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AN ESTIMATE OF CANONICAL DIMENSION OF GROUPS BASED ON SCHUBERT CALCULUS

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Abstract. We sketch the proof of a connection between the canonical (0-)dimension of semisimple split simply connected groups and cohomology of their full flag varieties. Using this connection, we get a new estimate of the canonical (0-)dimension of simply connected split exceptional groups of type E understood as a group. A full proof will be published later.

1. Introduction

To define the canonical (0-)dimension of an algebraic group understood as a group, we first need to define the canonical (0-)dimension of a scheme understood as a scheme (which is a different definition). Roughly speaking, the canonical (0-)dimension of a scheme is a number indicating how hard it is to get a rational point in the scheme. The canonical (0-)dimension of an algebraic group shows how hard it is to get rational points in torsors related to the group.

To be more precise, let us fix some conventions and give some definitions. We speak of algebraic schemes and use stacks project as the source of basic definitions. All schemes in the present text are of finite type over a field and separated. The base field is arbitrary.

Speaking of canonical dimension of schemes, there are two closely related notions in the literature: the *canonical 0-dimension of a scheme* defined in [14] and the *canonical dimension of a scheme* defined in [9]. These two definitions are not known to be always equivalent, but they are equivalent for two particular classes of schemes: for smooth complete schemes and for torsors of split reductive groups (see [13, Theorem 1.16, Remark 1.17, and Example 1.18]). The definition from [14] looks more motivated, so we are going to use it.

Definition 1.1 ([14, Section 4a, first paragraph of Section 4b, and the last paragraph of Section 2a])**.** Given a scheme X over a field K, the canonical 0-dimension of X understood as a scheme (notation: $\mathrm{cd}_0(X)$) is:

$$\mathrm{cd}_0(X) = \max_{\substack{L=\text{a field containing } K\\ X_L \text{ has a rational point}}} \min_{\substack{L_0=\text{a subfield of } L,\ K\subseteq L_0\\ X_{L_0} \text{ still has a rational point}}} \mathrm{trdeg}_K L_0.$$

A bit less formally, canonical dimension can be explained as follows. Suppose we have expanded the base field K to L, and got a rational point in X_L. How large can L be, compared to K? In general, it can be very large, this is unbounded. A related question with a finite answer is: how many algebraically independent generators do we have to keep, at worst (for the worst L), to still have a rational point after scalar expansion (*not necessarily the same* rational point that we found after expanding scalars to L)? This number of generators is the canonical dimension of X. For more properties of canonical dimension, see [14] in the case of general X and [8] in the case of smooth projective X.

We have underlined above that we want to get a rational point over a field between K and L, but not necessarily the same rational point. If we demanded to get the same rational point, we would get the definition of the *essential 0-dimension* of a scheme, which is known to coincide with the (standard in algebraic geometry) dimension, see [13, Proposition 1.2]. This can be viewed as a motivation for the word "dimension". (But essential dimension is not only defined for schemes, and in broader generality it becomes a much more nontrivial notion.)

Another motivation for canonical (0-)dimension comes from *incompressible varieties*, but this motivation is only valid for the canonical (0-)dimension of smooth complete schemes. The definition that we are going to give next, the canonical 0-dimension of an algebraic group, and that will be used in the main theorem of this text, does not involve the canonical (0-)dimension of smooth complete schemes, so this motivation will be useless for us. One can find details for this motivation in [8, Section 2].

The second object we need to define before we can define the canonical 0-dimension of a group is a torsor of a group. All algebraic groups in this text are affine. All reductive, semisimple, and simple groups in this text are smooth. Torsors of algebraic groups (over a point) are, informally speaking, homogeneous spaces that are "as large as the group itself". This notion is mostly interesting over non algebraically closed fields.

Definition 1.2 ([14, Section 3a])**.** Given an algebraic group G, a *G-torsor over a point* (or simply a *G-torsor*) is a scheme E with an action $\varphi\colon G\times E\to E$ such that $(\varphi,\mathrm{pr}_2)\colon G\times E\to E\times E$, where pr_2 is the projection to the second factor, is an isomorphism.

It is known that all torsors of affine algebraic groups over a point are affine.

Finally, the canonical 0-dimension of an algebraic group understood as a group measures how hard it is to get rational points in torsors, informally speaking, related to the group. Precisely:

Definition 1.3 ([14, Section 4g])**.** given an algebraic group G over a field F, the canonical 0-dimension of G understood as a group (notation: $\mathfrak{cd}_0(G)$) is

$$\mathfrak{cd}_0(G)=\max_{K=\text{a field containing }F}\ \max_{E=\text{a }G_K\text{-torsor}}\mathrm{cd}_0(E).$$

The definition of canonical dimension of an algebraic group understood as a group in [9, Introduction] repeats this definition almost exactly, with the only difference being that instead of $\mathrm{cd}_0(E)$ it uses the definition of canonical dimension of E understood as a scheme from the paper [9] itself. But as we already mentioned above, it is known that these two notions are known to be equivalent for torsors of split reductive groups. So, Definition 1.3 is also equivalent to the definition of canonical dimension of a group from [9, Introduction] for split reductive groups. All groups whose canonical dimension we are going to estimate in this text are split reductive (and even simply connected semisimple), so these results also estimate the canonical dimension in the sense of [9, Introduction].

To formulate the main goal of this text precisely, we need to introduce some more notation and terminology. Given a split semisimple algebraic group G and a Borel subgroup B, the corresponding Weyl group W, and the element $w_0\in W$ of maximal length, for each $w\in W$ we denote the Schubert variety $\overline{Bw_0w^{-1}B/B}\subseteq G/B$ by Z_w. This Z_w is a Schubert divisor if and only if w is a simple reflection, and we denote all Schubert divisors by $D_1,\dots,D_r$.

It is known that the classes $[Z_w]\in\mathrm{CH}(G/B)$ for all $w\in W$ form a free set of generators of $\mathrm{CH}(G/B)$ as of an abelian group. We say that a product of classes of Schubert divisors $[D_1]^{n_1}\dots[D_r]^{n_r}$ is *multiplicity-free* if there exists $w\in W$ such that the coefficient at $[Z_w]$ in the decomposition of $[D_1]^{n_1}\dots[D_r]^{n_r}$ into a linear combination of Schubert classes equals 1.

