The family of lines on the Fano threefold V_5

by

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Introduction. A smooth projective algebraic 3-fold V over the field $\mathbb{C}$ is called a Fano 3-fold if the anticanonical divisor $-K_V$ is ample. The integer $g = g(V) = \frac{1}{2}(-K_V)^3$ is called the genus of the Fano 3-fold V. The maximal integer $r \geq 1$ such that $\mathcal{O}(-K_V) \cong H^r$ for some (ample) invertible sheaf $H \in \mathrm{Pic}\, V$ is called the index of the Fano 3-fold V. Let V be a Fano 3-fold of the index $r = 2$ and the genus $g = 21$ which has the second Betti number $b_2(V) = 1$. Then V can be embedded in $\mathbb{P}^6$ with degree 5, by the linear system $|H|$, where $\mathcal{O}(-K_V) \cong H^2$ (see Iskovskih [5]). We denote this Fano 3-fold V by V_5.

V_5 can be also obtained as the section of the Grassmannian $G(2,5) \hookrightarrow \mathbb{P}^9$ of lines in $\mathbb{P}^4$ by 3 hyperplanes in general position.

There are some other constructions of the Fano 3-fold V_5 (cf. Fujita [1], Mukai-Umemura [9] and Furushima-Nakayama [3]). But so obtained V_5's are all projective equivalent (cf. [5]).

The remarkable fact on V_5 is that V_5 is a complex analytic compactification of $\mathbb{C}^3$ which has the second Betti number one (see Problem 28 in Hirzebruch [4]).

Now, in this paper, we will analyze in detail the universal family of lines on V_5 and determine the hyperplane sections which can be the boundary of $\mathbb{C}^3$ in V_5 .

In § 1, we will summarize some basic results about V_5 following to Iskovskih [5], Fujita [1] and Peternell-Schneider [6]. In § 2, we will construct a $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ over $\mathbb{P}^2$, where E is a locally free sheaf of rank 2 on $\mathbb{P}^2$, and a finite morphism $\psi : \mathbb{P}(E) \longrightarrow V_5 \hookrightarrow \mathbb{P}^6$ of $\mathbb{P}(E)$ onto V_5 applying the results by Mukai-Umemura [9]. Further, we will show that the $\mathbb{P}^1$ - bundle $\mathbb{P}(E)$ is in fact the universal family of lines on V_5 . In § 3, we will study the boundary of $\mathbb{C}^3$ in V_5 and the set $\{H \in |\mathcal{O}_V(1)| \ ; \ V_5 \backslash H \cong \mathbb{C}^3\}$.

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§ 1. Basic facts on V_5 .

Let $V := V_5$ be a Fano 3-fold of degree 5 in $\mathbb{P}^6$ (see Introduction) and $\ell \cong \mathbb{P}^1$ is a line on V . Then the normal bundle $N_{\ell|V}$ of ℓ in V can be written as follow:

(a) $\qquad N_{\ell|V} \cong \mathcal{O}_\ell \oplus \mathcal{O}_\ell \qquad$, or

(b) $\qquad N_{\ell|V} \cong \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1)$.

We will call a line ℓ of the type $(0, 0)$ (resp. $(-1, 1)$) if $N_{\ell|V}$ is of the type (a) (resp. type (b)) above.

Let $\sigma : V' \longrightarrow V$ be the blowing up of V along the line ℓ , and put $L' := \sigma^{-1}(\ell)$. Then $L' \cong \mathbb{P}^1 \times \mathbb{P}^1$ if ℓ is of type $(0, 0)$, and $L' \cong \mathbb{F}_2$ if ℓ is of type $(-1, 1)$. Let f_1, f_2 be respectively fibers of the first and second projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto $\mathbb{P}^1$, and let s , f be respectively the negative section and a fiber of $\mathbb{F}_2$. Let H be a hyperplane section of V . Since the linear system $|\sigma^*H - L'|$ on V' has no fixed component and no base point and since $h^0(\mathcal{O}(\sigma^*H - L')) = 5$ and $(\sigma^*H - L')^3 = (\sigma^*H - L')^2 \cdot L' = 2$, the linear system $|\sigma^*H - L'|$ defines a birational morphism $\varphi := \varphi_{|\sigma^*H-L'|} : V' \longrightarrow W \hookrightarrow \mathbb{P}^4$ of V' onto a quadric hypersurface W in $\mathbb{P}^4$, in particular, $Q := \varphi(L')$ is a hyperplane section of W . Let $E := E_\ell$ be the ruled surface swept out by lines which intersect the line ℓ and E' the proper transform of E in V'.

Lemma 1.1 (Iskovskih [5], Fujita [1]). W is a smooth quadric hypersurface in $\mathbb{P}^4$ and $Y := \varphi(E)$ is a twisted cubic curve contained in Q. In particular, $\varphi : V' \longrightarrow W$ is the blowing up of W along the curve Y. Further, we have the following.

(a) If ℓ is of type $(0, 0)$, then $\varphi|_{L'} : L' \xrightarrow{\sim} Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\bar{Y} \sim f_1 + 2f_2$ in L'.

(b) If ℓ is of type $(-1, 1)$, then $\varphi|_{L'} : L' \longrightarrow Q \cong \mathbb{Q}_0^2$ (a quadric cone) is the contraction of the negative section s of $L' \cong \mathbb{F}_2$, and $\bar{Y} \sim s + 3f$ in L'.

