these are related by the natural identification $\pi X = \pi_{CW}|X|$ where $|\cdot|$ denotes the geometric realization functor from simplicial sets to CW-complexes. See [Ton03] for further details and references.

Definition 4.1. A *crossed module* of groups is a group homomorphism $\partial\colon M \to N$ such that N acts (on the right) on M and the following equations are satisfied for $m, m' \in M$ and $n \in N$.

$$\partial(m^n) = -n + \partial(m) + n, \tag{4.2}$$

$$m^{\partial(m')} = -m' + m + m'. \tag{4.3}$$

Morphisms of crossed modules are defined by commutative squares of group homomorphisms which are compatible with the actions in the obvious way. Such a morphism is a *weak equivalence* if it induces isomorphisms between the kernels and cokernels of the homomorphisms ∂.

A crossed complex of groups (C,∂) is given by a crossed module $\partial_2\colon C_2\to C_1$ as above, a chain complex of modules C_n, $n\geq 3$, over $\operatorname{Coker}\partial_2$, and a connecting map $\partial_3\colon C_3\to C_2$ subject to certain compatibility axioms. Crossed modules and complexes over groups are the 'one object' cases of the more general crossed modules and crossed complexes over groupoids.

The category **ccplx** of crossed complexes is symmetric monoidal with respect to the tensor product $\otimes$, see [BH87] or [Ton03, Definition 1.4]. This tensor product satisfies the following crucial property: given two CW-complexes Y, Z there is a natural isomorphism $\pi_{CW}Y\otimes\pi_{CW}Z\cong\pi_{CW}(Y\times Z)$ [BH91, Theorem 3.1 (iv)] satisfying the usual coherence properties. As examples of monoids in the category of crossed complexes we can cite the fundamental crossed complex $\pi_{CW}M$ of a CW-monoid M, and the crossed cobar construction $\underline{\Omega}_{\mathbf{Crs}}X$ on a 1-reduced simplicial set X, see [BT97]. As a consequence of [Ton03] the fundamental crossed complex πN of a simplicial monoid N is also a monoid in **ccplx**.

For our purposes it will be convenient to have a small model for the fundamental crossed complex of the diagonal of a bisimplicial set. This is achieved by the following definition.

Definition 4.4. The *total crossed complex* $\Pi(X)$ of a bisimplicial set X is the coend

$$\Pi(X) \quad = \quad \int^{m,n}\pi(\Delta[m])\otimes\pi(\Delta[n])\ \cdot\ X_{m,n}\ . \tag{4.5}$$

Here $\Delta[k]$ is the k-simplex, $k\geq 0$, and $C\cdot E$ is the E-fold coproduct of a crossed complex C over an indexing set E; see [Mac71, IX.6] for more details on coend calculus.

Note that if $X_{0,0}=\{*\}$ then $\Pi(X)$ is a crossed complex of groups. The following lemma gives an explicit presentation in terms of generators and relations which is suitable for our purposes.

Lemma 4.6. *Suppose X is a horizontally-reduced bisimplicial set, in the sense that $X_{0,*}=\Delta[0]$. Then $\Pi(X)$ is the crossed complex of groups with one generator $x_{m,n}$ in $\Pi(X)_{m+n}$ for each $x_{m,n}\in X_{m,n}$ and subject to the following relations:*

$$\begin{aligned}
x_{m,n} &= 0 \quad \textit{if } x_{m,n} \textit{ is degenerate in } X_{m,n}\ ,\\
\partial_2x_{1,1} &= -d_0^vx_{1,1}+d_1^vx_{1,1}\ ,\\
\partial_2x_{2,0} &= -d_1^hx_{2,0}+d_0^hx_{2,0}+d_2^hx_{2,0}\ ,\\
\partial_3x_{1,2} &= -d_2^vx_{1,2}-d_0^vx_{1,2}+d_1^vx_{1,2}\ ,\\
\partial_3x_{2,1} &= d_2^hx_{2,1}+d_0^hx_{2,1}{}^{d_2^hd_1^vx_{2,1}}-d_1^vx_{2,1}-d_1^hx_{2,1}+d_0^vx_{2,1}\ ,\\
\partial_3x_{3,0} &= d_2^hx_{3,0}+d_0^hx_{3,0}{}^{(d_2^h)^2x_{3,0}}-d_3^hx_{3,0}-d_1^hx_{3,0}\ .
\end{aligned}$$