Now we can formulate the goal of this text precisely. Our goal is to sketch the proof of the following theorem.

Theorem 1.4. *Let G be a split semisimple simply connected algebraic group over an arbitrary field, let B be a Borel subgroup, let r be the rank of G, and let $D_1,\dots,D_r\subset G/B$ be the Schubert divisors corresponding to the r simple roots of G. If $[D_1]^{n_1}\dots[D_r]^{n_r}$ is a multiplicity-free product of Schubert divisors, then $\mathfrak{cd}_0(G)\le\dim(G/B)-n_1-\dots-n_r$.*

As a corollary of this theorem and [5, Theorem 11.5], we will immediately get the following:

Corollary 1.5. *Let G be a split semisimple simply connected algebraic group of type E_r. Then $\mathfrak{cd}_0(G)\le$ 17, 37, or 86 for $r=6$, 7, or 8, respectively.*

The canonical dimension of simply connected split groups of type A_r and C_r is known to be zero. For types B_r and D_r, the canonical dimension was estimated (and computed exactly if r is a power of 2) by N. Karpenko in [10]. The paper [10] also relates cohomology of flag varieties of orthogonal groups (more precisely, orthogonal Grassmannians, not full flag varieties) to canonical dimension, and in this text, we are going to follow the ideas of several proofs from [10]. For type G_2, the canonical dimension (of a split simply connected group) is known and equals 3, see [1, Example 10.7]. For type F_4, no nontrivial upper bounds on the canonical dimension are known.

In types E_r, the most difficult part of obtaining Corollary 1.5 was to understand which products of Schubert divisors are multiplicity-free (and this was understood in [5] by the author). The part of the argument establishing relation between Schubert calculus and canonical dimension (in other words, the proof of Theorem 1.4 itself) was known to the experts (or at least they believed that the argument is doable this way). However, we were unable to find an exposition suitable for more general mathematical audience. The present paper contains such an exposition.

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2. Isomorphism of Picard groups under scalar expansion

We always denote by $\mathrm{id}_X : X \to X$, where X is a scheme, the identity map.

To start proving Theorem 1.4, we first need to define the quotient of a torsor modulo a Borel subgroup. The definition we are going to use is not very intuitive, but it is used in papers on canonical dimension (for example, in [9]).

Definition 2.1. Let G be a semisimple split algebraic group over a field K, let B be a Borel subgroup, and let E be a G-torsor. The *quotient of the torsor modulo the Borel subgroup* (notation: E/B) is the categorical quotient (see [16, Definition 0.5]; "categorical" is in the category of all separated schemes of finite type over K) of $E \times G/B$ modulo the diagonal action of G.

In fact, it can be proved that such a quotient is also a categorical quotient of E modulo B, but we will not use this. The existence of such a categorical quotient $(E \times G/B)/G$ is known, is stated in [6, Proposition 12.2], and can be proved using Galois descent theory. It is known that such a quotient E/B is smooth, absolutely irreducible, and projective.

Given this definition, we can say that the first and the most technically difficult step in proving Theorem 1.4 is to prove the following proposition.

Proposition 2.2. *Let G be a semisimple split algebraic group over a field K, let B be a Borel subgroup, and let E be a G-torsor. Let K_1 be an extension of K. Then the map of Picard groups induced by field extension* $\mathrm{Pic}(E/B) \to \mathrm{Pic}((E/B)_{K_1})$ *is an isomorphism.*

The proof of this proposition makes a lot of use of Galois descent theory. Let us recall the basic notions and facts of this theory (or of the form of this theory that we need). We will need three categories. The first category, $\mathcal{S}ch_K$ is the category of (separated and of finite type, as everywhere in the text) schemes over a field K.

To define the second category, suppose we have two fields, $K \subseteq L$. First, we need to recall the definition of the functor of restriction of scalars from $\mathcal{S}ch_L$ to $\mathcal{S}ch_K$ (notation: $-|_K$). If X is an object of $\mathcal{S}ch_L$, we say that X *with scalars restricted from L to K* is the scheme that has the same topological space as X, the same ring of regular functions on each open subset *as an abstract ring*, but for the algebra structure, we view this ring as a K-algebra rather than an L-algebra (the multiplication by elements of K is given by the embedding $K \subseteq L$). We denote this scheme by $X|_K$. And if $f \in \mathrm{Mor}_{\mathcal{S}ch_L}(X, Y)$, then one can check directly that the same map of topological spaces as in f, together with the same map of abstract rings for each open subset of Y (= each open subset of $Y|_K$) as in f, satisfies the definition of a morphism of K-schemes from $X|_K$ to $Y|_K$.

Now we can say that the second category, which we will call *the category of K, L-schemes* (notation: $\mathcal{S}ch_{K,L}$), has schemes over L as objects, and the set $\mathrm{Mor}_{\mathcal{S}ch_{K,L}}(X, Y)$, where X and Y are L-schemes, is the set of morphisms of K-schemes from $X|_K$ to $Y|_K$.

The third category will be introduced a bit later.

Example 2.3. Let $K \subseteq L$ be a finite Galois extension of fields, and let $\sigma \in \mathrm{Gal}(L/K)$. Let $X = \operatorname{Spec} L$. Then $K[X|_K] = L$, and $\sigma^{-1}: L \to L$ is an automorphism of this K-algebra. So, we have a dual automorphism of the K-scheme $X|_K$, which we denote by $\sigma_* \in \mathrm{Mor}_{\mathcal{S}ch_{K,L}}(\operatorname{Spec} L, \operatorname{Spec} L)$.

We keep the notation σ_* until the end of the text.

