EISENSTEIN SERIES AND QUANTUM AFFINE ALGEBRAS

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The classical Eisenstein-Maass series is the sum

$$E(z,s)=\frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ (c,d)=1}}\frac{y^{s/2}}{|cz+d|^2},\quad z=x+yi\in\mathbf{C}, y>0, s\in\mathbf{C}. \tag{1}$$

It converges for $\mathrm{Re}(s)>1$ and analytically continues to a meromorphic function in s, which satisfies the functional equation

$$\zeta^*(s)E(z,s)=\zeta^*(1-s)E(z,2-s), \tag{2}$$

where $\zeta^*(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the full zeta function of $\mathrm{Spec}(\mathbf{Z})$.

This paper started from the observation that the simplest function field analog of (1), in which $\mathbf{Z}$ is replaced by the polynomial ring $\mathbf{F}_q[x]$, is related to the quantum affine algebra $U_q(\widehat{sl_2})$. More precisely, the Eisenstein series in this case is naturally a function on $\mathrm{Bun}(P^1)$, the set of isomorphism classes of vector bundles on $P^1_{\mathbf{F}_q}$, and the space R of such functions has a natural algebra structure (Hall algebra, as modified by Ringel [R3]), given by the parabolic induction. The analog of $E(z,s)$ is in fact the product $E(t_1)E(t_2)$ of two simpler elements of R, and the analog of (2) can be written as the commutation relation in R:

$$E(t_2)E(t_1)=q\frac{\zeta_{P^1}(t_2/t_1)}{\zeta_{P^1}(q^{-1}t_2/t_1)}E(t_1)E(t_2), \tag{3}$$

where $\zeta_{P^1}(t)=1/(1-t)(1-qt)$ is the zeta-function of $P^1_{\mathbf{F}_q}$.

Relations like this are familiar in the theory of vertex operators [FLM] [FJ]. In particular, (3), being written in the polynomial form, turns out to be identical to one of the relations written by Drinfeld [Dr1-2] for his "loop realization" of $U_q(\widehat{sl_2})$. So R is identified with the natural "pointwise uppertriangular" subalgebra $U_q(\widehat{\mathfrak{n}}^+)\subset U_q(\widehat{sl_2})$.

The main results of this paper (Theorems 3.3, 3.8.4 and 6.7) show that for an arbitrary smooth projective curve X over $\mathbf{F}_q$ the structure of the algebra R formed by unramified automorphic forms and of its natural extensions, is strikingly similar to the structure of the quantum affine algebras $U_q(\widehat{\mathbf{g}})$ for Kac-Moody algebras $\mathbf{g}$ and of their natural subalgebras. The role of the simple roots of $\mathbf{g}$ is played here by unramified cusp eigenforms, and the role of the Cartan matrix is played by the Rankin-Selberg convolution L-functions. Thus R is analogous to $U_q(\widehat{\mathfrak{n}}^+)$ with

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functional equations of general Eisenstein series providing commutation relations among the generating functions.

These results can be viewed as "continual" analogs of results of Ringel [R1-3] and Lusztig [Lu 1-3] on realizing quantum Kac-Moody algebras by means of Hall algebras associated to the category of representations of a quiver, instead of the category of coherent sheaves on a curve. The two types of categories have very similar properties, in particular, they have homological dimension 1.

The Langlands conjectures bring cusp forms (i.e., simple roots of our generalized root system) in correspondence with irreducible representations of the geometric fundamantal group of the curve X. This becomes therefore analogous to the well known McKay correspondence for finite subgroups in $SL(2, \mathbf{C})$. These subgroups are in correspondence with affine Dynkin graphs of type A, D, E, and for a subgroup G the vertices of the corresponding graph correspond to irreducible representations of G. Including all coherent sheaves (and not just vector bundles) into the general framework of Hall algebras amounts to considering Hecke operators acting in unramified automorphic forms. It turns out that some natural generating functions for these Hecke operators are analogous to Drinfeld's generators for the pointwise-Cartan subalgebra $U_q(\mathbf{h}[t]) \subset U_q(\widehat{\mathbf{g}})$.

One nice outcome of this analogy is that it finally provides some explanation of Drinfeld's construction, which since its publication 10 years ago, was reproduced and used many times, but without questioning its nature (i.e., asking why the formulas have exactly this form and not some other). From our point of view, each of these formulas has a clear conceptual meaning. Some of them express functional equations for Eisenstein series, others the fact that Eisenstein series are eigenfunctions of Hecke operators, still other express the commutativity of the unramified Hecke algebras and so on.

What seems more important, though, is the conclusion this analogy brings about the theory of automorphic forms. Namely, the algebra formed by all unramified automorphic forms (on all the GL_n) and by the Hecke operators, corresponds in our analogy, to just one half of $U_q(\widehat{\mathbf{g}})$, namely the quantization of the subalgebra $\mathbf{n}^+[t, t^{-1}] \oplus \mathbf{h}[t]$. It means that one should "double" the whole theory of automorphic forms by finding the automorphic analog of the missing subalgebra $\mathbf{n}^-[t, t^{-1}] \oplus \mathbf{h}[t^{-1}]$. In this paper we do this (in our unramified context) by applying Drinfeld's quantum double construction [CP] to the Hopf algebra structure given, essentially, by taking the constant terms of an automorphic form. Certainly, there should be a better and more conceptual definition of this double, and the author plans to address this in a future paper. However, the purely algebraic identification of the double given in Theorem 6.7, exhibits it as a self-contained quantum group-like structure involving all the automorphic L-functions at once, and one can begin to study its representation theory, which is of great interest because of the relation with L-functions. For instance, one can generalize Frenkel-Jing bosonization construction [FJ] to our automorphic case. This will be done elsewhere; here let us point out just one thing: the validity of the most non-trivial relation (6.7.5) for the bosonization operators is equivalent to the fact that the zeta-function $\zeta_X(t)$ has only two "trivial" poles at $t = 1, q^{-1}$ (and they produce the two summands in the RHS of (6.7.5)) and all the other L-functions associated with cusp forms have no poles.

There are two further contexts in which the described approach can be pursued. One is that of "geometric Langlands correspondence", in which we consider a curve X over the complex numbers rather than $\mathbf{F}_q$. Instead of functions on the discrete set $\mathrm{Bun}(X/F_q)$ one should consider here the cohomology of the moduli stack of all vector bundles (such stack-theoretic cohomology involves, for instance, the cohomology of classifying spaces of stabilizers of points, i.e., behaves like equivariant cohomology). Results of Grojnowski [Gro] on equivariant cohomology of the spaces of $\mathbf{C}$-representations of quivers suggest that in our case we should get quantized double-affine algebras, as in [GKV].

Another direction concerns automorphic forms over number fields. The results of this paper make it clear that there should be a number-theoretic analog of the theory of quantum affine algebras in which curves over $\mathbf{F}_q$ are replaced with spectra of the rings in number fields. For instance, the most immediate analog of $U_q(\widehat{sl_2})$ corresponds to (compactified) Spec($\mathbf{Z}$) instead of $P^1_{\mathbf{F}_q}$ and is generated by values of operator fields $E^{\pm}(s)$ (generating the analogs of $U_q(\widehat{\mathbf{n}^{\pm}})$) subject to relations like

$$E^+(s)E^+(t) = \frac{\zeta^*(s-t)}{\zeta^*(s-t+1)}E^+(t)E^+(s)$$

and similar other relations involving the Riemann zeta. The author hopes to be able to say more about these questions in the future.

Let us give a brief overview of the contents of the paper. In Section 1 we recall the general framework of Hall-Ringel algebras, including the recent result of Green [Gr] on the categorical description of the comultiplication in the case of homological dimension 1. In Green's formulation one gets, on the Ringel algebra, a structure of a bialgebra in a certain twisted sense (familiar from [Lu 1-2]). To get a bialgebra in the ordinary sense, we add the Cartan generators in the standard way. We also supply a formula (1.6.3) for the antipode in this bialgebra which seems to be new.

Section 2 serves to fix the notation related to unramified automorphic forms on an algebraic curve $X/\mathbf{F}_q$ (with respect to all the groups GL_n). The most important concept for us is that of the Rankin-Selberg convolution L-function associated to two cusp forms.

In Section 3 we formulate the main results about the Hopf algebra formed by the unramified automorphic forms on all the GL_n together with all the Hecke operators. We introduce the appropriate generating functions and state the main result (Theorem 3.3) describing multiplication and comultiplication of these generating functions in a completely explicit way. We also prove (Theorem 3.8.4) that the coefficients of our generating functions generate the subalgebra formed by unramified automorphic forms.

Section 4 contains proofs of the results stated but left unproved in Section 3.

In Section 5 we compare the results of Section 3 with the structure of quantum affine algebras in the "new realization" of Drinfeld. We observe an analogy between two theories; we show also that in the simplest instances (curve P^1, affine algebra $\widehat{sl_2}$) the analogy becomes the identity.

Finally in Section 6 we describe Drinfeld's quantum double of the Hopf algebra constructed in Section 3. This seems to be a very important "semisimple" object naturally appearing in the theory of automorphic forms. The main technical tool here is the use of Heisenberg doubles [AF] [ST] which are easier to handle. In particular, we get a very transparent formula for the multiplication in the Heisenberg double of the Ringel algebra in terms of long exact sequences. Then we find the relations in the Drinfeld double by using the recent work of Kashaev [Kas] who found an embedding of the Drinfeld double into the tensor product of two Heisenberg doubles.

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§1. Hall algebras.

(1.1) The Euler form. We will say that an Abelian category $\mathcal{A}$ is of finite type, if for any objects $A, B \in \mathrm{Ob}(\mathcal{A})$ all the groups $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$ have finite cardinality and are zero for almost all i. If $\mathcal{A}$ is an abelian category of finite type, and $A, B \in \mathrm{Ob}(\mathcal{A})$, we denote

$$(1.1.1) \qquad \langle A, B\rangle = \sqrt{\prod_{i>0} |\mathrm{Ext}^i_{\mathcal{A}}(A, B)|^{(-1)^i}}$$

For $A \in \mathcal{A}$ let $\bar{A}$ be the class of A in the Grothendieck group $\mathcal{K}_0\mathcal{A}$. Clearly, the quantity $\langle A, B\rangle$ depends only on $\bar{A}$ and $\bar{B}$ and descends to a bilinear form (called the Euler form) still denoted by

$$(1.1.2) \qquad \alpha, \beta \mapsto \langle \alpha, \beta\rangle, \quad \mathcal{K}_0\mathcal{A} \otimes \mathcal{K}_0\mathcal{A} \to \mathbf{Q}^*$$

(1.2) Hall and Ringel algebras. Let $\mathcal{A}$ be an Abelian category of finite type. Its Hall algebra $H(\mathcal{A})$ is the $\mathbf{C}$ -vector space with basis [A] for all isomorphism classes of objects $A \in \mathrm{Ob}(\mathcal{A})$. The multiplication is given by

$$(1.2.1) \qquad [A] \circ [B] = \sum_{[C]} g^C_{AB}[C]$$

for a fixed object C, where g^C_{AB} is the number of subjects $A' \subset C$ such that $A' \simeq A$ and $C/A \simeq B$, or equivalently, the number of exact sequences

$$0 \to A \xrightarrow{\alpha} C \xrightarrow{\beta} B \to 0$$

taken modulo the (free) action of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$. This multiplication is well known to be associative, the coefficient at C in $A_1 \circ \ldots \circ A_n$ being the number of filtrations of C with quotients $A_1, \ldots, A_n$.

The modified multiplication

$$(1.2.2) \qquad [A] * [B] = \langle B, A\rangle \cdot [A] \circ [B]$$

is still associative. We will call the Ringel algebra of $\mathcal{A}$ and denote $R(\mathcal{A})$ the same vector space as $H(\mathcal{A})$ but with $*$ as multiplication.

(1.2.3) Remark. It was C.M. Ringel [R3] who first drew attention to the twist (1.2.2). A little earlier and independently, G. Lusztig [Lu2-3] considered several twistings by bilinear forms, withour specially distinguishing the Euler form (1.1.1). With a certain hindsight, precursors of (1.2.2) can be traced as far back as the relabelling of the principal series representations so as to make the intertwiners to act between representations whose weights differ exactly by permutation, see, e.g. [GN].

(1.3) Moduli space point of view. Let $\mathcal{M}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A})/\mathrm{Iso}$ be the set of isomorphism classes of objects of $\mathcal{A}$. The algebras $H(\mathcal{A}), R(\mathcal{A})$ can be identified with the space of functions $f : M(\mathcal{A}) \to \mathbf{C}$ with finite support, the operations being

(1.3.1)
$$(f \circ g)(A) = \sum_{A' \subset A} f(A')g(A/A') \qquad (f * g)(A) = \sum_{A' \subset A} \langle A/A', A' \rangle f(A')g(A/A').$$

This point of view makes very natural the "orbifold" Hermitian scalar product on $H(\mathcal{A})$ and $R(\mathcal{A})$:

$$(f,g) = \sum_{A \in \mathcal{M}(A)} \frac{f(A)\overline{g(A)}}{|\mathrm{Aut}(A)|} \tag{1.3.2}$$

or, equivalently,

$$([A],[B]) = \delta_{[A][B]}/|\mathrm{Aut}(A)|. \tag{1.3.3}$$

(1.4) Green's comultiplication. Let $\mathcal{A}$ be an Abelian category of finite type, satisfying the following additional condition: every object of $\mathcal{A}$ has only finitely many subobjects. Let $r : R(\mathcal{A}) \to R(\mathcal{A}) \otimes R(\mathcal{A})$ be the map, adjoint to the multiplication map $m : R(\mathcal{A}) \otimes R(\mathcal{A}) \to R(\mathcal{A})$ with respect to the scalar product (1.3.2). It has the form

$$r([A]) = \sum_{A' \subset A} \langle A/A', A' \rangle \frac{|\mathrm{Aut}(A')| \cdot |\mathrm{Aut}(A/A')|}{|\mathrm{Aut}(A)|} [A'] \otimes [A/A'], \tag{1.4.1}$$

or, in the functional language (1.3), for a function $f : M(\mathcal{A}) \to \mathbf{C}$, the element $r(f)$ is a function on $\mathcal{M}(\mathcal{A}) \times \mathcal{M}(\mathcal{A})$ given by

$$r(f)(A', A'') = \langle A'', A' \rangle \sum_{\xi \in \mathrm{Ext}^1(A'', A')} f(\mathrm{Cone}(\xi)[-1]) \tag{1.4.2}$$

where $\mathrm{Cone}(\xi)[-1]$ is the middle term of the extension corresponding to ξ.

For two objects $A, B \in \mathcal{A}$ set

$$(A|B) = \langle A, B \rangle \cdot \langle B, A \rangle. \tag{1.4.3}$$

One easily verifies that the twisted multiplication on $R(\mathcal{A}) \otimes R(\mathcal{A})$ given by

$$([A] \otimes [B])([C] \otimes [D]) = (A|B)(([A] * [C]) \otimes ([B] * [D])) \tag{1.4.4}$$

is associative.

(1.5) Green's theorem. *Suppose that $\mathcal{A}$ satisfies the conditions of (1.4), and, in addition, has homological dimension ≤ 1, i.e., $Ext^i_{\mathcal{A}}(A,B) = 0$ for $i \geq 2$ and all A, B. Then $r : R(\mathcal{A}) \to R(\mathcal{A}) \otimes R(\mathcal{A})$ is an algebra homomorphism, if the multiplication on $R(\mathcal{A}) \otimes R(\mathcal{A})$ is given by (1.4.4).*

Note that because of the twist (1.4.4), the theorem does <u>not</u> mean that $R(\mathcal{A})$ is a bialgebra in the ordinary sense; it can be interpreted, however, by saying that $R(\mathcal{A})$ is a bialgebra in an appropriate braided monoidal category of $\mathcal{K}_0\mathcal{A}$ -graded vector spaces.

In [Gr], Green considered only the case when $\mathcal{A}$ consists of finite modules over an $\mathbf{F}_q$ -algebra. The modification to the case of finite modules over any ring (of homological dimension 1) is trivial. The case of general $\mathcal{A}$, as in (1.5), can be reduced to this by embedding finite pieces of $\mathcal{A}$ into the categories of modules over appropriate rings, as in Freyd's embedding theorem [Fr].

Sources of Green's result can be found in the works of Lusztig [Lu4] and Zelevinsky [Ze] and in more classical studies of the functors of parabolic induction and restriction in representation theory [BerZ], evaluation of constant terms of Eisenstein series [La3] and so on.

(1.6) Reformulation. As with $\langle A, B\rangle$, the quantity $(A|B)$ depends only on $\bar{A}, \bar{B} \in \mathcal{K}_0\mathcal{A}$, giving rise to the form

$$(\alpha|\beta) = \langle\alpha,\beta\rangle \cdot \langle\beta,\alpha\rangle : \mathcal{K}_0\mathcal{A} \otimes \mathcal{K}_0\mathcal{A} \to \mathbf{Q}^* \tag{1.6.1}$$

Let $\mathbf{C}[\mathcal{K}_0\mathcal{A}]$ be the group algebra of $\mathcal{K}_0\mathcal{A}$, with basis K_α, $\alpha \in \mathcal{K}_0\mathcal{A}$ and multiplication $K_\alpha K_\beta = K_{\alpha+\beta}$. Let us extend the algebra $R(\mathcal{A})$ by adding to it these symbols K_α which we make commute with $[A] \in R(\mathcal{A})$ by the rule

$$[A]K_\beta = (\bar{A}|\beta)K_\beta[A]. \tag{1.6.2}$$

Denote the resulting algebra $B(\mathcal{A})$. So as a vector space $B(\mathcal{A}) \simeq \mathbf{C}[\mathcal{K}_0\mathcal{A}] \otimes_{\mathbf{C}} R(\mathcal{A})$, with $K_\alpha \otimes [A] \mapsto K_\alpha A$ establishing the isomorphism.

(1.6.3) Green's theorem (strengthened form). *In the assumptions of (1.5), the map $\Delta : B(\mathcal{A}) \to B(\mathcal{A}) \otimes B(\mathcal{A})$ given by:*

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha,$$

$$\Delta([A]) = \sum_{A' \subset A} \langle A/A', A'\rangle \frac{|Aut(A')| \cdot |Aut(A/A')|}{|Aut(A)|}[A'] \otimes K_{A'}[A/A'] \tag{1.6.4}$$

makes $B(\mathcal{A})$ into a bialgebra in the ordinary sense, i.e., Δ is a homomorphism of algebras with respect to the standard (untwisted) multiplication in $B(\mathcal{A}) \otimes B(\mathcal{A})$. Moreover, $B(\mathcal{A})$ is a Hopf algebra with respect to the counit $\epsilon : B(\mathcal{A}) \to \mathbf{C}$ given by

$$\epsilon(K_\alpha[A]) = \begin{cases} 1, & \textit{if} \quad A = 0 \\ 0, & \textit{if} \quad A \neq 0 \end{cases} \tag{1.6.5}$$

and antipode $S : B(\mathcal{A}) \to B(\mathcal{A})$ *given by*

$$S(K_\alpha[A] = \sum_{n=1}^{\infty}(-1)^n \sum_{A_0\subset\ldots\subset A_n=A}\prod_{i=1}^{n}\langle A_i/A_{i-1}, A_{i-1}\rangle \frac{\prod_{j=0}^{n}|Aut(A_j/A_{j-1})|}{|Aut(A)|}\cdot$$

$$(1.6.6) \qquad \cdot[A_0]*[A_1/A_0]*\ldots*[A_n/A_{n-1}]\cdot K_\alpha^{-1}K_A^{-1}$$

where $A_0 \subset \ldots \subset A_n = A$ *runs over arbitrary chains of strict* $(A_i \neq A_{i+1})$ *inclusions of length* n.

Proof. The fact that Δ is a homomorphism of algebras, follows at once from Theorem 1.5 and from (1.6.2). To prove that ϵ is a counit, we must show that it is an algebra homomorphism and that the compositions

$$(Id\otimes\epsilon)\Delta, \quad (\epsilon\otimes Id)\Delta : B(\mathcal{A}) \to B(\mathcal{A})\otimes\mathbf{C} = B(\mathcal{A})$$

are the identity maps. Both these statements are obvious from the nature of multiplication in $H(\mathcal{A}), R(\mathcal{A})$ and $B(\mathcal{A})$. To prove that S is an antipode, we must show that the compositions

$$m(S\otimes Id)\Delta, \quad m(Id\otimes S)\Delta : B(\mathcal{A}) \to B(\mathcal{A})$$

coincide with $i\circ\epsilon$ where $i : \mathbf{C} \to B(\mathcal{A})$ is the embedding of the unit, and m is the multiplication in $B(\mathcal{A})$. Let us show this for the first composition, the second one being similar.

From (1.6.4) and (1.6.6), we find that

$$((S\otimes Id)\Delta)(K_\alpha[A]) = \sum_{A'\subset A}\sum_{n=1}^{\infty}\sum_{A'_0\subset\ldots\subset A'_n=A'}(-1)^n\cdot\langle A/A', A'\rangle\cdot\prod_{i=1}^{n}\langle A'_i/A'_{i-1}, A'_{i-1}\rangle\cdot$$

$$\cdot\frac{|\mathrm{Aut}(A/A')|\cdot\prod_{j=0}^{n}|\mathrm{Aut}(A'_j/A'_{j-1})|}{|\mathrm{Aut}(A)|}\cdot[A'_0]\ldots[A'_n/A'_{n-1}]K_A^{-1}K_\alpha^{-1}\otimes K_\alpha K_A[A/A']$$

where the first sum is over all subobjects $A' \subset A$. We can combine the first and third summations together and write the above quantity as

$$\sum_{m=1}^{\infty}(-1)^{m-1}\sum_{A_0\subset\ldots\subset A_m=A}\prod_{i=1}^{m}\langle A_i/A_{i-1}, A_{i-1}\rangle\frac{\prod_{j=0}^{m}|\mathrm{Aut}(A_j/A_{j-1})|}{|\mathrm{Aut}(A)|}\cdot$$

$$\cdot[A_0]\ldots[A_{m-1}/A_{m-2}]K_{A_{m-1}}^{-1}K_\alpha^{-1}\otimes K_\alpha K_{A_{m-1}}[A_m/A_{m-1}],$$

where $A_0 \subset \ldots \subset A_m = A$ runs over all chains of subobjects of length m, in which all the inclusions, except, maybe, $A_{m-1} \subset A_m$, are strict. Therefore

$$(m(S\otimes Id)\Delta)(K_\alpha[A]) =$$

$$= \sum_{m-1}^{\infty}(-1)^{m-1}\sum_{A_0\subset\ldots A_m=A}\prod_{i=1}^{m}\langle A_i/A_{i-1}, A_{i-1}\rangle\frac{\prod_{j=0}^{m}|\mathrm{Aut}(A_j/A_{j-1})|}{|\mathrm{Aut}(A)|}\cdot[A_0]\ldots[A_m/A_{m-1}]$$

Now notice that for $A \neq 0$ each summand in this sum will appear twice: once for a strictly increasing filtration $A_0 \subset \dots A_m$ and once for the filtration $A_0 \subset \dots \subset A_m = A_{m-1}$. These summands will enter with opposite signs and so will cancel each other, and the whole sum will be equal to $0 = i(\epsilon(K_\alpha[A]))$. If $A = 0$, we get the sum of the empty set of summands, which is equal to $1 = i(\epsilon(K_\alpha))$. Theorem is proved.

(1.7) The bilinear form on $B(\mathcal{A})$. Let us extend the Hermitian form (1.3.2-3) on the Ringel algebra $R(\mathcal{A})$ to $B(\mathcal{A})$ by putting

$$(K_\alpha[A], K_\beta[B]) = (\alpha|\beta)([A],[B]) = \frac{(\alpha|\beta)\delta_{[A],[B]}}{|\mathrm{Aut}(A)|} \tag{1.7.1}$$

In other words, we introduce the form on $B(\mathcal{A}) = \mathbf{C}[\mathcal{K}_0\mathcal{A}] \otimes_{\mathbf{C}} R(\mathcal{A})$ to be the tensor product of the old form on $R(\mathcal{A})$ and the form on $\mathbf{C}[\mathcal{K}_0\mathcal{A}]$ given by $(K_\alpha, K_\beta) = (\alpha|\beta)$.

(1.7.2) Proposition. *With respect to the form (,) the multiplication and comultiplication in the Hopf algebra $B(\mathcal{A})$ are adjoint to each other.*

Proof. In other words, we need to prove the equality

$$(K_\alpha[A]K_\beta[B], K_\gamma[C]) = (K_\alpha[A] \otimes K_\beta[B], \Delta(K_\gamma[C])) \tag{1.7.3}$$

where on the right stands the Hermitian bilinear form on $B(\mathcal{A}) \otimes B(\mathcal{A})$ given by tensoring (,) with itself. To prove (1.7.3), notice that the left hand side is

$$\begin{aligned}(\bar{A}|\beta)(K_{\alpha+\beta}[A][B], K_\gamma[C]) &= (\bar{A}|\beta)(\alpha+\beta|\gamma)([A][B],[C]) = \\ &= (\bar{A}|\beta)(\alpha+\beta|\gamma)([A]\otimes[B], r([C])) = \\ &= (\bar{A}|\beta)(\alpha+\beta|\gamma)\sum_{C'\subset C}\langle C/C', C'\rangle \frac{|\mathrm{Aut}(C')|\cdot|\mathrm{Aut}(C/C')|}{|\mathrm{Aut}(C)|}\cdot \\ &\quad \cdot(A,C')\cdot(B,C/C'),\end{aligned} \tag{1.7.4}$$

while the right hand side of (1.7.3) is

$$\begin{aligned}&\Big(K_\alpha[A]\otimes K_\beta[B], \quad \sum_{C'\subset C}\langle C/C', C'\rangle \frac{|\mathrm{Aut}(C')|\cdot|\mathrm{Aut}(C/C')|}{|\mathrm{Aut}(C)|}\cdot \\ &\qquad \cdot K_\gamma[C']\otimes K_{\bar{C}+\gamma}[C'']\Big) = \\ &= \sum_{C'\subset C}\langle C/C', C'\rangle \frac{|\mathrm{Aut}(C')|\cdot|\mathrm{Aut}(C/C')|}{|\mathrm{Aut}(C)|}(\alpha|\gamma)(\beta|\gamma)(\beta|\bar{C}')\cdot \\ &\qquad \cdot(A,C')\cdot(B,C/C').\end{aligned} \tag{1.7.5}$$

Notice now that in order that $(A, C') \neq 0$, we should have $A \simeq C'$, and under this assumption the corresponding summands in (1.7.4) and (1.7.5) coincide. Proposition is proved.

§2 Background material related to automorphic forms.

(2.1) Notations and conventions. Let X be a smooth projective algebraic curve over a finite field $\mathbf{F}_q$. In this paper we will be interested in the Hall algebra of $\mathcal{A} = \mathrm{Coh}_X$, the category of all coherent sheaves on X. Let us start by introducing some notations and conventions, to be used throughout the rest of the paper.

By g_X we denote the genus of X. By a point of X we always mean a 0-dimensional point x (notation: $x \in X$). For such a point x we denote by q_x the cardinality of $\mathbf{F}_q(x)$, the residue field of x, and by $\deg(x)$ the degree $[\mathbf{F}_q(x) : \mathbf{F}_q]$. Thus $q_x = q^{\deg x}$. By $\mathrm{Pic}(X)$ we denote the Picard group of line bundles on X (defined over $\mathbf{F}_q$). For $L \in \mathrm{Pic}(X)$ we denote by $\deg(L) \in \mathbf{Z}$ the degree (first Chern class) of L. Thus for $x \in X$ we have $\deg(x) = \deg(\mathcal{O}_X(x))$. For a vector bundle V on X of rank n we set $\deg(V) = \deg(\Lambda^n V)$. The kernel of the degree homomorphism $\mathrm{Pic}(X) \to \mathbf{Z}$ is denoted $\mathrm{Pic}^0(X)$. It is a finite Abelian group.

As usual, we identify $\mathrm{Pic}(X)$ with the group of divisors modulo linear equivalence, by associating to a divisor $D = \sum n_x \cdot x$ the line bundle $\mathcal{O}(D)$. Thus $\deg D = \deg(\mathcal{O}(D))$.

(2.2) The adelic language. Automorphic forms. Let $\mathrm{Bun}_n(X)$ be the set of isomorphism classes of rank n vector bundles on X and $\mathrm{Bun}_{n,d}(X) \subset \mathrm{Bun}_n(X)$ the set of isomorphism classes of bundles of degree d.