In (a) and (b), we denote the proper transform of $Y \hookrightarrow Q$ in L' by $\bar{Y}$.

Corollary 1.1. (a) If ℓ is of type $(0, 0)$, then $E' \cong \mathbb{F}_1$. (b) If ℓ is of type $(-1, 1)$, then $E' \cong \mathbb{F}_3$.

Proof. Let $N_{Y|W}$ be the normal bundle of Y in W. Then $N_{Y|W} \cong \mathcal{O}_Y(3) \oplus \mathcal{O}_Y(4)$ if ℓ is of the type $(0, 0)$, and $N_{Y|W} \cong \mathcal{O}_Y(2) \oplus \mathcal{O}_Y(5)$ if Y is of type $(-1, 1)$.

Q.E.D.

Corollary 1.2. (a) If ℓ is of type $(0, 0)$, then there are two points $q_1 \neq q_2$ of ℓ such that (i) there are two lines in V through the point q_i $(i = 1, 2)$, and (ii) there are three lines in V through every point q of $\ell \setminus \{q_1, q_2\}$.

(b) If ℓ is of type $(-1, 1)$, there is exactly one point q_0 of ℓ such that (i) ℓ is the unique line in V through the point q_0, and (ii) there are two lines in V through every point q of $\ell \setminus \{q_0\}$.

Proof. (a) Let $p_2 : Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the projection onto the second component. Since $\bar{Y} \sim f_1 + 2f_2$, $p_2|_Y : Y \longrightarrow \mathbb{P}^1$ is a double cover over $\mathbb{P}^1$. Thus there are two branched point $b_1 \neq b_2$ in $\mathbb{P}^1$. We put $q_i := \sigma \circ (\varphi|_{L^i})^{-1} \circ (p_2|_Y)^{-1}(b_i)$ $(i = 1, 2)$. Then $\ell = \sigma(\bar{Y})$ and $\ell_i := \sigma(\varphi^{-1}(p_2^{-1}(b_i)))$ $(i = 1, 2)$ are two lines through the point q_i for each i. For $b \in \mathbb{P}^1 \setminus \{b_1, b_2\}$, $\ell = \sigma(\bar{Y})$ and $\sigma(\varphi^{-1}(p_2^{-1}(b)))$ are three lines through the point $q \in \ell \setminus \{q_1, q_2\}$, since $p_2^{-1}(b)$ consists of two different points. This proves (a).

(b) We put $q_0 := \sigma(\bar{Y} \cap s) \in \ell$. Then $\ell = \sigma(\bar{Y}) = \sigma(s)$ is the unique line through the point $q_0 \in \ell$. For $y \in Y \setminus \varphi(s)$, $\ell = \sigma(\bar{Y})$ and $\sigma(\varphi^{-1}(y))$ are two lines through a point of $\ell \setminus \{q_0\}$. This proves (b).

Q.E.D.

Corollary 1.3 (Peternell-Schneider [6]). Let E be a non-normal hyperplane section of V_5. Then the singular locus of E is a line ℓ on V, in particular, E is a ruled surface swept out by lines which intersect the line ℓ. Further $V - E \cong \mathbb{C}^3$ if and only if the line ℓ is of type $(-1, 1)$.

Proof. By lemma (3.35) in Mori [8], the non-normal locus of E is a line ℓ on V. Since $h^0(\mathcal{O}_V(1) \otimes I_\ell^2) = 1$ and $\operatorname{Pic} V \cong \mathbb{Z}$, the linear system $|\mathcal{O}_V(1) \otimes I_\ell^2|$ consists of E, where I_ℓ is the ideal sheaf of ℓ. By Lemma 1, ℓ must be the singular locus of E. Assume ℓ is of type $(0, 0)$. Then, by Lemma 1, $V - E \cong \{(x, y, z, u) \in \mathbb{C}^4 ;\ x^2 + y^2 + z^2 + u^2 = 1\} \not\cong \mathbb{C}^3$.

Q.E.D.

§ 2. Construction of the universal family.

1. Let $(x:y)$, $(u:v)$ be respectively homogeneous coordinates of the first factor and the second factor of $S := \mathbb{P}^1 \times \mathbb{P}^1$. Let us consider the diagonal $SL(2;\mathbb{C})$ - action on S , namely, for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 := SL(2;\mathbb{C})$,

$$\begin{cases} x^\sigma = ax + by \\ y^\sigma = cx + dy \end{cases} , \qquad \begin{cases} u^\sigma = au + bv \\ v^\sigma = cu + dv \end{cases} .$$

Let $\tau : S \longrightarrow \mathbb{P}^2$ be the double covering of $\mathbb{P}^2$ given by

$$\begin{cases} \tau^* X_0 = x \otimes u \\ \tau^* X_1 = \frac{1}{2}(x \otimes v + y \otimes u) \\ \tau^* X_2 = y \otimes v \end{cases}$$

where $(X_0 : X_1 : X_2)$ be a homogeneous coordinate on $\mathbb{P}^2$. We can also define SL_2 - action on $\mathbb{P}^2$ as follow:

$$\begin{cases} X_0^\sigma = a^2 X_0 + 2abX_1 + b^2 X_2 \\ X_1^\sigma = acX_0 + (ad+bc)X_1 + bdX_2 \\ X_2^\sigma = c^2 X_0 + 2cdX_1 + d^2 X_2 \end{cases}$$

for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$.