Definition 2.4. Let $K \subseteq L$ be a finite Galois extension of fields with Galois group Γ, and let $\sigma \in \Gamma$. A morphism $f: X \to Y$ in $\mathcal{S}ch_{K,L}$ is called *σ-semilinear* if the following diagram (in $\mathcal{S}ch_{K,L}$) is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} L & \xrightarrow{\sigma_*} & \operatorname{Spec} L \end{array}$$

The vertical arrows are the restrictions of scalars of the structure morphisms.

Clearly, under the conditions of this definition, if f (resp. g) is a σ (resp. τ)-semilinear morphism, then $g \circ f$ is a $\tau\sigma$-semilinear morphism. It is also clear that 1-semilinear morphisms are exactly the restrictions of scalars of the morphisms in $\mathcal{S}ch_L$.

Definition 2.5. Let $K \subseteq L$ be a finite Galois extension of fields with Galois group Γ. We will say that we have a *Galois-semiaction* of Γ on an L-scheme X (or that Γ *Galois-semiacts* on X) if we have an action $\psi: \Gamma \times X|_K \to X|_K$ (here Γ is understood as an algebraic group over K) such that for each $\sigma \in \Gamma$, the automorphism $\psi_\sigma = \psi|_{\{\sigma\} \times X|_K}$ of $X|_K$, understood as an automorphism of X in $\mathcal{S}ch_{K,L}$, is σ-semilinear.

We say that a finite affine open covering of X is *Γ-stable* if Γ preserves (normalizes) each of the open sets.

Now we are ready to define the third category we need to formulate basic facts of Galois descent theory. Given a Galois extension of fields $K \subseteq L$ with Galois group Γ, we define the category of *stable L-schemes with semiaction of* Γ (notation: $(\mathcal{S}t\mathcal{S}ch_L, \Gamma)$). Its objects are pairs (X, ψ), where X is an L-scheme, and $\psi: \Gamma \times X|_K \to X|_K$ is a Galois-semiaction such that X admits a Γ-stable finite affine open covering. The morphisms are morphisms in $\mathcal{S}ch_L$ that become Γ-equivariant after the restriction of scalars to K.

Now recall that if a finite group acts on a scheme (now this is going to be a scheme over the smaller field, K), and there is a stable finite affine open covering for this action, then the categorical quotient always exists, and can be constructed, for example, as the orbit space of the action.

So, for a Galois extension $K \subseteq L$ with group Γ, we can define the *Galois descent functor* $\mathrm{Dec}_K: (\mathcal{S}t\mathcal{S}ch_L, \Gamma) \to \mathcal{S}ch_K$ as follows: an object (X, ψ) is mapped to the categorical quotient X/Γ, and the morphisms are mapped using the universal property of the categorical quotient. We can also define the *Galois upgrade functor* $\cdot_{L,\Gamma}: \mathcal{S}ch_K \to (\mathcal{S}ch_L, \Gamma)$. On the objects, it maps a K-scheme Y to (Y_L, φ), where the semiaction φ is defined on the affine charts as follows: if U is an open affine chart of Y, $\sigma \in \Gamma$, then $\varphi(\Gamma, (U_L)|_K) = (U_L)|_K$ (recall that the restriction of scalars does not change the topological space). And if $f \otimes \lambda \in L[U_L] = K[U] \otimes L$, then $(\varphi|_{\{\sigma\} \times (U_L)|_K})^*(f \otimes \lambda) = f \otimes \sigma^{-1}(\lambda)$. On the morphisms, the Galois upgrade functor is just expansion of scalars.

Given these two functors, let us state the main theorem of Galois descent theory

Theorem 2.6. *Let $K \subseteq L$ be a Galois extension with Galois group Γ. The Galois descent and upgrade functors are mutually quasiinverse equivalences of categories $\mathcal{S}ch_K \leftrightarrow (\mathcal{S}t\mathcal{S}ch_L, \Gamma)$.*

Proof. Well-known. Uses the explicit construction of the categorical quotient modulo Γ as an orbit space. □

To apply this theory to algebraic groups, we first need to understand how direct products work in $\mathcal{S}ch_{K,L}$ and in $(\mathcal{S}t\mathcal{S}ch_L, \Gamma)$. The direct products in $\mathcal{S}ch_L$ and in $\mathcal{S}ch_{K,L}$ are different. However, the following lemma shows that direct products from $\mathcal{S}ch_L$ are useful in $\mathcal{S}ch_{K,L}$ if we work with semilinear morphisms.

Lemma 2.7. *Let $K \subseteq L$ be a Galois extension of fields with Galois group Γ. Let X and Y be L-schemes, let Z be their product in $\mathcal{S}ch_L$, and let $p_1 \in \mathrm{Mor}_{\mathcal{S}ch_L}(Z, X)$ and $p_2 \in \mathrm{Mor}_{\mathcal{S}ch_L}(Z, Y)$ be the standard projections. Then for every L-scheme T, for every $\sigma \in \Gamma$, and for every two σ-semilinear morphisms $f\colon T \to X$ and $g\colon T \to Y$ there exists a unique σ-semilinear morphism $h\colon T \to Z$ such that $p_1|_K \circ h = f$ and $p_2|_K \circ h = g$.*

Proof. Well-known. □

Suppose, for K, L, Γ, X, Y, Z, p_1, and p_2 as in the lemma, we have two σ-semilinear morphisms: $f \in \mathrm{Mor}_{\mathcal{S}ch_{K,L}}(A, X)$ and $g \in \mathrm{Mor}_{\mathcal{S}ch_{K,L}}(B, Y)$. Let C be the product of A and B in $\mathcal{S}ch_L$, and let $q_1 \in \mathrm{Mor}_{\mathcal{S}ch_L}(C, A)$ and $q_2 \in \mathrm{Mor}_{\mathcal{S}ch_L}(C, B)$ be the standard projections. In this case we will denote by $f \times g \in \mathrm{Mor}_{\mathcal{S}ch_{K,L}}(C, Z)$ the unique σ-semilinear morphism such that $p_1|_K \circ (f \times g) = f \circ q_1|_K$ and $p_2|_K \circ (f \times g) = f \circ q_2|_K$. Informally speaking, this is a straightforward way to build a morphism $A \times B \to X \times Y$ out of morphisms $A \to X$ and $B \to Y$.

