SEMICLASSICAL FOCUSING NLS WITH BARRIER DATA

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ABSTRACT

We study the semiclassical behavior of the focusing nonlinear Schrödinger equation in 1+1 dimensions under discontinuous "barrier" initial data. We show that the description of the oscillations appearing in the barrier-data problem is very similar to the description of the case with real analytic data, as studied recently. We reduce the analysis of the associated Riemann-Hilbert factorization problem to the analysis of the Riemann-Hilbert problem arising in the real analytic data case.

0. INTRODUCTION

The semiclassical limit of the 1+1 dimensional, integrable nonlinear Schrödinger equation with cubic focusing nonlinearity has been the subject of recent investigations. Several numerical studies have appeared since 1998 (see [MK], [BK], [CT], [C]) and a rigorous analysis of the initial value problem under real analytic data has just appeared [KMM]. The present paper makes use of the method and results in [KMM] for the study of a very particular problem with discontinuous (barrier) data. It has been shown by A.Cohen and T.Kappeler [CK] that weak solutions exist for all time under barrier data and, even more, that the inverse scattering technique is still applicable to the integration of the problem. Here, we use the inverse scattering method of [CK] to pose an associated Riemann-Hilbert factorization problem, which we then asymptotically (as $h \to 0$) reduce to the Riemann-Hilbert factorization problem that describes the solution of the real analytic data problem (as in [KMM]).

The real aim of this paper is to indicate that a discontinuity in the initial data does not necessarily alter the behavior of the semiclassical focusing NLS problem. Some changes of course *have* to happen. For example, if the Euler system that appears as a formal limit of the focusing semiclassical NLS does not even admit a solution for small times, it is obvious that the genus zero ansatz cannot hold uniformly for small times.

A natural generalization of the barrier data problem is the problem of general step data. For such data, one can easily see that the eigenvalue density ρ^0 can no longer be analytically extended from the "spike" where the eigenvalues accumulate. So, the only way the analysis of [KMM] can be immediately applicable is if the contour of discontinuity of the g-function (see section 3) is forced to touch the spike at the points of non-analyticity of ρ^0. We conjecture that under this constraint, the appropriate contour still exists and the finite (or perhaps infinite) genus ansatz still applies. We plan to address the problem of general real step initial data, with complete rigorous proofs, in a future publication.

1. FOCUSING NLS WITH BARRIER DATA

We consider the nonlinear Schrödinger equation (1+1 dimensional, integrable, focusing case), on the half-line

$$
\begin{aligned}
ihu_t^h(x,t) + \frac{h^2}{2}u_{xx}^h(x,t) + |u^h(x,t)|^2 u^h(x,t) = 0,\\
u^h(x,0) = u_0(x),
\end{aligned}
\tag{1}
$$

under barrier-like initial data:

$$
\begin{aligned}
u_0(x) = A,\ -1/2 < x < 1/2,\\
= 0,\ otherwise.
\end{aligned}
\tag{2}
$$

Here A is a fixed positive constant and h is a small positive constant. Eventually we will take $h \to 0$. We will assume that h is not taking any of the values in the discrete set $\{\frac{2A}{(2k+1)\pi}\}, k = 0, 1, 2, ...$

Setting

$$
\begin{aligned}
\rho^h = |u^h|^2,\\
\mu^h = -i\frac{h}{2}(\bar{u}^h u_x^h - u^h \bar{u}_x^h),
\end{aligned}
$$

(1) is transformed to

$$
\begin{aligned}
\partial_t \rho^h + \partial_x \mu^h = 0,\\
\partial_t \mu^h + \partial_x(\frac{(\mu^h)^2}{\rho^h}) - \partial_x(\rho^h)^2/2 = \frac{h^2}{4}\partial_x(\rho^h \partial_x^2 log\rho^h).
\end{aligned}
$$

The formal limit as $h \to 0$ is the Euler system

$$
\begin{aligned}
\partial_t \rho + \partial_x \mu = 0,\\
\partial_t \mu + \partial_x(\frac{\mu^2}{\rho}) - \partial_x(\rho)^2/2 = 0.
\end{aligned}
\tag{1a}
$$

The initial data become $\rho = u_0^2(x), \mu = 0$.