For $m\geq 1$ and $m+n\geq 4$, the boundary relations are abelian:

$$\partial_{m+n}x_{m,n} \quad = \quad d_0^hx_{m,n}{}^{(d_2^h)^{m-1}(d_1^v)^nx_{m,n}}+\sum_{i=1}^{m}(-1)^id_i^hx_{m,n}+\sum_{j=0}^{n}(-1)^{m+j}d_j^vx_{m,n}.$$

The last summation is trivial if $n=0$; all the other terms are trivial if $m=1$.

Proof. This follows by using the presentations for $\pi\Delta[n]$ and the tensor product of crossed complexes in [Ton03, 1.2 and 1.4] for example. $\square$

The following results are natural generalizations of the Eilenberg–Zilber theorem for crossed complexes given in [Ton03].

Theorem 4.7. *There is a natural homotopy equivalence (in fact, a strong deformation retraction) of crossed complexes between the total crossed complex of a bisimplicial set X and the fundamental crossed complex of its diagonal,*

$$\begin{tikzcd} \pi\,\mathrm{Diag}(X) \arrow[loop left, "\phi'"] \arrow[r, shift left, "a'"] & \Pi(X). \arrow[l, shift left, "b'"] \end{tikzcd}$$

Proof. As observed for example in [BF78, Proposition B.1], the diagonal of a bisimplicial set X may be expressed as a coend

$$\mathrm{Diag}(X) \;\cong\; \int^{m} \Delta[m] \times X_{m,*}.$$

Since each $X_{m,*}$ is the coend of $\Delta[n]\cdot X_{m,n}$, and π preserves colimits,

$$\begin{aligned} \pi\,\mathrm{Diag}(X) &\cong \pi \int^{m,n} \Delta[m] \times \Delta[n] \;\cdot\; X_{m,n} \\ &\cong \int^{m,n} \pi(\Delta[m] \times \Delta[n]) \;\cdot\; X_{m,n}\,. \end{aligned}$$

The result therefore follows from the Eilenberg–Zilber equivalence

$$\begin{tikzcd} \pi(\Delta[m] \times \Delta[n]) \arrow[loop left, "\phi"] \arrow[r, shift left, "a"] & \pi\Delta[m] \otimes \pi\Delta[n] \arrow[l, shift left, "b"] \end{tikzcd}$$

given in [Ton03, Theorem 3.1] (see also [BGPT97, Section 3]). $\square$

Theorem 4.8. *Given two bisimplicial sets X, Y, there is a natural deformation retraction*

$$\begin{tikzcd} \Pi(X \times Y) \arrow[loop left, "\phi''"] \arrow[r, shift left, "a''"] & \Pi(X) \otimes \Pi(Y). \arrow[l, shift left, "b''"] \end{tikzcd} \tag{4.9}$$

Moreover, the following diagram of 'shuffle maps' commutes:

$$\begin{tikzcd} \Pi X \otimes \Pi Y \arrow[rr, "b' \otimes b'"] \arrow[d, "b''"'] & & \pi\,\mathrm{Diag}\,X \otimes \pi\,\mathrm{Diag}\,Y \arrow[d, "b"] \\ \Pi(X \times Y) \arrow[r, "b'"] & \pi\,\mathrm{Diag}(X \times Y) \arrow[r, "\cong"] & \pi(\mathrm{Diag}\,X \times \mathrm{Diag}\,Y) \end{tikzcd} \tag{4.10}$$