After we have this lemma, it is easy to construct a Galois-semiaction on a product of two L-schemes X and Y out of two semiactions on X and on Y. Precisely, if $\psi_1\colon \Gamma \times X|_K \to X|_K$ and $\psi_2\colon \Gamma \times Y|_K \to Y|_K$ are two Galois-semiactions, then the new semiaction on $Z = X \times Y$ (the product in $\mathcal{S}ch_L$), which we will call *the product of semiactions* and denote $\psi_1 \times \psi_2$, is defined as follows: $(\psi_1 \times \psi_2)|_{\{\sigma\} \times Z|_K} = (\psi_1)|_{\{\sigma\} \times X|_K} \times (\psi_2)|_{\{\sigma\} \times Y|_K}$. (In fact, $(Z, \psi_1 \times \psi_2)$ will then be the product of (X, ψ_1) and (Y, ψ_2) in $(\mathcal{S}t\mathcal{S}ch_L, \Gamma)$, but we will not need this.)

Using products of semiactions, we can speak about Γ-equivariant morphisms between products of varieties. In particular, if K, L, and Γ are as above, and G is an algebraic group over L, then we call a Galois-semiaction ψ on G *compatible with the group structure* if the multiplication and inversion map are Γ-equivariant, and the unit is a fixed point of Γ. And if, in addition, (E, φ) is a G-torsor with a Galois-semiaction ψ' on E (and ψ is still a semiaction compatible with the group structure), we call (ψ, ψ') *compatible with the torsor structure* if φ is Γ-equivariant.

Now, using Theorem 2.6 and these notions and still keeping K, L, and Γ as above, one can similarly construct mutually quasiinverse equivalences between the categories of (affine) algebraic groups over a field K and (affine) algebraic groups over L equipped with a compatible Galois-semiaction. By a slight abuse of notation, we also denote these functors by $\cdot_{L,\Gamma}$ and Dec_K. With torsors one should be a bit more careful, because for groups we have only quasiinverse equivalences, and if G is a group over K, then $\mathrm{Dec}_K(G_{L,\Gamma})$ is a different group, even though canonically isomorphic. So, for torsors we get equivalences of categories, but not mutually quasiinverse equivalences: their composition is a functor from G-torsors to $\mathrm{Dec}_K(G_{L,\Gamma})$-torsors. However, we still denote these equivalences by $\cdot_{L,\Gamma}$ and Dec_K.

This finishes the part of theory of Galois descent that we need. To apply this theory to torsors, we start with a few lemmas.

Lemma 2.8. *Let (E, φ) be a torsor of an algebraic group G. If e is a rational point of E, then the map $\mathrm{triv}_e = \varphi|_{G \times \{e\}}\colon G \to E$ is an isomorphism.*

Proof. Well-known. □

We keep the notation triv_e until the end of the text and call a torsor *trivial* if it has a rational point.

Lemma 2.9. *Let (E, φ) be a torsor of a* smooth *algebraic group G over a field K. Then there exists a finite Galois extension L of K such that (E_L, φ_L) is a trivial G_L-torsor.*

Idea of the proof. Find a finite extension L' such that $E_{L'}$ has a rational point. Then $E_{L'}$ is isomorphic to $G_{L'}$, therefore smooth, and E is smooth. Smooth schemes obtain a rational point after scalar expansion to a separable closure ([17, Prop. 3.2.20]). Using finite type, we can keep only a finite (automatically still separable) subextension and keep the rational point. □

So, instead of studying a torsor without rational points, we can make a finite Galois extension of scalars and study a torsor with a rational point and with a compatible Galois-semiaction.

From now on, we fix until the end of this section: a split semisimple algebraic group G over a field K, a Borel subgroup B of G, a G-torsor (E, φ), a finite Galois extension L of K such that E_L has a rational point, and a rational point $e \in E_L$. Denote $\Gamma = \mathrm{Gal}(L/K)$. It is known that G_L is also split semisimple, and that B_L is a Borel subgroup.

For any rational point $g \in G_L$, denote by $\operatorname{Inn} g\colon G_L \to G_L$ the conjugation by g (so that on rational points, $(\operatorname{Inn} g)h = ghg^{-1}$). Denote by ψ_1, ψ_2, $\overline{\psi}_2$ the semiactions such that $G_{L,\Gamma} = (G_L, \psi_1)$, $E_{L,\Gamma} = (E_L, \psi_2)$, and $(E/B)_{L,\Gamma} = ((E/B)_L, \overline{\psi}_2)$.

Recall that E/B is defined as a categorical quotient $(E \times G/B)/G$, denote the standard projection $E \times G/B \to E/B$ by p. It is known (stated in [6, Proposition 12.2]) and can be proved using Galois descent theory that then $p_L\colon (E_L \times (G/B)_L) \to (E/B)_L$ is also a categorical quotient modulo G_L. Denote $\operatorname{trivQuot}_e = p_L|_{\{e\}\times(G/B)_L}$. Denote by $\operatorname{inv}\colon G_L \to G_L$ the inversion map. Denote by $\overline{\varphi}\colon G_L \times (E/B)_L \to (E/B)_L$ the following map: $\overline{\varphi} = p_L \circ (\operatorname{triv}_e \times (\operatorname{trivQuot}_e)^{-1}) \circ (\operatorname{inv} \times \operatorname{id}_{(E/B)_L})$.

Lemma 2.10. *The map $\overline{\varphi}$ is an action of G_L on $(E/B)_L$. The map $\operatorname{trivQuot}_e$ is a G_L-equivariant isomorphism for the (expansion of scalars of) the standard action of G on G/B and for the action $\overline{\varphi}$ on $(E/B)_L$.*

Proof. To see that $\operatorname{trivQuot}_e$ is an isomorphism, use the universal property of categorical quotient. The rest is direct computation. Details omitted. □

We keep the notation $\overline{\varphi}$ and $\operatorname{trivQuot}_e$ until the end of the section. Note that in particular this lemma implies that $(E/B)_L$ has a rational point, and this is probably a good point to make a sort of side remark that we will need later about the converse statement.