This initial value problem admits a weak solution for all time, as shown by A.Cohen and T.Kappeler ([CK]). Furthermore, the inverse scattering theory is still applicable.

The associated linear system is

$$
h\psi_x = \begin{pmatrix} -i\lambda & u(x) \\ -u^*(x) & i\lambda \end{pmatrix} \psi,
\tag{3}
$$

where * denotes complex conjugation. Jost functions ϕ, ψ are defined as column vector solutions of (3) satisfying the asymptotic conditions

(4)
$$\psi \sim \begin{pmatrix} 0 \\ e^{i\lambda x/h} \end{pmatrix}, \ as\ x \to +\infty,$$
$$\phi \sim \begin{pmatrix} e^{-i\lambda x/h} \\ 0 \end{pmatrix}, \ as\ x \to -\infty.$$

One can define the reflection and transmission coefficient as follows.

Let ψ^0 be the transpose of $(\psi_2^*, -\psi_1^*)$. Then, following [CK] define a_+, b_+ by

$$\phi = a_+\psi^0 + b_+\psi.$$

The reflection and transmission coefficients are given by

$$r_+ = b_+/a_+,$$

$$t_+ = 1/a_+.$$

Similarly, one can define coefficients r_-, t_- by normalizing the Jost functions at the opposite infinities.

From [CK], we have

(5)
$$\psi(x,\lambda) = \begin{pmatrix} 0 \\ e^{i\lambda x/h} \end{pmatrix}, \ x < -1/2,$$
$$= \begin{pmatrix} Ae^{i\lambda/h}((x-1)/h)\frac{sin[(A^2+\lambda^2)^{1/2}((x-1)/h)]}{(A^2+\lambda^2)^{1/2}((x-1)/h)} \\ e^{i\lambda/h}cos[(A^2+\lambda^2)^{1/2}((x-1)/h)] + \lambda e^{i\lambda/h}\frac{sin[(A^2+\lambda^2)^{1/2}((x-1)/h)]}{(A^2+\lambda^2)^{1/2}((x-1)/h)} \end{pmatrix}, \ -1/2 < x < 1/2,$$
$$= \begin{pmatrix} \beta(\lambda,h)e^{i\lambda x/h} \\ \alpha(\lambda,h)e^{i\lambda x/h} \end{pmatrix}, \ x > 1/2,$$

where

(6)
$$\alpha = e^{i\lambda/h} \left(cos[(A^2+\lambda^2)^{1/2}/h] - i\lambda\frac{sin[(A^2+\lambda^2)^{1/2}(1/h)]}{(A^2+\lambda^2)^{1/2}} \right),$$
$$\beta = -Ae^{i\lambda/h}\frac{sin[(A^2+\lambda^2)^{1/2}(1/h)]}{(A^2+\lambda^2)^{1/2}}.$$

The coefficients a_+, b_+, r are given by

(7)
$$a_+ = \frac{\alpha}{|\alpha|^2+|\beta|^2},$$
$$b_+ = \frac{\beta}{|\alpha|^2+|\beta|^2},$$
$$r = \frac{\beta}{\alpha}.$$

The eigenvalues are the zeros of a_+. They lie on the imaginary interval $[-iA, iA]$ and are given by

$$\lambda = i\eta,$$

$$tan[\frac{(A^2-\eta^2)^{1/2}}{h}] = \frac{(A^2-\eta^2)^{1/2}}{\eta}. \tag{8}$$

In other words,

$$\frac{(A^2-\eta^2)^{1/2}}{h} = arctan[\frac{(A^2-\eta^2)^{1/2}}{\eta}] + k\pi,\ \ k \in \mathbb{Z}.$$