Let $k = \mathbf{F}_q(X)$ be the field of rational functions on X, $\mathbf{A}$ its ring of adeles and $\widehat{\mathcal{O}} \subset \mathbf{A}$ the ring of integer adeles. Then

$$\mathrm{Bun}_n(X) \simeq GL_n k \setminus GL_n \mathbf{A} / GL_n \widehat{\mathcal{O}}. \tag{2.2.1}$$

Let AF_n (resp. $\mathrm{AF}_{n,d}$) be the space of all complex valued functions on $\mathrm{Bun}_n(X)$ (resp. $\mathrm{Bun}_{n,d}(X)$). By (2.2.1) we can regard such functions as $GL_n\widehat{\mathcal{O}}$ -invariant automorphic forms on $GL_n\mathbf{A}$.

A function $f \in \mathrm{AF}_n$ is called a cusp form if for any vector bundles $V' \in \mathrm{Bun}_{n'}(X), V'' \in \mathrm{Bun}_{n''}(X), n' + n'' = n$, $n', n'' > 0$, we have

$$\sum_{\xi \in \mathrm{Ext}^1(V'', V')} f(\mathrm{Cone}(\xi)[-1]) = 0, \tag{2.2.2}$$

compare with (1.4.2). This is equivalent to the standard condition

$$\int_{U(k)\setminus U(\mathbf{A})} f(ug)du = 0 \tag{2.2.3}$$

where we view f, by (2.2.1), as a function on $GL_n\mathbf{A}$ and $U \subset GL_n\mathbf{A}$ is the unipotent radical of a minimal parabolic subgroup.

Let $\mathrm{AF}_n^{cusp} \subset \mathrm{AF}_n, \mathrm{AF}_{n,d}^{cusp} \subset \mathrm{AF}_{n,d}$ be the subspaces formed by cusp forms. The following fact is a well known consequence of the reduction theory of Harder [Ha2] [MW].

(2.2.4) Proposition. *Every function from* $AF_{n,d}^{cusp}$ *has finite support. The space* $AF_{n,d}^{cusp}$ *is finite-dimensional.*

(2.3) Hecke operators. Let $\mathrm{Coh}_{0,X}$ be the category of coherent sheaves on X with 0-dimensional support. By $\mathrm{Coh}_0^{\leq n}(X)$ we denote the set of isomorphism classes of such sheaves $\mathcal{F}$ which satisfy the additional property $\dim(\mathcal{F}\otimes\mathcal{O}_x)\leq n$ for all $x\in X$. We have an identification

$$\mathrm{Coh}_0^{\leq n}(X) \simeq GL_n\widehat{\mathcal{O}}\setminus GL_n\mathbf{A}\cap \mathrm{Mat}_n(\widehat{\mathcal{O}})/GL_n\widehat{\mathcal{O}}, \tag{2.3.1}$$

which takes $g\in GL_n\mathbf{A}\cap\mathrm{Mat}_n\widehat{\mathcal{O}}$ into the sheaf $\mathrm{Coker}\{g:\widehat{\mathcal{O}}^n\to\widehat{\mathcal{O}}^n\}$.

For $\mathcal{F}\in\mathrm{Coh}_{0,X}$ and $n>0$ we define the operator $T_{\mathcal{F}}:\mathrm{AF}_n\to\mathrm{AF}_n$ by

$$(T_{\mathcal{F}}f)(V)=\sum_{\substack{V'\subset V\\ V/V'\simeq\mathcal{F}}} f(V') \tag{2.3.2}$$

where the sum is over coherent subsheaves V' in $V\in\mathrm{Bun}_n(X)$ with quotient isomorphic to $\mathcal{F}$. (Since V' is locally free, $f(V')$ is defined.) Clearly, $T_{\mathcal{F}}=0$ on AF_n unless $\mathcal{F}\in\mathrm{Coh}_0^{\leq n}(X)$. If $\mathcal{F}\in\mathrm{Coh}_0^{\leq n}(X)$, one can describe $T_{\mathcal{F}}$ in the adelic language as the operator taking a function $f:GL_nk\setminus GL_n\mathbf{A}/GL_n\widehat{\mathcal{O}}\to\mathbf{C}$ into $T_{\mathcal{F}}f$ given by

$$(T_{\mathcal{F}}f)(g)=\int_{h\in GL_n\mathbf{A}} f(gh^{-1})\mathbf{1}_{\mathcal{F}}(h)dh \tag{2.3.3}$$

where $\mathbf{1}_{\mathcal{F}}$ is the characteristic function of the double coset corresponding to $\mathcal{F}$ by (2.3.1). For this reason, $T_{\mathcal{F}}$ is called the Hecke operator.

(2.3.4) Proposition. *The correspondence* $[\mathcal{F}]\mapsto T_{\mathcal{F}}$ *makes* AF_n *into a left module over the Hall algebra* $H(Coh_{0,X})$, *and* $AF_n^{cusp}\subset AF_n$ *is a submodule.*

Proof. Let $\mathcal{M}$ be the set of isomorphism classes of all coherent sheaves on X, and $\mathbf{C}[\mathcal{M}]$ be the space of all functions $\mathcal{M}\to\mathbf{C}$. This is just the vector space dual to the Hall algebra $H(\mathrm{Coh}_X)$ and is therefore an $H(\mathrm{Coh}_X)$ -bimodule. The left action of $[\mathcal{F}]\in H(\mathrm{Coh}_{0,X})$ on $\mathbf{C}[\mathcal{M}]$ (dual to its right action on $H(\mathrm{Coh}_X)$) is given by the formula identical to (2.3.2), but in which V,V' are arbitrary sheaves. If we view AF_n as a subspace in $\mathbf{C}[\mathcal{M}]$ (consisting of functions vanishing outside $\mathrm{Bun}_n(X)$), then it is preserved by this action, so is an $H(\mathrm{Coh}_{0,X})$ -module as claimed.

To see that $\mathrm{AF}_n^{cusp}\subset\mathrm{AF}_n$ is a submodule, note that in the adelic language the condition for $f\in\mathrm{AF}_n$ to the cuspidal involves left shifts of f, while the Hecke operators involve right shifts. □

Let $\mathrm{Coh}_{x,X}$ be the category of coherent sheaves on X supported at x (so each such sheaf has the form $\bigoplus\mathcal{O}_X/I_x^{\lambda_i}$, where $I_x\subset\mathcal{O}_X$ is the ideal of x). The following facts are well known.

(2.3.5) Proposition. *(a) For $\mathcal{F}, \mathcal{G} \in Coh_{0,X}$ we have $\langle \mathcal{F}, \mathcal{G} \rangle = 1$, so the multiplications $\circ, *$ in $H(Coh_{0,X})$ and $R(Coh_{0,X})$ coincide.*
(b) $H(Coh_{0,X}) = \bigotimes_{x \in X} H(Coh_{x,X})$ (the restricted tensor product in which almost all factors in any decomposable tensor are required to be 1).
(c) Each $H(Coh_{0,X})$ is a commutative polynomial algebra in either of the following two sets of generators:

$$(c1) \qquad [\mathcal{O}_x^{\oplus n}], n \geq 1,$$

$$(c2) \qquad [\mathcal{O}/I_x^n], n \geq 1.$$

(d) Let $\Lambda = \varprojlim \mathbf{C}[z_1, \ldots, z_n]^{S_n}$ be the ring of symmetric functions, and define an isomorphism

$$Ch : H(Coh_{x,X}) \to \Lambda, [\mathcal{O}_x^{\oplus n}] \mapsto q_x^{-n(n-1)/2} e_n(z_1, \ldots, z_n)$$

where e_n is the elementary symmetric function. Then for any integer sequence $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r \geq 0)$ the element $[\bigoplus \mathcal{O}_X/I_x^{\mu_i}]$ will go into $q^{-\Sigma(i-1)\mu_i} P_\mu(z_1, \ldots, z_n; q_x^{-1})$ where $P_\mu(z_1, \ldots, z_N, t)$ is the Hall-Littlewood polynomial.

Proof. Part (a) is easily obtained by reduction by devissage to the case $\mathcal{F} = \mathcal{O}_x, \mathcal{G} = \mathcal{O}_y$. Part (b) is obvious, while (c) and (d) can be found in Macdonald [Mac].

(2.4) Cusp eigenforms. The sets $\mathbf{Cusp}_n$. Let $\widehat{\mathrm{Pic}(X)}$ be the group of all homomorphisms (characters) $\mu : \mathrm{Pic}(X) \to \mathbf{C}^*$. There is an embedding

$$(2.4.1) \qquad \mathbf{C}^* \hookrightarrow \widehat{\mathrm{Pic}(X)}, t \mapsto t^{\deg} : L \mapsto t^{\deg L}$$

whose cokernel is a finite group of characters of $\mathrm{Pic}^0(X)$. Let us choose representatives $\mu_1, \ldots, \mu_h$, one in each coset by the image of (2.4.1), which are unitary, i.e., $|\mu_i(L)| = 1$ for any $L \in \mathrm{Pic}(X)$.

For any character $\mu : \mathrm{Pic}(X) \to \mathbf{C}^*$ we denote by $\mathrm{AF}_n(\mu)$ the space of functions (automorphic forms) $f : \mathrm{Bun}_n(X) \to \mathbf{C}$ satisfying the property

$$(2.4.2) \qquad f(V \otimes L) = \mu(L) f(V), \quad \forall L \in \mathrm{Pic}(X).$$

Let $\mathrm{AF}_n^{cusp}(\mu) \subset \mathrm{AF}_n(\mu)$ be the subspace formed by cusp forms. By (2.2.4) $\dim \mathrm{AF}_n^{cusp}(\mu) < \infty$. The Hecke operators $T_\mathcal{F}, \mathcal{F} \in \mathrm{Coh}_0(X)$, preserve $\mathrm{AF}_n(\mu)$ and $\mathrm{AF}_n^{cusp}(\mu)$. For any algebra homomorphism $\chi : H(\mathrm{Coh}_{0,X}) \to \mathbf{C}$ denote by $\mathrm{AF}_n^{cusp}(\mu)_\chi$ the corresponding eigenspace, i.e., the space of $f \in \mathrm{AF}_n^{cusp}(\mu)$ such that

$$(2.4.3) \qquad T_\mathcal{F} f = \chi([\mathcal{F}]) \cdot f, \qquad \forall \mathcal{F} \in \mathrm{Coh}_{0,X}$$

By the multiplicity one theorem for GL_n, see [Sh], the space $\mathrm{AF}_n^{cusp}(\mu)_\chi$ has dimension at most 1. Let $\mathfrak{X}_n(\mu)$ be the set of χ such that $\dim \mathrm{AF}_n^{cusp}(\mu)_\chi = 1$. If $\chi \in \mathfrak{X}_n(\mu)$, then for any $x \in X$ we have

$$\chi([\mathcal{O}_x^{\oplus n}]) = \mu(\mathcal{O}_X(-x)). \tag{2.4.4}$$

We have a direct sum decomposition

$$\mathrm{AF}_n^{cusp}(\mu) = \bigoplus_{\chi \in \mathfrak{X}_n(\mu)} \mathrm{AF}_n^{cusp}(\mu)_\chi. \tag{2.4.5}$$

Choose a non-zero vector f_χ in each summand in (2.4.5) (in virtue of (2.4.4), μ is determined by χ so it can be dropped from the notation). Let Cusp_n be the set $\bigcup_{i=1}^h \{f_\chi, \chi \in \mathfrak{X}_n(\mu_i)\}$ where $\{\mu_1, \dots, \mu_h\}$ are our unitary representatives (see above). Let also

$$\mathrm{Cusp} = \coprod_{n \geq 1} \mathrm{Cusp}_n.$$

(2.5) Rankin L-functions. For $f = f_\chi \in \mathrm{Cusp}_n$ and $x \in X$ introduce the numbers $\lambda_{i,x}(f), i = 1, \dots, n$ defined up to permutation by the condition

$$e_l\big(\lambda_{1,x}(f)^{-1}, \dots, \lambda_{n,x}(f)^{-1}\big) = q_x^{l(l-n)/2} \chi([\mathcal{O}_x^l]) \tag{2.5.1}$$

where e_l is the elementary symmetric function. By (2.3.5) (d) we have, for any $\mu = (\mu_1 \geq \dots \geq \mu_r \geq 0)$:

(2.5.2)

$$\chi\left(\left[\bigoplus \mathcal{O}/I_x^{\mu_i}\right]\right) = q^{-\Sigma(i-1)\mu_i} P_\mu\left(q_x^{\frac{n-1}{2}} \lambda_{1,x}(f)^{-1}, \dots, q_x^{\frac{n-1}{2}} \lambda_{n,x}(f)^{-1}; q_x^{-1}\right)$$

The L-function of f is defined by the product

$$L(f,t) = \prod_{x \in X} \prod_{i=1}^n \frac{1}{1 - \lambda_{i,x}(f) t^{\deg(x)}} \tag{2.5.3}$$

Our normalization of the $\lambda_{i,x}(f)$ is chosen so as to make $L(f,t)$ satisfy the functional equation exchanging t and $1/qt$ rather than t and $1/q^n t$, as is sometimes done in the theory of automorphic forms.

We will be interested, however, in a more general class of L-functions, which we call Rankin L-functions. Given two cusp eigenforms $f \in \mathrm{Cusp}_n, g \in \mathrm{Cusp}_m$, their Rankin L-function $\mathrm{LHom}(f, g, t)$ is defined by the product

$$\text{(2.5.4)}\qquad \mathrm{LHom}(f,g,t)=\prod_{x\in X}\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-\frac{\lambda_{j,x}(g)}{\lambda_{i,x}(f)}t^{\deg(x)}}$$

The notation LHom is explained as follows. The Langlands correspondence predicts that to any $f\in \mathrm{Cusp}_n$ one can associate a local system $\mathcal{L}_f$ on X. The function $\mathrm{LHom}(f,g,t)$ is the automorphic counterpart of the L-function of the local system $\underline{\mathrm{Hom}}(\mathcal{L}_f,\mathcal{L}_g)$.

The following result can be found in [JPS] (see [Bu] for a general survey of the Rankin-Selberg method).

(2.5.5) Theorem. *The product (2.5.4) converges for $|t|<q^{-1}$ and defines a rational function in t, still denoted $LHom(f,g,t)$. If $f\neq g$, then $LHom(f,g,t)$ is a polynomial of degree $mn(2g_X-2)$. If $f=g$, then*

$$\text{(2.5.6)}\qquad LHom(f,g,t)=\frac{P_{f,g}(t)}{(1-t)(1-qt)}$$

where $P_{f,g}$ is a polynomial of degree $(2g_X-2)mn+2$. In any event, $LHom(f,g,t)$ satisfies the functional equation

$$\text{(2.5.7)}\qquad LHom(f,g,1/qt)=\epsilon_{f,g}t^{(2-2g_X)mn}LHom(f,g,t)$$

where $\epsilon_{f,g}=\prod_{x\in X}\left(\frac{\prod_j\lambda_{i,x}(g)}{\prod_i\lambda_{i,x}(f)}\right)^{\mathrm{ord}_x\omega}$ for any rational differential form ω on X (this product is independent of ω, see [De]).

(2.6) Scalar products and dual Hecke operators. For two automorphic forms $f,g\in \mathrm{AF}_{n,d}$, of which at least one has finite support, put

$$\text{(2.6.1)}\qquad (f,g)_d=\sum_{V\in \mathrm{Bun}_{n,d}(X)}\frac{f(V)\overline{g(v)}}{|\mathrm{Aut}(V)|}.$$

Thus $(f,g)_d$ is the degree d part of the orbifold scalar product (1.3.2). If f,g are automorphic forms defined on all $\mathrm{Bun}_n(X)$, not just $\mathrm{Bun}_{n,d}(X)$, we denote by $(f,g)_d$ the scalar product of their restrictions to $\mathrm{Bun}_{n,d}(X)$. We also write

$$\text{(2.6.2)}\qquad \| f\|_d^2=(f,f)_d$$

Note that cusp forms have finite support on each $\mathrm{Bun}_{n,d}(X)$ by (2.2.4), so their scalar products are defined.

We will be interested in the adjoints of the Hecke operators with respect to these scalar products. To describe them, we introduce the concept of an overbundle.

Let V be a vector bundle on X. By an overbundle of V we mean a vector bundle U of the same rank as V containing V as a coherent subsheaf. So the sheaf U/V lies in $\mathrm{Coh}_{0,X}$. Its isomorphism class is called the cotype of the overbundle, and the isomorphism class of U is called its type. Two overbundles $U,U'\supset V$ are called equivalent, if there is an isomorphism $U\to U'$ identical on V.

(2.6.3) Proposition. *The number of equivalence classes of overbundles of V of type U and cotype $\mathcal{F}$ is equal to*

$$g_{V\mathcal{F}}^{U} \cdot \frac{|Aut(V)|}{|Aut(U)|},$$

where $g_{V\mathcal{F}}^{U}$ are the same as in (1.2).

Proof. Let $e_{V\mathcal{F}}^{U}$ be the number of all exact sequences

$$0 \to V \to U \to \mathcal{F} \to 0.$$

Then $g_{V\mathcal{F}}^{U} = e_{V\mathcal{F}}^{U}/|\mathrm{Aut}(V)| \cdot |\mathrm{Aut}(\mathcal{F})|$, while the number of equivalence classes of overbundles is equal to $e_{V\mathcal{F}}^{U}/|\mathrm{Aut}(U)| \cdot |\mathrm{Aut}(\mathcal{F})|$, whence the statement.

For $\mathcal{F} \in \mathrm{Coh}_{0,X}$ we define the dual Hecke operator $T_{\mathcal{F}}^{\vee} : \mathrm{AF}_n \to \mathrm{AF}_n$ by

$$(T_{\mathcal{F}}^{\vee} f)(V) = \sum_{\substack{U \supset V \\ U/V \simeq \mathcal{F}}} f(U) = \sum_{U \in \mathrm{Bun}_n(X)} g_{V\mathcal{F}}^{U} \frac{|\mathrm{Aut}(V)|}{|\mathrm{Aut}(U)|} f(U) \tag{2.6.4}$$

where the first sum is over equivalence classes of overbundles of V of cotype $\mathcal{F}$, and the second sum is over all isomorphism classes of bundles. It is clear that $T_{\mathcal{F}}$ takes $\mathrm{AF}_{n,d}$ into $\mathrm{AF}_{n,d+h^0(\mathcal{F})}$ and $T_{\mathcal{F}}^{\vee}$ takes $\mathrm{AF}_{n,d}$ into $\mathrm{AF}_{n,d-h^0(\mathcal{F})}$. Here $h^0(\mathcal{F}) = \dim_{F_q} H^0(X, \mathcal{F})$.

(2.6.5) Proposition. *For each $d \in \mathbf{Z}$ the operators*

$$T_{\mathcal{F}} : AF_{n,d} \to AF_{n,d+h^0(\mathcal{F})}, \quad T_{\mathcal{F}}^{\vee} : AF_{n,d+h^0(\mathcal{F})} \to AF_{n,d}$$

are adjoint to each other with respect to the scalar product (2.6.1), i.e., for $f \in AF_{n,d}$, $g \in AF_{n,d+h^0(\mathcal{F})}$ with finite support, we have

$$(T_{\mathcal{F}} f, g)_{d+h^0(\mathcal{F})} = (f, T_{\mathcal{F}}^{\vee} g)_d.$$

Proof. This follows at once from Proposition 2.6.3 and the definition of the orbifold scalar product.

Let now f be a cusp eigenform, $f \in \mathrm{Cusp}_n$, and let $\chi_f : H(\mathrm{Coh}_{0,X}) \to \mathbf{C}$ be the algebra homomorphism describing the action of Hecke operators on f:

$$\chi_f([\mathcal{F}]) \cdot f = T_{\mathcal{F}} f. \tag{2.6.6}$$

Define a new homomorphism $\chi_f^{\vee} : H(\mathrm{Coh}_{0,X}) \to \mathbf{C}$ by

$$\chi_f^{\vee}([\mathcal{O}_x^i]) = \begin{cases} \chi_f([\mathcal{O}_x^{n-i}])\chi_f([\mathcal{O}_x^n])^{-1}, i \leq n \\ 0, i > n \end{cases} \tag{2.6.7}$$

(2.6.8) Proposition. *In the above assumptions the action of the dual Hecke operators on f is given by*

$$T^\vee_{\mathcal{F}} f = \chi^\vee_f([\mathcal{F}]) \cdot f.$$

Proof. Let $V(x), x \in X$, be the sheaf whose sections are sections of V which are allowed a first order pole at x. Then each equivalence class of overbundles of V of cotype $\mathcal{O}^i_x$ can be realized by a unique subsheaf $U \subset V(x)$ such that $V \subset U \subset V(x)$. In other words, such equivalence classes are in bijection with subsheaves in $V(x)$ of cotype $\mathcal{O}^{n-i}_x$. This means that

$$T^\vee_{\mathcal{O}^i_x} = T_{\mathcal{O}^{n-i}_x} T^{-1}_{\mathcal{O}^n_x} \tag{2.6.9}$$

and our statement follows from the definitions.

Note that it follows from (2.6.7) and (2.5.2) that

$$\chi^\vee_f([\oplus \mathcal{O}/I^{\mu_i}_x]) = q_x^{-\Sigma(i-1)\mu_i} P_\mu\left(q^{\frac{n-1}{2}}\lambda_{1,x}(f), \ldots, q_x^{\frac{n-1}{2}}\lambda_{n,x}(f); q_x^{-1}\right). \tag{2.6.10}$$

(2.6.11) Proposition. *Let $f, g \in Cusp_n$. Then:*

(a) *For each $\mathcal{F} \in Coh_{0,X}$ we have $\chi^\vee_f([\mathcal{F}]) = \overline{\chi_f([\mathcal{F}])}$.*
(b) *The number $\| f \|^2_d$ is independent on $d \in \mathbf{Z}$.*
(c) *If $f \neq g$, then $(f,g)_d = 0$ for any d.*

Proof. (a) Fix $h \in \mathbf{Z}_+$. Then for every $\mathcal{F}$ with $h^0(\mathcal{F}) = h$, we have

$$\chi_f([\mathcal{F}])(f,f)_{d+h} = (T_{\mathcal{F}} f, f)_{d+h} = (f, T^\vee_{\mathcal{F}} f)_d = \overline{\chi^\vee_f([\mathcal{F}])}(f,f)_d. \tag{2.6.12}$$

Note that $(f,f)_d > 0$ and depends only on d modulo n, since $f(V \otimes L) = \mu(L) f(V)$ for a line bundle L, and μ is a unitary character. Thus we conclude that for any $\mathcal{F}$ with $h^0(\mathcal{F})$ divisible by n we indeed have the desired equality $\chi^\vee_f([\mathcal{F}]) = \overline{\chi_f([\mathcal{F}])}$. However, any two characters of the Hall algebra coinciding an $[\mathcal{F}]$ with $h^0(\mathcal{F}) \equiv 0 (\mathrm{mod}\, n)$ should be equal.

(b) Apply (a) and (2.6.12) with h now being arbitrary.

(c) If $f \neq g$ then by the multiplicity one theorem for GL_n, see [Sh], there is $x \in X$ such that the set of the $\lambda_{i,x}(f)$ (with multiplicities) is not equal to the set of the $\lambda_{i,x}(g)$. It follows that we can find $\mathcal{F} \in \mathrm{Coh}_{0,x}(X)$ such that $\chi_f([\mathcal{F}]) \neq \chi_g([\mathcal{F}])$ and, in addition, $h^0(\mathcal{F}) \equiv 0 (\mathrm{mod}\, n)$. Thus

$$\begin{aligned}\chi_f([\mathcal{F}])(f,g)_d = \chi_f([\mathcal{F}])(f,g)_{d+h^0(\mathcal{F})} = (T_{\mathcal{F}} f, g)_{d+h^0(\mathcal{F})} = \\ = (f, T^\vee_{\mathcal{F}} g)_d = \chi_g([\mathcal{F}])(f,g)_d\end{aligned}$$

whence $(f,g)_d = 0$.

(2.6.13) Remark. Part (a) of the above proposition means that for each $x \in X$ the value of any symmetric function on $\lambda_{1,x}(f)^{-1}, \ldots, \lambda_{n,x}(f)^{-1}$ is equal to its value on $\overline{\lambda_{1,x}(f)}, \ldots, \overline{\lambda_{n,x}(f)}$, in other words, that the set of the $\lambda_{i,x}(f)^{-1}$ is equal to the set of the $\overline{\lambda_{i,x}(f)}$. This is not to be confused with the generalized Ramanujan-Petersson conjecture [FK] which, in our normalization, asserts that $\lambda_{i,x}(f)^{-1} = \overline{\lambda_{i,x}(f)}$ for each i, i.e., that $|\lambda_{i,x}(f)| = 1$.

(2.6.14) Normalization convention. Recall (2.4.5) that the set Cusp_n was obtained by choosing a nonzero vector f_χ in each 1-dimensional vector space $\mathrm{AF}_n^{cusp}(\mu)_\chi$. By Proposition 2.6.11 (b) we can choose f_χ so that $\| f_\chi \|_d^2 = 1$ for any d. So in the sequel we will always assume that $\| f \|_d^2 = 1$ for any $f \in \mathrm{Cusp}_n, d \in \mathbf{Z}$.

(2.7) Dualization of bundles and conjugation of forms.

(2.7.1) Proposition. *For any $\mathcal{F} \in Coh_{0,X}$ equivalence classes of overbundles of V of cotype $\mathcal{F}$ are in bijection with (locally free) subsheaves in the dual bundle V^*, of the same cotype $\mathcal{F}$.*

Proof. Let an overbundle U be given. From the short exact sequence

$$0 \to V \to U \to \mathcal{F} \to 0$$

we get a long exact sequence for $\underline{\mathrm{Ext}}^\bullet(-, \mathcal{O}_X)$, a part of which has the form

$$0 \to U^* \to V^* \to \underline{\mathrm{Ext}}^1(\mathcal{F}, \mathcal{O}_X) \to 0$$

Since X is a curve and $\mathcal{F} \in \mathrm{Coh}_{0,X}$, the sheaf $\underline{\mathrm{Ext}}^1(\mathcal{F}, \mathcal{O}_X)$ is (non-canonically) isomorphic to $\mathcal{F}$. So U^* is a subsheaf in V^* of the same cotype $\mathcal{F}$. By applying th dualization twice, we find that our correspondence is a bijection.

(2.7.2) Corollary. *If $V, W \in Bun_n(X)$ and $\mathcal{F} \in Coh_{0,X}$, then*

$$g_{W^*\mathcal{F}}^{V^*} = g_{V\mathcal{F}}^{W} \frac{|Aut(V)|}{|Aut(W)|}.$$

For an automorphic form $f \in AF_n$ define $f^D \in AF_n$ by

$$f^D(V) = f(V^*).$$

(2.7.3) Proposition. *For any $\mathcal{F} \in Coh_{0,X}$ and $f \in AF_n$ we have*

$$T_\mathcal{F}(f^D) = (T_\mathcal{F}^\vee f)^D.$$

Proof. This is an immediate consequence of (2.7.1)

(2.7.4) Corollary. *If $f \in Cusp_n$, then there is $\epsilon \in \{\pm 1\}$ such that $f(V^*) = \epsilon \overline{f(V)}$ for any $V \in Bun_n(X)$.*

Proof. From (2.7.3), both f^D and $\bar{f}$ are eigenforms of the Hecke algebra with the same character $\chi_f^\vee = \bar{\chi}_f$. So by the multiplicity one theorem $f^D = \epsilon \bar{f}$ for some constant $\epsilon \neq 0$. Since the dualization and conjugation are involutive, $\epsilon^2 = 1$.

§3. The Hopf algebra of automorphic forms.

(3.1) The setup. We keep the notation of (2.1), and are going to apply the general formalism of Section 1 to the Abelian category $\mathcal{A} = \mathrm{Coh}_X$. We denote $H = H(\mathcal{A}), R = R(\mathcal{A})$ and $B = B(\mathcal{A})$ its Hall, Ringel and extended Ringel algebra (see Section 1).

The Grothendieck group $\mathcal{K}_0\mathcal{A} = \mathcal{K}_0 X$ is identified with $\mathbf{Z} \oplus \mathrm{Pic}(X)$ via the map

$$(3.1.1) \qquad \left(n, \sum_{x\in X} m_x \cdot x\right) \quad \mapsto \quad n \cdot \bar{\mathcal{O}}_X + \sum_{x\in X} m_x \cdot \bar{\mathcal{O}}_x$$

where we view $\mathrm{Pic}(X)$ as the quotient of the group of divisors modulo principal divisors.