First, let us recall a definition from [9]. Let X be a scheme over an arbitrary field F. The *determination function associated with* X (see [9, Section 2]) is the following functor from the category of fields containing F to the category consisting of $\varnothing$ and a fixed one-element set $\{0\}$: A field F_1 is mapped to $\{0\}$ if and only if X_{F_1} has a rational point, otherwise $F_1 \mapsto \varnothing$.

Second, recall that an algebraic group H over K is called *special* if all torsors of all groups H_{K_1}, where K_1 is a field extension of K, are trivial. It is known (see, for example, [11, Section 3 and Theorem 2.1]) that B is special.

Now, with these two definitions, we can say that the following lemma becomes a particular case of [9, Lemma 6.5], namely, for the special group P there being equal to B:

Lemma 2.11. *For any field extension K'/K, $E_{K'}$ has a rational point if and only if $(E/B)_{K'}$ has a rational point.* □

Now let us return to the map $\operatorname{trivQuot}_e$ and to the action $\overline{\varphi}$. The map $\overline{\varphi}$ (as well as triv_e and $\operatorname{trivQuot}_e$) is, however, not Γ-equivariant for the semiactions of Γ we already have. Let us introduce another semiaction of Γ on G_L, denoted by ψ_3. Namely, for each $\sigma \in \Gamma$, set

$$\psi_3|_{\{\sigma\}\times G_L} = (\operatorname{Inn} \operatorname{triv}_e^{-1}(\psi_2(\sigma, e)))^{-1} \circ \psi_1|_{\{\sigma\}\times G_L}. \quad (2.12)$$

Lemma 2.13. *The semiaction ψ_3 is compatible with the group structure.*

Proof. Computation. Details omitted. □

We keep the notation ψ_3 until the end of this section.

Lemma 2.14. *The map $\overline{\varphi}$ is Γ-equivariant for the semiactions $\psi_3 \times \overline{\psi}_2$ and $\overline{\psi}_2$.*

Proof. Computation. Uses Γ-equivariance of φ_L for the semiactions $\psi_1 \times \psi_2$ and ψ_2. Also uses G_L- and Γ-equivariance of p_L. Details omitted. □

Now we are going to use the notion of *varieties of Borel subgroups of a non-split semisimple algebraic group.* It was introduced in [2, 4.9 and 4.10] in higher generality, as variety of subgroups that belong to a certain class, and in [3, 5.24] for parabolic (including Borel) subgroups. It is known (see the previous two references) that if H is a not necessarily split semisimple algebraic group over an arbitrary field, then its variety of Borel subgroups exists. The precise definition involves a certain Galois descent quotient, but instead of following it precisely, it is more convenient to use properties of varieties of Borel subgroups and then prove that all varieties[1] with these properties are isomorphic. First, recall that for every semisimple

[1]We use the word "variety" now as it is used in [2], namely as an absolutely reduced scheme ([2, 2.11]), we don't mean the elementary version of algebraic geometry where affine varieties are just subsets of vector spaces, without any irrational points.

algebraic group over an arbitrary field, there exists a finite separable extension of the field where the group splits ([15, 2.11 and 17.82]) and therefore obtains a Borel subgroup ([15, 21.30]).

Now, to be more specific, given a not necessarily split semisimple algebraic group H over K, the variety of Borel subgroups of H is a certain variety Y over K with an action $\xi\colon H\times Y\to Y$. It is known to have the following property.

(∗) Let K_s be a separable closure of K. Then there is a bijection between the set of Borel subgroups of H_{K_s} and the set of rational points of X_{K_s}, equivariant for the action of (the set of rational points of) H_{K_s} on the set of its Borel subgroups by conjugation.

Lemma 2.15. *Let H be a semisimple algebraic group over K. Let Y be an arbitrary variety satisfying (∗). Then there exists a finite separable extension K' of K such that $Y_{K'}$ has a rational point. Moreover, for every such extension K', $H_{K'}$ has a Borel subgroup (denote it by B_0), and $Y_{K'}$ is $H_{K'}$-equivariantly isomorphic to $H_{K'}/B_0$.*

Proof. Easy to see. Consider the stabilizer of a rational point of $Y_{K'}$. The surjectivity of the orbit map follows from the completeness of $H_{K'}/B_0$ and from the absolute reducedness of Y, see [17, 3.2.20]. Details omitted. □

Lemma 2.16. *Let H be a semisimple algebraic group over K. Let Y be the variety of Borel subgroups of H and let X be a scheme over K with an action of H satisfying the following property:*

(∗∗) *There exists a finite separable extension K' of K such that $H_{K'}$ is split (denote a Borel subgroup by B_0), and $X_{K'}$ is $H_{K'}$-equivariantly isomorphic to $H_{K'}/B_0$.*

Then there exists a unique H-equivariant isomorphism between X and Y.

Idea of proof. By Lemma 2.15, w.l.o.g., $X_{K'}$ has a rational point. Also w.l.o.g., K'/K is finite Galois. Using $N_{H_{K'}}(B_0)=B_0$ and [15, 25.9], check that $H_{K'}/B_0$ has no equivariant automorphisms except identity. Then use Theorem 2.6 for K'/K. □

In other words, this lemma says that up to a unique equivariant isomorphism, (∗∗) can be used as a definition of the variety of Borel subgroups.

Now set $H=\mathrm{Dec}_K(G_L,\varphi_3)$ and keep this notation until the end of the section. Using Lemma 2.10, we get an action of H on E/B via $\mathrm{Dec}_K(\overline{\varphi})$ and the canonical isomorphism $E/B\cong \mathrm{Dec}_K((E/B)_{L,\Gamma})$.

Lemma 2.17. *Under this notation, H is a semisimple algebraic group, and E/B with the action of H described above is H-equivariantly isomorphic to the variety of Borel subgroups of H.*

Proof. The fact that H is semisimple is well-known. The rest follows from Lemmas 2.10 and 2.16. □

From now on, we suppose that G is simply connected. Then G_L and H are known to be simply connected as well. We also assume that L is large enough for all points of $Z(G_L)$ (and therefore of $Z(H_L)$) to be rational.