As $h \to 0$, we get an asymptotic expression for the eigenvalues $\lambda_k = i\eta_k$.

$$(A^2-\eta_k^2)^{1/2} \sim hk\pi,\ \ k \in \mathbb{Z}. \tag{8'}$$

The limiting density of eigenvalues is then given by

$$\frac{\eta}{\pi(A^2-\eta^2)^{1/2}}. \tag{9}$$

or, in terms of λ,

$$\rho^0(\lambda) = \frac{\lambda}{\pi i(A^2+\lambda^2)^{1/2}}. \tag{9'}$$

It is easily seen that the associated norming constants can only take the values $-1, 1$ (by symmetry in x) and that in fact they have to oscillate between these two values (by a Sturm-Liouville oscillation argument; see [KMM]).

2. THE RIEMANN-HILBERT PROBLEM

We can now state the following Riemann-Hilbert problem, following [KMM]. Let C be a loop encircling all eigenvalues in the upper half-plane and lying entirely in the upper half-plane except for the point 0. Let C^* be its conjugate, encircling the eigenvalues in the lower half-plane.

Also give the following orientation:

(i) the real axis is oriented from left to right,

(ii) the loop C is oriented counterclockwise,

(iii) the loop C^* is also oriented counterclockwise.

We use the following convention: the +-side of an oriented contour is always to its left, according to the given orientation.

We also choose C so that it approaches 0 (from left and right) at a non-zero, non-straight angle with the real axis.

THEOREM 1. Let $d\mu = \Sigma_k h(\delta_{\lambda_k^*} - \delta_{\lambda_k})$ be a sum of Dirac measures supported at the eigenvalues of the system (3), as given by (8). Let

$$X(\lambda) = -(A^2 + \lambda^2)^{1/2}. \tag{10}$$

Letting M_+ and M_- denote the limits of M on Σ from left and right respectively, we define the Riemann-Hilbert factorization problem

$$M_+(\lambda) = M_-(\lambda)J(\lambda),$$

$$where$$

$$\begin{aligned} J(\lambda) = \begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x - 2i\lambda^2 t}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x + 2i\lambda^2 x}{h}} & 1 + |r(\lambda)|^2 \end{pmatrix}, \lambda \in \mathbb{R}, \\ = v(\lambda), \lambda \in C, \\ = \sigma_2 v(\lambda^*)^* \sigma_2, \lambda \in C^*, \\ lim_{\lambda \to \infty} M(\lambda) = I, \end{aligned} \tag{11}$$

where

(12)
$$v(\lambda) = \begin{pmatrix} 1 & -i\ exp(\frac{1}{h}\int log(\lambda - \eta)d\mu(\eta))exp(-\frac{1}{h}(2i\lambda x + 2i\lambda^2 t - X(\lambda))) \\ 0 & 1 \end{pmatrix},$$

and r is the reflection coefficient defined in (7).

In fact,

$$r(\lambda) = -A \frac{1}{cot[(A^2+\lambda^2)^{1/2}(1/h)](A^2+\lambda^2)^{1/2} - i\lambda}. \tag{13}$$

Note that $r(\lambda)$ can have a real singularity at $\lambda = 0$ if A/h is an odd multiple of $\pi/2$. By assumption, we have excluded such values of h.

The above Riemann-Hilbert problem admits a solution and the solution of (1) can be recovered from the solution of (11) as follows.

$$u(x,t) = 2ih \; lim_{\lambda\to\infty}(\lambda M^{12}(\lambda)), \tag{14}$$

where the index 12 here denotes the (12)-entry of a matrix.

PROOF: See Chapter 2 of [KMM]. The quantity $-i\ exp(\frac{1}{h}(X(\lambda_k)))$ oscillates between -1 and 1. The function $-i\ exp(\frac{1}{h}(X(\lambda)))$ is thus an extrapolation of the norming constants.