Proof. The natural homotopy equivalence of the objects

$$\begin{aligned} \Pi(X \times Y) &\cong \int^{p,p',q,q'} \pi(\Delta[p] \times \Delta[p']) \otimes \pi(\Delta[q] \times \Delta[q']) \cdot X_{p,q} \times Y_{p',q'}, \\ \Pi(X) \otimes \Pi(Y) &\cong \int^{p,p',q,q'} \pi\Delta[p] \otimes \pi\Delta[q] \otimes \pi\Delta[p'] \otimes \pi\Delta[q'] \cdot X_{p,q} \times Y_{p',q'}, \end{aligned}$$

is defined using the symmetry $\pi\Delta[q] \otimes \pi\Delta[p'] \cong \pi\Delta[p'] \otimes \pi\Delta[q]$ and the Eilenberg–Zilber equivalence, see [Ton03]. The commutativity of the diagram (4.10) follows from standard properties of the shuffle map. $\square$

Example 4.11. Suppose X, Y are bisimplicial sets, with $x \in X_{1,0}$ and $y \in Y_{1,0}$ corresponding to generators in degree one of ΠX and ΠY respectively. Then by [Ton03, 2.6] we have $b''(x \otimes y) \in \Pi(X \times Y)_2$ given by

$$b''(x \otimes y) \quad = \quad -(s_0^h x, s_1^h y) + (s_1^h x, s_0^h y).$$

The category of crossed modules inherits a monoidal structure $\otimes$ from the category of crossed complexes, since it may be regarded as the full reflective subcategory of crossed complexes concentrated in degrees one and two. We denote the reflection functor by $\psi\colon \mathbf{ccplx} \to \mathbf{cross}$.

The following lemma illustrates the rigidity of monoids in the category of crossed modules of groups.

Lemma 4.12. (1) *Let C be a crossed complex of groups and $\mu : C \otimes C \to C$ a unital morphism. Then the induced morphism $\psi\mu : \psi C \otimes \psi C \to \psi C$ is a monoid structure.*

(2) *Let $f : C \to C'$ be a morphism of crossed complexes of groups which preserves given unital morphisms $\mu : C \otimes C \to C$ and $\mu' : C' \otimes C' \to C'$ up to some homotopy. Then $\psi f : \psi C \to \psi C'$ is a strict monoid homomorphism.*

Proof. (1) Since the only degree 0 element of C is the unit, and μ is unital, the associativity relation $\mu(\mu(a \otimes b) \otimes c) = \mu(a \otimes \mu(b \otimes c))$ holds if the degree of a, b or c is 0. If not, the total degree is at least 3 and the relation is trivial on ψC.

(2) Write a_i, b_i, a'_i, b'_i for elements of C_i and C'_i, $i \geq 0$. Since all the maps are unital, $\mu'(fa_i \otimes fb_j) = f\mu(a_i \otimes b_j)$ if i or $j = 0$. It remains to show that this relation holds in the crossed module $\psi C'$ for $i = j = 1$ also.

The homotopy will be given by a degree one function $h : C \otimes C \to C'$ satisfying a certain derivation formula and an analogue of $\partial h + h\partial = \mu'(f \otimes f) - f\mu$, see e.g. [Whi49, BH87].

Clearly $\partial h(a_i \otimes b_j) = 0$ for $\{i, j\} = \{0, 1\}$, and furthermore the tensor product relations in C' say that $\partial'(a'_2 \otimes b'_1) = \partial' a'_2 \otimes b'_1 - a'_2 + {a'_2}^{b'_1}$. In $\psi C'$ we can therefore deduce that C'_1 acts trivially on the elements $a'_2 = h(a_i \otimes b_j)$ for $\{i, j\} = \{0, 1\}$. By the derivation property it now follows that in fact $h\partial(a_1 \otimes b_1) = 0$ in $\psi C'$, and so $\mu'(fa_1 \otimes fb_1) = f\mu(a_1 \otimes b_1)$ here also. □

Corollary 4.13. (1) *Let $M \times M \to M$ be a strictly unital multiplication, where M is one of the following:*

- *a reduced simplicial set,*
- *a reduced CW-complex,*
- *a bisimplicial set with $M_{0,0} = \{point\}$.*

Then $\psi\pi M$, $\psi\pi_{CW} M$ or $\psi\Pi M$ respectively is a monoid in the category of crossed modules.