We denote the generator $K_{\bar{\mathcal{O}}_X} \in \mathbf{C}[\mathcal{K}_0\mathcal{A}]$ simply by K and denote $K_{\bar{\mathcal{O}}_x}, x \in X$ simply by c_x. For a divisor $D = \sum m_x \cdot x$ we set $c_D = \prod c_x^{m_x}$. By the above, $c_D = 1$ for principal D. So for a line bundle L on X there is a well-defined element $c_L = c_D$ where D is any divisor such that $L \simeq \mathcal{O}_X(D)$.

Note that each $\bar{\mathcal{O}}_x$ lies in the kernel of the bilinear form $(\alpha|\beta)$ on $\mathcal{K}_0(X)$. Thus each c_x, c_L is a central element in B. We have there fore a character

$$(3.1.2) \qquad c : \mathrm{Pic}(X) \to B^*, \quad L \mapsto c_L$$

of $\mathrm{Pic}(X)$ with values in the multiplicative group of B. As for the generator K, for any vector bundle V on X we have, by Riemann-Roch theorem

$$(3.1.3) \qquad [V]K = q^{rk(V)(1-g_X)}K[V].$$

Any coherent sheaf $\mathcal{F}$ on X can be written as a direct sum $\mathcal{F} = \mathcal{F}_{tors} \oplus \mathcal{F}_{lf}$, where $\mathcal{F}_{lf}$ is locally free and $\mathcal{F}_{tors}$ is a torsion sheaf (i.e., has 0-dimensional support). The isomorphism classes of $\mathcal{F}_{tors}, \mathcal{F}_{lf}$ depend on $\mathcal{F}$ only. If $\mathcal{F}_{lf} \neq 0$, there are infinitely many subsheaves in $\mathcal{F}$, so the formula (1.6.3) for the comultiplicaton in B produces an infinite sum, i.e., an element of a certain completion of $B \otimes B$. More precisely, let $B\widehat{\otimes}B$ be the space of possibly infinite sums $\sum b_i' \otimes b_i''$ where

$$b_i' = [\mathcal{F}_i']\kappa_i', \quad b_i'' = [\mathcal{F}_i'']\kappa_i'', \quad \mathcal{F}_i', \mathcal{F}_i'' \in \mathrm{Coh}(X), \kappa_i', \kappa_i'' \in \mathbf{C}[\mathcal{K}_0 X],$$

satisfying the following condition:

(3.1.4) For each $d \in \mathbf{Z}$ the number of i such that $\deg \mathcal{F}'_{i,lf} = d$, is finite, and for $d \gg 0$ this number is 0.

The following fact is easily proved by applying the main lemma of Green [Gr] plus the fact that the number of coherent subsheaves of given degree in a vector bundle is finite.

(3.1.5) Proposition. *$B\widehat{\otimes}B$ is an algebra, and $\Delta : B \to B\widehat{\otimes}B$ given by (1.6.3), is a homomorphism of algebras.*

So we shall say that B is a topological Hopf algebra.

(3.2) Generating functions associated to cusp forms. Let $f \in \mathrm{Cusp}_n$ be a cusp eigenform on $\mathrm{Bun}_n(X)$, and $\chi = \chi_f : H(\mathrm{Coh}_{0,X}) \to \mathbf{C}$ be the algebra homomorphism giving the action of Hecke operators on f:

$$T_{\mathcal{F}} f = \chi([\mathcal{F}]) \cdot f \tag{3.2.1}$$

Consider the following formal power series with coefficients in the Ringel algebra R:

$$E_f(t) = \sum_{V \in \mathrm{Bun}_n(X)} f(V) t^{\deg(V)} \quad \in \quad R[[t, t^{-1}]], \tag{3.2.2}$$

the sum over all isomorphism classes of rank n vector bundles on X. Note that the coefficients at each power of t in $E_f(t)$ is a finite sum by (2.2.4).

More generally, for any quasi-character $\mu : \mathrm{Pic}(X) \to B^*$ taking values in the multiplicative group of the center of B we can form the series

$$E_f(\mu t) = \sum_{V \in \mathrm{Bun}_n X} f(V) \mu(\det V) t^{\deg V} \quad \in \quad B[[t, t^{-1}]]. \tag{3.2.1'}$$

The notation becomes unambigious once we agree to identify t itself with the quasicharacter $L \mapsto t^{\deg L}$ of $\mathrm{Pic}(X)$. If μ takes values in $\mathbf{C}^* \subset B^*$, then $\mu(L) = \mu_i(L)\lambda^{\deg L}$ for some $\lambda \in \mathbf{C}^*$ and $\mu_i \in \{\mu_1, \dots, \mu_h\}$, our set of unitary representatives (2.4), so $E_f(\mu t) = E_{f'}(\lambda t)$ for some $f' \in \mathrm{Cusp}_n$ and we don't get anything new. However, taking $\mu = c$, the character defined by (3.1.2), we get new elements, to be used later in the formulas for comultiplication.

Let also

$$\psi_f(t) = \sum_{\mathcal{F} \in \mathrm{Coh}_0(X)} \bar{\chi}_f([\mathcal{F}]) t^{h^0(\mathcal{F})} |\mathrm{Aut}(\mathcal{F})| \cdot [\mathcal{F}] \quad \in \quad R[[t]], \tag{3.2.2}$$

where the sum is over isomorphism classes of all sheaves with 0-dimensional support and $h^0(\mathcal{F}) = \dim H^0(X, \mathcal{F})$. Thus ψ_f is a generating function for Hecke operators. If W is any rank n vector bundle such that $f(W) \neq 0$, we have, by (2.7.4):

$$\psi_f(t) = \frac{t^{\deg W}}{q^{\deg W} f(W^*)} \sum_{\substack{V \subset W \\ rk(V) = n}} f(V^*) t^{-\deg V} |\mathrm{Aut}(W/V)| \cdot [W/V] \tag{3.2.3}$$

where the sum is over all subsheaves in W of full rank n, i.e., over "effective matrix divisors". As with $E_f(t)$, we will use the series $\psi_f(\mu t) \in B[[t]]$ for any central quasicharacter $\mu : \mathrm{Pic}(X) \to B^*$. It is given by

$$\Psi_f(\mu t) = \sum_{\mathcal{F} \in \mathrm{Coh}_0(X)} \bar{\chi}_f([\mathcal{F}]) t^{h^0(\mathcal{F})} \mu(\bar{\mathcal{F}}) |\mathrm{Aut}(\mathcal{F})| \cdot [\mathcal{F}] \tag{3.2.4}$$

where $\bar{\mathcal{F}}$ is the class of $\mathcal{F}$ in $\mathcal{K}_0(X)$ which lies in the subgroup $\mathrm{Pic}(X) \subset \mathcal{K}_0(X)$, see (3.1.1). The space $R[[t, t^{-1}]]$ of series $a(t) = \sum_{i=-\infty}^{+\infty} r_i t^i, r_i \in R$, infinite in both directions, is not a ring, but we do have a well-defined multiplication

$$R[[t_1, t_1^{-1}]] \otimes R[[t_2, t_2^{-1}]] \to R[[t_1^{\pm 1}, t_2^{\pm 1}]], \quad a(t_1) \otimes b(t_2) \mapsto a(t_1) b(t_2).$$

Also, the use of generating functions is well-suited to the study of the topological Hopf algebra $B \supset R$. More precisely, we have the following theorem which is the main result of this section.

(3.3) Theorem. *Let $f \in Cusp_n, g \in Cusp_m$ be the two cusp eigenforms. Then:*
(a) For each $\mathcal{F} \in Coh(X)$ the coefficient at the basis vector $[\mathcal{F}] \in R$ in each of the products

$$E_f(t_1) * E_g(t_2), \quad E_f(t_1) * \psi_g(t_2), \quad \psi_f(t_1) * E_g(t_2) \in R[[t_1^{\pm 1}, t_2^{\pm 1}]]$$

is a power series in t_1, t_2 which converges for $|t_1| \gg |t_2|$ to a rational function.
(b) These rational functons satisfy the following relations:

$$E_f(t_1) * E_g(t_2) = q^{mn(1-g_X)} \frac{LHom(f, g, t_2/t_1)}{LHom(f, g, t_2/qt_1)} E_g(t_2) * E_f(t_1) \tag{3.3.1}$$

$$E_f(t_1) * \psi_g(t_2) = \frac{LHom(f, g, q^{\frac{m}{2}} t_2/t_1)}{LHom(f, g, q^{\frac{m}{2}-1} t_2/t_1)} \psi_g(t_2) * E_f(t_1) \tag{3.3.2}$$

(c) In the topological Hopf algebra $B \supset R$ we have the identities:

$$\Delta \psi_f(t) = \psi_f(t \otimes c)(1 \otimes \psi_f(t)) \tag{3.3.3}$$

$$\Delta E_f(t) = 1 \otimes E_f(t) + E_f(t \otimes c)(1 \otimes K^n \psi_f(q^{-\frac{n}{2}} t)) \tag{3.3.4}$$

where $E_f(t \otimes c) = \sum_V f(V) t^{\deg(V)} \otimes c_{\det(V)}$.

$$\epsilon(\psi_f(t)) = 1, \quad \epsilon(E_f(t)) = 0, \quad \epsilon(c_L) = 1, \quad \epsilon(K) = 1. \tag{3.3.5}$$

$$(3.3.6) \qquad S(\psi_f(t)) = \psi_f(c^{-1}t)^{-1}, \quad S(E_f(t)) = -E_f(c^{-1}t)\psi_f(q^{-\frac{n}{2}}t)^{-1}K^{-n}.$$

In this section we will prove only the equality (3.3.1), relegating the rest to Section 4.

(3.4) Eisenstein series. Proof of (3.3.1). Let $f \in \mathrm{Cusp}_n, g \in \mathrm{Cusp}_m$. Let us write, for the product in the Hall algebra

$$(3.4.1) \qquad E_f(t_1) \circ E_g(t_2) = \sum_{V \in \mathrm{Bun}_{n+m}(X)} \mathcal{E}_V(f, g, t_1, t_2)[V].$$

Then

$$(3.4.2) \qquad \mathcal{E}_V(f, g, t_1, t_2) = \sum_{\substack{V' \subset V \\ \mathrm{rk}(V)=n}} f(V')g(V/V')t_1^{\deg(V')}t_2^{\deg(V/V')}$$

where V' runs over all subbundles (i.e., subsheaves which are locally direct summands) in V of rank n. This is nothing but the unramified Eisenstein series associated to cusp forms f, g, see [Hal] [Mor] [MW]. We recall the following result.

(3.4.3) Proposition. *(a) For any $V \in Bun_{n+m}(X)$ the series (3.4.2) converges for $|t_1| \gg |t_2|$ to a rational function.*
(b) These rational function satisfy the functional equations

$$\mathcal{E}_V(f, g, t_1, t_2) = q^{mn(1-g_X)} \frac{LHom(f, g, qt_2/t_1)}{LHom(f, g, t_2/t_1)} \mathcal{E}_V(g, f, q^n t_2, q^{-m}t_1)$$

(c) The poles of the rational function $\mathcal{E}_V(f, g, t_1, t_2)$ are precisely the poles of the function

$$\frac{LHom(f, g, qt_2/t_1)}{LHom(f, g, t_2/t_1)}$$

and the orders of poles are the same.

Proof. (a) See, for instance, [Hal], [Mor] or [MW], Prop. IV.1.12 .

(b) Let us briefly recall the general framework of functional equations of Eisenstein series [MW] and show what it yields in our particular case. For a space S let $\mathrm{Fun}(S)$ denote the space of locally constant functions on S (we ignore the growth conditions in this formal reminder). Let $\Xi_{n,m} \subset GL_{n+m}\mathbf{A}$ be the subgroup

$$\Xi_{n,m} = \begin{pmatrix} GL_n(k) & \mathrm{Mat}_{n,m}(\mathbf{A}) \\ 0 & GL_m(k) \end{pmatrix}$$

The Eisenstein series construction defines a map

$$\mathrm{Eis}_{n,m}: \mathrm{Fun}(\Xi_{n,m} \setminus GL_{n+m}(\mathbf{A})) \to \mathrm{Fun}(GL_{n+m}(k) \setminus GL_{n+m}(\mathbf{A})),$$

$$(\mathrm{Eis}\,\varphi)(g) = \sum_{\gamma \in P_{n,m}(k) \setminus GL_{n+m}(k)} \varphi(\gamma g), \quad P_{n,m} = \begin{pmatrix} GL_n & * \\ 0 & GL_m \end{pmatrix}$$

(when converges). In general, $\mathrm{Eis}_{n,m}$ is defined by analytic continuation over auxiliary parameters. The functional equation, formally, has the form

$$\mathrm{Eis}_{n,m}(\varphi) = \mathrm{Eis}_{m,n}(M\varphi)$$

where

$$M: \mathrm{Fun}(\Xi_{n,m} \setminus GL_{n+m}\mathbf{A}) \to \mathrm{Fun}(\Xi_{m,n} \setminus GL_{n+m}\mathbf{A})$$

is defined by

$$(M\varphi)(g) = \int_{Z \in \mathrm{Mat}_{n,m}(\mathbf{A})} \varphi\left(\begin{pmatrix} Z & 1_m \\ 1_n & 0 \end{pmatrix} g \right) dZ,$$

where $dZ = \prod dz_{ij}$ and $\int_{\mathbf{A}/k} dz_{ij} = 1$. Our case is obtained from here as follows. For

$$f \in \mathrm{AF}_n = \mathrm{Fun}(GL_n k \setminus GL_n \mathbf{A} / GL_n \widehat{\mathcal{O}}), \quad g \in \mathrm{AF}_m$$

let $f \odot g \in Fun(\Xi_{n,m} \setminus GL_{n+m}\mathbf{A} / GL_{n+m}\widehat{\mathcal{O}})$ be the function defined uniquely (in virtue of the Iwasawa decomposition) by

$$(f \odot g) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = f(A) g(B), \quad A \in GL_n\mathbf{A}, B \in GL_m\mathbf{A}.$$

For $A \in GL_n(\mathbf{A})$ let $\deg(A) = \sum_{x \in X} \deg(x) \cdot \mathrm{ord}_x(a)$. Then for $f \in \mathrm{Cusp}_n, g \in \mathrm{Cusp}_m$ we have

$$\mathcal{E}_V(f, g, t_1, t_2) = \mathrm{Eis}_{n,m}((f t_1^{\deg}) \odot (g t_2^{\deg}))$$

so (3.4.3) (b) follows from the identification

(3.4.4)

$$M((f t_1^{\deg}) \odot (g t_2^{\deg})) = q^{mn(1-g_X)} \frac{\mathrm{LHom}(f, g, q t_2/t_1)}{\mathrm{LHom}(f, g t_2/t_1)} (g \cdot (q^m t_2)^{\deg}) \odot (f \cdot (q^{-m} t_1)^{\deg})$$

This can be established by using the fact that the representation of $GL_n\mathbf{A}$ corresponding to f is, at every $x \in X$, a principal series representation, and same for g. Thus the operator M which has the form $\bigotimes_{x \in X} M_x$ can be calculated by splitting each M_x into mn 1-dimensional intertwiners, each evaluated by integration over $\widehat{k}_x$. Each such intertwiner contributes a factor

$$\frac{1 - \lambda_{j,x}(g) \lambda_{i,x}(f)^{-1} q_x t_2^{\deg(x)} / t_1^{\deg(x)}}{1 - \lambda_{j,x}(g) \lambda_{i,x}(f)^{-1} t_2^{\deg(x)} / t_1^{\deg(x)}} \int_{\widehat{\mathcal{O}}_x} dz$$

which, being all multiplied, give precisely (3.4.4), once we recall, that $\int_{\hat{\mathcal{O}}} dz = q^{1-g_X}$ if the Haar measure dz on $\mathbf{A}$ is normalized by $\int_{\mathbf{A}/k} dz = 1$.

(c) This follows from (3.4.4) and the general fact about Eisenstein series of relative rank 1([MW], § IV.3.10, Remark) which says that the singularities of Eisenstein series are in this case precisely the singularities of the intertwiner M.

Proposition 3.4.3 is proved.

Now, part (a) of (3.4.3) can be written as an equality in the Hall algebra of the category Coh(X) :

$$(3.4.5) \qquad E_f(t_1) \circ E_g(t_2) = q^{mn(1-g_X)} \frac{\mathrm{LHom}(f, g, qt_2/t_1)}{\mathrm{LHom}(f, g, t_2/t_1)} E_g(q^n t_2) \circ E_f(q^{-m} t_1)$$

By using the definition of the Ringel product $*$ and the Riemann-Roch theorem, we find

$$E_f(t_1) * E_g(t_2) = q^{mn(1-g_X)} E_f(q^{\frac{m}{2}} t_1) \circ E_g(q^{-\frac{n}{2}} t_2)$$

whence the validity of (3.3.1).

(3.5) Algebraic relations in B. For $f \in \mathrm{Cusp}_n$ we define elements $E_{f,d}, d \in \mathbf{Z}$ and $a_{f,d}, d \in \mathbf{N}$, by

$$(3.5.1) \qquad E_f(t) = \sum_{d \in \mathbf{Z}} E_{f,d} t^d; \quad a_f(t) := \log \psi_f(t) = \sum_{d=1}^{\infty} a_{f,d} t^d$$

Thus $E_{f,d}, a_{f,d}$ are some (finite) elements of B. We now proceed to find some algebraic relations among them by using the relations (3.3.1-2) for generating functions. For two different cusp forms $f \in \mathrm{Cusp}_n, g \in \mathrm{Cusp}_m$ let

$$(3.5.2) \qquad Q_{f,g}(t_1, t_2) = t_1^{(2g_X - 2)mn} \mathrm{LHom}(f, g, t_2/t_1)$$

be the homogeneization of their Rankin L-function, and

(3.5.3)

$$Q_f(t_1, t_2) = t_1^{(2g_X - 2)n^2 + 2} P_f(t_2/t_1), \quad \text{where} \quad \mathrm{LHom}(f, f, t) = \frac{P_f(t)}{(1-t)(1-qt)},$$

be the homogeneization of the numerator of $\mathrm{LHom}(f, f, t)$. The relations (3.3.1) can be written in the polynomial form:

(3.5.4)

$$(t_1 - qt_2) Q_f(qt_1, t_2) E_f(t_1) * E_f(t_2) = q^{1/mn(1-g_X)} (qt_1 - t_2) Q_f(t_1, t_2) \cdot E_f(t_2) * E_f(t_1),$$

$$(3.5.5) \quad Q_{f,g}(qt_1, t_2) E_f(t_1) * E_g(t_2) = q^{-mn(1-g_X)} Q_{f,g}(t_1, t_2) E_g(t_2) * E_f(t_1), f \neq g$$

where the equality is understood in the same sense as in Theorem 3.3: as the equality of rational functions constituting the coefficients of the LHS and the RHS at any given $[V]$, $V \in \mathrm{Bun}_{n+m}(X)$.

(3.5.6) Theorem. *(a) For each $i, j \in \mathbf{Z}$ comparing the coefficients at $t_1^i t_2^j$ in both sides of (3.5.4) or (3.5.5) gives a valid relation among the elements $E_{f,d}, E_{g,d} \in R \subset B$.*
(b) If we write

$$LHom(f,g,t) = \prod_i (1 - \alpha_i(f,g)t)^{\nu_i}, \tag{3.5.7}$$

then we have an equality in B:

$$[a_{g,d}, E_{f,l}] = \frac{1}{d}\left(\sum_i \nu_i \alpha_i(f,g)^d (q^d - 1) q^{(n+m-\frac{3}{2})}\right) E_{f,l+d}. \tag{3.5.8}$$

Proof. (a) Consider, say, the equality (3.5.4), and let some $V \in Bun_{n+m}(X)$ be fixed. The coefficients at V of the two sides of (3.5.4) are power series in t_1, t_2 converging to the same rational function, denote it $\varphi_V(t_1,t_2)$, but in different regions: $|t_1| \gg |t_2|$ for the left hand side and $|t_1| \ll |t_2|$ for the right hand side. So the coefficient at $t_1^i t_2^j$ in the left hand side is $\int_{|t_1|=R, |t_2|=r} \varphi_V(t_1,t_2) t_1^{-i-1} t_2^{-j-1} dt, dt_2$, $R \gg r$, while the coefficient at $t_1^i t_2^j$ in the right hand side is similar integral, but taken over the torus $|t_1| = r, |t_2| = R$. However, Proposition 3.4.3 (c) shows that $\varphi_V(t_1,t_2)$ has no singularities and thus is a Laurent polynomial in t_1, t_2. So the two integrals coincide, and comparing coefficients indeed gives a valid relation.

(b) From (3.3.2) we find

$$[a_g(t_2), E_f(t_1)] = \log \frac{\mathrm{LHom}(f,g,q^{n+m-\frac{3}{2}} t_2/t_1)}{\mathrm{LHom}(f,g,q^{n+m-\frac{1}{2}}(t_2/t_1)} E_f(t_1) \tag{3.5.9}$$

Moreover, the series $a_g(t_2)$ going only in one direction, the coefficients at each $[V]$ in both sides of (3.5.9) are Laurent polynomials in t_1, t_2, so we can proceed to comparing coefficients at each $t_1^i t_2^j$. This is done by applying the formula $\log(1 - z) = -\sum_{d \geq 1} z^d/d$ to (3.5.9) and (3.5.7) and yields the claimed answer (3.5.8).

(3.6) The Hermitian form on $B(\mathrm{Coh}_X)$. We now proceed to describe the values of the Hermitian form (1.7.1) on our generating functions $E_f(t), \psi_f(t)$ (or $a_f(t) = \log \psi_f(t)$). This will automatically give us scalar products of any products of the generating functions because of the identities

$$(xy, z) = (x \otimes y, \Delta(z)), \quad (x, yz) = (\Delta(x), y \otimes z) \tag{3.6.1}$$

expressing the fact that the multiplication and the comultiplication in $B(\mathrm{Coh}_X)$ are conjugate to each other.

In the following we will make use of the formal power series

$$\delta(z) = \sum_{n=-\infty}^{+\infty} z^n \tag{3.6.2}$$

representing (in the sense of distribution theory) the Dirac δ-function at $z = 1$.

(3.6.3) Proposition. *The scalar products of the generating functions $E_f(t), a_f(t) \in B(Coh_X)$ are given by*

$$(E_f(t_1), a_g(t_2)) = 0, \tag{3.6.4}$$

$$(E_f(t_1), E_g(t_2)) = \delta_{f,g} \cdot \delta(t_1 \bar{t}_2), f \in Cusp_n, g \in Cusp_m. \tag{3.6.5}$$

$$(a_f(t_1), a_g(t_2)) = \log \frac{LHom(g, f, q^{\frac{n+m}{2}-1} t_1 \bar{t}_2)}{LHom(g, f, q^{\frac{n+m}{2}} t_1 \bar{t}_2)} \tag{3.6.6}$$

$$(c_L, x) = 0, \quad \forall x \in B(Coh_X) \tag{3.6.7}$$

$$(K^i, K^j) = q^{ij(1-g_X)}, \quad (K^i, E_f(t)) = (K^i, a_f(t)) = 0 \tag{3.6.8}$$

The proof will be given in § 4.

(3.7) More general scalar products. A general element of the subalgebra in $R(\mathrm{Coh}_X)$ generated by the coefficients $E_{f,d}$, has the form

$$E(\mathbf{f}, \varphi) = \int_{|t_i|=a_i} E_{f_1}(t_1) \dots E_{f_r} \varphi(t_1, \dots, t_r) \prod \frac{dt_i}{t_i}, \quad a_1 \gg \dots \gg a_r \tag{3.7.1}$$

where $\mathbf{f} = (f_1, \dots, f_r), f_i \in$ Cusp, is a sequence of cusp forms, and φ is a Laurent polynomial. In the standard terminology of the theory of automorphic forms, such elements are called theta-series [Go] or pseudo-Eisenstein series [MW]. The well-known formula [La3] MW] for the scalar product of two pseudo-Eisensetin seris can be easily deduced from Theorem 3.3, Proposition 3.6.5 and general properties of Hopf algebras. Let us recall this formula in our notation.

Let $\bar{B}(\mathrm{Bun}_X)$ be the quotient algebra of $B(\mathrm{Coh}_X)$, obtained by putting each $[\mathcal{F}]$, where $\mathcal{F}$ is not a vector bundle, to be equal to 0, and each c_L to be equal to 1, and let

$$p_{\mathrm{Bun}} : B(\mathrm{Coh}_X) \to \bar{B}(\mathrm{Bun}_X) \tag{3.7.2}$$

be the natural projection. Notice that if $b \in B(\mathrm{Coh}_X)$ is a linear combination of $[V]$ with V being vector bundles, then for any $a \in B(\mathrm{Coh}_X)$ the scalar product (b, a) depends only on $p_{\mathrm{Bun}}(a)$.

For any Hopf algebra A with comultiplication Δ we denote $\Delta^{(r)} : A \to A^{\otimes r}$ the $(r-1)$ fold iteration of Δ, i.e., $\Delta^{(2)} = \Delta$ and $\Delta^{(r+1)} = Id_{A^{\otimes(r-1)}} \otimes \Delta$. If a scalar product $(\ ,\ )$ on A satisfies the identity (3.6.1), then by iterating these identities we find, in particular, that

$$(x, y_1 \ldots y_r) = (\Delta^{(r)}(x), y_1 \otimes \ldots \otimes y_r) \tag{3.7.3}$$

For a sequence $\mathbf{f} = (f_1, \ldots, f_r), f_i \in \mathrm{Cusp}_{n_i}$ of cusp forms and a permutation $\sigma \in S_r$ let

$$M_\sigma^{\mathbf{f}}(t_1, \ldots, t_r) = \prod_{\substack{i<j \\ \sigma(i)>\sigma(j)}} q^{n_i n_j (1-g_X)} \frac{\mathrm{LHom}(f_i, f_j, t_j/t_i)}{\mathrm{LHom}(f_i, f_j, t_j/qt_i)}, \tag{3.7.4}$$

so that the functional equation for Eisenstein series yields

$$E_{f_1}(t_1) \ldots E_{f_r}(t_r) = M_\sigma^{\mathbf{f}}(t_1, \ldots, t_r) E_{f_{\sigma(1)}}(t_{\sigma(1)}) \ldots E_{f_{\sigma(r)}}(t_{\sigma(r)}). \tag{3.7.5}$$

The classical formula of Langlands for the constant term of a (pseudo) Eisenstein series has, in our notation, the form

$$p_{\mathrm{Bun}}^{\otimes r'} \Delta^{r'} (E_{f_1}(t_1) \ldots E_{f_r}(t_r)) = \tag{3.7.6}$$

$$= \delta_{rr'} \sum_{\sigma \in S_r} M_\sigma^{\mathbf{f}}(t_1, \ldots, t_r) K^n E_{f_{\sigma^{-1}(1)}}(t_{\sigma^{-1}(1)}) \otimes \ldots \otimes K^n E_{f_{\sigma^{-1}(r)}}(t_{\sigma^{-1}(r)}),\ n = \Sigma n_i.$$

This formula follows at once from (3.3.2) and (3.3.4). By using (3.6.7) and (3.7.3), we deduce that for $\mathbf{g} = (g_1, \ldots, g_{r'}), g_j \in \mathrm{Cusp}_{m_j}$, we have

$$\Big(E_{f_1}(t_1) \ldots E_{f_r}(t_r),\ E_{g_1}(t_1') \ldots E_{g_{r'}}(t_{r'}')\Big) = \tag{3.7.7}$$

$$= \delta_{rr'} \sum_{\substack{\sigma \in S_r \\ g_j = f_{\sigma^{-1}(j)}, \forall j}} M_\sigma^{\mathbf{f}}(t_1, \ldots, t_r) \prod_{j=1}^r \delta(t_{\sigma^{-1}(j)} \cdot \bar{t}_j').$$

Then by integrating (3.7.7) against a Laurent polynomial $\psi(t_1' \ldots t_r')$ we find

$$\Big(E_{f_1}(t_1) \ldots E_{f_r}(t_r),\ E(\mathbf{g}, \psi)\Big) = \tag{3.7.8}$$

$$= \sum_{\substack{\sigma \in S_r \\ g_j = f_{\sigma^{-1}(j)}}} M_\sigma^{\mathbf{f}}(t_1, \ldots, t_r) \cdot \bar{\psi}(\bar{t}_{\sigma^{-1}(1)}^{-1}, \ldots, \bar{t}_{\sigma^{-1}(r)}^{-1})$$

(3.8) The algebras $\tilde{\mathcal{B}}$ and $\mathcal{B}$. Let $\mathcal{B} \subset B(\mathrm{Coh}_X)$ be the subalgebra generated by K, c_L and the coefficients $E_{f,d}, a_{f,d}, f \in \mathrm{Cusp}$. One would like to have a complete description of $\mathcal{B}$ by generators and relations. To address this problem, let us introduce an algebra $\tilde{\mathcal{B}}$ generated by formal symbols $\tilde{E}_{f,d}, \tilde{a}_{f,d}, \tilde{K}, \tilde{c}_L$ which are subject only to the relations that the $\tilde{c}_L$ are central, that

$$\tilde{c}_L\tilde{c}_M = \tilde{c}_{L\otimes M}, \quad \tilde{K}\tilde{c}_L = \tilde{c}_L\tilde{K}, \quad \tilde{K}\tilde{E}_{f,d} = q^{-n(1-g)}\tilde{E}_{f,d}K, \tag{3.8.1}$$

$$\tilde{K}\tilde{a}_{f,d} = \tilde{a}_{f,d}\tilde{K}, \quad f \in \mathrm{Cusp}_n,$$

plus the relations obtained from (3.4.4), (3.4.5), (3.4.8) by replacing E with $\tilde{E}$ and a with $\tilde{a}$. So we have a natural surjection

$$\pi_{\mathcal{B}} : \tilde{\mathcal{B}} \to \mathcal{B}, \tag{3.8.2}$$

and generating functions

$$\tilde{E}_f(t) = \sum_{d\in\mathbf{Z}} \tilde{E}_{f,d}t^d, \quad \tilde{a}_f(t) = \sum_{d\geq 1} \tilde{a}_{f,d}t^d, \quad \tilde{\psi}_f(t) = \exp(\tilde{a}_f(t))$$

satifying (3.3.1) and (3.4.2). The problem of explicit determination of the ideal $\mathrm{Ker}(\pi_{\mathcal{B}})$ seems very interesting. Basically, elements of $\mathrm{Ker}(\pi_{\mathcal{B}})$ are some subtle relations between residues of Eisenstein series. As we will see in § 5, these elements should be thought of analogs of Serre relations in quantum affine algebras which makes it plausible that one can give a completely explicit description of the generators of the ideal. Here we will use an approach similar to one used by G. Lusztig [Lu1] for ordinary quantum groups.