REMARKS. 1. In [KMM] the Riemann-Hilbert problem jump involves some parameters denote by K, σ, J. Here we are simply choosing $K = -1, \sigma = 1, J = 1$. This is compatible with the discussion in [KMM] as long as we focus our attention to the case $x \geq 0$. By the obvious symmetry $x \to -x$ this is acceptable.

2. Obviously, the contour C can be deformed anywhere in the upper half-plane, as long as it passes through 0 (non-tangent to either the real axis or the imaginary axis at 0) and does not touch the linear segment $[0, iA]$. It will be eventually fixed by the choice of the "g-function" in the next section. Similarly for C^*.

We note the following factorization of the jump $J(\lambda)$ on the real line.

$$\begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x+2i\lambda^2 t}{h}} & 1+|r(\lambda)|^2 \end{pmatrix} = L(\lambda)U(\lambda), \quad \textit{where} \tag{15+}$$

$$L(\lambda) = \begin{pmatrix} 1 & 0 \\ r^*(\lambda^*)e^{\frac{2i\lambda x+2i\lambda^2 t}{h}} & 1 \end{pmatrix}, \; U(\lambda) = \begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix}.$$

A different factorization is also possible.

$$\begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x+2i\lambda^2 t}{h}} & 1+|r(\lambda)|^2 \end{pmatrix} = S(\lambda)D(\lambda)T(\lambda),$$

where

$$S(\lambda) = \begin{pmatrix} 1 & \frac{r(\lambda)}{1+|r(\lambda)|^2}e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix}. \tag{15-}$$

$$D(\lambda) = \begin{pmatrix} (1+|r(z)|^2)^{-1} & 0 \\ 0 & 1+|r(z)|^2 \end{pmatrix},$$

$$T(\lambda) = \begin{pmatrix} 1 & 0 \\ 1\frac{r^*(\lambda^*)}{1+|r(\lambda)|^2}e^{\frac{2i\lambda x+2i\lambda^2 t}{h}} & 1 \end{pmatrix}.$$

Both factorizations will be useful later.

3. ASYMPTOTIC ANALYSIS OF THE RIEMANN-HILBERT PROBLEM.

We begin with the following observation. Consider the reflection coefficient given by (13). For λ in the upper half-plane we have

$$r(\lambda) \sim \frac{-iA}{(A^2+\lambda^2)^{1/2}+\lambda}. \tag{16}$$

while for λ in the lower half-plane we have

$$r(\lambda) \sim \frac{iA}{(A^2+\lambda^2)^{1/2}-\lambda}. \tag{16*}$$

Note that, whatever our choice for the branch of the square root of $(A^2+\lambda^2)$, there is a sign change across the real line. We do not specify the choice of the actual branch because it is irrelevant for our purposes.

This means that $r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}}$ is exponentially decaying (growing) in the upper half-plane, as long as $Re(\lambda) < -x/2t$ ($Re(\lambda) > -x/2t$), while $r^*(\lambda^*)e^{\frac{2i\lambda x+2i\lambda^2 t}{h}}$ is exponentially decaying (growing) in the lower half-plane, as long as $Re(\lambda) < -x/2t$ ($Re(\lambda) > -x/2t$).

The above suggests that using the factorizations of the Riemann-Hilbert problem defined by (15-), (15+) and applying the obviously suggested deformations, the jump across the real line should be reduced to a diagonal matrix independent of x, t.

To do this of course, we must ensure that the solution of the Riemann-Hilbert problem is uniformly bounded (in x, t) as $h \to 0$. This is actually not true as will be seen later. What is true however, is that a judicious conjugation of the Riemann-Hilbert problem will deform it to a new Riemann-Hilbert problem whose solution *is* uniformly bounded (in x, t) as $h \to 0$. At *that* point we can ignore *all* terms involving r. Furthermore, the very same conjugation, suggested by the one in [KMM] Chapter 5, will ensure that the new Riemann-Hilbert is explicitly solvable (asymptotically).