Then, the morphism τ is SL_2 - linear, that is, $\tau(p^\sigma) = \tau(p)^\sigma$ for $p \in S$ and $\sigma \in SL_2$. Further, τ is branched along the smooth conic $C := \{X_1^2 = X_0X_2\} = \tau(\Delta)$, where $\Delta := \Delta_{\mathbb{P}^1}$ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1 = S$. Let f_i be a fiber of the projection $P_i : S \longrightarrow \mathbb{P}^1$ onto i-th factor $(i = 1, 2)$. Let $\pi : M := \mathbb{P}(E) \longrightarrow \mathbb{P}^2$ be the $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ associated with the vector bundle $E := \tau_*\mathcal{O}_S(4f_1)$ of rank 2 on $\mathbb{P}^2$.

Lemma 2.1. (1) $\det(\pi_*\mathcal{O}_S(kf_1)) \cong \mathcal{O}_{\mathbb{P}^2}(k-1)$ and $c_2(\pi_*\mathcal{O}_S(kf_1)) = \frac{1}{2}k(k-1)$ for all $k \geq 0$.

(2) $E \otimes \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$, where $C = \tau(\Delta)$.

(3) The natural morphism $S \longrightarrow M$ corresponding to the homomorphism $\tau^*E \longrightarrow \mathcal{O}_S(4f_1)$ is a closed embedding, hence, S can be considered as a divisor on M .

(4) $\mathcal{O}_M(S) \cong \mathcal{O}_E(2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-2)$, where $\mathcal{O}_E(1)$ is the tautological line bundle on M with respect to E .

(5) $\mathcal{O}_E(1)$ is nef , i.e., E is a semi-positive vector bundle

(6) We put $\mathcal{O}_M(1) := \mathcal{O}_E(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then

$$H^0(M, \mathcal{O}_M(1)) \cong H^0(S, \mathcal{O}_S(5f_1 + f_2))$$

$$\cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(5)) \otimes_{\mathbb{C}} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) .$$

Proof. (1) Let us consider the exact sequence:

$$0 \longrightarrow \tau_*\mathcal{O}_S(kf_1) \longrightarrow \tau_*\mathcal{O}_S((k+1)f_1) \longrightarrow \tau_*\mathcal{O}_{f_1} \longrightarrow 0 .$$

Now $\ell_1 = \tau(f_1)$ is a line on $\mathbb{P}^2$ and $\mathcal{O}_{\ell_1} \cong \tau_*\mathcal{O}_{f_1}$. Thus, $\det(\tau_*\mathcal{O}_S((k+1)f_1)) \cong \det(\tau_*\mathcal{O}_S(kf_1)) \otimes \mathcal{O}(1)$ and $c_2(\tau_*\mathcal{O}_S((k+1)f_1)) = (\det(\tau_*\mathcal{O}_S(kf_1)) \cdot \mathcal{O}(1)) + c_2(\tau_*\mathcal{O}_S(kf_1))$. Since $\tau_*\mathcal{O}_S \cong \mathcal{O} \oplus \mathcal{O}(-1)$, we are done.

(2) Let us consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{O}_\Delta(3f_1 - f_2) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_S(2f_1 - 2f_2) & \longrightarrow & \mathcal{O}_S(4f_1) & \longrightarrow & \mathcal{O}_\Delta(4f_1) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_S(3f_1 - f_2) & \longrightarrow & \mathcal{O}_S(4f_1) & \longrightarrow & \mathcal{O}_{2\Delta}(4f_1) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{O}_\Delta(3f_1 - f_2) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Since $\tau^*C = 2\Delta$, we have $\tau_*\mathcal{O}_{2\Delta}(4f_1) \cong E \otimes \mathcal{O}_C$ and the exact sequence:

$$0 \longrightarrow \tau_*\mathcal{O}_\Delta(3f_1 - f_2) \longrightarrow E \otimes \mathcal{O}_C \longrightarrow \tau_*\mathcal{O}_\Delta(4f_1) \longrightarrow 0$$

$$\tau_*\mathcal{O}_\Delta(3f_1 - f_2) \cong \mathcal{O}_{\mathbb{P}^1}(2), \qquad \tau_*\mathcal{O}_\Delta(4f_1) \cong \mathcal{O}_{\mathbb{P}^1}(4) .$$

To show that $E \otimes \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$, it is enough to prove that

$$H^0(C, (E \otimes \mathcal{O}_C) \otimes \mathcal{O}_{\mathbb{P}^1}(-4)) \cong H^0(\mathcal{O}_{2\Delta}(2f_1 - 2f_2)) = 0 .$$