Now we will apply the results of [18] and [12] to H and E/B. First, recall that given any two groups G_0 and H_0 over K (in our case they will be G and H) that become isomorphic over L, there is a standard way to define an element of $H^1(\Gamma,\mathrm{Aut}((G_0)_L))$ called the cohomology class *twisting G_0 into H_0*. More precisely, if $(G_0)_{L,\Gamma}=((G_0)_L,\zeta_1)$ and $(H_0)_{L,\Gamma}$ is isomorphic to $((G_0)_L,\zeta_2)$, then this cohomology class maps each $\sigma\in\Gamma$ to $\zeta_2|_{\{\sigma\}\times(G_0)_L}\circ(\zeta_1|_{\{\sigma\}\times(G_0)_L})^{-1}$. If this class turns out to be in $H^1(\Gamma,(G_0)_L/Z((G_0)_L))$, where $(G_0)_L/Z((G_0)_L)$ acts on $(G_0)_L$ by conjugation, then H_0 is called an *inner form* of G_0.

It follows from formula (2.12) that H is an inner form of G. Denote the cohomology class twisting G to H by $\xi\in H^1(\Gamma,(G_0)_L/Z((G_0)_L))$. The paper [18] defines a map $\beta_{H,K}\colon((Z(H_L))^*)^\Gamma\to\mathrm{Br}(K)$ for a semisimple group H and gives a cohomological interpretation of this map if H is an inner form of a quasi-split (including split) semisimple group G.

Lemma 2.18. *The map $\beta_{H,K}\colon((Z(H_L))^*)^\Gamma\to\mathrm{Br}(K)$ defined in* [18, 3.5] *is zero for the group H we have constructed*[2]*.*

[2] In [18], the base field is expanded to a separable closure instead of a finite Galois extension. But since both G and H split over L and all points of $Z(H_L)$ are rational, the absolute Galois group of K actually acts via its quotient Γ.

Idea of proof. Following [18, 4.3], consider the exact sequence

$$1 \to {}'T_L \to ({}'T_L \times G_L)/Z(G_L) \to G_L/Z(G_L) \to 1,$$

where T is a split maximal torus of G, and ${}'T_L$ is a copy of T_L. A direct computation using formula (2.12) shows that for the long exact sequence map $\delta\colon H^1(\Gamma, G_L/Z(G_L)) \to H^2(\Gamma, {}'T_L)$, we have $\delta(\xi) = 0$. The rest follows from [18, End of 4.3]. □

Proposition 2.19. *The map of Brauer groups* $\cdot \otimes_K K(E/B)\colon \operatorname{Br}(K) \to \operatorname{Br}(K(E/B))$ *is injective.*

Proof. By Lemma 2.17, we can use [12, Theorem B] for the group H. This theorem says that the kernel $\ker(\cdot \otimes_K K(E/B))$ equals what is denoted in [12] by $A(\Sigma)$, i.e., the subgroup of $\operatorname{Br}(K)$ generated by the images of certain characters on $Z(H_L)$ under $\beta_{H,K}$. Lemma 2.18 completes the proof. □

Now we will need a few general lemmas about extension of scalars on schemes (not necessarily on torsors). First, note that for any Galois-semiaction on an irreducible scheme Y there is a straightforward way to extend this semiaction to an action on the set of open subsets of Y, on the field of rational functions on Y, and therefore on the Picard group of Y. Let us recall a well-known result about Picard and Brauer groups. For any two fields $K' \subseteq L'$, denote $\operatorname{Br}_{L'}(K') = \ker(\cdot \otimes_{K'} L'\colon \operatorname{Br}(K') \to \operatorname{Br}(L'))$.

Lemma 2.20. *Let X be a complete smooth connected scheme over a field K', and let L' be a finite Galois extension of K'. Then:*

(1) *The image of the map* $\cdot_{L'}\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X_{L'})$ *is contained in* $\operatorname{Pic}(X_{L'})^\Gamma$.
(2) *There is an exact sequence*

$$0 \to \operatorname{Pic}(X) \xrightarrow{\cdot_{L'}} \operatorname{Pic}(X_{L'})^\Gamma \to \operatorname{Br}_{L'}(K') \xrightarrow{\cdot \otimes K'(X)} \operatorname{Br}(K'(X))$$

Proof. Well-known. □

Lemma 2.21. *Let $F_1 \subseteq F_2$ be a finite Galois extension of fields. Let F_3 be another extension of F_1 such that $F_4 = F_2 \otimes_{F_1} F_3$ (understood as a tensor product of algebras) is a field. Then $F_3 \subseteq F_4$ is a finite Galois extension, $\Gamma = \operatorname{Gal}(F_4/F_3)$ preserves (normalizes) F_2, and the action of Γ on F_2 is exactly the action of the Galois group of the extension $F_1 \subseteq F_2$.*

Moreover, let X be an irreducible and reduced scheme over F_1 such that X_{F_2}, X_{F_3}, and X_{F_4} is also irreducible and reduced. Then the composition $\operatorname{Pic}(X_{F_2}) \xrightarrow{\cdot_{F_4}} \operatorname{Pic}((X_{F_2})_{F_4}) \cong \operatorname{Pic}(X_{F_4}) \cong \operatorname{Pic}((X_{F_3})_{F_4})$ *is Γ-equivariant.*

Proof. The first part is well-known. The second part is direct check. □

Now we are ready to prove Proposition 2.2

Lemma 2.22. *Proposition 2.2 is true if E is a trivial torsor.*

Proof. Follows from Lemma 2.10 and the explicit description of $\operatorname{Pic}(G/B)$ as a free abelian group in Schubert divisors. □

Lemma 2.23. *Proposition 2.2 is true when K_1 equals L (the field we fixed earlier in this section).*

Proof. One checks easily that $(E/B)_{K(E/B)}$ has a rational point and that $L \otimes_K K(E/B) = L((E/B)_L)$. By Lemma 2.21,

$$\Gamma = \operatorname{Gal}(L((E/B)_L)/K(E/B)).$$

By Lemma 2.11, $E_{K(E/B)}$ has a rational point. By Lemma 2.22 for the torsor $E_{K(E/B)}$ and the extension $L((E/B)_L)/K(E/B)$, we see that

$$\cdot_{L((E/B)_L)}\colon \operatorname{Pic}((E/B)_{K(E/B)}) \to \operatorname{Pic}((E/B)_{L((E/B)_L)})$$

is an isomorphism. By Lemma 2.20 (1) for $K' = K(E/B)$ and $L' = L((E/B)_L)$, the Γ-action on $\operatorname{Pic}((E/B)_{L((E/B)_L)})$ is trivial.