(2) *Let $N \times N \to N$ be another such structure and $f\colon M \to N$ a morphism which preserves multiplication up to a homotopy. Then f induces a strictly multiplicative homomorphism between the respective monoids in the category of crossed modules.*

Monoids in the category of crossed modules of groups are also termed reduced 2-modules, reduced 2-crossed modules and strict braided categorical groups, see [Con84, BC97, BC91, JS93]. Commutative monoids are similarly termed stable crossed modules, stable 2-modules and strict symmetric categorical groups, see [Con84, BC97, BCC93].

We recall now the usual definition of these concepts, following [BC91] and [BCC93] up to a change of conventions.

Definition 4.14. A *reduced* 2*-module* is a crossed module $\partial\colon M \to N$ together with a map
$$\langle\cdot,\cdot\rangle\colon N\times N \longrightarrow M$$
satisfying the following identities for any $m, m' \in M$ and $n, n', n'' \in N$.

(1) $\partial\langle n, n'\rangle = [n', n]$,
(2) $m^n = m + \langle n, \partial(m)\rangle$,
(3) $\langle n, \partial(m)\rangle + \langle \partial(m), n\rangle = 0$,
(4) $\langle n, n' + n''\rangle = \langle n, n'\rangle^{n''} + \langle n, n''\rangle$,
(5) $\langle n + n', n''\rangle = \langle n', n''\rangle + \langle n, n''\rangle^{n'}$.

Moreover, $(\partial, \langle\cdot,\cdot\rangle)$ is a *stable* 2*-module* if (1), (2), (4) and

(6) $\langle n, n'\rangle + \langle n', n\rangle = 0$

are satisfied.

By (2), the action of N on M is completely determined by the bracket $\langle\cdot,\cdot\rangle$. The first crossed module axiom (4.2) is now redundant, and (4.3) is equivalent to

(7) $\langle \partial(m), \partial(m')\rangle = [m', m]$,

If we take (2) as a definition it is straightforward to check that it does define a group action. Therefore we do not need to require that ∂ is a crossed module, but just a homomorphism of groups.

Comparing (1), (4), (6), (7) with Definition 1.1 one readily obtains the following.

Lemma 4.15. *The category of stable quadratic modules is a full reflective subcategory of the category of stable* 2*-modules, given by those objects*
$$N\times N \xrightarrow{\langle\cdot,\cdot\rangle} M \xrightarrow{\partial} N$$
which satisfy $\langle n, [n', n'']\rangle = 0$ *for all* $n, n', n'' \in N$.

The reflection functor will be denoted by ϕ: **s2mod** $\to$ **squad**.

Another nice feature of monoids in the category of crossed modules of groups is that the property of being commutative is preserved by weak equivalences.

Lemma 4.16. *Let* $C \xrightarrow{\sim} D$ *be a morphism of reduced* 2*-modules which is a weak equivalence. Then* C *is stable if and only if* D *is.*

Proof. The operation $\langle\cdot,\cdot\rangle$ induces a natural quadratic function
$$\operatorname{Coker}\partial \longrightarrow \operatorname{Ker}\partial\colon x \mapsto \langle x, x\rangle,$$
the k-invariant of C. By using the properties of $\langle\cdot,\cdot\rangle$ it is easy to see that C is stable if and only if this quadratic function is indeed a group homomorphism. Therefore the property of being stable is preserved under weak equivalences between reduced 2-modules. □

Remark 4.17. One can obtain a stable 2-module from an $(n-1)$-reduced simplicial group G, $n \geq 2$, by using the following truncation of the Moore complex N_*G
$$N_{n+1}G/d_0(N_{n+2}G) \xrightarrow{d_0} N_nG = G_n.$$
The bracket is defined by
$$\langle x, y\rangle = [s_1(x), s_0(y)] + [s_0(y), s_0(x)], \quad x, y \in G_n.$$

This stable quadratic module will be denoted by $\mu_{n+1}G$. If G is only 0-reduced this formula defines a reduced 2-module $\mu_2 G$. Compare [Con84, BC91, BCC93].