By the above diagram, we have the exact sequences:

$$0 \longrightarrow P_{2*}\mathcal{O}_S(-4f_2) \xrightarrow{\varphi} P_{2*}\mathcal{O}_S(2f_1-2f_2) \longrightarrow P_{2*}\mathcal{O}_{2\Delta}(2f_1-2f_2) \longrightarrow 0 ,$$

$$P_{2*}\mathcal{O}_S(-4f_2) \cong \mathcal{O}_{\mathbb{P}^1}(-4), \qquad P_{2*}\mathcal{O}_S(2f_1-2f_2) \cong \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 3}$$

and

$$0 \longrightarrow P_{2*}\mathcal{O}_\Delta(f_1-3f_2) \longrightarrow P_{2*}\mathcal{O}_{2\Delta}(2f_1-2f_2) \longrightarrow P_{2*}\mathcal{O}_\Delta(2f_1-2f_2) \longrightarrow 0 .$$

$$P_{2*}\mathcal{O}_\Delta(f_1-3f_2) \cong \mathcal{O}_{\mathbb{P}^1}(-2), \qquad P_{2*}\mathcal{O}_\Delta(2f_1-2f_2) \cong \mathcal{O}_{\mathbb{P}^1}$$

Hence $P_{2*}\mathcal{O}_{2\Delta}(2f_1 - 2f_2)$ is locally free and the dual homomorphism $\varphi^* : \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(4)$ is surjective. Therefore φ^* is obtained from the natural surjection $H^0(\mathbb{P}^1, \mathcal{O}(2)) \otimes \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(2)$ by tensoring $\mathcal{O}_{\mathbb{P}^1}(2)$. Thus we have $P_{2*}\mathcal{O}_{2\Delta}(2f_1 - 2f_2) \cong$ $\cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore we have $H^0(\mathcal{O}_{2\Delta}(2f_1 - 2f_2)) = 0$.

(3) It is enough to show that the natural homomorphism $\mathrm{Sym}^k E \longrightarrow \tau_*\mathcal{O}_S(4kf_1)$ is surjective for $k >> 0$. Since τ is finite morphism, $\tau_*\mathcal{O}_S(4kf_1) \otimes \tau_*\mathcal{O}_S(4f_1) \longrightarrow \tau_*\mathcal{O}_S(4(k+1)f_1)$ is always surjective. Thus we are done.

(4) Since $\tau : S \longrightarrow \mathbb{P}^2$ is a double covering, there is a line bundle L on $\mathbb{P}^2$ such that $\mathcal{O}_E(2) \otimes \mathcal{O}_M(-S) \cong \pi^*L$. By the exact sequence:

$$0 \longrightarrow \pi^*L \longrightarrow \mathcal{O}_E(2) \longrightarrow \mathcal{O}_E(2) \otimes \mathcal{O}_S \cong \mathcal{O}_S(8f_1) \longrightarrow 0 ,$$

we have $\det(\mathrm{Sym}^2 E) \cong L \otimes \det(\tau_*\mathcal{O}_S(8f_1))$. Therefore, by (1), $L \cong \mathcal{O}_{\mathbb{P}^2}(2)$, hence, $\mathcal{O}_M(S) \cong \mathcal{O}_E(2) \otimes \tau^*\mathcal{O}_{\mathbb{P}^2}(-2)$.

(5) We put $D := \pi^{-1}(C)$. Then, by (2), $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_E(1) \otimes \mathcal{O}_D \cong \mathcal{O}_D(s_1 + 3s_2)$, where s_2 is a fiber of $D \longrightarrow C$ and s_1 is a fiber of another projection $D \rightarrow \mathbb{P}^1$. By (4), we have $\mathcal{O}_E(2) \cong \mathcal{O}_M(S + D)$. Assume that there is an irreducible curve γ on M such that $(\mathcal{O}_E(1) \cdot \gamma) < 0$. Then, $\gamma \subseteq D$ or $\gamma \subseteq S$. Since $\mathcal{O}_E(1) \otimes \mathcal{O}_S \cong \mathcal{O}_S(4f_1)$ and $\mathcal{O}_E(1) \otimes \mathcal{O}_D \cong \mathcal{O}_D(s_1 + 3s_2)$, this is a contradiction.

(6) By the exact sequence

$$0 \longrightarrow \mathcal{O}_M(1) \otimes \mathcal{O}_M(-S) \longrightarrow \mathcal{O}_M(1) \longrightarrow \mathcal{O}_M(1) \otimes \mathcal{O}_S \longrightarrow 0 ,$$

$$\mathcal{O}_M(1) \otimes \mathcal{O}_M(-S) \cong \mathcal{O}_E(-1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(3) \qquad \mathcal{O}_M(1) \otimes \mathcal{O}_S \cong \mathcal{O}_S(5f_1+f_2)$$

we have $\pi_*\mathcal{O}_M(1) \cong \tau_*\mathcal{O}_S(5f_1+f_2)$. Therefore $H^0(M , \mathcal{O}_M(1)) \cong$ $\cong H^0(S , \mathcal{O}_S(5f_1+f_2))$.

Q.E.D.

Remark 2.1. There is a SL_2 - action on $(M , \mathcal{O}_M(1))$ compatible to $\tau : S \longrightarrow \mathbb{P}^2$. The last isomorphism in (6) is an isomorphism as a SL_2 - module .