Denote by $\tilde{\mathcal{R}}$ (resp., $\tilde{\mathcal{R}}_{\mathrm{Bun}}, \tilde{\mathcal{R}}_0$) the subalgebra in $\tilde{\mathcal{B}}$ generated only by the $\tilde{E}_{f,d}$ and $\tilde{a}_{f,d}$ (resp. only by $\tilde{E}_{f,d}$, only by $\tilde{a}_{f,d}$) and by $\mathcal{R}$ (resp. $\mathcal{R}_{\mathrm{Bun}}, R_0$) the image of $\tilde{\mathcal{R}}$ (resp. $\tilde{\mathcal{R}}_{\mathrm{Bun}}, \tilde{\mathcal{R}}_0$) under $\pi_{\mathcal{B}}$.

Note that formulas (3.3.4-6) (modified by putting tildes over generating functions) make $\tilde{\mathcal{B}}$ into a topological Hopf algebra, and $\pi_{\mathcal{B}}$ into a Hopf homomorphism. Note also that formulas (3.6.6-10) define a Hermitian scalar product on $\tilde{\mathcal{B}}$, with respect to which the multiplication and comultiplication are conjugate, i.e., (3.6.1) holds. Let $N \subset \tilde{\mathcal{B}}$ be the kernel of this scalar product. Since $(\pi_{\mathcal{B}}(x), \pi_{\mathcal{B}}(y)) = (x, y)$ for any $x, y \in \tilde{\mathcal{B}}$, we have $\mathrm{Ker}(\pi_{\mathcal{B}}) \subset N$.

(3.8.3) Proposition. *$Ker(\pi_{\mathcal{B}}) = \mathbf{C}[\mathcal{K}_0 X] \cdot (N \cap \tilde{\mathcal{R}})$.*

Proof. This follows from the fact that the form $(\ , \ )$ on $\mathcal{R} \subset R(\mathrm{Coh}_X)$, being the restriction of a positive definite Hermitian form, is itself positive definite and hence non-degenerate. Thus $(\mathrm{Ker}\, \pi_{\mathcal{B}}) \cap \tilde{\mathcal{R}} = N \cap \tilde{\mathcal{R}}$, and we should only use the fact that both $\tilde{\mathcal{B}}$ and $\mathcal{B}$ have the form $\tilde{\mathcal{B}} = \mathbf{C}[\mathcal{K}_0 X] \otimes \tilde{\mathcal{R}}, \mathcal{B} = \mathbf{C}[\mathcal{K}_0 X] \otimes \mathcal{R}$.

This proposition shows that the algebra $\mathcal{B}$ can be defined entirely in terms of the L-function data made explicit in Theorem 3.3 and Proposition 3.6.5. We want to finish this section by discussing the relationship between $\mathcal{B}$, the "free" algebra $\tilde{\mathcal{B}}$ and the bigger algebra $B(\mathrm{Coh}_X)$ a little bit more closely.

Let $R(\mathrm{Bun}_X) \subset R(\mathrm{Coh}_X)$ be the subalgebra generated by elements $[V]$ where V is a vector bundle. By restricting $\pi_{\mathcal{B}}$ to the two subalgebras below and composing it with natural embeddings, we get homomorphisms

$$\pi_0 : \tilde{\mathcal{R}}_0 \to R(\mathrm{Coh}_{0,X}), \quad \pi_{\mathrm{Bun}} : \tilde{\mathcal{R}}_{\mathrm{Bun}} \to R(\mathrm{Bun}_X).$$

(3.8.4) Theorem. *The homomorphism π_{Bun} is surjective, i.e. $\mathcal{R}_{\mathrm{Bun}} = R(Bun_X)$.*

In other words, any element $[V]$, $V \in \mathrm{Bun}_n(X)$ can be expressed as a polynomial in the $E_{f,d}$, i.e., in the form (3.7.1). This statement is similar to the spectral decomposition theorem [MW] but is different since here we consider the coefficients of the Laurent expansion of the rational functions $E_{f_1}(t_1)...E_{f_r}(t_r)$ in the domain of convergence of the series defining these functions (and these coefficients are automorphic forms with finite support on each $\mathrm{Bun}_{n,d}(X)$), while spectral decomposition theorem deals, in our notation, with the coefficients of Laurent expansions of the same functions but on the unit torus $|t_i| = 1$ (and these coefficients are, in general, only square-integrable).

We will prove (3.8.4) a little later. As for the homomorphism π_0, we would like to state the following conjecture.

(3.8.5) Conjecture. *The map π_0 is injective, i.e., the (commuting) elements $a_{f,d}, f \in Cusp, d \geq 1$, are algebraically independent over* $\mathbf{C}$.

Note that since $\langle \mathcal{F}, \mathcal{G} \rangle = 1$ for any $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}_{0,X}$, the formula (1.4.1) makes $R(\mathrm{Coh}_{0,X}) = H(\mathrm{Coh}_{0,X})$ into a Hopf algebra (with comultiplication denoted by r). This is just the tensor product over all $x \in X$ of the Hopf algebras studied by Zelevinsky [Ze]. With respect to r every $\psi_f(t)$ is group-like: $r(\psi_f(t)) = \psi_f(t) \otimes \psi_f(t)$, so $a_f(t)$ is primitive: $r(a_f(t)) = a_f(t) \otimes 1 + 1 \otimes a_f(t)$, and the same is true for each $a_{f,d}$. Now, primitive elements in a commutative and cocommutative Hopf $\mathbf{C}$ -algebra are algebraically dependent if and only if they are linearly dependent. By using (3.6.6), we get the following reformulation of the conjecture. Let $\mathrm{Cusp}_{\leq N} = \coprod_{i \leq N} \mathrm{Cusp}_i$.

(3.8.6) Reformulation. *Let N be fixed, and ϵ be small enough. Consider the self-adjoint matrix integral operator on the circle $|t| = \epsilon$ whose matrix indices run over the set $Cusp_{\leq N}$ and whose kernel is given by*

$$\mathbf{K}_{f,g}(t_1, t_2) = \log \frac{LHom(g, f, q^{\frac{m+n}{2}-1} t_1 \bar{t}_2)}{LHom(g, f, q^{\frac{m+n}{2}} t_1 \bar{t}_2)}, \quad f \in Cusp_n, g \in Cusp_m, m, n \leq N$$

Then this operator is strictly positive definite.

This conjectural property can be regarded as a certain strengthening of the multiplicity one theorem for GL_n.

(3.9) Proof of Theorem 3.8.4. We will proceed in three steps.

Step 1: Use of the reduction theory. We recall some basic facts about stable vector bundles, see [HN], [Stu] for more details. For any vector bundle V on X let $\mu(V) = \deg(V)/\mathrm{rk}(V)$ denote its "slope". A bundle V is called semistable, if for any subbundle $V' \subset V$ we have $\mu(V') \leq \mu(V/V')$. An arbitrary bundle V possesses a canonical Harder-Narasimhan filtration

$$V_\bullet = (V_1 \subset \ldots \subset V_r = V)$$

with the properties that for each i the quotient $\mathrm{gr}_i(V) = V_i/V_{i-1}$ is semistable and $\mu(\mathrm{gr}_i(V)) > \mu(\mathrm{gr}_{i+1}(V))$. Thus V is semistable if and only if its Harder-Narasimhan filtration consists of only one layer.

Let $\lambda \in \mathbf{Z}$ be a positive integer. Let us say that a vector bundle V is λ-unstable, if it is not semistable and for at least one i we have $\mu(\mathrm{gr}_i(V)) + \lambda < \mu(\mathrm{gr}_{i-1}(V))$. Denote by $\mathrm{Bun}_{n,d}^{>\lambda}$ the set of isomorphism classes of λ-unstable bundles on X of rank n and degree d, and by $\mathrm{Bun}_{n,d}^{\leq\lambda} = \mathrm{Bun}_{n,d}(X) - \mathrm{Bun}_{n,d}^{>\lambda}$ its complement.

(3.9.1) Lemma. *(a) For each $\lambda > 0$ the set $Bun_{n,d}^{\leq\lambda}$ is finite.*
(b) For fixed n, d we can find $\lambda > 0$ such that whenever $V \in Bun_{n,d}^{>\lambda}$ and i is such that $\mu(gr_i(V)) + \lambda < \mu(gr_{i-1}(V))$, then in the Hall algebra we have the equality $[V] = [V_i] \circ [V/V_i]$.

Proof of (3.9.1). (a) This is an easy consequence of the reduction theory, see, e.g., [Stu].

(b) If E is any vector bundle on X, then there is $\delta \in \mathbf{Z}$ such that for any line bundle on X of degree $\geq \delta$ we have $H^1(X, E \otimes L) = 0$, $H^0(X, (E \otimes L)^*) = 0$ (Serre's theorem). Moreover, if we have an algebraic family of bundles parametrized by a scheme S of finite type, we can find δ good for all the bundles in the family.

We apply this to the "universal" situation. Let $\mathcal{B}un_{n,d}(X)$ be the moduli space of semistable vector bundles over X of rank n and degree d. It is a projective algebraic variety defined over $\mathbf{F}_q$. Tensoring with a line bundle of degree 1 defines an isomorphism $\mathcal{B}un_{n,d}(X) \to \mathcal{B}un_{n,n+d}(X)$. We conclude therefore the following:

(3.9.2) Lemma. *For each m_1, m_2 there exists a number $\delta_{m_1,m_2} > 0$ with the following property: If W_i, $i = 1, 2$, is a semistable bundle on X of rank m_i and $\mu(W_1) + \delta_{m_1,m_2} < \mu(W_2)$, then*

$$Ext^1_X(W_2, W_1) = Hom_X(W_1, W_2) = 0.$$

To return to the proof of (3.9.1), take λ greater than all the δ_{m_1,m_2}, $m_i \leq n$. If V satisfies the condition of (3.9.1), then by (3.9.2) we have

$$\mathrm{Ext}^1(\mathrm{gr}_j(V), \mathrm{gr}_k(V)) = \mathrm{Hom}(\mathrm{gr}_k(V), \mathrm{gr}_j(V)) = 0, \quad j > i, k \leq i$$

from which we deduce that

$$\mathrm{Ext}^1(V/V_i, V_i) = \mathrm{Hom}(V_i, V/V_i) = 0.$$

The vanishing of Ext^1 means that $V \simeq V_i \oplus (V/V_i)$ and the vanishing of Hom means that there exists a unique subbundle in $V_i \oplus (V/V_i)$ of type V_i and cotype

V/V_i. Using the vanishing of Ext^1 one more time, we find that $[V_i] \circ [V/V_i] = [V_i \oplus (V/V_i)] = [V]$. Lemma (3.9.1) is proved.

Step 2: Representation of a function as an infinite sum of Eisenstein coefficients. Let $R(\mathrm{Bun}_{n,X})$ be the subspace in $R(\mathrm{Coh}_X)$ spanned by basis vectors $[V]$ where V is a rank n vector bundle. We are going to use the formula for the scalar product of two Eisenstein series in order to represent an element $h \in R(\mathrm{Bun}_{n,X})$ as a possibly infinite linear combination of coefficients of Eisenstein series.

Fix $n > 0$, fix some real numbers $a_1 \gg \ldots \gg a_n > 0$ (enough to take $a_i > qa_{i+1}$) and a total order $\leq$ on the set $\mathrm{Cusp}_{\leq n} = \coprod_{i \leq n} \mathrm{Cusp}_i$. If $f \in \mathrm{Cusp}_i$, write $n(f) = i$. Denote also $\mathbf{Cusp}_n$ the set of sequences $\mathbf{f} = (f_1 \leq \ldots \leq f_r), f_i \in \mathrm{Cusp}_{\leq n}, \Sigma n(f_i) = n$. We will also write $E_{\mathbf{f}}(t)$ for $E_{f_1}(t_1) \ldots E_{f_r}(t_r), t = (t_1, \ldots, t_r)$.

(3.9.3) Lemma. *If $h \in R(Bun_{n,X})$ is orthogonal to each element $E_{\mathbf{f}}(t), f = (f_1, \ldots, f_r) \in \mathbf{Cusp}_n, |t_i| = a_i$, then $h = 0$.*

Proof. The assumptions imply that $(E_{\mathbf{f}}(t), h) = 0$ identically as a rational function in $t_1 \ldots t_r$. Thus from the functional equations (3.7.5) we find that for any permutation $\sigma \in S_r$ and $|t_i| = a_i$ the product $E_{f_{\sigma(1)}}(t_1) \ldots E_{f_{\sigma(r)}}(t_r)$ is orthogonal to h as well. This means that h is orthogonal to all pseudo-Eisenstein series and thus $h = 0$ by [MW], Th. II.11.2.

Now, given $h \in R(\mathrm{Bun}_{n,X})$, we can try to find an element $h' = \sum_{\mathbf{f} \in \mathbf{Cusp}_n} E(\mathbf{f}, \varphi_{\mathbf{f}})$ such that $(E_{\mathbf{f}}(t), h - h') = 0$ for any $f \in \mathbf{Cusp}_n$. For a given $\mathbf{f}$ we denote by $S(\mathbf{f}) \subset S_r$ the subgroup of permutations σ such that $f_{\sigma(i)} = f_i, \forall i$, and will look for $\varphi_{\mathbf{f}}$ invariant under $S(\mathbf{f})$. By (3.7.9) the condition for $(E_{\mathbf{f}}(t), h) = (E_{\mathbf{f}}(t), h')$ is that

$$(3.9.4) \qquad \Big(\sum_{\sigma \in S(\mathbf{f})} M_\sigma^{\mathbf{f}}(t) \Big) \bar{\varphi}_{\mathbf{f}}(\bar{t}^{-1}) \quad = \quad (E_{\mathbf{f}}(t), h)$$

Thus $\varphi_{\mathbf{f}}(t) = \bar{\Phi}_{\mathbf{f}}(\bar{t}^{-1})$, where

$$(3.9.5) \qquad \Phi_{\mathbf{f}}(t) = \frac{(E_{\mathbf{f}}(t), h)}{\sum_{\sigma \in S(\mathbf{f})} M_\sigma^{\mathbf{f}}(t)}.$$

The function $\varphi_{\mathbf{f}}(t)$ is indeed invariant under $S(\mathbf{f})$, but it is only a rational function, not a Laurent polynomial, because of the zeros of the denominator in $\Phi_{\mathbf{f}}(t)$. So in order to get a representation of h' as a sum of coefficients of Eisenstein series, we should first expand $\Phi_{\mathbf{f}}$.

Fix $\mathbf{f} = (f_1 \leq \ldots \leq f_r)$ and consider the coordinate vector space $\mathbf{R}^r$ with the standard basis $e_1, \ldots, e_r$. Let $\Gamma_{\mathbf{f}} \subset \mathbf{R}^r$ be the convex cone with apex 0 generated by the vectors $e_j - e_i$ for $i < j$ such that $f_i = f_j$. For a Laurent series $\sum_{\omega \in \mathbf{Z}^r} a_\omega t^\omega$ in r variables we call the set of ω such that $a_\omega \neq 0$ the support of the series.

(3.9.6) Lemma. *The function $\Phi_{\mathbf{f}}(t)$ can be expanded into a Laurent series*

$$\sum_{\omega\in\mathbf{Z}^r\cap(\tau_{\mathbf{f}}+\Gamma_{\mathbf{f}})} a^{\mathbf{f}}_\omega t^\omega$$

whose support is contained in some translation of the cone $\Gamma_{\mathbf{f}}$.

Proof. First, the numerator $(E_{\mathbf{f}}(t), h)$ is a finite linear combination of $(E_{\mathbf{f}}(t), [V]), V \in \mathrm{Bun}_n(X)$, i.e., of Eisenstein series $\mathcal{E}_V(f_1, ..., f_r, t_1, ..., t_r)$. The poles of any such series come from the poles of

$$M^{\mathbf{f}}_\sigma(t) = \prod_{i<j:\,\sigma(i)>\sigma(j)} q^{n_i n_j(1-g_X)} \frac{\mathrm{LHom}(f_i, f_j, t_j/t_i)}{\mathrm{LHom}(f_i, f_j, t_j/qt_i)}, \quad \sigma\in S(\mathbf{f}) \tag{3.9.7}$$

Each factor here is a series in non-negative powers of t_j/t_i, starting with 1. So each $M^{\mathbf{f}}_\sigma(t)$ has support in a translation of $\Gamma_{\mathbf{f}}$. Now, for the denominator $\sum M^{\mathbf{f}}_\sigma(t)$ we have, by the same reason, an expansion $\sum_{\omega\in\Gamma_{\mathbf{f}}\cap\mathbf{Z}^r} c_\omega t^\omega$ with $c_0\neq 0$. Thus its inverse admits the geometric series expansion

$$\frac{1}{\sum M^{\mathbf{f}}_\sigma(t)} = \frac{1}{c_0}\sum_{m=0}^{\infty}\Big(-\sum_{\omega\in(\Gamma_{\mathbf{f}}\cap\mathbf{Z}^r)-\{0\}} (c_\omega/c_0)t^\omega\Big)^m$$

which is a series supported in $\Gamma_{\mathbf{f}}$. Lemma 3.9.6 is proved.

Thus we can write h' as an infinite series

$$h' = \sum_{\mathbf{f}\in\mathbf{Cusp}_n} E(\mathbf{f}, \varphi_{\mathbf{f}}) = \sum_{\mathbf{f}\in\mathbf{Cusp}_n}\sum_{\omega\in\mathbf{Z}^r\cap(\tau_{\mathbf{f}}+\Gamma_{\mathbf{f}})} \bar{a}^{\mathbf{f}}_\omega E_{f_1,\omega_1}...E_{f_r,\omega_r}, \tag{3.9.8}$$

where $E_{f,d}$ is the d-th coefficient of $E_f(t)$.

(3.9.9) Lemma. *For any $\lambda > 0$ all but finitely many terms of the series (3.9.8), regarded as functions on $Bun_n(X)$, vanish outside $Bun^{\leq\lambda}_{n,d}$.*

This implies, in particular, that the series converges, as a series of functions on $\mathrm{Bun}_n(X)$.

Proof. It is enough to conisder one $\mathbf{f}$ at a time. In order that $E_{f_1,\omega_1}...E_{f_r,\omega_r}$ be nonzero on V, there should be a flag $V_1 \subset ... \subset V_r = V$ of subbundles with $\deg(V_i/V_{i-1}) = \omega_i$. Thus, if for at least one i we have

$$\frac{\omega_i}{n_i} + \lambda < \frac{\omega_{i+1}}{n_{i+1}} \quad (\text{where} \quad f_i\in \mathrm{Cusp}_{n_i}), \tag{3.9.10}$$

then V must have a destabilizing flag with at least one gap in the slopes greater than λ, which implies that in the Harder-Narasimhan filtration at least one gap in the slopes will be greater than λ, i.e., that $V\in \mathrm{Bun}^{>\lambda}_{n,d}$. It remains to notice that if $f_i = f_j$ then, of course $n_i = n_j$ so for all but finitely many integer points lying in a translation of $\Gamma_{\mathbf{f}}$, the condition (3.9.10) will be satisfied for some i. Lemma 3.9.10 is proved.

So h' is a well defined function on $\mathrm{Bun}_n(X)$ and the scalar product $(E_{\mathbf{f}}(t), h')$ is indeed defined for any $\mathbf{f}\in\mathbf{Cusp}_n$ by (3.7.9) and is equal to $(E_{\mathbf{f}}(t), h)$ by construction. So we get the following fact.

(3.9.11) Lemma. *The series in the right hand side of (3.9.8) converges to h.*

Step 3: Completion of the proof. Fix n and assume, by induction, that any $h \in R(\mathrm{Bun}_{m,X})$ with $m < n$ lies in the image of π_{Bun}. Fix $d \in \mathbf{Z}$ and a big enough $\lambda \in \mathbf{Z}$. By Step 1, all $[V]$ with $V \in \mathrm{Bun}_{n,d}^{>\lambda}$ lie in $\mathrm{Im}(\pi_{\mathrm{Bun}})$. So we h to be equal to $[V]$, $V \in \mathrm{Bun}_{n,d}^{\leq\lambda}$. By Lemmas 3.9.9 and 3.9.11, we can, by truncating the series (3.9.8), find a finite linear combination h'' of elements from $\mathrm{Im}(\pi_{\mathrm{Bun}})$ which coincides with h on $\mathrm{Bun}_{n,d}^{\leq\lambda}$. But then $h - h''$ lies in $\mathrm{Im}(\pi_{\mathrm{Bun}})$ by Step 1. This proves the theorem.

§4. Proof of Theorem 3.3 and Proposition 3.6.3.

Recall that the equality (3.3.1) has already been proved. So we first prove the rest of the assertions of Theorem 3.3.

(4.1) Proof of (3.3.2). Let $\hat{H}(\mathrm{Coh}_X)$ be the completion of the Hall algebra $H(\mathrm{Coh}_X)$ consisting by all, possibly infinite, sums $\sum_{A\in\mathrm{Coh}_X} f(A)[A]$. Clearly $\hat{H}(\mathrm{Coh}_X)$ is a bimodule over the algebra $H(\mathrm{Coh}_X)$. We denote the bimodule structure by $\circ$.

(4.1.1) Proposition. *For any $\mathcal{F} \in Coh_{0,X}$ and any $f : Bun_n(X) \to \mathbf{C}$ we have equalities in $\hat{H}(Coh_X)$:*

$$[\mathcal{F}] \circ \sum_{V\in\mathrm{Bun}_n(X)} f(V)[V] = \sum_{V\in\mathrm{Bun}_n(X)} f(V)[V\oplus\mathcal{F}], \tag{4.1.1a}$$

$$\left(\sum_{V\in\mathrm{Bun}_n(X)} f(V)[V])\right)\circ[\mathcal{F}] = \sum_{\mathcal{F}'\subset\mathcal{F}}\sum_{W\in\mathrm{Bun}_n(X)} (T_{\mathcal{F}/\mathcal{F}'}f)(W)\cdot \tag{4.1.1b}$$
$$\cdot\frac{|Aut(\mathcal{F}')|\cdot|Aut(\mathcal{F}/\mathcal{F}')|}{|Aut(\mathcal{F})|}\cdot q^{nh^0(\mathcal{F}')}[W\oplus\mathcal{F}'].$$

Proof. The statement (a) follows from the equality $[\mathcal{F}]\circ[V] = [V\oplus\mathcal{F}]$ holding for any $\mathcal{F}\in\mathrm{Coh}_{0,X}, V\in\mathrm{Bun}_n(X)$, because $\mathrm{Ext}^1(V,\mathcal{F}) = \mathrm{Hom}(\mathcal{F},V) = 0$. To see part (b), recall (1.2.1) the notation g^C_{AB} for the structure constants in the Hall algebra. Our statement follows from the next lemma about these constants.

(4.1.2) Lemma. *Let $V,W\in Bun_nX$ and $\mathcal{F},\mathcal{F}'\in Coh_{0,X}$. Then*

$$g^{W\oplus\mathcal{F}'}_{V,\mathcal{F}} = \sum_{\mathcal{F}''\in\mathrm{Coh}_{0,X}} g^W_{V\mathcal{F}''}\frac{|Aut(\mathcal{F}')|\cdot|Aut(\mathcal{F}'')|}{|Aut(\mathcal{F})|}\cdot q^{nh^0(\mathcal{F}')}$$

Indeed, supposing the lemma true, we have

$$\left(\sum_v f(V)[V])[\mathcal{F}]\right) = \sum_{\substack{V,W\in\mathrm{Bun}_nX\\ \mathcal{F}'\in\mathrm{Coh}_{0,X}}} g^{W\oplus\mathcal{F}'}_{V\mathcal{F}} f(V)[W\oplus\mathcal{F}'] =$$

$$= \sum_{V,W}\sum_{\mathcal{F}',\mathcal{F}''}\frac{|\mathrm{Aut}(\mathcal{F}')|\cdot|\mathrm{Aut}(\mathcal{F}'')|}{|\mathrm{Aut}(\mathcal{F})|}f(V)[W\oplus\mathcal{F}']\cdot q^{nh^0(\mathcal{F}')}$$

$$= \sum_{\mathcal{F}'\subset\mathcal{F}}\sum_W\sum_{\substack{V\subset W\\ W/V\simeq\mathcal{F}'/\mathcal{F}'}} f(V)\frac{|\mathrm{Aut}(\mathcal{F}')|\cdot|\mathrm{Aut}(\mathcal{F}/\mathcal{F}')|}{|\mathrm{Aut}(\mathcal{F})|}\cdot[W\oplus\mathcal{F}']\cdot q^{nh^0(\mathcal{F})}$$

$$= \sum_{\mathcal{F}'\subset\mathcal{F}}\sum_W(T_{\mathcal{F}/\mathcal{F}'}f)(W)\frac{|\mathrm{Aut}(\mathcal{F}')|\cdot|\mathrm{Aut}(\mathcal{F}/\mathcal{F}')|}{|\mathrm{Aut}(\mathcal{F})|}\cdot q^{nh^0(\mathcal{F}')}[W\oplus\mathcal{F}'],$$

as claimed.

Proof of Lemma 4.1.2. If G is a finite group acting on a finite set S, we call the orbifold number of elements in S modulo G the quantity

$$\frac{|S|}{|G|} = \sum_{\{s\}} \frac{1}{|\mathrm{Stab}(s)|}$$

where $\{s\}$ are all the G-orbits in S, with $s \in \{s\}$ being a chosen representative and $\mathrm{Stab}(s) \subset G$ being its stabilizer.