Now, also by Lemma 2.22, this time for the torsor E_L and the extension $L((E/B)_L)/L$, we get that

$$\cdot_{L((E/B)_L)}\colon \operatorname{Pic}((E/B)_L) \to \operatorname{Pic}((E/B)_{L((E/B)_L)})$$

is an isomorphism. By Lemma 2.21, the Γ-action on $\mathrm{Pic}((E/B)_L)$ is trivial. Finally, by Proposition 2.19 and by Lemma 2.20 (2) for $K' = K$, $L' = L$, we get[3] that $\cdot_L\colon \mathrm{Pic}(E/B) \to \mathrm{Pic}((E/B)_L)$ is an isomorphism. □

Idea of proof of Proposition 2.2 in the general case. We omit the details regarding commutativity of the diagrams of Picard groups for consecutive field extensions. First, prove the proposition for K_1 containing L using Lemma 2.22 for the torsor E_L and for the extension K_1/L.

Then, for a completely arbitrary K_1 containing K, we first find a finite Galois extension L_1 of K_1 for the G_{K_1}-torsor E_{K_1} in the same way as we found and fixed L for K, G, and E. Since L is a finite Galois extension of K, we can construct a field L_2 admitting embeddings of L and L_1. By the previous step for E_{K_1} instead of E, $\mathrm{Pic}((E/B)_{K_1}) \cong \mathrm{Pic}((E/B)_{L_2})$. By the previous step for the original E, $\mathrm{Pic}(E/B) \cong \mathrm{Pic}((E/B)_{L_2})$. Therefore, $\mathrm{Pic}(E/B) \to \mathrm{Pic}((E/B)_{K_1})$ is an isomorphism. □

3. Estimate of canonical dimension

The next steps of the proof of Theorem 1.4 follows the idea of proof of [10, Proposition 5.1]. First, we will need a well-known fact about the Chow ring of a smooth scheme.

Proposition 3.1. *Let X be a smooth scheme over a field K, and let L be an extension of K. The map of Chow rings $\mathrm{CH}_L\colon \mathrm{CH}(X) \to \mathrm{CH}(X_L)$, $[Y] \mapsto [Y_L]$ for each irreducible*[4] *and reduced subscheme Y of X is well-defined and is a morphism of rings.*

Moreover, the isomorphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$ commutes with expansion of scalars.

Proof. Well-known. □

We will also need the following theorem. It is stated in [10, Theorem 2.3] and follows from [7, Corollary 12.2], the preceding commutative diagram, and the definition of distinguished varieties in [7]. More precisely, this definition implies that in the particular case of the commutative diagram, the distinguished varieties are subvarieties of the intersection of supports of the cycles. Recall that a cycle (a formal linear combination of irreducible subvarieties) is called *nonnegative* if the coefficients in this linear combination are nonnegative, and an element of the Chow ring is called *nonnegative* if it can be represented by a nonnegative cycle.

Theorem 3.2. *Let X be a smooth scheme over an arbitrary field K such that the tangent bundle is generated by global sections. Let α and β be nonnegative elements of $\mathrm{CH}(X)$. If α (resp. β) is represented by a nonnegative cycle with support on $A \subseteq X$ (resp. $B \subseteq X$), then $\alpha\beta$ can be represented by a nonnegative cycle with support on $A \cap B$.* □

We need two more facts from [10]:

Lemma 3.3 ([10, Remark 2.4])**.** *Let G be a split simple simply connected algebraic group over an arbitrary field K, let B be a Borel subgroup of G, let E be a G-torsor. Then the tangent bundle of E/B is generated by global sections.* □

Lemma 3.4 ([10, Corollary 2.2])**.** *Let X be a smooth absolutely irreducible scheme over an arbitrary field K, and let L be an extension of K. Let $\alpha \in \mathrm{CH}^1(X)$. If $\mathrm{CH}_L(\alpha) \in \mathrm{CH}^1(X_L)$ is nonnegative, then $\alpha \in \mathrm{CH}^1(X)$ is nonnegative.* □

The following proposition is like Proposition 5.1 in [10], but in a different situation. It is known that if an algebraic group G over a field F is semisimple, split, and simply connected, and B is a Borel subgroup, then for every extension K of F, G_K is also semisimple, split, and simply connected, and B_K is a Borel subgroup.

Proposition 3.5. *Let G be a semisimple split simply connected algebraic group over an arbitrary field F. Let B be a Borel subgroup, and let $D_1, \dots, D_r \subset G/B$ be the Schubert divisors. Suppose that the product $[D_1]^{n_1} \dots [D_r]^{n_r}$ is multiplicity-free.*

---

[3] The idea of using the exact sequence of Brauer and Picard groups to prove the isomorphism between Picard groups is present in [10, Proof of Theorem 1.4].

[4] We will not need this, but the map defined this way actually maps the class of any subscheme Y of X to $[Y_L] \in \mathrm{CH}(X_L)$.

Let K be a field extension of F, and let E be a G_K-torsor. Then there exists a closed, irreducible, and reduced subscheme Y of E/B_K of codimension $n_1 + \ldots + n_r$ such that $Y_{K(E/B_K)}$ has a rational point.