2. Let us consider the subvector space $L \subsetneqq H^0(S , \mathcal{O}_S(5f_1+f_2))$ generated by the following 7 elements (cf. Lemma (1.6) in [9]) :

$$\begin{cases} e_0 := x^5 \otimes u \\ e_1 := x^4y \otimes u + \frac{1}{5} x^5 \otimes v \\ e_2 := x^3y^2 \otimes u + \frac{1}{2} x^4y \otimes v \\ e_3 := x^2y^3 \otimes u + x^3y^2 \otimes v \\ e_4 := \frac{1}{2} xy^4 \otimes u + x^2y^3 \otimes v \\ e_5 := \frac{1}{2} y^5 \otimes u + xy^4 \otimes v \\ e_6 := y^5 \otimes v \end{cases}$$

Then L is a SL_2 - invariant subspace. By the isomorphism $H^0(M, \mathcal{O}_M(1)) \cong H^0(S, \mathcal{O}_S(5f_1 + f_2))$, L can be considered as a subspace of $H^0(M, \mathcal{O}_M(1))$.

Lemma 2.2. (1) The homomorphism $L \otimes \mathcal{O}_M \longrightarrow \mathcal{O}_M(1)$ is surjective. Especially, we have a morphism $\psi : M \longrightarrow \mathbb{P}(L) \cong \mathbb{P}^6$, which is SL_2 - linear.

(2) The image $V := \psi(M)$ is isomorphic to the Fano 3-fold V_5 of degree 5 in $\mathbb{P}^6$.

Proof. (1) We have only to show that $g : L \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow$ $\longrightarrow E \otimes \mathcal{O}_{\mathbb{P}^2}(1)$ is surjective. Since SL_2 acts on g, the support of Coker(g) is SL_2 - invariant. Now SL_2 acts on $\mathbb{P}^2$ with two orbits $\mathbb{P}^2 \setminus C$ and C. First, take a point $p \in \mathbb{P}^2 \setminus C$. Then $g \otimes \mathbb{C}(p) : L \longrightarrow (E \otimes \mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathbb{C}(p)$ is described as follow:

Let $\alpha : L \otimes \mathcal{O}_S \longrightarrow \mathcal{O}_S(5f_1 + f_2)$ be the natural homomorphism and let $\alpha(q) : L \longrightarrow \mathcal{O}_S(5f_1 + f_2) \otimes \mathbb{C}(q) \cong \mathbb{C}$ be the evaluation map for $q \in S$. Then $g \otimes \mathbb{C}(p) : L \longrightarrow \mathbb{C}^{\oplus 2}$ is nothing but $\alpha(q_1) \oplus \alpha(q_2) : L \longrightarrow \mathbb{C}^{\oplus 2}$, where $\{q_1, q_2\} := \psi^{-1}(p)$. For example, take a point $p = (0:1:0) \in \mathbb{P}^2$. Then $q_1 = ((1:0), (0:1))$ and $q_2 = ((0:1), (1:0))$ in $S = \mathbb{P}^1 \times \mathbb{P}^1$. Then the calculation is as follow:

$$\begin{cases} \alpha_1(e_0) = \alpha_1(e_2) = \dots = \alpha_1(e_6) = 0, & \alpha_1(e_1) = \frac{1}{5} \\ \alpha_2(e_0) = \dots = \alpha_2(e_4) = \alpha_2(e_6) = 0, & \alpha_2(e_5) = \frac{1}{5}, \end{cases}$$

where $\alpha_1 := \alpha_1(q_1)$, $\alpha_2 := \alpha_2(q_2)$.

Therefore $g \otimes \mathbb{C}(p)$ is surjective for any $p \in \mathbb{P}^2 \setminus C$.

Next take $p := (1 : 0 : 0) \in C$, $q = ((1 : 0), (1 : 0)) \in S$. Let $z_1 = \frac{y}{x}$, $z_2 = \frac{v}{u}$ be the local coordinate around q . Then $m_p \mathcal{O}_S = (z_1 + z_2, z_1 \cdot z_2) \subseteq m_q$. The evaluation map $q \otimes \mathbb{C}(p) : L \longrightarrow \mathbb{C}^{\oplus 2}$ is now the composition

$$\beta : L \longrightarrow L \otimes \mathcal{O}_S \longrightarrow \mathcal{O}_S / m_p \mathcal{O}_S \cong \mathbb{C}1 \oplus \mathbb{C}\bar{z}_1 .$$

Since we have isomorphisms

$$\begin{array}{cccccccc} \mathcal{O}_S(f_1)_q & \cong & \mathcal{O}_{S,q} & & \mathcal{O}_S(f_2)_q & \cong & \mathcal{O}_{S,q} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ x & \longmapsto & 1 & & u & \longmapsto & 1 \\ y & \longmapsto & 0 & & v & \longmapsto & 0 \end{array} ,$$

$\beta : g \otimes \mathbb{C}(p)$ is calculated by evaluating $x = u = 1$ and $y = \bar{z}_1 = -v = -\bar{z}_2$. Therefore $\beta(e_0) = 1$, $\beta(e_1) = \frac{4}{5}\bar{z}_1$, $\beta(e_2) = 0$, $\beta(e_3) = 0$, $\beta(e_4) = 0$, $\beta(e_5) = 0$, $\beta(e_6) = 0$. Thus $g \otimes \mathbb{C}(p)$ is surjective for any $p \in C$.

(2) Let $h_0, h_1, \ldots, h_6 \in L^\vee$ be the dual basis of $\{e_0, e_1, \ldots, e_6\}$. Since $\mathbb{P}(L) \cong L^\vee \setminus \{0\}/\mathbb{C}^*$, we denote the point of $\mathbb{P}(L)$ corresponding to $\sum_{j=0}^{6} \lambda_j h_j \in L^\vee \setminus \{0\}$ by $[\sum_{j=0}^{6} \lambda_j h_j]$.