Now, let $V, W, \mathcal{F}', \mathcal{F}$ be given. The subsheaf $\mathcal{F}' \subset W \oplus \mathcal{F}'$ is defined intrinsically, as the maximal torsion subsheaf, so $g^{W\oplus\mathcal{F}'}_{\mathcal{F}',W} = 1$. It follows that

$$g^{W\oplus\mathcal{F}'}_{V,\mathcal{F}} = |\mathrm{Aut}(W \oplus \mathcal{F}')| \cdot C,$$

where C is the orbifold number of diagrams ("crosses")

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ & & & & W & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & V & \xrightarrow{\alpha} & W \oplus \mathcal{F}' & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle\gamma} & & & & \\ & & & & \mathcal{F}' & & & & \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array} \tag{4.1.3}$$

modulo the product of the groups of automorphisms of all five objects. By elementary homological algebra, every cross can be completed to a 3×3 diagram with exact rows and columns, from which we want to retain the outer frame

$$\begin{array}{ccccccccc} & & 0 & & & & 0 & & \\ & & \uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & V'' & \longrightarrow & W & \longrightarrow & \tilde{\mathcal{F}}'' & \longrightarrow & 0 \\ & & \uparrow & & & & \uparrow & & \\ & & V & & & & \mathcal{F} & & \\ & & \uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & V' & \longrightarrow & \mathcal{F}' & \longrightarrow & \tilde{\mathcal{F}}' & \longrightarrow & 0 \\ & & \uparrow & & & & \uparrow & & \\ & & 0 & & & & 0 & & \end{array} \tag{4.1.4}$$

Here, for instance, $\tilde{\mathcal{F}}' = \mathrm{Im}(\beta\gamma)$ etc. Notice that we have $V' = 0$ since it should be a torsion sheaf embedded in V. Thus $\tilde{\mathcal{F}}' \cong \mathcal{F}$ and $V'' \simeq V$. For each $\mathcal{F}''$ let $C(\mathcal{F}'')$ be the orbifold number of crosses giving frames with the upper right corner $\tilde{\mathcal{F}}''$ isomorphic to $\mathcal{F}''$, so that $C = \sum_{\mathcal{F}''} C(\mathcal{F}'')$. Let also $F(\mathcal{F}'')$ be the orbifold number of frames, i.e., arbitrary diagrams of the form (4.1.4) in which $\tilde{\mathcal{F}}'' = \mathcal{F}'', \tilde{\mathcal{F}}' = \mathcal{F}', V'' = V$, (modulo the product of the groups of automorphisms of all 8 objects constituting the frame).

The main result of Green ([Gr], Theorem 2) says that $C(\mathcal{F}'') = \langle \mathcal{F}'', V' \rangle^2 \cdot F(\mathcal{F}'')$ which is equal to just $F(\mathcal{F}'')$ since $V' = 0$ and thus $\langle \mathcal{F}'', V' \rangle = 1$. On the other hand,

$$F(\mathcal{F}'') = g^{\mathcal{F}}_{\tilde{\mathcal{F}}'\tilde{\mathcal{F}}''} \cdot g^{W}_{V''\tilde{\mathcal{F}}''} \cdot \frac{|\mathrm{Aut}(V'')| \cdot |\mathrm{Aut}(\tilde{\mathcal{F}}'')| \cdot |\mathrm{Aut}(\tilde{\mathcal{F}}')| \cdot |\mathrm{Aut}(V')|}{|\mathrm{Aut}(V)| \cdot |\mathrm{Aut}(W)| \cdot |\mathrm{Aut}(\mathcal{F})| \cdot |\mathrm{Aut}\mathcal{F}'|}$$

as it follows from the general fact that g^C_{AB} is the orbifold number of exact sequences

$$0 \to A \to C \to B \to 0$$

modulo $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$.

Recalling our identifications, we find that

$$F(\mathcal{F}'') = g^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''} g^{W}_{V\mathcal{F}''} \cdot \frac{|\mathrm{Aut}(\mathcal{F}'')|}{|\mathrm{Aut}(W)| \cdot |\mathrm{Aut}(\mathcal{F})|}.$$

Notice also that

$$\begin{aligned} |\mathrm{Aut}(W \oplus \mathcal{F}')| &= |\mathrm{Aut}(W)| \cdot |\mathrm{Aut}(\mathcal{F}')| \cdot |\mathrm{Hom}(W, \mathcal{F}')| = \\ &= q^{nh^0(\mathcal{F}')} |\mathrm{Aut}(W)| \cdot |\mathrm{Aut}(\mathcal{F}')|. \end{aligned}$$

Thus

$$\begin{aligned} g^{W\oplus\mathcal{F}'}_{V,\mathcal{F}} &= |\mathrm{Aut}(W \oplus \mathcal{F}')| \cdot C = |\mathrm{Aut}(W \oplus \mathcal{F}')| \cdot \sum_{\mathcal{F}''} C(F'') = \\ &= |\mathrm{Aut}(W \oplus \mathcal{F})| \cdot \sum_{\mathcal{F}''} F(\mathcal{F}'') = \\ &= q^{nh^0(\mathcal{F}')} |\mathrm{Aut}(W)| \cdot |\mathrm{Aut}(\mathcal{F}')| \cdot \sum_{\mathcal{F}''} g^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''} g^{W}_{V\mathcal{F}''} \frac{|\mathrm{Aut}(\mathcal{F}'')|}{|\mathrm{Aut}(W)| \cdot |\mathrm{Aut}(\mathcal{F})|} = \\ &= q^{nh^0(\mathcal{F}')} \sum_{\mathcal{F}''} g^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''} g^{W}_{V\mathcal{F}''} \frac{|\mathrm{Aut}(\mathcal{F}')| \cdot |\mathrm{Aut}(\mathcal{F}'')|}{|\mathrm{Aut}(\mathcal{F})|} \end{aligned}$$

as claimed. Lemma 4.1.2 and thus Proposition 4.1.1 are proved.

For the product in the Ringel algebra R we find from (4.1.1):

$$(4.1.5a)\qquad [\mathcal{F}] * \sum_{V\in \mathrm{Bun}_n X} f(V)[V] = q^{nh^0(\mathcal{F})/2}\sum_V f(V)[V\oplus \mathcal{F}]$$

$$(4.1.5b)\qquad \left(\sum_{V\in \mathrm{Bun}_n X} f(V)[V]\right) * [\mathcal{F}] = q^{-nh^0(\mathcal{F})/2}\cdot \left(\sum_V f(V)[V]\right)\circ [\mathcal{F}] =$$

$$= \sum_{\mathcal{F}'\subset\mathcal{F}} q^{-nh^0(\mathcal{F}\mathcal{F}')/2}(T_{\mathcal{F}/\mathcal{F}'}f)(W)\frac{|\mathrm{Aut}(\mathcal{F}')|\cdot|\mathrm{Aut}(\mathcal{F}/\mathcal{F}')|}{|\mathrm{Aut}(\mathcal{F})|}\cdot q^{\frac{nh^0(\mathcal{F}')}{2}}[W\oplus\mathcal{F}'].$$

Passing now from just one basis vector $[\mathcal{F}]$ to the generating function

$$\psi_g(t) = \sum_{\mathcal{F}} \bar{\chi}_g([\mathcal{F}])t^{h^0(\mathcal{F})}|\mathrm{Aut}(\mathcal{F})|\cdot[\mathcal{F}],\quad g\in \mathrm{Cusp}_m$$

we find, for $f\in\mathrm{Cusp}_n$:

$$(4.1.6)\qquad \psi_g(t_2) * E_f(t_1) =$$

$$\sum_{\substack{\mathcal{F}\in\mathrm{Coh}_{0,X}\\ V\in \mathrm{Bun}_n(X)}} \bar{\chi}_g([\mathcal{F}])f(V)q^{+nh^0(\mathcal{F})/2}|\mathrm{Aut}(\mathcal{F})|\cdot\cdot t_2^{h^0(\mathcal{F})}t_1^{\deg(V)}[V\oplus\mathcal{F}].$$

To find the product in the opposite order, we use (4.1.5b) together with the fact that f is a Hecke eigenform, and find

$$(4.1.7)\quad E_f(t_1)*\psi_g(t_2) = \sum_{\mathcal{F}\in\mathrm{Coh}_{0,X}}\ \sum_{W\in\mathrm{Bun}_n X}\ \sum_{\mathcal{F}',\mathcal{F}''\in\mathrm{Coh}_{0,X}} g^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''}\chi_f([\mathcal{F}''])f(W)\cdot$$

$$\cdot\bar{\chi}_g([\mathcal{F}])q^{-n(h^0(\mathcal{F}'')-h^0(\mathcal{F}'))/2}|\mathrm{Aut}(\mathcal{F}')|\cdot|\mathrm{Aut}(\mathcal{F}'')|\cdot$$

$$\cdot t_2^{h^0(\mathcal{F})}t_1^{\deg W-h^0(\mathcal{F}'')}[W\oplus\mathcal{F}'],$$

where we have also replaced the summation over subsheaves $\mathcal{F}'\subset\mathcal{F}$ by the summation over arbitrary pairs of isomorphism classes $\mathcal{F}',\mathcal{F}''$, with the factor $g^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''}$ counting the number of subsheaves of type $\mathcal{F}'$ and cotype $\mathcal{F}''$. Since $\bar{\chi}_g$ is a character of the Hall algebra, and $\sum_{\mathcal{F}} g^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''}[\mathcal{F}] = [\mathcal{F}']\circ[\mathcal{F}'']$, we can replace the RHS of (4.1.7) by

$$(4.1.8)\qquad \sum_{W\in\mathrm{Bun}_n X}\ \sum_{\mathcal{F}',\mathcal{F}''\in\mathrm{Coh}_{0,X}} \chi_f([\mathcal{F}''])f(W)\bar{\chi}_g([\mathcal{F}'])\bar{\chi}_g([\mathcal{F}''])\cdot$$

$$\cdot q^{nh^0(\mathcal{F}')/2-nh^0(\mathcal{F}'')/2}|\mathrm{Aut}(\mathcal{F}')|\cdot|\mathrm{Aut}(\mathcal{F}'')|\cdot t_2^{h^0(\mathcal{F}')+h^0(\mathcal{F}'')}\cdot[W\oplus\mathcal{F}'].$$

This can be written as the product

(4.1.9)

$$\left\{ \sum_{W\in \mathrm{Bun}_n X} \sum_{\mathcal{F}'\in \mathrm{Coh}_{0,X}} \bar{\chi}_g([\mathcal{F}'])f(W)q^{nh^0(\mathcal{F}')/2} \cdot |\mathrm{Aut}(\mathcal{F}')| \cdot t_2^{h^0(\mathcal{F}')} t_1^{\deg(W)} \cdot [W\oplus \mathcal{F}'] \right\} \times$$

$$\times \left\{ \sum_{\mathcal{F}''\in \mathrm{Coh}_{0,X}} \chi_f([\mathcal{F}''])\bar{\chi}_g([\mathcal{F}''])q^{nh^0(\mathcal{F}'')/2} \cdot |\mathrm{Aut}(\mathcal{F}'')|(t_2/t_1)^{h^0(\mathcal{F}'')} \right\},$$

of which the first factor is identical, up to renaming the summation arguments, with the RHS of (4.1.6), i.e., it is equal to $\psi_g(t_2) * E_f(t_1)$. The second factor has the form $\Lambda(t_2/q^{\frac{n}{2}} t_1)$, where

$$\Lambda(t) = \sum_{\mathcal{F}\in \mathrm{Coh}_{0,X}} \chi_f([\mathcal{F}])\bar{\chi}_g([\mathcal{F}]) \cdot |\mathrm{Aut}(\mathcal{F})| t^{h^0(\mathcal{F})} \tag{4.1.10}$$

For $x \in X$ and an integer sequence $\mu = (\mu_1 \geq \ldots \geq \mu_r \geq 0)$ denote $\mathcal{F}_{x,\mu} = \bigoplus \mathcal{O}/I_x^{\mu_i}$. Since every $\mathcal{F} \in \mathrm{Coh}_{0,X}$ has a unique decomposition into the direct sum of sheaves of the form $\mathcal{F}_{x,\mu}$ (one for each $x \in X$), we can decompose $\Lambda(t)$ into the Euler product:

$$\Lambda(t) = \prod_{x\in X} \Lambda_x(t^{\deg(x)}), \tag{4.1.11}$$

$$\Lambda_x(t) = \sum_{\mu} \chi_f([\mathcal{F}_{x,\mu}])\bar{\chi}_g([\mathcal{F}_{x,\mu}]) \cdot |\mathrm{Aut}(\mathcal{F}_{x,\mu})| t^{|\mu|},$$

where $|\mu| = \sum \mu_i$. By (2.5.2) and (2.6.13),

$$\chi_f([\mathcal{F}_{x,\mu}]) = q_x^{-\sum(i-1)\mu_i} P_\mu \left(q_x^{\frac{n-1}{2}} \lambda_{1,x}(f)^{-1}, \ldots, q_x^{\frac{n-1}{2}} \lambda_{n,x}(f)^{-1}; q_x^{-1} \right) \tag{4.1.12}$$

$$\bar{\chi}_g([\mathcal{F}_{x,\mu}]) = q_x^{-\sum(i-1)\mu_i} P_\mu \left(q_x^{\frac{m-1}{2}} \lambda_{1,x}(g), \ldots, q_x^{\frac{m-1}{2}} \lambda_{m,x}(g); q_x^{-1} \right)$$

where P_μ is the Hall-Littlewood polynomial. It is known ([Mac], Ch. III, formula (4.4)) that

$$\sum_\mu b_\mu P_\mu(z_1 \ldots z_n; q_x^{-1}) P_\mu(w_1, \ldots, w_m; q_x^{-1}) = \prod_{i,j} \frac{1 - q_x^{-1} z_i w_j}{1 - z_i w_j}, \tag{4.1.13}$$

where

$$b_\mu = q_x^{-|\mu| - 2\Sigma(i-1)\mu_i} \cdot |\mathrm{Aut}(\mathcal{F}_{x,\mu})|. \tag{4.1.14}$$

Therefore

$$\Lambda_x(t) = \prod_{i,j} \frac{1 - q_x^{\frac{n+m}{2}-1} \frac{\lambda_{j,x}(g)}{\lambda_{i,x}(f)} t}{1 - q_x^{\frac{n+m}{2}} \frac{\lambda_{j,x}(g)}{\lambda_{i,x}(f)} t},$$

and so

$$\Lambda(t) = \frac{\mathrm{LHom}(f, g, q^{\frac{n+m}{2}} t)}{\mathrm{LHom}(f, g, q^{\frac{n+m}{2}-1} t)} \tag{4.1.15}$$

Since, as we already saw,

$$E_f(t_1) * \psi_g(t_2) = \Lambda(t_2/q^{\frac{n}{2}} t_1) \Psi_g(t_2) * E_f(t_1),$$

we have proved the formula (3.3.2).

(4.2) Proof of (3.3.3). Recalling that the bilinear form $\langle \alpha, \beta \rangle$ vanishes on $\mathcal{K}_0(\mathrm{Coh}_{0,X})$ and that $\bar{\chi}_f$ is a homomorphism $H(\mathrm{Coh}_{0,X}) \to \mathbf{C}$, we find, from (1.6.3):

$$\Delta\psi_f(t) = \sum_{\mathcal{F} \in \mathrm{Coh}_{0,X}} \bar{\chi}_f([\mathcal{F}]) t^{h^0(\mathcal{F})} |\mathrm{Aut}(\mathcal{F})| \cdot \Delta([\mathcal{F}]) =$$

$$= \sum_{\mathcal{F}} \sum_{\mathcal{F}', \mathcal{F}''} \bar{\chi}_f([\mathcal{F}]) t^{h^0(\mathcal{F})} |\mathrm{Aut}(\mathcal{F})| g^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''} \frac{|\mathrm{Aut}(\mathcal{F}')| \cdot |\mathrm{Aut}(\mathcal{F}'')|}{|\mathrm{Aut}(\mathcal{F})|} \cdot [\mathcal{F}'] \otimes c_{\mathcal{F}'}[\mathcal{F}''] =$$

$$= \sum_{\mathcal{F}', \mathcal{F}''} \bar{\chi}_f([\mathcal{F}']) \bar{\chi}_f([\mathcal{F}'']) t^{h^0(\mathcal{F}') + h^0(\mathcal{F}'')} \cdot |\mathrm{Aut}(\mathcal{F}'| \cdot |\mathrm{Aut}(\mathcal{F}'') \cdot$$

$$\cdot [\mathcal{F}'] \otimes c_{\mathcal{F}'}[\mathcal{F}''] = \psi_f(t \otimes c)(1 \otimes \Psi_f(t))$$

as claimed.

(4.3) Proof of (3.3.4). Let $r_0 : AF_n \to \bigoplus_{i+j=n} AF_i \otimes AF_j$ be the linear map dual to the Hall multiplication $AF_i^0 \otimes AF_j^0 \to AF_{i+j}^0$ (with respect to the orbifold scalar product (1.3.2) on each AF_i^0). Thus for $f \in AF_n$ and vector bundles W', W'' of ranks i, j we have

$$(r_0 f)(W', W'') = \sum_{V \in \mathrm{Bun}_n X} g^V_{W'W''} \frac{|\mathrm{Aut}(W')| \cdot |\mathrm{Aut}(W'')|}{|\mathrm{Aut}(V)|} f(V), \tag{4.3.1}$$

and f is a cusp form if $r_0(f) = 1 \otimes f + f \otimes 1$. Let now $f \in \mathrm{Cusp}_n$ be given. Then, by definition

$$\Delta E_f(t) = \sum_{V \in \mathrm{Bun}_n(X)} f(V) t^{\deg(V)} \Delta([V]) = \tag{4.3.2}$$

$$= \sum_V \sum_{U,\mathcal{G}} \langle \mathcal{G}, U\rangle\, g^V_{U\mathcal{G}} f(V) t^{\deg(V)} \frac{|\mathrm{Aut}(U)|\cdot|\mathrm{Aut}(\mathcal{G})|}{|\mathrm{Aut}(V)|}[U]\otimes K_U[\mathcal{G}],$$

where U and $\mathcal{G}$ can be, a priori, any coherent sheaves on X. However, if $g^V_{U\mathcal{G}} \neq 0$, U can be embedded as a subsheaf into V and so is locally free. The sheaf $\mathcal{G}$ may not be locally free, and we write it as $\mathcal{G} = W \oplus \mathcal{F}$ where W is a vector bundle, $\mathcal{F}$ is a torsion sheaf (their isomorphism classes are determined by $\mathcal{G}$). Further, if we have an exact sequence

$$0 \to U \to V \to W \oplus \mathcal{F} \to 0 \tag{4.3.3}$$

we can consider the unqiue subbundle (i.e., subsheaf which is locally a direct summand) $\bar{U} \supset U$, $\mathrm{rk}(\bar{U}) = \mathrm{rk}\, U$. The sequence (4.3.3) gives thus two sequences

$$0 \to \bar{U} \to V \to W \to 0$$
$$0 \to U \to \bar{U} \to \mathcal{F} \to 0$$

and we conclude that

$$g^V_{U,W\oplus\mathcal{F}} = \sum_{\bar{U}\in\mathrm{Bun}(X)} g^{\bar{U}}_{U\mathcal{F}} g^{\mathcal{F}}_{\bar{U}W} \tag{4.3.4}$$

Note also that

$$|\mathrm{Aut}(W\oplus\mathcal{F})| = |\mathrm{Aut}(W)|\cdot|\mathrm{Aut}(\mathcal{F})|\cdot q^{\mathrm{rk}(W)\cdot h^0(\mathcal{F})} \tag{4.3.5}$$

Keeping these two formulas in mind and substituting $\mathcal{G} = W \oplus \mathcal{F}$ into (4.3.2), we find

$$\Delta E_f(t) = \sum_{V,U,W,\bar{U},\mathcal{F}} \langle W\oplus\mathcal{F}, U\rangle g^{\bar{U}}_{U\mathcal{F}} g^V_{\bar{U}W} \frac{|\mathrm{Aut}(\bar{U})|\cdot|\mathrm{Aut}(W)|\cdot|\mathrm{Aut}(U)|\cdot|\mathrm{Aut}(\mathcal{F})|}{|\mathrm{Aut}(V)|\cdot|\mathrm{Aut}(\bar{U})|}\cdot$$
$$\cdot q^{rk(W)\cdot h^0(\mathcal{F})}\cdot f(V)t^{\deg(V)}[U]\otimes K_U[W\oplus\mathcal{F}]$$

where, in addition, we have multiplied and divided by $|\mathrm{Aut}(\bar{U})|$. In this sum, $V \in \mathrm{Bun}_n(X)$, while $U, W, \bar{U}$ are isomorphism classes of vector bundles of arbitrary rank, and $\mathcal{F} \in \mathrm{Coh}_{0,X}$. By (4.3.1) we can write the result of summation over V in terms of r_0, getting

$$\Delta E_f(t) = \sum_{U,W,\bar{U},\mathcal{F}} g^{\bar{U}}_{U\mathcal{F}} (r_0 f)(\bar{U}, W) \frac{|\mathrm{Aut}(U)|\cdot|\mathrm{Aut}(\mathcal{F})|}{\mathrm{Aut}(\bar{U})|}. \tag{4.3.7}$$

$$\cdot \langle W \oplus \mathcal{F}, U\rangle q^{rk(W)\cdot h^0(\mathcal{F})} t^{\deg \bar{U} + \deg W}[U] \otimes K_U[W \oplus \mathcal{F}]$$

Since f is a cusp form, $(r_0 f)(\bar{U}, W) = 0$ unless $\bar{U} = 0$ or $W = 0$. Summation with $\bar{U} = 0$ gives $1 \otimes E_f(t)$. If $W = 0$, the summation gives

$$\text{(4.3.8)} \qquad \sum_{U,\bar{U},\mathcal{F}} g^{\bar{U}}_{U\mathcal{F}} f(\bar{U}) \frac{|\mathrm{Aut}(U)| \cdot |\mathrm{Aut}(\mathcal{F})|}{|\mathrm{Aut}\bar{U}|} \langle \mathcal{F}, U\rangle t^{\deg \bar{U}} [U] \otimes K_U[\mathcal{F}] =$$

$$= \sum_{U,\mathcal{F}} (T^V_{\mathcal{F}} f)(U) \cdot |\mathrm{Aut}(\mathcal{F})| q^{\frac{n}{2} h^0(\mathcal{F})} t^{h^0(\mathcal{F}) + \deg U} [U] \otimes K_U[\mathcal{F}] =$$

$$= \sum_{U,\mathcal{F}} \bar{\chi}_f([\mathcal{F}]) f(U) |\mathrm{Aut}(\mathcal{F})| q^{-\frac{n}{2} h^0(\mathcal{F})} t^{h^0(\mathcal{F}) + \deg U} [U] \otimes K_U[\mathcal{F}]$$

where we used Propositions (2.6.8) and 2.6.11 to identify the action of $T^V_{\mathcal{F}}$. Now, the last expression in (4.3.8) factors into the product

$$\left(\sum_{U \in \mathrm{Bun}_n(X)} f(U) t^{\deg(U)} [U] \otimes K_U \right) \left(1 \otimes \sum_{\mathcal{F} \in \mathrm{Coh}_{0,X}} \bar{\chi}_f([\mathcal{F}]) |\mathrm{Aut}(\mathcal{F})| (q^{-\frac{n}{2}} t)^{h^0(\mathcal{F})} [\mathcal{F}] \right)$$

By using the equality $K_U = K^n c_{\det(U)}$ we can rewrite this as

$$E_f(t \otimes c)(1 \otimes K^n \psi_f(q^{-\frac{n}{2}} t))$$

thus proving (3.3.4).

(4.4) End of the proof of Theorem 3.3. It remains to prove the equalities (3.3.5) and (3.3.6) describing the counit and the antipode. Now, (3.3.5) is obvious from the definition (1.6.5) of ϵ. To see the first equality in (3.3.6), notice that

$$\text{(4.4.1)} \qquad S(\psi_f(t)) = \sum_{\mathcal{F} \in \mathrm{Coh}_{0,X}} \bar{\chi}_f([\mathcal{F}]) \cdot |\mathrm{Aut}(\mathcal{F})| \cdot t^{h^0(\mathcal{F})} S([\mathcal{F}]) =$$

$$= \sum_{\mathcal{F}} \sum_{m=1}^{\infty} (-1)^m \sum_{\mathcal{F}_0 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}} \bar{\chi}_f([\mathcal{F}]) \prod_i |\mathrm{Aut}(\mathcal{F}_i/\mathcal{F}_{i-1})| \cdot \prod_i t^{h^0(\mathcal{F}_i/\mathcal{F}_{i+1})}.$$

$$\cdot \prod_i [\mathcal{F}_i/\mathcal{F}_{i-1}] \cdot \prod_i c^{-1}_{\mathcal{F}_i/\mathcal{F}_{i-1}}.$$

By replacing the summation over $\mathcal{F}$ and then over flags of subsheaves $\mathcal{F}_0 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$ in a given $\mathcal{F}$ by the summation over (independent) isomorphism classes of $\mathcal{G}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ with coefficient $g^{\mathcal{F}}_{\mathcal{G}_0 \ldots \mathcal{G}_m}$ (the number of filtrations on $\mathcal{F}$ with quotients $\mathcal{G}_0, \ldots, \mathcal{G}_m$) and using the fact that $\bar{\chi}_f$ is a character of the Hall algebra, we bring (4.6.1) to the form

$$1+\sum_{m=1}^{\infty}(-1)^m(\psi_f(c^{-1}t)-1)^m$$

i.e., to the geometric series for $\psi_f(c^{-1}t)^{-1}$, as claimed. To prove the second equality in (3.3.6), we write, for $f\in \mathrm{Cusp}_n$:

$$S(E_f(t))=\sum_{V\in \mathrm{Bun}_n(X)} f(V)t^{\deg(V)}S([V])=$$

$$=\sum_{V\in \mathrm{Bun}_n(X)}\sum_{n=1}^{\infty}(-1)^m\sum_{\mathcal{G}_0\subset\ldots\subset\mathcal{G}_m=V} f(V)t^{\deg(V)}\prod_{i=1}^{m}\langle \mathcal{G}_i/\mathcal{G}_{i-1},\mathcal{G}_{i-1}\rangle\cdot$$

$$\cdot\frac{\prod_{j=0}^{m}|\mathrm{Aut}(\mathcal{G}_j/\mathcal{G}_{j-1})|}{|\mathrm{Aut}(V)|}[\mathcal{G}_0]\ldots[\mathcal{G}_m/\mathcal{G}_{m-1}]\cdot K^{-n}c^{-1}_{\det(V)}$$

Since f is a cusp form, the reasoning similar to that in (4.3), shows that only flags $\mathcal{G}_0\subset\ldots\subset\mathcal{G}_m=V$ with $\mathrm{rk}(\mathcal{G}_i)=\mathrm{rk}(V),\forall i$, contribute to the total sum. But when we restrict the summation to such flags (in which we thus have $\mathcal{G}_i/\mathcal{G}_{i-1}\in \mathrm{Coh}_{0,X}$ for $i>0$) we immediately get $-E_f(c^{-1}t)\psi_f(q^{-\frac{n}{2}}t)^{-1}K^{-n}$ by the same reason as above (summation of geometric series) plus the application of the Riemann-Roch theorem to account for $\prod\langle \mathcal{G}_i/\mathcal{G}_{i-1},\mathcal{G}_{i-1}\rangle$.

Theorem 3.3 is completely proved.

(4.5) Proof of Proposition 3.6.3. The equality (3.6.4) is obvious: non-isomorphic objects give orthogonal elements in the Hall algebra. To see (3.6.5) notice that cusp eigenforms with different eigenvalues of Hecke operators are orthogonal, by (2.6.11), so $(E_f(t_1),E_g(t_2))=0$ for $f\neq g$. If $f=g$, the equality (3.6.5) follows at once from the definition and the assumption (2.6.14) that $\| f \|_d^2=1$. So we concentrate on the proof of (3.6.6). Notice that $\psi_f(t)$ can be written as the Euler product

(4.5.1)
$$\psi_f(t)=\prod_{x\in X}\psi_{f,x}(t^{\deg(x)}),\quad \psi_{f,x}(t)=\sum_{\mu}\bar{\chi}_f([\mathcal{F}_{x,\mu}])\cdot|\mathrm{Aut}(\mathcal{F}_{x,\mu})|\cdot t^{|\mu|}\cdot[\mathcal{F}_{x,\mu}]$$

where μ runs over all partitions $\mu_1\geq\ldots\geq\mu_n\geq 0$, with $|\mu|=\sum\mu_i$ and $\mathcal{F}_{x,\mu}=\bigoplus\mathcal{O}_x/I_x^{\mu_i}$ having the same meaning as in (4.1), more precisely, after (4.1.10). Thus we have:

(4.5.2)
$$a_f(t)=\sum_{x\in X}a_{f,x}(t^{\deg(x)}),\quad a_{f,x}(t)=\log\psi_{f,x}(t),$$

and therefore

(4.5.3)
$$(a_f(t_1),a_g(t_2))=\sum_{x\in X}\left(a_{f,x}(t_1^{\deg(x)}),a_{g,x}(t_2^{\deg(x)})\right).$$

We will evaluate each summand in this sum. Let

$$\text{(4.5.4)} \qquad \mathrm{Ch}: H(\mathrm{Coh}_{x,X}) \to \Lambda, \quad [\mathcal{F}_{x,\mu}] \mapsto q^{-\sum(i-1)\mu_i} P_\mu(z_1, \ldots, z_N; q_x^{-1})$$

be the isomorphism discussed in (2.3.5d).