Proof. Denote $X = E/B_K$ and $L = K(X)$. Write

$$[D_1]^{n_1}[D_2]^{n_2}\ldots[D_r]^{n_r} = \sum C_{w,n_1,\ldots,n_r}[Z_w].$$

Fix an element $v \in W$ such that $C_{v,n_1,\ldots,n_r} = 1$. Set $v' = vw_0$. Then it follows from [4, §3.3, Proposition 1a] that $[D_1]^{n_1}\ldots[D_r]^{n_r}[Z_{v'}] = [\mathrm{pt}]$. By Proposition 3.1, we have $[(D_1)_L]^{n_1}\ldots[(D_r)_L]^{n_r}[(Z_{v'})_L] = [\mathrm{pt}] \in \mathrm{CH}((G/B)_L)$.

It is easy to see that X_L has a rational point. By Lemma 2.11, E_L also has a rational point. Then by Lemma 2.10, X_L is isomorphic to $(G/B)_L$. Fix one such isomorphism (it depends on the choice of a rational point of E_L) and denote it by $f\colon X_L \to (G/B)_L$.

Denote the composition $f_* \circ \mathrm{CH}_L\colon \mathrm{CH}(X) \to \mathrm{CH}(X_L) \to \mathrm{CH}((G/B)_L)$ by g. Denote $g_1 = g|_{\mathrm{CH}^1(X)}$. By Proposition 2.2 (and by Proposition 3.1), g_1 is an isomorphism between $\mathrm{CH}^1(X)$ and $\mathrm{CH}^1((G/B)_L)$ For each i ($1 \le i \le r$), denote $\alpha_i = g_1^{-1}((D_1)_L) \in \mathrm{CH}^1(X)$. By Lemma 3.4, these are nonnegative classes (although we don't claim that each α_i is representable by a single irreducible and reduced divisor).

By Theorem 3.2, the class $\alpha_1^{n_1}\ldots\alpha_r^{n_r}$ is nonnegative. Choose irreducible subvarieties $Y_i \subseteq X$ of codimension $n_1+\ldots+n_r$ such that $\alpha_1^{n_1}\ldots\alpha_r^{n_r}$ can be written as their linear combination with nonnegative coefficients. Denote these coefficients by $c_i \ge 0$:

$$\alpha_1^{n_1}\ldots\alpha_r^{n_r} = \sum c_i[Y_i].$$

It is clear from the definitions that for each i, $g([Y_i])$ is a linear combination of the irreducible components of $f((Y_i)_L)$ with nonnegative coefficients. Since g is a morphism of rings (Proposition 3.1), we have

$$g\left(\sum c_i[Y_i]\right)[(Z_{v'})_L] = [(D_1)_L]^{n_1}\ldots[(D_r)_L]^{n_r}[(Z_{v'})_L] = [\mathrm{pt}].$$

On the other hand, $g(\sum c_i[Y_i])[(Z_{v'})_L] = \sum(c_i g([Y_i])[(Z_{v'})_L])$, and by Theorem 3.2, each $g([Y_i])[(Z_{v'})_L]$ is (can be written as) a linear combination of (reduced) 0-dimensional subvarieties (i. e. closed points) of $f((Y_i)_L) \cap (Z_{v'})_L$ with nonnegative coefficients.

So, a rational point of $(G/B)_L$ is equivalent in the Chow ring to a linear combination of some closed points with nonnegative coefficients. Then it follows from the well-definedness of the degree map $\mathrm{CH}^{\dim(G/B)}((G/B)_L) \to \mathbb{Z}$ (see [7, Definition 1.4]) that the linear combination actually consists of just one point with coefficient 1, and this point is rational. Recall that this was a point in some intersection $f((Y_i)_L) \cap (Z_{v'})_L$. In particular, we see that for one of the schemes Y_i, $(Y_i)_L$ has a rational point, and we can set $Y = Y_i$.

(We don't need this, but for this index i we also get $c_i = 1$, and for all other indices i we get $g([Y_i])[(Z_{v'})_L] = 0$ or $c_i = 0$.) □

(Last steps of the) Proof of Theorem 1.4. Let F be the base field of G. It is known that for any extension K of F, G_K is also semisimple, split, and simply connected, and B_K is a Borel subgroup of G_K.

We are going to use some results from [9]. As we already mentioned in the Introduction, the definitions of canonical dimension in [9] are not literally the same as here, so for simplicity of notation, we write cd without subscript for the canonical dimension of a scheme in the sense of [9, Section 2] and $\mathfrak{cd}$ (also without subscript) for the canonical dimension of a group in the sense of [9, Introduction].

As we also mentioned in Introduction, if E is a torsor of a split reductive group, then $\mathrm{cd}_0(E) = \mathrm{cd}(E)$ by [13, Theorem 1.16 and Example 1.18]. Therefore it follows from the statements of Definition 1.3 and the definition of $\mathfrak{cd}$ in [9, Introduction], that $\mathfrak{cd}(G) = \mathfrak{cd}_0(G)$ since G_K is (in particular) a split reductive group for any extension K of F.

Until the end of this paragraph, let K be an extension of F, let E be a G_K-torsor. Denote $L = K(E/B_K)$. By Proposition 3.5, there exists a subscheme $Y \subseteq E/B_K$ of codimension $n_1 + \ldots + n_r$ such that Y_L has a rational point. By [9, Corollary 4.7], we have[5] $\mathrm{cd}(E/B_K) \le \dim(G/B) - n_1 - \ldots - n_r$.

[5]We don't need this fact directly, but since E/B_K is smooth and projective, by [13, Theorem 1.16 and Remark 1.17], we have $\mathrm{cd}(E/B_K) = \mathrm{cd}_0(E/B_K)$. So, we could write "$\mathrm{cd}_0(E/B_K)$" here.

Now, [9, discussion after Lemma 6.7] says that $\mathfrak{cd}(G)$ can be computed as the supremum of $\mathrm{cd}(E/B_K)$ for all extensions K of F and all G_K-torsors E. Therefore, $\mathfrak{cd}(G) \leq \dim(G/B) - n_1 - \ldots - n_r$ and also $\mathfrak{cd}_0(G) \leq \dim(G/B) - n_1 - \ldots - n_r$. □

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