If $C \cong \mu_{n+1}G$ for an $(n-1)$-reduced free simplicial group G, $n \geq 2$, then the natural morphism $C \to \phi C$ is a weak equivalence. This is a consequence of Curtis's connectivity result [Cur65], which implies that we can divide out weight three commutators in G and still obtain the same π_n and π_{n+1}, compare also [Bau91, IV.B]. In order for C to be such a truncation it is enough that the lower-dimensional group of C is free. Indeed suppose that E is the basis of the lower-dimensional group of C. By [Con84] there exists an $(n-1)$-reduced simplicial group G whose Moore complex is given by C concentrated in dimensions n and $n+1$. In particular, $G_n = \langle E \rangle$. By "attaching cells" one can construct a free resolution of G (i.e. a cofibrant replacement) given by a weak equivalence $G' \overset{\sim}{\to} G$ in the category of simplicial groups which is the identity in dimensions $\leq n$. Then $C \cong \mu_{n+1}G \cong \mu_{n+1}G'$. As a consequence we observe that the reflection ϕ preserves weak equivalences between objects with a free low-dimensional group.

Let $\mathrm{Ho}\,\mathbf{Spec}_0$ be the homotopy category of connective spectra of simplicial sets, and let $\mathrm{Ho}\,\mathbf{Spec}_0^1$ be the full subcategory of spectra with trivial homotopy groups in dimensions other than 0 and 1.

Lemma 4.18. *There is a functor*

$$\lambda_0 \colon \mathrm{Ho}\,\mathbf{Spec}_0 \longrightarrow \mathrm{Ho}\,\mathbf{squad}$$

together with natural isomorphisms

$$\pi_i \lambda_0 X \cong \pi_i X, \quad i = 0, 1,$$

which induces an equivalence of categories

$$\lambda_0 \colon \mathrm{Ho}\,\mathbf{Spec}_0^1 \overset{\sim}{\longrightarrow} \mathrm{Ho}\,\mathbf{squad}.$$

Moreover, for any connective spectrum the first Postnikov invariant of X coincides with the k-invariant of $\lambda_0 X$.

Proof. Stable quadratic modules, stable crossed modules and stable 2-modules are known to be algebraic models of the $(n+1)$-type of an $(n-1)$-reduced simplicial set X for $n \geq 3$, see [Bau91, Con84, BC97, BCC93]. All these approaches are essentially equivalent, and they encode the first k-invariant as stated above. For example, if X is an $(n-1)$-reduced simplicial set, $n \geq 3$, then $\mu_n G(X)$ is such a model for the $(n+1)$-type of X. Here we use the Kan loop group $G(X)$. Its projection to stable quadratic modules $\phi\mu_n G(X)$ is also a model for the $(n+1)$-type of X since $G(X)$ is free, see Remark 4.17 above.

The 1-type of a connective spectrum X of simplicial sets is completely determined by the 4-type of the third simplicial set Y_3 of a fibrant replacement (in particular an Ω-spectrum) Y of X. We can always assume that Y_3 is 2-reduced. Therefore we can define the functor λ_0 above as follows. Each spectrum X is sent by λ_0 to $\phi\mu_3 G(Y_3)$. □

Lemma 4.19. *Given a 1-reduced simplicial set X there is a natural isomorphism of monoids in crossed modules $\psi\underline{\Omega}_{\mathbf{Crs}} X \cong \mu_2 G(X)$.*

Proof. This lemma is not surprising, since both $\psi\underline{\Omega}_{\mathbf{Crs}} X$ and $\mu_2 G(X)$ are models for the 2-type of the loop space of X, and moreover they have the same low-dimensional group $\langle X_2 - * \rangle$.