If $\psi(M)$ contains the point $[h_1 - h_5] \in \mathbb{P}(L)$, then $\psi(M)$ contains the SL_2 - orbit $SL_2[h_1 - h_5]$ and its closure $\overline{SL_2[h_1 - h_5]}$. On the other hand, we know that the closure $\overline{SL_2[h_1 - h_5]}$ is isomorphic to V_5 by [§ 3, 7]. Here $h_1 - h_5$ corresponds to $f_6(x , y) = xy(x^4 - y^4)$ in their notation. Therefore we have only to show that $\psi(M)$ contains $[h_1 - h_5] \in \mathbb{P}(L)$. Let $P := (0 : 1 : 0) \in \mathbb{P}^2$. Then by (1), the evaluation map $g \otimes \mathbb{C}(p) : L \longrightarrow \mathbb{C} \oplus \mathbb{C}$ with $(g \otimes \mathbb{C}(p))(e_1) = (\frac{1}{5}, 0)$, $(g \otimes \mathbb{C}(p))(e_5) = (0 , \frac{1}{5})$, and $(g \otimes \mathbb{C}(p))(e_j) = (0 , 0) (j \neq 1,5)$. Therefore the point $q \in \pi^{-1}(p) \cong \mathbb{P}^1$ corresponding to the linear function $\mathbb{C} \oplus \mathbb{C} \ni (a , b) \longmapsto a - b \in \mathbb{C}$ is mapped to $[h_1 - h_5]$ by ψ .

Q.E.D.

Remark 2.2. (1) By Lemma (1.5) in [8], $V := \psi(M)$ has three SL_2 - orbits $\psi(M) \setminus \psi(S)$, $\psi(S) \setminus \psi(\Delta_{\mathbb{P}^1})$, and $\psi(\Delta_{\mathbb{P}^1})$, in particular, $\psi(\Delta_{\mathbb{P}^1})$ is a smooth rational curve of degree 6 in V .

(2) $\psi|_S : S \longrightarrow \psi(S)$ is the same morphism as in Lemma (1.6) in [8]. Especially, $\psi|_S$ is one to one and $\mathrm{Sing}\, \psi(S) = \psi(\Delta_{\mathbb{P}^1})$, where $\mathrm{Sing}\, \psi(S)$ is the singular locus of $\psi(S)$.

Let us denote $\psi(S)$ and $\psi(\Delta_{\mathbb{P}^1})$ by B and Σ .

Lemma 2.3. (1) ψ is a finite morphism of degree 3.

(2) ψ is étale outsides B

(3) $\psi^*B = S + 2D$, hence ψ is not Galois.

(4) We put $M_t := \pi^{-1}(t)$ for $t \in \mathbb{P}^2$. Then $\ell_t := \psi(M_t)$ is a line of $V \subsetneqq \mathbb{P}^6$ and $\psi|_{M_t} : M_t \longrightarrow \ell_t$ is an isomorphism.

(5) For $t_1 \neq t_2 \in \mathbb{P}^2$, we have $\ell_{t_1} \neq \ell_{t_2}$.

(6) Let ℓ be a line in $V \subsetneqq \mathbb{P}^6$. Then there is a point $t \in \mathbb{P}^2$ such that $\ell = \ell_t$.

Proof. (1) By Lemma (1.1) - (5), $\mathcal{O}_M(1)$ is ample. Therefore ψ is a finite morphism and $\psi^*\mathcal{O}_V(1) \cong \mathcal{O}_M(1)$. Thus $\deg \psi = (\mathcal{O}_M(1))^3/(\mathcal{O}_V(1))^3 = 15/5 = 3$.

(2) Since $V \setminus B$ is an open orbit of SL_2 , ψ is étale over $V - B$.

(3) Since $(\mathcal{O}_V(1)^2 \cdot B) = (\mathcal{O}_M(1)^2 \cdot S) = (\mathcal{O}_S(5f_1 + f_2))_S^2 = 10$, we have $\mathcal{O}_V(B) \cong \mathcal{O}_V(2)$. Therefore $\mathcal{O}_M(\psi^*B - S) \cong \pi^*\mathcal{O}_{\mathbb{P}^2}(4)$. Since $\psi^*B - S$ is a SL_2 - invariant effective divisor, its support must be D . Thus $\psi^*B = S + 2D$.

(4) It is clear since $(\psi^*\mathcal{O}_V(1) \cdot M_t) = (\mathcal{O}_M(1) \cdot M_t) = 1$.

(5) Assume that $\ell_{t_1} = \ell_{t_2}$. Since $\psi|_S : S \longrightarrow B$ is one to

one , we have $M_{t_1} \cap S = M_{t_2} \cap S$. Hence $t_1 = t_2$.