In [Mac], Ch. III, formula (4.8), Macdonald defined a certain scalar product on $\Lambda[t]$ with values in $\mathbf{Q}[t, t^{-1}]$. We denote by $(\ , \ )_{Macd}$ the $\mathbf{Q}$ -valued scalar product on Λ obtained from this $\mathbf{Q}[t, t^{-1}]$ -valued scalar product by specializing to $t = q_x^{-1}$. Let us, for the time being, abbreviate $H(\mathrm{Coh}_{x,X})$ to simply H and introduce on the algebra H the grading $H = \bigoplus_{d \geq 0} H_d$, where H_d is linearly spanned by $\mathcal{F}_{x,\mu}$, $|\mu| = d$.

(4.5.5) Lemma. *Let $u_i \in H_{d_i}, i = 1, 2$. If $d_1 \neq d_2$, then $(u_1, u_2) = 0$, and if $d_1 = d_2 = d$, then*

$$(u_1, u_2) = q_x^{-d}(Ch(u_1), Ch(u_2))_{Macd}.$$

Proof. The first statement is obvious. The second follows from the equality (formula III (4.9) of [Mac])

$$\left(P_\mu(z; q_x^{-1}), Q_{\mu'}(z; q_x^{-1})\right)_{Macd} = \delta_{\mu\mu'},$$

where $Q_{\mu'}(z; q_x^{-1}) = b_{\mu'} P_{\mu'}(z, q_x^{-1})$ and b_μ is defined in (4.1.14), while on the other hand,

$$(\mathcal{F}_{x,\mu}, \mathcal{F}_{x,\mu'}) = \frac{\delta_{\mu\mu'}}{|\mathrm{Aut}(\mathcal{F}_{x,\mu})|}.$$

The lemma implies that

$$\text{(4.5.6)} \qquad \left(a_{f,x}(t_1), a_{g,x}(t_2)\right) = \left(\mathrm{Ch}(a_{f,x}(q_x^{-1} t_1)), \mathrm{Ch}(a_{g,x}(t_2))\right)_{Macd}.$$

Let us therefore find $\mathrm{Ch}(a_{f,x}(t))$. Let us omit the mention of the parameter in the Hall polynomials, which we will always implicitly understand to be equal to q_x^{-1}. We start by writing

$$\mathrm{Ch}\,(\psi_{f,x}(t)) = \sum_\mu q_x^{-2\sum(i-1)\mu_i} P_\mu\left(q_x^{\frac{n-1}{2}} \lambda_x(f)\right) P_\mu(z) |\mathrm{Aut}(\mathcal{F}_{x,\mu})| t^{|\mu|} =$$

$$= \sum_\mu q_x^{|\mu|} b_\mu P_\mu\left(q_x^{\frac{n-1}{2}} \lambda_x(f)\right) P_\mu(z) |t|^\mu$$

where we used (4.1.12) and set $\lambda_x(f) = (\lambda_{x,1}(f), \ldots, \lambda_{x,n}(f))$ as well as $z = (z_1, \ldots, z_N)$. By the formula quoted in (4.1.13), we find:

$$\mathrm{Ch}\,(\psi_{f,x}(t)) \quad = \quad \prod_{i,j} \frac{1 - q_x^{\frac{n-1}{2}} \lambda_{i,x}(f) z_j\, t}{1 - q_x^{\frac{n-1}{2}} \lambda_{i,x}(f) z_j\, qt}$$

and hence

$$\begin{aligned}\mathrm{Ch}(a_{f,x}(t)) &= \log\ \mathrm{Ch}(\psi_{f,x}(t)) = \sum_{d=1}^{\infty} \frac{1-q^d}{d} t^d \sum_{i,j} \left(q_x^{\frac{n-1}{2}} \lambda_{i,x}(f) z_j\right)^d = \\ &= \sum_d \frac{1-q^d}{d} p_d(q_x^{\frac{n-1}{2}} \lambda_x(f)) p_d(z) t^d\end{aligned}$$

where p_d is the $d-th$ power sum symmetric function. Now, by formula III (4.11) of [Mac],

$$(p_d, p_{d'})_{Macd} = \frac{d}{1-q_x^{-d}} \delta_{dd'}.$$

By using (4.5.6), we now find

$$\begin{aligned}(a_{f,x}(t_1), a_{g,x}(t_2)) &= \sum_{d=1}^{\infty} \frac{q^d-1}{d} p_d(q_x^{\frac{n-1}{2}} \lambda_x(f)) p_d(q_x^{\frac{m-1}{2}} \overline{\lambda_x(g)}) t_1^d \bar{t}_2^d = \\ &= \log \prod_{i,j} \frac{1 - q_x^{\frac{n+m}{2}} \lambda_{x,i}(f) \overline{\lambda_{x,j}(g)} t_1 \bar{t}_2}{1 - q_x^{\frac{n+m}{2}-1} \lambda_{x,i}(f) \overline{\lambda_{x,j}(g)} t_1 \bar{t}_2},\end{aligned}$$

and (3.6.8) is obtained by performing the summation over $x \in X$, and noticing (2.6.13) that the set of the $\overline{\lambda_{x,j}(g)}$ is the same as the set of the $\lambda_{x,j}(g)^{-1}$.

§5. Quantum affine algebras.

In this section we compare Hopf algebras formed by automorphic forms with quantum affine algebras. For our purposes it is convenient to treat the quantization parameter q as a constant rather than as an indeterminate variable.

(5.1) Drinfeld's realization of $U_q(\hat{\mathcal{G}})$. Fix a nonzero complex number q. Let $\mathbf{g}$ be a finite-dimensional semisimple Lie algebra whose Cartan matrix $A = \| a_{ij} \|_{i,j=1,\dots,r}$ is symmetric and let $\widehat{\mathbf{g}}$ be the corresponding Kac-Moody algbera (central extension of $\mathbf{g}[t,t^{-1}]$). The quantization $U_q(\widehat{\mathbf{g}})$ can be defined in two ways: the first (root realization) uses the system of simple roots for the affine root system of $\widehat{\mathbf{g}}$ and proceeds directly from the affine Cartan matrix $\hat{A}$ of $\widehat{\mathbf{g}}$ (of size $(r+1)\times(r+1)$). The other, the so-called loop realization of Drinfeld [Dr1-2] (see also [CP]) is the $\mathbf{C}$-algebra generated by the symbols

$$x_i^+(n), x_i^-(n), i = 1, \dots, r, n \in \mathbf{Z}, \quad k_{i,n}^{\pm}, i = 1, \dots, r, n \geq 0$$

and the central element c. They are subject to some relations which are best written in terms of formal generating functions

$$F_i^{\pm}(t) = \sum_{n\in\mathbf{Z}} x_i^{\pm}(n) t^{+n}, \quad \varphi_i^{\pm}(t) = \sum_{n=0}^{\infty} k_{i,n}^{\pm} t^{\pm n}.$$

The relations have the form:

$$k_{i,0}^+ k_{i,0}^- = k_{i,0}^- k_{i,0}^+ = 1, \quad [\varphi_i^{\pm}(t_1), \varphi_j^{\pm}(t_2)] = 0, \tag{5.1.1}$$

$$(t_1 - \sqrt{q}^{\pm a_{ij}} t_2) X_i^{\pm}(t_1) X_j^{\pm}(t_2) = (\sqrt{q}^{\pm a_{ij}} t_1 - t_2) X_j^{\pm}(t_2) X_i^{\pm}(t_1), \tag{5.1.2}$$

$$F_i^{\pm}(t_1)\varphi_j^+(t_2) = \left(\frac{c^{\mp 1/2} t_2/t_1 - \sqrt{q}^{a_{ij}}}{\sqrt{q}^{a_{ij}} c^{\mp 1/2}(t_2/t_1) - 1} \right)^{\pm 1} \varphi_j^+(t_2) F_i^{\pm}(t_1), \tag{5.1.3}$$

$$F_i^{\pm}(t_1)\varphi_j^-(t_2) = \left(\frac{c^{\mp 1/2} t_1/t_2 - \sqrt{q}^{a_{ij}}}{\sqrt{q}^{a_{ij}} c^{\mp 1/2}(t_1/t_2) - 1} \right)^{\mp 1} \varphi_j^-(t_2) F_i^+(t_1), \tag{5.1.4}$$

(5.1.5)

$$[F_i^+(t_1), F_j^-(t_2)] = \delta_{ij} \left\{ \delta\left(\frac{t_1}{ct_2}\right) \varphi_i^-(c^{1/2} t_2) - \delta\left(\frac{ct_1}{t_2}\right) \varphi_i^+(c^{1/2} t_1) \right\} \cdot (q^{1/2} - q^{-1/2})^{-1},$$

$$(5.1.6) \qquad \varphi_i^+(t_1)\varphi_j^-(t_2) = \frac{(\sqrt{q}^{a_{ij}}c^{-1}t_1/t_2 - 1)(ct_1/t_2 - \sqrt{q}^{a_{ij}})}{(c^{-1}t_1/t_2 - \sqrt{q}^{a_{ij}})(\sqrt{q}^{a_{ij}}ct_1/t_2 - 1)}\varphi_j^-(t_2)\varphi_i^+(t_1),$$

$$(5.1.7) \qquad \operatorname*{Sym}_{t_1,\dots,t_m} \sum_{l=0}^{m} (-1)^l \begin{bmatrix} m \\ l \end{bmatrix}_q F_i^\pm(t_1)\dots F_i^\pm(t_l)F_j^\pm(s)F_i^\pm(t_{l+1})\dots F_i^\pm(t_m) = 0,$$

where $m = 1 - a_{ij}$ and Sym stands for the sum over all the permutations of $t_1, \dots, t_m$.

The relations (5.1.7) are the analogs of the Serre relations in finite-dimensional semisimple Lie algebras.

Let $U_q(\widehat{\mathbf{n}^+})$, (resp. $U_q(\widehat{\mathbf{n}^-})$) denote the subalgebra in $U_q(\widehat{\mathbf{g}})$ generated by the elements $x_i^+(n)$ (resp. $x_i^-(n)$) only, which are subject to relations (5.1.2), (5.1.7). Let also $U_q(\widehat{\mathbf{b}^+})$ denote the subalgebra generated by the $x_i^+(n), n \in \mathbf{Z}$ and by the elements $k_i^+(n), n \geq 0$ Similarly for $U_q(\widehat{\mathbf{b}^-})$. They are the quantizations of the enveloping algebras of the Lie algebras

$$\widehat{\mathbf{n}^\pm} = \mathbf{n}^\pm[t, t^{-1}], \quad \widehat{\mathbf{b}^\pm} = \mathbf{n}^\pm[t, t^{-1}] \oplus \mathbf{h}[t^{\pm 1}],$$

where $\mathbf{g} = \mathbf{n}^+ \oplus \mathbf{h} \oplus \mathbf{n}^-$ is the standard decomposition of $\mathbf{g}$ into the nilpotent and Cartan subalgebras.

(5.2) The case $\mathbf{g} = sl_2$ and sheaves on P^1. Consider the simplest case $\mathbf{g} = sl_2$. In this case there is only one root, so we will denote the generators and generating functions as $x^\pm(n), F^\pm(t)$ etc. The algebra $U_q(\widehat{\mathbf{b}^+})$ is generated by c and by the coefficients of the two power series $F^+(t), \varphi^+(t)$.

On the other hand, let us suppose that q is a prime power and specialize the theory of § 3 to the case of the simplest algebraic curve X, namely the projective line $P^1_{\mathbf{F}_q}$. Let $B = B(\mathrm{Coh}_{P^1})$ be the extended Ringel algebra of the category of coherent sheaves on P^1. The group $\mathrm{Pic}(X)$ consists of sheaves $\mathcal{O}(n), n \in \mathbf{Z}$. Thus the algebra B is obtained from the algebra $R(\mathrm{Coh}_{P^1})$ by adding two elements $K = K_{\mathcal{O}}$ and $c = c_{\mathcal{O}(1)}$. The set Cusp consists of one element: the trivial character $\mathbf{1}$ of $\mathrm{Pic}(P^1) = \mathbf{Z}$. Thus there are only two generating functions

$$E(t) = \sum_{n \in \mathbf{Z}} [\mathcal{O}(n)]t^n \quad \in \quad B[[t, t^{-1}]],$$

$$\psi(t) = \sum_{D \in \mathrm{Div}^+(P')} |\mathrm{Aut}\, \mathcal{O}_D| \cdot t^{\deg D}[\mathcal{O}_D] \quad \in \quad 1 + tB[[t]]$$

where $\mathrm{Div}^+(P^1)$ is the set of effective divisors on P^1 and $\mathcal{O}_D$ is the torsion sheaf $\mathcal{O}_{P^1}/\mathcal{O}_{P^1}(-D)$.

Let $R(\mathrm{Bun}(P^1)) \subset B$ be the subalgebra generated by elements $[V]$, where V is a vector bundle. Let $\mathcal{B} \subset B$ be the subalgebra generated by $[V]$ as above together with K, c and the coefficients of $\psi(t)$.

(5.2.1) Theorem. *The algebra $R(Bun(P^1))$ is isomorphic to $U_q(\widehat{\mathbf{n}^+}) \subset U_q(\widehat{sl}_2)$, and $\mathcal{B}$ is isomorphic to $U_q(\widehat{\mathbf{b}^+})$.*

Proof. The function $\mathrm{LHom}(\mathbf{1}, \mathbf{1}, t)$, i.e., the zeta-function of P^1, has the form

$$\zeta(t) = \frac{1}{(1-t)(1-qt)}. \tag{5.2.2}$$

Thus the functional equation for the Eisenstein series (3.3.1), when brought to the polynomial form (3.5.4) gives:

$$(t_1 - qt_2)E(t_1)E(t_2) = (qt_1 - t_2)E(t_2)E(t_1), \tag{5.2.3}$$

which is identical to the defining relation (5.1.2) for $U_q(sl_2)$ (in which $i = j = 1, a_{11} = 2$). Thus the correspondence $x_i^+(n) \mapsto [\mathcal{O}(n)]$ gives a homomoprhism $\gamma : U_q(\widehat{\mathbf{n}^+}) \to R(\mathrm{Bun}(P^1))$. To show that γ is an isomorphism, note that every vector bundle V on P^1 can be represented as $V = \bigoplus_{i\in\mathbf{Z}} \mathcal{O}(i)^{m_i}$. Moreover, the filtration $V^j = \bigoplus_{i\geq j} \mathcal{O}(i)^{m_i}$ is defined intrinsically (this is the Harder-Narasimhan filtration, already discussed in (3.9)). It follows that

$$[V] = q^{-\frac{1}{2}\sum_{i<j} m_i m_j (i-j+1)}[\mathcal{O}(b)^{m_b}] * \ldots * [\mathcal{O}(a)^{m_a}]$$

where $\{a, a+1, \ldots, b\}$ is any interval in $\mathbf{Z}$ containing all i such that $m_i \neq 0$. Note also that

$$[\mathcal{O}(i)^m] * [\mathcal{O}(i)^{m'}] = q^{\frac{mm'}{2}} \begin{bmatrix} m+m' \\ m \end{bmatrix}_q \mathcal{O}(i)^{m+m'},$$

which implies that $[V]$ is actually a monomial in the $[\mathcal{O}(i)]$. Thus γ is surjective. To see that γ is injective, i.e., that there are no further relations among the $[\mathcal{O}(i)]$ except those following from (5.2.2), remark that we just have shown that the monomials

$$[\mathcal{O}(i_1)]^{m_1}[\mathcal{O}(i_2)]^{m_2} \ldots [\mathcal{O}(i_r)]^{m_r}, \quad i_1 > \ldots > i_r \tag{5.2.4}$$

constitute a $\mathbf{C}$-basis in $R(\mathrm{Bun}(P^1))$. On the other hand, the relations (5.2.3) (after passing to the coefficients) can be used to express any monomial as a linear combination of monomials (5.2.4). Since the latter monomials are linearly independent in $R(\mathrm{Bun}(P^1))$, there can be no further relations and so γ is an isomorphism.

Let us now turn to the algebra $\mathcal{B}$. The commutation relation (3.3.2) reads in our case (again, use (5.2.2)) as follows:

$$E(t_1)\psi(t_2) = \frac{1 - q^{-1/2}t_2/t_1}{1 - q^{3/2}t_2/t_1}\psi(t_2)E(t_1), \tag{5.2.5}$$

while in $U_q(\widehat{\mathbf{b}^+})$ we have, by (5.1.3):

$$F^+(t_1)\varphi^+(t_2) = \frac{q - c^{-1/2}t_2/t_1}{1 - q\, c^{-1/2}t_2/t_1}\varphi^+(t_2)F^+(t_1) \tag{5.2.6}$$

Thus the correspondence

$$\varphi^+(t) \mapsto K\,\psi(c^{-1/2}q^{-1/2}t), \quad F^+(t) \mapsto E(t)$$

defines a homomorphism $U_q(\widehat{\mathbf{b}^+}) \to \mathcal{B}$. It is surjective by the definition of $\mathcal{B}$. Its injectivity follows from the injectivity of the homomorphism $\gamma : U_q(\widehat{\mathbf{n}^+}) \to R(\mathrm{Bun}(P^1))$ above and the fact that the coefficients of $\psi(t)$ are algebraically independent over $\mathbf{C}$. This latter fact is proved by the same argument as given in (3.8.5): the coefficients of $\log\psi(t)$, being linearly independent and primitve with respect to a Hopf algebra structure, are algebraically independent. Theorem (5.2.1) is proved.

(5.2.7) Remark. It is instructive to compare Theorem 5.2.1 with results of Ringel [R1-3] and Lusztig [Lu1-3] on representations of quivers. Namely, let Γ be a quiver, i.e., a finite oriented graph without edges-loops. Let Γ – mod be the category of representations of Γ over $\mathbf{F}_q$, i.e., rules which associate to any vertex $i \in \mathrm{Vert}(\Gamma)$ a finite-dimensional $\mathbf{F}_q$ -vector space V_i, and to every oriented edge $i \xrightarrow{e} j$ a linear map $V_e : V_i \to V_j$. This is an Abelian category satisfying our finiteness conditions (1.1), so its Ringel algebra $R(\Gamma - \mathrm{mod})$ is defined. For $i \in \mathrm{Vert}(\Gamma)$ let $V(i)$ be the representation assigning $\mathbf{F}_q$ to the vertex i and 0 to other vertices. Let $A = \| a_{ij} \|_{i,j\in Vert(\Gamma)}$ be the Cartan matrix associated to Γ, i.e., $a_{ii} = 2$ and a_{ij} is minus the number of edges (regardless of the orientation) joining i and j, if $i \neq j$. Let $\mathfrak{g}_\Gamma$ be the Kac-Moody Lie algebra with the Cartan matrix A and $U_q(\mathfrak{g}_\Gamma)$ be its q-quantization. This is (see, e.g., [Lu 1]) the algebra generated by symbols $e_i^\pm, K_i$ (with K_i being invertible) subject to the relations:

$$K_i e_j^\pm K_i^{-1} = q^{\pm a_{ij}/2}e_j, \quad [e_i^+, e_j^-] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{5.2.8}$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_q (e_i^\pm)^l e_j^\pm (e_i^\pm)^{1-a_{ij}-l} = 0. \tag{5.2.9}$$

Let $U_q(N_\Gamma)$ be the subalgebra generated by the e_i^+. Then the subalgebra in $R(\Gamma -$ mod) generated by the $[V(i)]$, is isomorphic to $U_q(N_\Gamma)$ so that $[V(i)]$ corresponds to e_i^+.

Now, $\widehat{sl}_2$ is the Kac-Moody algebra associated to the Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, and its Dynkin graph is $A_1^{(1)} = \{\bullet \rightrightarrows \bullet\}$. So the general results about representations of quivers are applicable to this case and realize $U_q(N_{A_1^{(1)}}) \subset U_q(\widehat{sl}_2)$ inside $R(A_1^{(1)} - \mathrm{mod})$. So we have realizations of two "nilpotent" subalgebras,

$U_q(\widehat{\mathbf{n}^+}), U_q(N) \subset U_q(\widehat{sl}_2)$ in terms of Ringel algebras of two Abelian categories: Coh_{P^1} (sheaves on $P^1_{F_q}$) and $A_1^{(1)}$ – mod. One may wonder what these two categories have in common, and the answer is that their derived categories are equivalent. This is a particular case of a theorem of Beilinson [Be], as reformulated by Bondal [Bo] and Geigle-Lenzing [GL]. This suggests a deeper relation between derived categories and quantum group-like objects, which will be studied in a separate paper.

(5.3) The case of a general curve X. Considering now the case of an arbitrary smooth projective curve $X/\mathbf{F}_q$ we find, by comparing Theorem 3.5 with formulas of (5.1), that the algebra $B(\mathrm{Coh}_X)$ (or, rather, its subalgebra $\mathcal{B}$ defined in (3.8)) is analogous to the subalgebra $U_q(\widehat{\mathbf{b}^+}) \subset U_q(\widehat{\mathbf{g}})$ for a huge Kac-Moody algebra $\mathbf{g}$. Let us sketch this analogy in the following table:

Theory of quantum affine algebras	**Theory of automorphic forms**
The set of positive roots of $\mathbf{g}$	The set Cusp of cusp eigenforms
The entries of the Cartan matrix of $\mathbf{g}$	The coeffcients of Rankin L-functions $\mathrm{LHom}(f, g, t)$
Symmetry of Cartan matrix	Functional equations of L-functions
Pointwise-uppertriangular subalgebra $U_q(\widehat{\mathbf{n}}^+)$	The algebra $R(\mathrm{Bun}(X))$, i.e., the algebra of unramified forms
Pointwise-Cartan subalgebra $U_q(\mathbf{h}[t])$	The algebra of classical Hecke operators
Root decomposition of $U_q(\mathbf{n}^+)$	Spectral decomposition of the space of automorphic forms
Components of $x^+_{i_1}(t_1)\dots x^+_{i_n}(t_n)$ at a given basis vector of $U_q(\hat{\mathbf{g}})$	Eisenstein series
Serre relations	?
?	Selberg trace formula
Full algebra $U_q(\widehat{\mathbf{g}})$	?
Mc Kay correspondence	Langlands correspondence

We have also included in this table several concepts whose counterparts are not immediately clear. Thus, Serre relations should correspond to "extra" functional equations for Eisenstein series, i.e., elements of the kernel of the map $\pi_{\mathcal{B}} : \tilde{\mathcal{B}} \to \mathcal{B}$

in (3.8.2), which was described in (3.8) only implictly, by means of the kernel of a suitable quadratic form on $\tilde{\mathcal{B}}$. Note that the presence of such relations does not contradict the spectral decomposition theorem for automorphic forms [MW] which, as we already pointed out (3.8.4), is only concerned with the values of the (analytically continued) products $E_{f_1}(t_1)\ldots E_{f_r}(t_r)$ for $|t_i| = 1$, while the extra relations may look like distributions whose support does not meet the torus $|t_i| = 1$ at all. This is exactly the case with Serre relations (5.1.7). Namely, if we allow ourselves to divide equalities involving generating functions by any polynomials in $t_1, \ldots, t_n$, then (5.1.7) would "follow" from (5.1.2), as one can easily check by "bringing" the LHS of (5.1.7) to a normal form in which the factor $F_j^{\pm}(s)$ goes last in every monomial (of course, such a manipulation is illegal!). Moreover, in this particular case it is enough to divide by polynomials of the form $(t_i - q^{\alpha}t_j), \alpha \in \frac{1}{2}\mathbf{Z}$, so the need for (5.1.7) cannot be observed after restriction to the torus $|t_i| = 1$ where such polynomials do not vanish. However, there is no way to deduce (5.1.7) from the quadratic relations obtained by expanding (5.1.2). This just means that in the algebra defined by (5.1.2) alone the right hand side of (5.1.7) is a distribution with support of positive codimension.

The analog of the Selberg formula in the theory of affine Lie algebras is completely unclear at the moment. According to our analogy, it should be a statement about the trace of a Cartan element acting on the enveloping algebra of the upper-triangular subalgebra.

As for the analog of the $U_q(\widehat{\mathfrak{g}})$, we will construct such an analog in the next section by following Drinfeld's quantum double construction and using Green's comultiplication.

§6. The quantum double of the algebra of automorphic forms.

(6.0) Motivation. Our aim in this section is to construct the automorphic analog of the full quantum affine algebra $U_q(\widehat{\mathbf{g}})$ out of the Hopf algebra $B(\mathcal{A})$ which should be, morally, just one half of it. This will be done by the Drinfeld double construction. For any Hopf algebra Ξ its Drinfeld double [Dr 3] [CP] is the tensor product $\Xi \otimes_{\mathbf{C}} \Xi^*$ with certain twisted multiplication (see below).

The reason for using this construction is that it is known to solve, in a certain sense, the following model problem: recover the quantum Kac-Moody algebra $U_q(\mathbf{g})$ from its Hopf subalgebra $U_q(\mathbf{b}^+)$. Recall (5.2.7) that $U_q(\mathbf{g})$ is generated by symbols $e_i^{\pm}, K_i$. The subalgebra $U_q(\mathbf{b}^+)$ is, by definition, generated by e_i^+ and K_i. The comultiplication in it has the form:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(e_i^+) = 1 \otimes e_i^+ + e_i^+ \otimes K_i. \tag{6.0.1}$$

However, the recovery of $U_q(\mathbf{g})$ from the $U_q(\mathbf{b}^+)$ by this method is not so straightforward. To help motivate the constructions of this section, let us recall the principal features of (and subtleties involved in) this recovery.

(6.0.2) The subalgebras $U_q(\mathbf{n}^{\pm})$ generated by the $e_i^{\pm}$, are in natural duality, so adding $(U_q(\mathbf{n}^+))^*$ as a part of $(U_q(\mathbf{b}^+))^*$ supplies the missing subalgebra $U_q(\mathbf{n}^-)$.

(6.0.3) However, we get also the dual of the "Cartan" subalgebra $U_q(\mathbf{h})$ generated by the K_i. So by formally applying the double construction to $U_q(\mathbf{b}^+)$, we get an algebra bigger that $U_q(\mathbf{g})$ because of this dual. So in order to get $U_q(\mathbf{g})$, we need to impose certain identifications in the double. (The construction of the restricted double later in this section does just that, in the context of Hall algebras.)

(6.0.4) In addition, the Hopf algebra $U_q(\mathbf{h})$ is not self-dual. It is the algebra of functions on T, the maximal torus in the Lie group corresponding to $\mathbf{g}$, and the dual algebra will be algebra of functions on $\widehat{T}$ (the lattice of characters of T) with pointwise multiplication. Formal dualization of $U_q(\mathbf{h})$ with respect to the basis of monomials in the K_i gives a basis of delta-functions on $\widehat{T}$, in particular, the unit element in the algebra of functions on $\widehat{T}$ will be represented by an infinite sum of such delta-functions. If the Cartan matrix A is non-degenerate, one can realize elements of $U_q(\mathbf{h})$ by certain infinite sums in the completion of the algebra of functons on $\widehat{T}$. After this one can perform the identification mentioned in (6.0.3).

(6.0.5) If the Cartan matrix is degenerate (as is the case with affine algebras), then one has to add to $U_q(\mathbf{h})$ some essentially new elements from $U_q(\mathbf{h})^*$ which pair with the center (the kernel of the Cartan matrix) in a nondegenerate way. In the affine case the new element (which is essentially one) has important geometric meaning: it corresponds to the "rotation of the loop" and is responsible for the appearance of elliptic functions in the theory of representations of affine algebras.

In what follows, we will encounter the analogs of all these phenomena for the Hopf algebras of automorphic forms.

(6.1) Heisenberg and Drinfeld doubles: generalities. Along with the just mentioned quantum double of Drinfeld it will be convenient for us to use, as an intermediate step, the so-called Heisenberg double, introduced in [AF] [ST]. So we start by recalling some basic properties of these doubles. Since Hall algebras come with a natural basis, we will use the coordinate-dependent approach, as in [Kas] (see [ST] for a more standard exposition). In this reminder we will ignore subtleties related to dualization of infinite-dimensional spaces, since we will pay due attention to them in the concrete situations in which we will use the doubles.