Using the presentation of $\underline{\Omega}_{\mathbf{Crs}}X$ as a monoid in the category of crossed complexes given in [BT97, Theorem 2.8] and the convention followed by May [May67, Definition 2.6.3] for the definition of $G(X)$, a natural isomorphism $\chi\colon \psi\underline{\Omega}_{\mathbf{Crs}}X \cong \mu_2 G(X)$ can be described on the monoid generators as follows. Given $x_2 \in X_2$, let $\chi(x_2) = x_2$, and given $x_3 \in X_3$,

$$\chi(x_3) = -s_1 d_2(x_3) + x_3 - s_2 d_3(x_3) + s_1 d_3(x_3).$$

This is the identity in low-dimensional groups. In order to check that it indeed defines an isomorphism in the upper groups one can use the presentation of $\underline{\Omega}_{\mathbf{Crs}}X$ in [BT97], and a computation of the Moore complex of $G(X)$ in low dimensions by using the Reidemeister-Schreier method, see [Kan58, 18] and [MKS66]. $\square$

In the statement of the following lemma we use the Moore loop complex functor Ω on the category of fibrant simplicial sets. Given a 1-reduced Kan complex X, define ΩX by

$$(\Omega X)_n \quad = \quad \operatorname{Ker}[d_{n+1}\colon X_{n+1} \to X_n]$$

in the category of pointed sets; compare [Cur71, 2.9], [May67, Definition 23.3]. The face and degeneracy operators are restrictions of the operators in X. If X is a simplicial group then so is ΩX.

Recall that the natural simplicial map

$$\tau_X : \Omega X \to GX$$

given by $(\Omega X)_n \subset X_{n+1} \to \langle X_{n+1}\rangle \twoheadrightarrow \langle X_{n+1} - s_0 X_n\rangle$ is a homotopy equivalence when X is a 1-reduced Kan complex. The composite of $\pi_n\tau_X$ with $\pi_{n+1}X \cong \pi_n\Omega X$ coincides with the connecting map $\delta\colon \pi_{n+1}X \cong \pi_n GX$ in the path–loop group fibration $GX \to EX \to X$.

Lemma 4.20. *For any 2-reduced Kan complex X there is a natural weak equivalence of simplicial groups $\sigma\colon G(\Omega X) \xrightarrow{\sim} \Omega G(X)$.*

Proof. For all $n \geq 0$ we have

$$G_n(\Omega X) \cong \langle (\Omega X)_{n+1} - s_0(\Omega X)_n\rangle,$$
$$(\Omega G(X))_n \subset G_{n+1}(X) \cong \langle X_{n+2} - s_0 X_{n+1}\rangle.$$

The homomorphisms $\sigma_n : G_n(\Omega X) \to (\Omega G(X))_n$ are the unique possible homomorphisms compatible with the inclusions $(\Omega X)_k \subset X_{k+1}$, $k \geq 0$, in the obvious way. Since $\sigma \circ \tau_{\Omega X} = \Omega\tau_X : \Omega\Omega X \to \Omega GX$, the map σ is a weak equivalence. $\square$

Now we are ready for the proof of the main theorem of this paper.

Proof of Theorem 1.5. The coproduct in $\mathbf{C}$ gives rise to a Γ-space A in the sense of Segal [Seg74] with $A(\mathbf{1}) = |\operatorname{Diag}\operatorname{Ner} wS.\mathbf{C}|$, see [Wal78, Section 4, Corollary]. The spectrum of topological spaces $A(\mathbf{1})$, $BA(\mathbf{1})$, $B^2A(\mathbf{1})$,... associated to A is an Ω-spectrum since $\operatorname{Diag}\operatorname{Ner} wS.\mathbf{C}$ is reduced. The Ω-spectrum defining $K\mathbf{C}$ is obtained from the spectrum of A by shifting the dimensions by $+1$, i.e. $K\mathbf{C}$ is given by

$$\Omega A(\mathbf{1}), A(\mathbf{1}), BA(\mathbf{1}), B^2A(\mathbf{1}), \dots.$$