(6) Let ℓ be a line of V . If $\ell \not\subset B$, then ℓ contains a point $p \in V \setminus B$. By Corollary (2.1) in § 2, we have #{lines through p} ≤ 3 . Thus by (4), (5) above, {lines through p} = $= \{\ell_{t_1}, \ell_{t_2}, \ell_{t_3}\}$, where $\{t_1, t_2, t_3\} = \pi(\psi^{-1}(p))$. Therefore $\ell = \ell_{t_2}$. If $\ell \subseteqq B$, then $\ell = \ell_t$ for some $t \in C$, because $\psi|_D : D \longrightarrow B$ is one to one by (3) and $\mathcal{O}_M(1) \otimes \mathcal{O}_D \cong \mathcal{O}_D(s_1 + 5s_2)$ by Lemma 2.1 - (2).

Theorem I. The $\mathbb{P}^1$-bundle $\pi : M \longrightarrow \mathbb{P}^2$ is the universal family of lines on $V = V_5$.

Proof. Let T be the space of lines on V , that is, T is a subscheme of the Grassmanian $G(2,7)$ parametrizing lines of $V \subseteqq \mathbb{P}^6$. Since $N_{\ell|V} \cong \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ for any line ℓ on V , we have $H^1(\ell, N_{\ell|V}) = 0$ and $H^0(\ell, N_{\ell|V}) \cong \mathbb{C}^2$. Therefore T is smooth surface. By the universal property of T , we have a morphism $\delta : \mathbb{P}^2 \longrightarrow T$ corresponding to the family $(\pi, \psi) : M \hookrightarrow \mathbb{P}^2 \times V$. By Lemma (1.3) - (5), (6), δ is one to one surjective. Therefore δ must be isomorphic.

We put $U_n := \{x \in V$; there is at most n lines through $x\}$. Then,

Corollary 2.1. $U_3 = V$, $U_2 = B$ and $U_1 = \Sigma$.

§ 3. Compactifications of $\mathbb{C}^3$

Take any point $t \in C \hookrightarrow \mathbb{P}^2$ and put $\ell_t := \psi(\pi^{-1}(t))$. Then ℓ_t is a line of type $(-1, 1)$. Let $\sigma : V' \longrightarrow V$ be blowing up of V along the line ℓ_t and $\bar{E}_t$ be the proper transform in V' of the ruled surface E_t swept out by lines which intersect the line ℓ_t. Then, by Lemma 1.1 - (b), we have the birational morphism $\varphi : V' \longrightarrow W_t$ of V' onto a smooth quadric hypersurface $W_t \cong \mathbb{Q}^3$ in $\mathbb{P}^4$, a quadric cone $Q_t := \varphi(\sigma^{-1}(\ell_t)) \cong \mathbb{Q}_0^2$, and a twisted cubic curve $Y_t := \varphi(\bar{E}_t) \hookrightarrow Q_t$. Let g_t be the unique generating line of Q_t such that $Y_t \cap g_t = \{v_t\}$, where v_t is the vertex of Q_t. Take any point $v \in g_t \setminus \{v_t\} \cong \mathbb{C}$. Let Q_v be the quadric cone in W_t with the vertex v, and put $H_t^v := \sigma(\varphi^{-1}(Q_v))$.

Then, by (4.3) in [2] and [6] (see also § 1), we have the following

Lemma 3.1. (1) For any $t \in C$, (V, E_t) is a compactification of $\mathbb{C}^3$ with the non-normal boundary E_t. Conversely, let (V, H) be a compactification of $\mathbb{C}^3$ with a non-normal boundary H. Then there is a point $t \in C$ such that $H = E_t$.

(2) For any $t \in C$ and any $v \in g_t \{v_t\} \cong \mathbb{C}$, (V, H_t^v) is a compactification of $\mathbb{C}^3$ with the normal boundary H_t^v. Conversely, let (V, H) be a compactification of $\mathbb{C}^3$ with a normal boundary H. Then there is a point $t \in C$ and a point

$v \in g_t \setminus \{v_t\}$ such that $H = H_t^v$.

Remark 3.1. Let Z_t be the line $\mathbb{P}^2$ which is tangent to C at the point $t \in C$. Then $E_t = \psi(\pi^{-1}(Z_t))$ and $\pi^{-1}(Z_t) \setminus (s_t \cup \pi^{-1}(t)) \cong E_t \setminus \ell_t$, where s_t is the negative section of $\pi^{-1}(Z_t) \cong \mathbb{F}_3$.

We put

$\Lambda_1 := \{\lambda \in \check{\mathbb{P}}^6 ; H_\lambda$ is a non-normal hyperplane section of V such that $V \setminus H_\lambda \cong \mathbb{C}^3\}$, and

$\Lambda_2 := \{\lambda \in \check{\mathbb{P}}^6 ; H_\lambda$ is a normal hyperplane section of V such that $V \setminus H_\lambda \cong \mathbb{C}^3\}$,

where $\check{\mathbb{P}}^6 := \mathbb{P}(\check{L})$.

Then we have

Corollary 3.1. $\dim_{\mathbb{C}} \Lambda_1 = 1$ and $\dim_{\mathbb{C}} \Lambda_2 = 2$.

Corollary 3.2. For each $t \in C$, $\{\lambda \in \Lambda_1 ; \ell_t \subsetneqq H_\lambda\} =$ {one point} and $\{\lambda \in \Lambda_2 ; \ell_t \subsetneqq H_\lambda\} \cong \mathbb{C}$.