Let Ξ be a Hopf algebra over $\mathbf{C}$, with comultiplication Δ, counit ϵ and antipode S. Let $\{e_i\}, i \in I$, be a basis of Ξ and $\{e^i\}$ be the dual basis of Ξ^*. Let us write the structure constants of Ξ with respect to our basis:

$$e_i e_j = \sum_k m_{ij}^k e_k, \quad \Delta(e_k) = \sum_{i,j} \mu_k^{ij} e_i \otimes e_j, \tag{6.1.1}$$

$$1 = \sum_i \epsilon^i e_i, \quad \epsilon(e_i) = \epsilon_i, \quad S(e_i) = \sum_j S_i^j e_j, \quad S^{-1}(e_i) = \sum_j \sigma_i^j e_j.$$

As usual, Δ makes Ξ^* into an algebra by

$$e^i e^j = \sum_k \mu_k^{ij} e^k. \tag{6.1.2}$$

The Heisenberg double $HD(\Xi)$ is, by definition, the algebra generated by the symbols $Z_i, Z^i, i \in I$, subject to the relations:

$$Z_i Z_j = \sum_k m_{ij}^k Z_k, \quad Z^i Z^j = \sum_k \mu_k^{ij} Z^k, \tag{6.1.3}$$

$$Z_i Z^j = \sum_{a,b,c} m_{ab}^j \mu_i^{bc} Z^a Z_c. \tag{6.1.4}$$

Thus the map $\Xi^* \otimes_{\mathbf{C}} \Xi \to HD(\Xi)$ given by the multiplication, i.e., by $e^i \otimes e_j \mapsto Z^i Z_j$, is an isomorphism of vector spaces: any other product can be brought by (6.1.4) to the normal form in which the Z^i stand on the left.

There is a certain asymmetry in the definition of the Heisenberg double: replacing Ξ with Ξ^* gives a different algebra. We will denote $HD(\Xi^*)$ by $\check{H}D(\Xi)$. Explicitly, it is generated by the symbols $\check{Z}_i, \check{Z}^i, i \in I$ subject to the relations:

$$\check{Z}_i \check{Z}_j = \sum_k m_{ij}^k \check{Z}_k, \quad \check{Z}^i \check{Z}^j = \sum_k \mu_k^{ij} \check{Z}^k, \tag{6.1.3'}$$

$$\check{Z}^i \check{Z}_j = \sum_{a,b,c} \mu_j^{ab} m_{bc}^i \check{Z}_a \check{Z}^c \tag{6.1.4'}$$

The Drinfeld double $DD(\Xi)$ is the algebra generated by the symbols $W_i, W^i, i \in I$, subject to the relations

$$W_i W_j = \sum_k m_{ij}^k W_k, \quad W^i W^j = \sum_k \mu_k^{ik} W^k, \tag{6.1.5}$$

$$\sum_{a,b,c} \mu_i^{ab} m_{bc}^j W_a W^c = \sum_{a,b,c} m_{ab}^j \mu_i^{bc} W^a W_c. \tag{6.1.6}$$

By using the antipode one can transform (6.1.6) into a relation giving a kind of normal form for elements of $DD(\Xi)$:

$$W_i W^j = \sum_{a,b,c,d,e,f,g} \sigma_b^a m_i^{bc} m_c^{de} \mu_j^{af} \mu_f^{ge} W^g W_d. \tag{6.1.7}$$

This is the standard description [Dr 3] [CP] of the Drinfeld double.

The algebra $DD(\Xi)$ is a Hopf algebra with respect to the comultiplication given by

$$\Delta(W_k) = \sum_{i,j} \mu_k^{ij} W_i \otimes W_j, \quad \Delta(W^k) = \sum_{i,j} m_{ji}^k W^i \otimes W^j, \tag{6.1.8}$$

(note the transposition in the second formula), the counit

$$\epsilon(W_k) = \epsilon_k, \quad \epsilon(W^k) = \epsilon^k, \tag{6.1.9}$$

and the antipode given by

$$S(W_i) = \sum_j S_i^j W_j, \quad S(W^j) = \sum_i S_i^j W^i. \tag{6.1.10}$$

The algebras $HD(\Xi), \check{H}D(\Xi)$ are not Hopf algebras. However, there is the following result due to R. Kashaev [Kas].

(6.1.11) Proposition. *The correspondence*

$$W_k \to \sum_{i,j} \mu_k^{ij} Z_i \otimes \check{Z}_j, \quad W^k \to \sum_{i,j} m_{ji}^k Z^i \otimes \check{Z}^j$$

defines an embedding $\kappa : DD(\Xi) \hookrightarrow HD(\Xi) \otimes HD(\Xi^*)$.

It is this embedding that makes the Heisenberg doubles important to us.

Proof. Let us first prove the existence of the homomorphism κ and then its injectivity. To prove the existence, we need to check that the relations (6.1.5-6) defining $DD(\Xi)$ are satisfied for the proposed images of W_i, W^i. For (6.1.5) it is obvious since $\Delta : \Xi \to \Xi \otimes \Xi$ and $m^* : \Xi^* \to \Xi^* \otimes \Xi^*$ (the map dual to the multiplication) are algebra homomorphisms. To verify (6.1.6) denote by L and R its left and right hand sides. Then:

$$L = \sum_{a,b,c} \mu_i^{ab} m_{bc}^j \kappa(W_a)\kappa(W^c) = \sum_{a,b,c,p,q,r,s} \mu_i^{ab} m_{bc}^j \mu_a^{pq} m_{sr}^c Z_p Z^r \otimes \check{Z}_q \check{Z}^s =$$

$$= \sum_{\substack{a,b,c,p,q,r,s\\u,v,w}} \mu_i^{ab} m_{bc}^j \mu_a^{pq} m_{sr}^c m_{uv}^r \mu_p^{vw} Z^u Z_w \otimes \check{Z}_q \check{Z}^s = \sum_{b,s,u,w,q,v} \mu_i^{vwqb} m_{bsuv}^j Z^u Z_w \otimes \check{Z}_q \check{Z}^s,$$

where μ_i^{vwqb} and m_{bsuv}^j are the structure constants for the 3-fold comultiplication and multiplication. Similarly,

$$R = \sum_{a,b,c} m_{ab}^j \mu_i^{bc} \kappa(W^a)\kappa(W_c) = \sum_{a,b,c,d,p,q,r,s} m_{ab}^j \mu_i^{bc} m_{qp}^a \mu_c^{rs} Z^p Z_r \otimes \check{Z}^q \check{Z}_s =$$

$$= \sum_{\substack{a,b,c,d,p,q,r,s \\ u,v,w}} m_{ab}^j \mu_i^{bc} m_{qp}^a \mu_c^{rs} \mu_s^{uv} m_{vw}^q \, Z^p Z_r \otimes \check{Z}_u \check{Z}^w = \sum_{b,r,u,w,p,v} \mu_i^{bruv} m_{vwpb} \, Z^p Z_r \otimes \check{Z}_u \check{Z}^w,$$

which is the same as L up to renaming the summation indices. So we indeed have the claimed homomorphism κ. To see that κ is injective, note that $\Delta : \Xi \to \Xi \otimes \Xi$ and $m^* : \Xi^* \to \Xi^* \otimes \Xi^*$ are injective because of the unit and counit in Ξ. Since $DD(\Xi) = \Xi \otimes \Xi^*$ as a vector space, and $\kappa|_\Xi = \Delta$ while $\kappa|_{\Xi^*}$ is the composition of m^* and the permutation, κ is injective.

(6.2) The restricted Heisenberg doubles of the Ringel algebra. We now specialize to the case when $\Xi = B(\mathcal{A}) = \mathbf{C}[\mathcal{K}_0\mathcal{A}] \otimes R(\mathcal{A})$ is the extended Ringel algebra of an Abelian category $\mathcal{A}$ satisfying the assumptions of (1.4) (we assume, in particular, that $\mathcal{A}$ has homological dimension 1). The bilinear form $(\alpha|\beta)$ on $\mathcal{K}_0\mathcal{A}$ may be degenerate, and we denote by I its kernel. Since $\mathbf{Q}^*$, the target of our form, has no torsion, I has the following property: if $n\alpha \in I$ for some $n \in \mathbf{Z}$, $\alpha \in \mathcal{K}_0\mathcal{A}$, then $\alpha \in I$. This implies that I has a complement, a subgroup $J \subset \mathcal{K}_0\mathcal{A}$ such that $\mathcal{K}_0\mathcal{A} = I \oplus J$. In the sequel we fix such a complement and denote π_I, π_J the projections to I or J along the other summand.

A natural basis of $\Xi = B(\mathcal{A})$ is given by $e_{\alpha A} = K_\alpha[A]$, $\alpha \in \mathcal{K}_0\mathcal{A}$, $A \in \mathcal{A}$. In this basis, the multiplication and comlutiplication have the form:

$$e_{\alpha,A} \cdot e_{\beta,B} = (\bar{A}|\beta)\langle B, A\rangle \sum_C g_{AB}^C e_{\alpha+\beta,C}, \tag{6.2.1}$$

where g_{AB}^C is the same as in (1.2),

$$\Delta(e_{\gamma,C}) = \sum_{A,B} \langle B, A\rangle \frac{|\mathrm{Aut}(A)| \cdot |\mathrm{Aut}(B)|}{|\mathrm{Aut}(C)|} g_{AB}^C e_{\gamma,A} \otimes e_{\gamma+\bar{A},B}. \tag{6.2.2}$$

Thus the structure constants are given by

$$m_{\alpha A,\beta B}^{\gamma C} = \begin{cases} (\bar{A}|\beta)\langle B, A\rangle g_{AB}^C, & \text{if} \quad \gamma = \alpha+\beta \\ 0, \quad \text{if} \quad \gamma \neq \alpha+\beta \end{cases} \tag{6.2.3}$$

$$\mu_{\gamma C}^{\alpha A,\beta B} = \begin{cases} \langle B, A\rangle \dfrac{|\mathrm{Aut}(A)| \cdot |\mathrm{Aut}(B)|}{|\mathrm{Aut}(C)|} g_{AB}^C, & \text{if} \quad \alpha = \gamma, \beta = \gamma + \bar{A}, \\ 0, \quad \text{otherwise} \end{cases} \tag{6.2.4}$$

This gives the multiplication of the dual generators in $HD(\Xi)$ in the form

$$(6.2.5)\qquad Z^{\alpha A}Z^{\beta B} = \begin{cases} \langle B, A\rangle \sum_C \frac{|\mathrm{Aut}(A)|\cdot|\mathrm{Aut}(B)|}{|\mathrm{Aut}(C)|} g_{AB}^C Z^{\alpha C}, & \text{if } \beta = \gamma + \bar{A}, \\ 0, & \text{otherwise} \end{cases}$$

In particular, the $Z^{\alpha 0}$ are orthogonal idempotents. They form the algebra dual to the coalgebra $\mathbf{C}[\mathcal{K}_0\mathcal{A}]$, i.e., the algebra of functions on $\mathcal{K}_0\mathcal{A}$ with pointwise multiplication. More precisely, $Z^{\alpha 0}$ corresponds to the function equal to 1 at α and to 0 elsewhere. In particular, the unit element is represented by the infinite sum $\sum_\alpha Z^{\alpha 0}$.

This shows that we need to work with certain infinite sums lying in the completion of the algebra $HD(B(\mathcal{A}))$ generated by the $Z_{\alpha A}, Z^{\alpha A}$. For $A \in \mathrm{Ob}(\mathcal{A})$ and a homomorphism $\chi : \mathcal{K}_0\mathcal{A} \to \mathbf{C}^*$ we introduce the elements

$$(6.2.6)\qquad Z_A^- = \sum_{\alpha\in\mathcal{K}_0\mathcal{A}} \frac{Z^{\alpha A}}{|\mathrm{Aut}(A)|}, \quad K^\chi = \sum_{\mathcal{K}_0\mathcal{A}} \chi(\alpha) Z^{\alpha 0}.$$

We set also $K_\alpha = Z_{\alpha 0}, Z_A^+ = Z_{0A}$.

If $\alpha \in \mathcal{K}_0\mathcal{A}$, we let $\chi_\alpha : \mathcal{K}_0\mathcal{A} \to \mathbf{Q}^*$ be the character taking β to $(\alpha|\beta)$, and write K^α for K^{χ_α}. Note that we have the following identities:

$$(6.2.7)\qquad Z_A^\pm Z_B^\pm = \sum_C g_{AB}^C Z_C^\pm,$$

$$(6.2.8)\qquad Z_A^\pm K_\beta = (\bar{A}|\beta)^{\pm 1} K_\beta Z_A^\pm, \quad Z_A^\pm K^\chi = \chi(\bar{A})^{\mp 1} K^\chi Z_A^\pm,$$

$$(6.2.9)\qquad K^\chi K^{\chi'} = K^{\chi\chi'}, \quad K^\chi K_\alpha = K_\alpha K^\chi,$$

which show that we have two copies of the Ringel algebra $R(\mathcal{A})$ inside the completion of $HD(B(\mathcal{A}))$.

(6.2.10) Proposition. *The subspace in the completion of $HD(B(\mathcal{A}))$ spanned by elements of the form $Z_A^- K_\alpha K^\chi Z_B^+$, is an algebra, i.e., the product of any two such elements is a linear combination of finitely many elements of this form.*

We will call this algebra the algebraic Heisenberg double of $B(\mathcal{A})$ and denote it $HD^{alg}(B(\mathcal{A}))$.

To prove (6.2.10), note that the identities (6.2.7-9) give an almost complete recipe for multiplying any $Z_A^- K_\alpha K^\chi Z_B^+$ with any $Z_{A'}^- K_{\alpha'} K^{\chi'} Z_{\beta'}^+$, except for the product $Z_B^+ Z_{A'}^-$ which we need to express through $Z_M^- Z_N^+$. In order to find this expression, denote, for any objects A, B, M, N of $\mathcal{A}$, by g_{AB}^{MN} the orbifold number of exact sequences

$$(6.2.11)\qquad 0 \to M \xrightarrow{u} B \xrightarrow{\varphi} A \xrightarrow{v} N \to 0$$

modulo automorphisms of A and B, i.e., (see the proof of Lemma 4.2) the total number of such sequences divided by $|\mathrm{Aut}(A)|\cdot|\mathrm{Aut}(B)|$. Proposition 6.2.10 follows from the next fact which is also useful by itself.

(6.2.12) Proposition. *In the completion of $HD(B(\mathcal{A}))$ we have the identities:*

$$Z^+_A Z^-_B = \sum_{M,N} g^{MN}_{AB} \langle \bar{N} - \bar{M},\, \bar{B} - \bar{M} \rangle K_{\bar{B}-\bar{M}} Z^-_M Z^+_N.$$

Proof. Let $\mathcal{F}^{MN}_{AB}$ be the set of all exact sequences (6.2.11). Also, for any three objects A, B, C let $\mathcal{E}^C_{AB}$ be the set of exact sequences

$$0 \to A \to C \to B \to 0.$$

Thus

$$g^{MN}_{AB} = \frac{|\mathcal{F}^{MN}_{AB}|}{|\mathrm{Aut}(A)| \cdot |\mathrm{Aut}(B)|}, \quad g^C_{AB} = \frac{|\mathcal{E}^C_{AB}|}{|\mathrm{Aut}(A)| \cdot |\mathrm{Aut}(B)|}.$$

Notice now that

$$\mathcal{F}^{MN}_{AB} \quad = \coprod_{L \in \mathrm{Ob}(\mathcal{A})/\mathrm{Iso}} (\mathcal{E}^L_{MB} \times \mathcal{E}^N_{LA})/\mathrm{Aut}(L), \tag{6.2.13}$$

with $\mathrm{Aut}(A)$ acting freely. This just means that any long exact sequence (6.2.10) can be split into two short sequences with $L = \mathrm{Im}(\varphi)$. From the cross-symmetry relations (6.1.4) in the Heisenberg double and the explicit form of the structure constants given above, we find:

$$Z_{0A} Z^{\beta B} = \sum_{M,L,N} \langle L, M \rangle \cdot \langle N, L \rangle \cdot \frac{|\mathrm{Aut}(L)| \cdot |\mathrm{Aut}(N)|}{|\mathrm{Aut}(A)|}.$$

$$\frac{|\mathcal{E}^B_{ML}|}{|\mathrm{Aut}(L)| \cdot |\mathrm{Aut}(M)|} \cdot \frac{|\mathcal{E}^A_{LN}|}{|\mathrm{Aut}(L)| \cdot |\mathrm{Aut}(N)|} \cdot Z^{\beta M} Z_{\bar{L},N} =$$

$$= \sum_{M,N} \langle \bar{B} - \bar{M}, \bar{M} \rangle \cdot \langle \bar{N}, \bar{B} - \bar{M} \rangle \cdot \frac{|\mathcal{F}^{MN}_{AB}|}{|\mathrm{Aut}(A)| \cdot |\mathrm{Aut}(M)|} Z^{\beta M} Z_{\bar{B}-\bar{M},N},$$

where we used (6.2.12) and the identity $\bar{L} = \bar{B} - \bar{M}$ holding whenever $\mathcal{E}^B_{ML} \neq \emptyset$. From this we deduce that

$$Z_{0A} \cdot \frac{Z^{\beta B}}{|\mathrm{Aut}(B)|} = \sum_{M,N} \langle \bar{B} - \bar{M}, \bar{M} \rangle \cdot \langle \bar{N}, \bar{B} - \bar{M} \rangle (\bar{M} | \bar{B} - \bar{M})^{-1} g^{MN}_{AB} K_{\bar{B}-\bar{M}} \frac{Z^{\beta M}}{|\mathrm{Aut}(M)|} \cdot Z_{0,N},$$

whence the statement of the proposition.

Finally, we define the algebra $\mathrm{Heis}(\mathcal{A})$, called the restricted Heisenberg double of $B(\mathcal{A})$, by imposing the following identifications in $HD^{alg}(B(\mathcal{A}))$:

$$K^\alpha = K^{-1}_{\pi_J(\alpha)}, \tag{6.2.14}$$

where π_J was defined in (6.1). Thus, in the case when the form $(\alpha|\beta)$ is non-degenerate, this identifies each K^χ with some product of (complex) powers of the

K_α, while in the degenerate case we have also elements K^χ for characters χ of $\mathcal{K}_0\mathcal{A} = I \oplus J$ trivial on J, i.e., characters of I. These elements are analogs of the "rotation of the loop" generators in the (quantum) Kac-Moody algebras.

The other , "checked", Heisenberg double $\check{H}D(B(\mathcal{A}))$ can be treated in a similar way. Its generators are denoted $\check{Z}_{\alpha A}, \check{Z}^{\alpha A}$, and we set
(6.2.15)

$$\check{Z}_A^+ = \check{Z}_{0A}, \quad \check{Z}_A^- = \sum_{\alpha \in \mathcal{K}_0\mathcal{A}} \frac{\check{Z}^{\alpha A}}{|\mathrm{Aut}(A)|}, \quad \check{K}_\alpha = \check{Z}_{\alpha 0}, \quad \check{K}^\chi = \sum_{\alpha \in \mathcal{K}_0\mathcal{A}} \chi(\alpha)\check{Z}^{\alpha 0}.$$

The subspace spanned by the $\check{Z}_A^- \check{K}_\alpha \check{K}^\chi \check{Z}_B^+$ is again a subalgebra, denoted $\check{H}D^{alg}(B(\mathcal{A}))$. Its generators satisfy the same identities as in (6.2.7-9), while (6.2.12) is now replaced by

$$(6.2.16) \qquad \check{Z}_A^- \check{Z}_B^+ = \sum_{M,N} \langle \bar{B} - \bar{M}, \bar{M} \rangle \cdot \langle \bar{N}, \bar{B} - \bar{M} \rangle \cdot g_{AB}^{MN} K^{\bar{B}-\bar{M}} \check{Z}_M^+ \check{Z}_N^-.$$

As before, we define the restricted double $\mathrm{Heis}^\vee(\mathcal{A})$ by quotienting $\check{H}D^{alg}(B(\mathcal{A}))$ by the relations $\check{K}^\alpha = \check{K}_{\pi_J(\alpha)}^{-1}$.

(6.3) The restricted Drinfeld double of the Ringel algebra. As in the case of Heisenberg doubles, let $W_{\alpha A}, W^{\alpha A}$ be the generators of $DD(B(\mathcal{A}))$ corresponding to the basis vector $e_{\alpha A} = K_\alpha[A]$. Similarly to the above, we set

$$W_A^+ = W_{0A}, \quad W_A^- = \sum_{\alpha \in \mathcal{K}_0\mathcal{A}} \frac{W^{\alpha A}}{|\mathrm{Aut}(A)|}, \quad K_\alpha = W_{\alpha 0}, \quad K^\chi = \sum_{\alpha \in \mathcal{K}_0\mathcal{A}} \chi(\alpha) W^{\alpha 0}.$$

We also write $K^\alpha = K^{\chi_\alpha}$, where $\chi_\alpha(\beta) = (\alpha|\beta)$. We denote by $DD^{alg}(B(\mathcal{A}))$ the subspace in the completion of $DD(B(\mathcal{A}))$ spanned by elements of the form $W_A^+ K_\alpha K^\chi W_B^-$.

(6.3.1) Proposition. *(a) $DD^{alg}(B(\mathcal{A}))$ is an algebra.*

(b) $DD^{alg}(B(\mathcal{A}))$ is a Hopf algebra with respect to the comultiplication given on generators by

$$(6.3.2) \qquad \Delta(W_A^+) = \sum_{A' \subset A} \langle A/A', A' \rangle \frac{|Aut(A')| \cdot |Aut(A/A')|}{|Aut(A)|} W_{A'}^+ \otimes K_{\bar{A}'} W_{A/A'}^+$$

$$(6.3.3) \qquad \Delta(W_A^-) = \sum_{A' \subset A} \langle A/A', A' \rangle \frac{|Aut(A')| \cdot |Aut(A/A')|}{|Aut(A)|} W_{A/A'}^- K^{\bar{A}'} \otimes W_{A'}^-,$$

$$(6.3.4) \qquad \Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \Delta(K^\chi) = K^\chi \otimes K^\chi.$$

the counit given by

(6.3.5) $$\epsilon(K_\alpha) = 1, \quad \epsilon(W^{\pm}_{\mathbf{A}}) = 0, \quad \epsilon(K^\chi) = 1,$$

and the antipode given by

(6.3.6) $$S(K_\alpha) = K_\alpha^{-1} = K_{-\alpha}, \quad S(K^\chi) = K^{\chi^{-1}},$$

(6.3.7)
$$S(W^+_A) = \sum_{n=1}^{\infty}(-1)^n \sum_{A_0 \subset \ldots \subset A_n = A} \prod_{i=1}^{n} \langle A_i/A_{i-1}, A_{i-1}\rangle \frac{\prod_{j=0}^{n} |Aut(A_j/A_{j-1})|}{|Aut(A)|} \cdot$$
$$\cdot W^+_{A_0} W^+_{A_1/A_0} \ldots W^+_{A_n/A_{n-1}} \cdot K^{-1}_{\bar{A}},$$

(6.3.8)
$$S(W^-_A) = \sum_{n=1}^{\infty}(-1)^n \sum_{A_0 \subset \ldots \subset A_n = A} \prod_{i=1}^{n} \langle A_i/A_{i-1}, A_{i-1}\rangle \frac{\prod_{j=0}^{n} |Aut(A_j/A_{j-1})|}{|Aut(A)|} \cdot$$
$$K^{-\bar{A}} \cdot W^-_{A_n/A_{n-1}} \ldots W^-_{A_1/A_0} W^-_{A_0}.$$

(c) The correspondence

(6.3.9) $$W^+_A \mapsto \sum_{A' \subset A} \langle A/A', A'\rangle \frac{|Aut(A')| \cdot |Aut(A/A')|}{|Aut(A)|} Z^+_{A'} \otimes \check{K}_{\bar{A}'} \check{Z}^+_{A/A'},$$

(6.3.10) $$W^-_A \mapsto \sum_{A' \subset A} \langle A/A', A'\rangle \frac{|Aut(A')| \cdot |Aut(A/A')|}{|Aut(A)|} Z^-_{A/A'} K^{\bar{A}} \otimes Z^-_{A'},$$

(6.3.11) $$K_\alpha \mapsto K_\alpha \otimes \check{K}_\alpha, \quad K^\chi \mapsto K^chi \otimes \check{K}^\chi$$

defines an algebra homomorphism

$$\kappa : DD^{alg}(B(\mathcal{A})) \hookrightarrow HD^{alg}(B(\mathcal{A})) \otimes \check{H}D^{alg}(B(\mathcal{A})).$$

Proof. (a) This follows from (6.1.7) together with the identities

$$W^{\pm}_A W^{\pm}_B = \sum_C \langle B, A\rangle g^C_{AB} W^{\pm}_C, \quad W^{\pm}_A K_\beta = (\bar{A}|\beta)^{\mp 1} K_\beta W^{\pm}_A, \quad W^{\pm}_A K^\chi = \chi(\bar{A})^{\mp 1} K^\chi W^{\pm}_A,$$

$$K_\alpha K^\chi = K^\chi K_\alpha, \quad K^\chi K^{\chi'} = K^{\chi\chi'},$$

which are verified in the same way as (6.2.7-9). The statements (b), (c) follow by a straightforward application of definitions, checking that they indeed make sense on the elements given by infinite sums we consider and applying the fact that, in general, $DD(\Xi)$ is a Hopf algebra and the map κ from proposition 6.2.12 is a homomorphism.

We now define the restricted Drinfeld double $U(\mathcal{A})$ by imposing in $DD^{alg}(B(\mathcal{A}))$ the relations $K^\alpha = K^{-1}_{\pi_J(\alpha)}$. The notation is chosen to suggest the analogy with the quantum universal enveloping algebra $U_q(\mathbf{g})$ of a Lie algebra $\mathbf{g}$. In fact, when $\mathcal{A} = \Gamma - \text{mod}$ is the category of $\mathbf{F}_q$-representations of a Dynkin quiver Γ, then $U(\mathcal{A}) = U_q(\mathbf{g})$, where $\mathbf{g}$ is the semisimple Lie algebra corresponding to Γ.

(6.3.12) Proposition. *(a) The Hopf algebra structure on $DD^{alg}(B(\mathcal{A}))$ descends to $U(\mathcal{A})$.*

(b) The homomorphism κ defined in (6.3.9-11) descends to a homomorphism

$$\varkappa : U(\mathcal{A}) \hookrightarrow Heis(\mathcal{A}) \otimes Heis^{\vee}(\mathcal{A}).$$

The proof is straightforward, by checking the compatibility of the relations $K^{\alpha} = K^{-1}_{\pi_J(\alpha)}$ with the Hopf algebra structure and with κ.

(6.4) The Heisenberg double of the algebra of automorphic forms. We now further specialize to the case when $\mathcal{A}$ is the category of coherent sheaves on a curve $X/\mathbf{F}_q$ and we keep the notations and assumptions of §2-3. Recall that we have the exact sequence

$$0 \to \mathrm{Pic}(X) \xrightarrow{i} \mathcal{K}_0(X) \xrightarrow{\mathrm{rk}} \mathbf{Z} \to 0, \tag{6.4.1}$$

where rk is the generic rank homomorphism, and $i(L) = \bar{L} - 1$. The image of i is precisely I, the kernel of the form $(\alpha|\beta)$. A natural splitting of the sequence (6.4.1) and thus a natural choice of a complement J to I, is provided by the maps

$$\det : \mathcal{K}_0(X) \to \mathrm{Pic}(X), \quad \varepsilon : \mathbf{Z} \to \mathcal{K}_0(X)$$

given by $\det(\bar{V}) = \overline{\bigwedge^{\mathrm{rk}(V)}(V)}$, if V is a vector bundle, and by $\varepsilon(1) = \bar{\mathcal{O}}$. Thus the Cartan elements of the restricted Heisenberg double are:

$$K = K_{\bar{\mathcal{O}}}, \quad c_L = K_{\bar{L}-\bar{\mathcal{O}}}, \, L \in \mathrm{Pic}(X), \quad d_\chi = K^{\chi(\det)}, \, \chi : \mathrm{Pic}(X) \to \mathbf{C}^*.$$

The elements c_L are central, while the d_χ are the "rotation of the loop" elements. They commute with the other generators by the rule:

$$d_\chi W^{\pm}_{\mathcal{F}} = \chi(\det(\bar{\mathcal{F}}))^{\mp 1} W^{\pm}_{\mathcal{F}} d_\chi, \quad \mathcal{F} \in \mathrm{Coh}_X.$$

In particular, for $\lambda \in \mathbf{C}^*$ we will denote $d(\lambda) = d_{\chi_\lambda}$, where $\chi_\lambda : \mathrm{Pic}(X) \to \mathbf{C}^*$ is the character given by $\chi_\lambda(L) = \lambda^{\deg(L)}$. This element $d(\lambda)$ is a familiar degree operator:

$$d(\lambda) W^{\pm}_V = \lambda^{\mp \deg(V)} W^{\pm}_V d(\lambda), \quad V \in \mathrm{Bun}(X),$$

$$d(\lambda) W^{\pm}_{\mathcal{F}} = \lambda^{\pm h^0(\mathcal{F})} W^{\pm}_{\mathcal{F}} d(\lambda), \quad \mathcal{F} \in \mathrm{Coh}_{0,X}.$$

We write Heis for $\mathrm{Heis}(\mathrm{Coh}_X)$ and $\mathrm{Heis}^\vee$ for $\mathrm{Heis}^\vee(\mathrm{Coh}_X)$. The Cartan elements of $\mathrm{Heis}^\vee$ will be denoted by $\check{K}, \check{c}_L, \check{d}_\chi$, according to our general conventions.