A particular choice of the coproduct $A \vee B$ of any pair of objects A, B in $\mathbf{C}$ induces a product in $\operatorname{Ner} wS.\mathbf{C}$. We choose $A \vee * = A = * \vee A$ so that this product is strictly unital as in Corollary 4.13. The structure weak equivalence

(a) $$|\operatorname{Diag}\operatorname{Ner} wS.\mathbf{C}| \xrightarrow{\sim} \Omega BA(\mathbf{1})$$

is a morphism of H-spaces up to homotopy.

We can replace $BA(\mathbf{1})$ and $B^2A(\mathbf{1})$ by homotopy equivalent spaces $|Y_2|$, $|Y_3|$ which are realizations of a 1-reduced fibrant simplicial set Y_2 and a 2-reduced fibrant simplicial set Y_3, respectively. As a consequence we obtain a replacement for (a) consisting of a homotopy equivalence of pointed CW-complexes

$$|\mathrm{Diag}\,\mathrm{Ner}\,wS.\mathbf{C}| \xrightarrow{\sim} \underline{\Omega}_{\mathbf{FTop}}Y_2. \tag{b}$$

Here $\underline{\Omega}_{\mathbf{FTop}}Y_2$ is the model for $\Omega\,|Y_2|$ in [BT97, Theorem 2.7]. The CW-complex $\underline{\Omega}_{\mathbf{FTop}}Y_2$ is a monoid and the map (b) is in the conditions of the statement of Corollary 4.13.

In order to define $\lambda_0K\mathbf{C}$ as $\phi\mu_3G(Y_3)$ we choose an Ω-spectrum Y in the category of simplicial sets representing $K\mathbf{C}$ with Y_2 and Y_3 the simplicial sets chosen above.

Combining the results above we obtain the following weak equivalences of stable 2-modules.

$$\begin{array}{llll}
\psi\Pi\,\mathrm{Ner}\,wS.\mathbf{C} & \xrightarrow{\sim} & \psi\pi\,\mathrm{Diag}\,\mathrm{Ner}\,wS.\mathbf{C} & \text{(Theorems 4.7 and 4.8)} \\
 & = & \psi\pi_{CW}\,|\mathrm{Diag}\,\mathrm{Ner}\,wS.\mathbf{C}| & \\
 & \xrightarrow{\sim} & \psi\pi_{CW}\underline{\Omega}_{\mathbf{FTop}}Y_2 & \text{(b)} \\
 & \cong & \psi\underline{\Omega}_{\mathbf{Crs}}Y_2 & \text{[BT97, proof of Proposition 2.11]} \\
 & \cong & \mu_2G(Y_2) & \text{(Lemma 4.19)} \\
 & \xrightarrow{\sim} & \mu_2G(\Omega Y_3) & \text{(Induced by } Y_2 \xrightarrow{\sim} \Omega Y_3\text{)} \\
 & \xrightarrow{\sim} & \mu_2\Omega G(Y_3) & \text{(Lemma 4.20)} \\
 & = & \mu_3G(Y_3). &
\end{array}$$

Here we use Lemma 4.16 to derive that not only $\mu_3G(Y_3)$ but all these reduced 2-modules are indeed stable.

Since ϕ preserves weak equivalences between stable 2-modules with free lower-dimensional group, see Remark 4.17, we obtain

$$\phi\psi\Pi\,\mathrm{Ner}\,wS.\mathbf{C} \xrightarrow{\sim} \lambda_0K\mathbf{C}.$$

Finally the formulas in Lemma 4.6 and Example 4.11 together with the laws of a stable quadratic module show that

$$\mathcal{D}_*\mathbf{C} = \phi\psi\Pi\,\mathrm{Ner}\,wS.\mathbf{C}.$$

In fact this was how we obtained the definition of $\mathcal{D}_*\mathbf{C}$. □

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