Now, take a point $t_0 = (1 : 0 : 0) \in C$. Then $\ell_{t_0} \hookrightarrow \mathbb{P}^6$ is written as follow:

$$\ell_{t_0} = \{h_2 = h_3 = h_4 = h_5 = h_6 = 0\}$$

(see the proof of Lemma 2.2 - (1)).

Since V is SL_2 - invariant, Λ_1 and Λ_2 are also SL_2 - invariant

By Lemma (1.4) of [9], the 2-dimension SL_2 - orbits are $SL_2x^3y^3$, $SL_2x^4y^2 = SL_2x^2y^4$, $SL_2x^5y = SL_2xy^5$, and $SL_2y^6 = SL_2x^6$ is the only one SL_2 - orbit of dimensional one on $\mathbb{P}^6$. Therefore we have $\Lambda_1 = SL_2y^6$. By an easy calculation, we have

$$\{\lambda \in SL_2x^3y^3 \ ; \ \ell_{t_0} \subsetneqq H_\lambda\} \cong \mathbb{C} \cup \mathbb{C} ,$$

$$\{\lambda \in SL_2x^2y^4 \ ; \ \ell_{t_0} \subsetneqq H_\lambda\} \cong \mathbb{C} \cup \mathbb{C} ,$$

$$\{\lambda \in SL_2xy^5 \ ; \ \ell_{t_0} \subsetneqq H_\lambda\} \cong \mathbb{C} .$$

Thus, by Corollary 3.2, we must have $\Lambda_2 = SL_2xy^5$. We put $\Lambda := \Lambda_1 \cup \Lambda_2$. Then $\Lambda = \overline{SL_2xy^5}$. Therefore, by Lemma (1.6) of [9], Λ is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ with diagonal SL_2 - operations by a linear system L of bidegree (5,1) on $\mathbb{P}^1 \times \mathbb{P}^1$.

Thus we have

Theorem 3.1. $\Lambda_1 = SL_2y^6$, $\Lambda_2 = SL_2xy^5$ and $\Lambda = \overline{SL_2xy^5}$.

In particular, $\Lambda_1 \cong \mathbb{P}^1$ and $\Lambda_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{\text{diagonal}\}$.

We will show explicitly below that for any $\lambda \in \Lambda$, $V \setminus H_\lambda \cong \mathbb{C}^3$.

By p. 505 in [9], $V := V_5 \hookrightarrow \mathbb{P}^6$ can be written as follow:

$$\begin{cases} h_0h_4 - 4h_1h_3 + 3h_2^2 = 0 \\ h_0h_5 - 3h_1h_4 + 2h_2h_3 = 0 \\ h_0h_6 - 9h_2h_4 + 8h_3^2 = 0 \\ h_1h_6 - 3h_2h_5 + 2h_3h_4 = 0 \\ h_2h_6 - 4h_3h_5 + 3h_4^2 = 0 \end{cases},$$

where $(h_0 : h_1 : h_2 : h_3 : h_4 : h_5 : h_6)$ is the homogeneous coordinate of $\mathbb{P}^6$.

First, $(0:0:0:0:0:0:1) \in SL_2Y^6$. In $V \cap \{h_6 \neq 0\}$, we consider the following coordinate transformation.

$$\begin{cases} x_0 = h_0 - 9h_2h_4 + 8h_3^2 \\ x_1 = h_1 - 3h_2h_5 + 3h_3h_4 \\ x_2 = h_2 - 4h_3h_5 + 3h_4^2 \\ x_3 = h_3 \\ x_4 = h_4 \\ x_5 = h_5 \end{cases}, \qquad h_6 = 1$$

Then we have

$$V \cap \{h_6 \neq 0\} \cong \{x_0 = x_1 = x_2 = 0\} \cong \mathbb{C}^3 ,$$

and the line $\{h_2 = h_3 = h_4 = h_5 = h_6 = 0\}$ is the singular locus of the boundary $V \cap \{h_6 = 0\}$.

Next, $(0:0:0:0:0:1:0) \in SL_2xy^5$. In $V \cap \{h_5 \neq 0\}$, we consider the coordinate transformation

$$\begin{cases} x_0 = h_0 - 3h_1h_4 + 2h_2h_3 \\ x_1 = h_1 \\ x_2 = 3h_2 - h_1h_6 - 2h_3h_4 \\ x_3 = 4h_3 - h_2h_6 - 3h_4^2 \\ x_4 = h_4 \\ x_6 = h_6 \end{cases} , \quad h_5 = 1 .$$

Then we have

$$V \cap \{h_5 \neq 0\} \cong \{x_0 = x_2 = x_3 = 0\} \cong \mathbb{C}^3 ,$$

and the boundary $V \cap \{h_5 = 0\}$ has a singularity of A_4-type at the point $(1:0:0:0:0:0:0)$.

Therefore, for any $\lambda \in SL_2y^6$ (resp. SL_2xy^5) , H_λ is non-normal (resp. normal with a rational double point of A_4-type), and further $V \setminus H_\lambda \cong \mathbb{C}^3$.

Since Λ_1 and Λ_2 are SL_2 - orbits, we have the following

Corollary 3.3 (Peternell-Schneider [6]). Let (V, H) and (V, H') be two compactifications of $\mathbb{C}^3$ with normal (resp. non-normal) boundaries H and H' . Then there is an automorphism α of V such that $H' = \alpha(H)$.

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