For a cusp form $f \in \mathrm{Cusp}_n$ we introduce the generating functions

$$E^+_f(t) = \sum_{V \in \mathrm{Bun}_n(X)} f(V) t^{\deg(V)} Z^+_V \quad \in \quad \mathrm{Heis}[[t, t^{-1}]], \tag{6.4.2}$$

$$E^-_f(t) = \sum_{V \in \mathrm{Bun}_n(X)} f(V^*) t^{-\deg(V)} Z^-_V \quad \in \quad \mathrm{Heis}[[t, t^{-1}]],$$

$$\Psi_f^+(t) = \sum_{\mathcal{F}\in \mathrm{Coh}_{0,x}} \overline{\chi_f}([\mathcal{F}]) t^{h^0(\mathcal{F})} |\mathrm{Aut}(\mathcal{F})| Z_{\mathcal{F}}^+ \quad \in \quad \mathrm{Heis}[[t]],$$

$$\Psi_f^-(t) = \sum_{\mathcal{F}\in \mathrm{Coh}_{0,x}} \chi_f([\mathcal{F}]) t^{-h^0(\mathcal{F})} |\mathrm{Aut}(\mathcal{F})| Z_{\mathcal{F}}^- \quad \in \quad \mathrm{Heis}[[t^{-1}]].$$

Similar generating functions in $\mathrm{Heis}^\vee$ will be denoted by $\check{E}_f^\pm(t), \check{\Psi}_f^\pm(t)$. Notice that, as in (4.5.1), we have Euler product expansions

$$\Psi_f^\pm(t) = \prod_{x\in X} \Psi_{f,x}^\pm(t^{\deg(x)}), \quad \Psi_{f,x}^+(t) = \sum_{\mu=(\mu_1\geq\cdots\geq\mu_n\geq 0)} \overline{\chi_f}([\mathcal{F}_{x,\mu}]) |\mathrm{Aut}(\mathcal{F}_{x,\mu})| t^{|\mu|} Z_{\mathcal{F}_{x,\mu}}^-$$

and similarly for $\Psi_{f,x}^-(t)$. We will also use the logarithmic generating functions

$$a_f^\pm(t) = \log\, \Psi_f^\pm(t) = \sum_{x\in X} a_{f,x}^\pm(t), \quad a_{f,x}^\pm(t) = \log\, \Psi_{f,x}^\pm(t).$$

The relations between generating functions with the same superscript (+ or −) are the same as given in Theorem 3.3, so we concentrate on relations involving generating functions with different signs in the superscript.

(6.5) Theorem. *(a) In the algebra Heis we have the following identities for any $f \in Cusp_n$, $g \in Cusp_m$:*

$$E_f^+(t_1)\Psi_g^-(t_2) = \Psi_g^-(t_2)E_f^+(t_1), \tag{6.5.1}$$

$$\Psi_g^+(t_1)E_f^-(t_2) = \frac{LHom(f,g,q^{m/2}t_1c/t_2)}{LHom(f,g,q^{(m/2)-1}t_1c/t_2)} E_f^-(t_2)\Psi_g^+(t_1), \tag{6.5.2}$$

$$\left[E_f^+(t_1), E_g^-(t_2)\right] = \delta_{f,g}\delta(t_1c/t_2)K^n\Psi_f^+(q^{-n/2}t_1), \tag{6.5.3}$$

$$\Psi_f^+(t_1)\Psi_g^-(t_2) = \frac{LHom(g,f,q^{(m+n/2)-1}t_1c/t_2)}{LHom(g,f,q^{(m+n/2}t_1c/t_2)} \Psi_g^-(t_2)\Psi_f^+(t_1). \tag{6.5.4}$$

(b) In $\mathrm{Heis}^\vee$ we have the following identities:

$$\check{\Psi}_g^-(t_1)\check{E}_f^+(t_2) = \frac{LHom(g,f,q^{m/2}t_2/t_1)}{LHom(g,f,q^{(m/2)-1}t_2/t_1)} \check{E}_f^+(t_2)\check{\Psi}_g^-(t_1), \tag{6.5.5}$$

$$\check{E}_f^-(t_1)\check{\Psi}_g^+(t_2) = \check{\Psi}_g^+(t_2)\check{E}_f^-(t_1), \tag{6.5.6}$$

$$\left[\check{E}_f^-(t_1), \check{E}_g^+(t_2)\right] = \delta_{f,g}\delta(t_2/t_1)K^{-n}\check{\Psi}_f^-(q^{n/2}t_1), \tag{6.5.7}$$

$$(6.5.8)\qquad \check{\Psi}_g^-(t_1)\check{\Psi}_f^+(t_2) = \frac{LHom(g,f,q^{(m+n/2)-1}t_2/t_1)}{LHom(g,f,q^{(m+n)/2}t_2/t_1)}\check{\Psi}_f^+(t_2)\check{\Psi}_g^-(t_1).$$

Proof. We will prove only part (a), part (b) being similar. The equality (6.5.1) follows from Proposition 6.2.12 since the only morphism from a torsion sheaf to a vector bundle is the zero morhism.

Let us prove (6.5.2). By Proposition 6.2.12,

$$(6.5.9)\qquad \Psi_g^+(t_1)E_f^-(t_2) = \sum_{\mathcal{F}\in \mathrm{Coh}_{0,X}}\ \sum_{V\in \mathrm{Bun}_n(X)} \overline{\chi_g}([\mathcal{F}])f(V^*)\cdot |\mathrm{Aut}(\mathcal{F})|\cdot$$

$$\cdot t_1^{h^0(\mathcal{F})}t_2^{-\deg(V)} \sum_{W,\mathcal{F}''\in \mathrm{Coh}_X} g_{\mathcal{F}V}^{W\mathcal{F}''} K_{\bar{\mathcal{F}}-\bar{\mathcal{F}}''}\langle \bar{\mathcal{F}}''-\bar{W},\, \bar{\mathcal{F}}-\bar{\mathcal{F}}''\rangle\cdot Z_W^- Z_{\mathcal{F}''}^+.$$

In order that $g_{\mathcal{F}V}^{W\mathcal{F}''}\neq 0$, i.e., that there exist at least one exact sequence

$$(6.5.10)\qquad 0\to W\to V\xrightarrow{\varphi}\mathcal{F}\to\mathcal{F}''\to 0,$$

$\mathcal{F}''$ must lie in $\mathrm{Coh}_{0,X}$, and W in $\mathrm{Bun}_n(X)$. Therefore $K_{\bar{\mathcal{F}}-\bar{\mathcal{F}}''}$ lies in the center and is the same as $c_{\bar{\mathcal{F}}-\bar{\mathcal{F}}''}$. Also, we have

$$\langle \bar{\mathcal{F}}''-\bar{W},\, \bar{\mathcal{F}}-\bar{\mathcal{F}}''\rangle = \langle \bar{W}, \bar{\mathcal{F}}-\bar{\mathcal{F}}''\rangle^{-1} = q^{-n(h^0(\mathcal{F})-h^0(\mathcal{F}''))/2}.$$

By using (6.2.13), we now write (introducing $\mathcal{F}'$ to be $\mathrm{Im}(\varphi)$ in (6.5.10)):

$$\Psi_g^+(t_1)E_f^-(t_2) = \sum_{V,W\in \mathrm{Bun}_n(X)}\ \sum_{\mathcal{F},\mathcal{F}',\mathcal{F}''\in \mathrm{Coh}_{0,X}} g_{W\mathcal{F}'}^{V} g_{\mathcal{F}'\mathcal{F}''}^{\mathcal{F}}\cdot\frac{|\mathrm{Aut}(\mathcal{F}')|\cdot|\mathrm{Aut}(\mathcal{F}'')|\cdot|\mathrm{Aut}(W)|}{|\mathrm{Aut}(V)|}\cdot$$

$$\cdot\overline{\chi_g}([\mathcal{F}])f(V^*)t_1^{h^0(\mathcal{F})}t_2^{-\deg(V)}c_{\bar{\mathcal{F}}'}q^{-nh^0(\mathcal{F})/2}Z_W^- Z_{\mathcal{F}''}^+.$$

We now use the fact that $\overline{\chi_g}$ is a character of the Hall algebra, together with the properties of the (dual) Hecke operators (4.3) to transform this sum into

$$\sum_{W,\mathcal{F}',\mathcal{F}''} \overline{\chi_g}([\mathcal{F}'])\overline{\chi_g}([\mathcal{F}''])|\mathrm{Aut}(\mathcal{F}')|\cdot|\mathrm{Aut}(\mathcal{F}'')|\cdot(T_{\mathcal{F}'}f)(W^*)\cdot$$

$$\cdot(t_1 c/q^{n/2})^{h^0(\mathcal{F}')}t_1^{h^0(\mathcal{F}'')}t_2^{-h^0(\mathcal{F}')}t_2^{-\deg(W)}Z_W^- Z_{\mathcal{F}''}^+.$$

Since f is an eigenfunction of Hecke operators, this can be factored into the product

$$\left(\sum_{W,\mathcal{F}''}\overline{\chi_g}([\mathcal{F}''])\cdot|\mathrm{Aut}(\mathcal{F}'')|\cdot f(W^*)t_1^{h^0(\mathcal{F}'')}t_2^{-\deg(W)}Z_W^- Z_{\mathcal{F}''}^+\right)\times$$

$$\times \left(\sum_{\mathcal{F}'} \overline{\chi_g}([\mathcal{F}'])\chi_f([\mathcal{F}']) \cdot |\mathrm{Aut}(\mathcal{F}')| \cdot \left(t_1 c/t_2 q^{n/2})\right)^{h^0(\mathcal{F}')} \right) =$$

$$= E_f^-(t_2)\Psi_g^+(t_1) \cdot \Lambda(t_1 c/t_2 q^{n/2}),$$

where $\Lambda(t)$, given by (4.1.10), is equal to the ratio

$$\frac{\mathrm{LHom}\left(f, g, q^{\frac{n+m}{2}} t\right)}{\mathrm{LHom}\left(f, g, q^{\frac{n+m}{2}-1} t\right)},$$

as we saw in (4.1.14). This proves the equality (6.5.2).

We now prove (6.5.3). By definition,

$$E_f^+(t_1)E_g^-(t_2) = \sum_{\substack{V \in \mathrm{Bun}_n(X) \\ W \in \mathrm{Bun}_m(X)}} f(V)g(W^*)t_1^{\deg(V)} t_2^{-\deg(W)} Z_V^+ Z_W^- =$$

$$\sum_{V,W,M,\mathcal{F}} g_{VW}^{M\mathcal{F}} K_{\bar{W}-\bar{M}} \cdot \langle \bar{\mathcal{F}} - \bar{M}, \bar{W} - \bar{M} \rangle \cdot f(V)g(W^*)t_1^{\deg(V)} t_2^{-\deg(W)} Z_M^- Z_{\mathcal{F}}^+,$$

where $M, \mathcal{F}$ run over all the isomorphism classes of coherent sheaves. However, since f and g are cusp forms, the reasoning similar to that in the proof of (3.3.4) (see (4.4)) shows that we can restrict the summation to M equal either W or 0, without changing the sum. The contribution from $M = W$ gives $E_g^-(t_2)E_f^+(t_1)$, while computing the contribution from $M = 0$ we find (note that $m = n$ because $M = 0$):

$$[E_f^+(t_1), E_g^-(t_2)] = \sum_{V,W,\mathcal{F}} g_{W\mathcal{F}}^{W} \frac{|\mathrm{Aut}(\mathcal{F})|}{|\mathrm{Aut}(V)|} K_{\bar{W}} f(V)g(W^*) \cdot t_1^{\deg(V)} t_2^{-\deg(W)} \langle \mathcal{F}, W \rangle Z_{\mathcal{F}}^+$$

$$= \sum_{W,\mathcal{F}} (T_{\mathcal{F}}^{\vee} f)(W) \cdot \frac{|\mathrm{Aut}(\mathcal{F})|}{|\mathrm{Aut}(V)|} \cdot g(W^*) K^m c_{\det(W)} t_1^{\deg(W)+h^0(\mathcal{F})} t_2^{-\deg(W)} q^{-mh^0(\mathcal{F})/2} Z_{\mathcal{F}}^+$$

$$= \left(\sum_{\mathcal{F}} \chi_f^{\vee}([\mathcal{F}]) \cdot |\mathrm{Aut}(\mathcal{F})| \cdot (t_1/q^{m/2})^{h^0(\mathcal{F})} Z_{\mathcal{F}}^+ \right) \cdot K^m \cdot$$

$$\cdot \left(\sum_{W} \frac{f(W)g(W^*)}{|\mathrm{Aut}(W)|} \left(\frac{t_1}{t_2}\right)^{\deg(W)} c_{\det(W)} \right).$$

If $f \neq g$, then for any d we have, by (2.6.11) and (2.7.4):

$$\sum_{W \in \mathrm{Bun}_{n,d}(X)} \frac{f(W)g(W^*)}{|\mathrm{Aut}(W)|} = 0, \qquad (6.5.11)$$

and when $f = g$ we get the formula (6.5.3), because of the assumptions (2.6.14) we made on the normalization of $f \in \mathrm{Cusp}_n$.

Finally, let us prove (6.5.4). Keeping f and g fixed, let

$$l(t) \quad = \quad \log \frac{\mathrm{LHom}\left(g, f, q^{\frac{m+n}{2}-1}t\right)}{\mathrm{LHom}\left(g, f, q^{\frac{m+n}{2}}t\right)} \quad = \quad \sum_{x \in X} l_x(t^{\deg(x)}),$$

where l_x is the logarithm of the ratio of the Euler factors corresponding to x. By taking logarithm of (6.5.4) and using the notation (6.4.3), we find that it is enough to prove, for each $x \in X$, the equality

$$[a^+_{f,x}(t_1),\, a^-_{g,x}(t_2)] = l_x(ct_1/t_2).$$

By introducing the Taylor coefficients of $a^{\pm}_{f,x}$ and l_x:

$$a^{\pm}_{f,x}(t) = \sum_{d=0}^{\infty} a^{\pm}_{f,x,d} t^{\pm d}, \quad l_x(t) = \sum_{d=0}^{\infty} l_{x,d} t^d, \tag{6.5.12}$$

we can rewrite the desired equality as

$$[a^+_{f,x,d},\, a^-_{g,x,d'}] = \delta_{dd'} c^d l_{x,d}. \tag{6.5.13}$$

This is a statement about the Heisenberg double of the Hopf subalgebra $\mathcal{H}_x \subset B(\mathrm{Coh}_X)$ generated by $[\mathcal{F}]$, $\mathcal{F} \in \mathrm{Coh}_{0,x}(X)$ and by the element c_x. The function $\psi_{f,x}(t)$ taking values in this subalgebra, has the coproduct

$$\Delta(\psi_{f,x}(t)) = \psi_{f,x}(t \otimes c_x)(1 \otimes \psi_{f,x}(t)).$$

This is proved in the same way as (3.3.3). So for $a_{f,x}(t) = \log(\psi_{f,x}(t))$ we have:

$$\Delta(a_{f,x}(t)) = a_{f,x}(t \otimes c_x) + 1 \otimes a_{f,x}(t),$$

or, in the coefficient language,

$$\Delta(a_{f,x,d}) = a_{f,x,d} \otimes c_x^d \,+\, 1 \otimes a_{f,x,d}.$$

So let us look at the following general situation. Let V be a $\mathbf{Z}$-graded vector space. Consider the commutative Hopf algebra $\Xi = S^\bullet(V) \otimes \mathbf{C}[c, c^{-1}]$ with comultiplication

$$\Delta(v) = v \otimes c^{\deg(v)} \,+\, 1 \otimes v, \quad \Delta(c) = c \otimes c.$$

For $v \in V$ let Z_v^+ be the corresponding element of $HD(\Xi)$. For $\phi \in V^*$ let Z_ϕ^- be the element of $HD(\Xi)$ corresponding to the linear form $\Xi \to \mathbf{C}$ which on each $S^\bullet(V) \otimes c^m$ is the derivation $\partial/\partial\phi$ in the symmetric algebra. let also $\check{Z}_v^+, \check{Z}_\phi^-$ be the similar elements of of $\check{H}D(\Xi)$.

(6.5.14) Lemma. *In the described situation we have, for homogeneous $v \in V, \phi \in V^*$,*

$$[Z_v^+, Z_\phi^-] = \phi(v)c^{\deg(v)}, \quad [\check{Z}_\phi^-, \check{Z}_v^+] = \phi(v).$$

The first formula in this lemma implies the equality (6.5.13) (we have added the second formula to give a hint of the proof of (6.5.8)). Indeed, take V to be the space of primitive elements of the commutative and cocommutative Hopf algebra $\mathcal{H}_x/(c-x-1)$. It is graded by $\deg[\mathcal{F}] = h^0(\mathcal{F})/\deg(x)$. The algebra Ξ defined above is just $\mathcal{H}_x$ itself. Taking $v = a_{f,x,d}$, we find that $Z_v^+ = a_{f,x,d}^+$, while $a_{f,x,d}^- = Z_\phi^-$ where ϕ is the linear functional on V given by

$$\phi(w) = \sum_{\mathcal{F}\in \mathrm{Coh}_{0,x}(X)} \frac{a_{f,x,d}(\mathcal{F})w(\mathcal{F})}{|\mathrm{Aut}(\mathcal{F})|} = (w, \overline{a_{f,x,d}}).$$

To deduce (6.5.13), it remains to recall the formula (4.5.7) for the scalar product of the $a_{f,x}(t)$.

As for Lemma 6.5.14, it immediately reduces to the particular case when $\dim(V) = 1$, i.e., $\Xi = \mathbf{C}[x, c, c^{-1}]$ with $\Delta(x) = x \otimes c^d + 1 \otimes x$ and $\Delta(c) = c \otimes c$. The treatment of this case is elementary and is left to the reader.

Theorem 6.5 is proved.

(6.6) The Drinfeld double of the algebra of automorphic forms. Let $U = U(\mathcal{A})$ be the restricted Drinfeld double of the Hopf algebra $B(\mathcal{A})$, $\mathcal{A} = \mathrm{Coh}_X$, see (6.3). To make the formulas below more symmetric, we extend U by adding square roots of the central elements $c_L, L \in \mathrm{Pic}(X)$. This means that we choose an identification $\mathrm{Pic}(X) \simeq \mathbf{Z} \oplus \bigoplus Z/m_i$ and embed $\mathrm{Pic}(X)$ into the group $\sqrt{\mathrm{Pic}(X))} = \frac{1}{2}\mathbf{Z} \oplus \bigoplus Z/2m_i$ in an obvious way. We set

$$U(c^{1/2}) = U \otimes_{\mathbf{C}[\mathrm{Pic}(X)]} \mathbf{C}\left[\sqrt{\mathrm{Pic}(X))}\right].$$

Thus, for any $L \in \mathrm{Pic}(X)$ there is a central element $c_L^{1/2} \in U(c^{1/2})$.

For a cusp for $f \in \mathrm{Cusp}_n$ we introduce the following generating functions with coefficients in $U(c^{1/2})$:

(6.6.1)

$$Y_f^+(t) = \sum_{V\in \mathrm{Bun}_n(X)} f(V)t^{\deg(V)}W_V^+, \quad Y_f^-(t) = \sum_{V\in \mathrm{Bun}_n(X)} f(V^*)t^{-\deg(V)}W_V^-,$$

$$\Phi^+(t) = \sum_{\mathcal{F}\in \mathrm{Coh}_{0,X}} \overline{\chi_f}([\mathcal{F}])t^{h^0(\mathcal{F})} \cdot |\mathrm{Aut}(\mathcal{F})| \cdot c_{\mathcal{F}}^{1/2} W_{\mathcal{F}}^+,$$

$$\Phi^-(t) = \sum_{\mathcal{F}\in \mathrm{Coh}_{0,X}} \chi_f([\mathcal{F}])t^{-h^0(\mathcal{F})} \cdot |\mathrm{Aut}(\mathcal{F})| \cdot c_{\mathcal{F}}^{1/2} W_{\mathcal{F}}^-.$$

Now we can formulate the main result of this paper.

(6.7) Theorem. *The above generating functions satisfy the following identities, for any $f \in Cusp_n$, $g \in Cusp_m$:*

$$Y_f^{\pm}(t_1)Y_g^{\pm}(t_2) = q^{mn(1-g_X)}\frac{LHom(f,g,t_2/t_1)}{LHom(f,g,t_2/qt_1)}Y_g^{\pm}(t_2)Y_f^{\pm}(t_1), \tag{6.7.1}$$

$$\left[\Phi_f^{\pm}(t_1), \Phi_g^{\pm}(t_2)\right] = 0, \tag{6.7.2}$$

$$Y_f^{\pm}(t_1)\Phi^{+}(t_2) = \left(\frac{LHom\left(f,g,q^{\frac{m}{2}}c^{\mp 1/2}t_2/t_1\right)}{LHom\left(f,g,q^{\frac{m}{2}-1}c^{\mp 1/2}t_2/t_1\right)}\right)^{\pm 1}\Phi_g^{+}(t_2)Y_f^{\pm}(t_1), \tag{6.7.3}$$

$$Y_f^{\pm}(t_1)\Phi^{-}(t_2) = \left(\frac{LHom\left(g,f,q^{\frac{m}{2}}c^{\pm 1/2}t_1/t_2\right)}{LHom\left(g,f,q^{\frac{m}{2}-1}c^{\pm 1/2}t_1/t_2\right)}\right)^{\mp 1}\Phi_g^{-}(t_2)Y_f^{\pm}(t_1), \tag{6.7.4}$$

(6.7.5)
$$\left[Y_f^{+}(t_1), Y_g^{-}(t_2)\right] = \delta_{f,g}\left\{\delta\left(\frac{ct_1}{t_2}\right)K^n\Phi_f^{+}(q^{-n/2}c^{1/2}t_1) - \delta\left(\frac{t_2}{t_1}\right)K^{-n}\Phi_f^{-}(q^{n/2}c^{-1/2}t_2)\right\},$$

(6.7.6)
$$\Phi_g^{+}(t_1)\Phi_g^{-}(t_2) = \frac{LHom\left(g,f,q^{\frac{m+n}{2}-1}c^2t_1/t_2\right)LHom\left(g,f,q^{\frac{m+n}{2}}ct_1/t_2\right)}{LHom\left(g,f,q^{\frac{m+n}{2}}c^2t_1/t_2\right)LHom\left(g,f,q^{\frac{m+n}{2}-1}ct_1/t_2\right)}\Phi_g^{-}(t_2)\Phi_f^{+}(t_1),$$

$$KY_f^{\pm}(t) = q^{\pm n(g_X-1)}Y_f^{\pm}(t)K, \quad K\Phi_f^{\pm}(t) = \Phi_f^{\pm}(t)K. \tag{6.7.7}$$

Proof. We use the embedding $\varkappa : U(\mathcal{A}) \hookrightarrow \mathrm{Heis}(\mathcal{A}) \otimes \mathrm{Heis}^{\vee}(\mathcal{A})$ given by Proposition 6.3.12 (b). From Theorem 3.5 we find that on the generating functions the embedding is as follows:

$$Y_f^{+}(t) \mapsto E_f^{+}(t\otimes\check{c})\left(1\otimes\check{K}^n\check{\Psi}_f^{+}(q^{-n/2}t)\right) \;+\; 1\otimes\check{E}_f^{+}(t),$$

$$Y_f^{-}(t) \mapsto E_f^{-}(t)\otimes 1 \;+\; K^{-n}\Psi_f^{-}(q^{-n/2}t)\otimes\check{E}_f^{-}(t),$$

$$\Phi_f^{+}(t) \mapsto \Psi_f^{+}(c^{1/2}t\otimes\check{c}^{3/2})\check{\Psi}_f^{+}(c^{1/2}\otimes\check{c}^{1/2}t),$$

$$\Phi_f^{-}(t) \mapsto \Psi_f^{-}(c^{-1/2}t\otimes\check{c}^{-1/2})\check{\Psi}_f^{-}(c^{-1/2}\otimes\check{c}^{-1/2}t),$$

$$c_L \mapsto c_L \otimes \check{c}_L, \quad K \mapsto K \otimes \check{K}.$$

Our result follows from these formulas and from Theorem 6.5 giving relations in the Heisenberg doubles.

(6.5) Final remarks. Comparing (6.7) with the relations (5.1) describing quantum affine algebras, we see that there is indeed an almost complete similarity. The only difference worth mentioning is related to the zero-modes of the Cartan generators. In quantum affine algebras there are as many such zero modes $\varphi_i^+(0) = \varphi_i^-(0)^{-1}$ as there are simple roots. In the automorphic case there is only one such generator K, coming from the K-theory of the curve X (the generating functions $\Phi_f^\pm(t)$ have constant term 1). This suggests that it may be more natural to consider, as, e.g., in [MW], the space Π of all cusp eigenforms on $\mathrm{Bun}(X)$, i..e., of functions

$$f \cdot t^{\deg}, \quad V \mapsto f(V)t^{\deg(V)}, \quad f \in \mathrm{Cusp}, t \in \mathbf{C}^*$$

and make it into a 1-dimensional complex manifold (disjoint union of copies of $\mathbf{C}^*$ paramterized by the set Cusp). Then instead of writing $E_f^\pm(t), \Phi_f^\pm(t)$, we can write $E^\pm(\pi), \Phi^\pm(\pi)$ where $\pi = f \cdot t^{\deg} \in \Pi$. So the whole algebra will look more like an sl_2-current algebra but with currents being defined on the space Π. We have therefore a kind of conformal field theory on the space of cusp forms, which strengthens certain analogies from [Kap].

(b) As with the generating functions $\Psi_f^\pm(t)$ of Heis, the $\Phi_f^\pm(t)$ can be written as Euler products $\prod_{x\in X} \Phi_{f,x}^\pm(t)$. In the same way as we proved (6.7.6), one obtains that for different x the $\Phi_{f,x}^\pm(t)$ commute with each other, while

$$\Phi_{f,x}^+(t_1)\Phi_{g,x}^-(t_2)\Phi_{f,x}^+(t_1)^{-1}\Phi_{g,x}^-(t_2)^{-1}$$

is the ratio similar to that in (6.7.6) but formed by the Euler factors at x of the L-functions appearing there. This means that for any x, f the coefficients $\mathbf{a}_{f,x,d}^\pm$ of the expansion of $\mathbf{a}_{f,x}^\pm(t) := \log(\Phi_{f,x}^\pm(t))$ form, apart from normalization, free bosons:

$$\left[\mathbf{a}_{f,x,d}^+, \mathbf{a}_{d,x,d'}^-\right] = \delta_{d+d',0} \cdot \left(\sum_{i,j} \left(\frac{\lambda_{x,j}(g)}{\lambda_{x,i}(f)}\right)^d\right) \cdot q^{d(m+n-2)/2} c_x^d (q^d - 1)(c_x^d - 1).$$

According to the point of view going back to Y.I. Manin and B. Mazur, one should visualize any 1-dimensional arithmetic scheme X as a kind of 3-manifold and closed points $x \in X$ as oriented circles in this 3-manifold. Thus the Frobenius element (which is only a conjugacy class in the fundamental group) is visualized as the monodromy around the circle (which, as an element of the fundamental group, is also defined only up to conjugacy since no base point is chosen on the circle), Legendre symbols as linking numbers and so on. From this point of view, it is natural to think of the operators (algebra elements) $\mathbf{a}_{f,x,d}$ for fixed f and varying x, d as forming a free boson field A_f on the "3-manifold" X; more precisely, for $\pm d > 0$, the operator

$\mathbf{a}^{\pm}_{f,x,d}$ is the dth Fourier component of A_f along the "circle" $\mathrm{Spec}(\mathbf{F}_q(x))$. The bosons $\mathbf{a}^{\pm}_{f,x,d}$ and their sums over $x \in X$ (i.e., the Taylor components of $\log \Phi^{\pm}_f(t)$) will be used in a subsequent paper to construct representations of U in the spirit of [FJ].

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