More precisely, following [KMM] we introduce the change of variables

$$Q(\lambda) = M(\lambda) e^{\frac{g\sigma_3}{h}}, \tag{17}$$

where $\sigma_3 = diag(1, -1)$ (a Pauli matrix) and g is exactly the "g-function" determined in [KMM]. See in particular Definition (4.1.1), (4.8), (5.34) and the whole discussion on Chapter 5 of [KMM].

This function g is analytic in the complex plane except across $C \cup C^*$, such that $g(\lambda) = O(1/\lambda)$ as $\lambda \to \infty$ away from $C \cup C*$ (in case C needs to go through the point infinity) and satisfying several conditions that enable one to deform the Riemann-Hilbert problem defined by (11), (12), to one that is explicitly solvable via theta functions.

The contour C of section 2 is now fixed by the conditions defining the function g. We do not wish to spell out these conditions, since they are not important for our discussion. See Chapter 4 of [KMM].

By (17) the Riemann-Hilbert problem for Q is $Q_+(\lambda) = Q_-(\lambda) v_Q(\lambda)$, where

(18)

$$v_Q(\lambda) = \begin{pmatrix} 1 & r(\lambda) e^{\frac{2g - 2i\lambda x - 2i\lambda^2 t}{h}} \\ r^*(\lambda^*) e^{\frac{-2g + 2i\lambda x + 2i\lambda^2 t}{h}} & 1 + |r(\lambda)|^2 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

$$v_Q(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & -i\ exp(\frac{1}{h} \int log(\lambda - \eta) d\mu(\eta)) exp(-\frac{1}{h}(-g_+ - g_- + 2i\lambda x + 2i\lambda^2 t - X(\lambda))) \\ 0 & e^{\frac{g_- - g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C,$$

$$v_Q(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & 0 \\ i\ exp(\frac{-1}{h} \int log(\lambda - \eta) d\mu(\eta)) exp(\frac{1}{h}(-g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*))) & e^{\frac{g_- - g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C^*.$$

Also $lim_{\lambda\to\infty}Q(\lambda)=I$.

Now, we *will* be able to treat the terms involving the reflection coefficient r by arguing as follows. First we note that g is purely imaginary for real λ; this follows from the definitions in [KMM]. Let's write $g=i\psi$, so that $\psi(\lambda)\in\mathbb{R}$ for $\lambda\in\mathbb{R}$. Let $\zeta(x,t,\lambda)=\psi-\lambda x-\lambda^2 t$. Clearly, $\zeta(\lambda)\in\mathbb{R}$ for $\lambda\in\mathbb{R}$.

Next, divide the real line into a union of (finitely many) intervals, say J_k, such that for $\lambda\in J_k^o$, either $\frac{d\zeta}{d\lambda}>0$ or $\frac{d\zeta}{d\lambda}<0$. Denote by J_l^+ the intervals in which $\frac{d\zeta}{d\lambda}>0$ and by J_l^- the intervals in which $\frac{d\zeta}{d\lambda}<0$. Naturally, the (nonzero) endpoints of the intervals J_k are given by the condition $\frac{d\zeta}{d\lambda}=0$, but we also make sure that 0 is such an endpoint, thus dividing the interval J_k containing 0 into two intervals J_l^-, J_{l+1}^-, as we shall see right away. Note that ζ is real analytic in $\mathbb{R}\setminus 0$, and $\zeta\sim-\lambda^2 t$, as $\lambda\to\pm\infty$. In fact, ζ even admits an analytic continuation in a small disc around $\lambda=0$ (from the left and from the right, possibly making ζ multivalued). Hence, zeros of $\frac{d\zeta}{d\lambda}$ can have no accumulation points, which proves that in fact only finitely many intervals J_k exist.

We now show that when $\lambda=0$, then $\frac{d\zeta}{d\lambda}<0$. Indeed, from Chapter 4 of [KMM] we have the following representation of g for $x\geq 0$ (if $x<0$ there is a sign change):

$$g(\lambda)=\frac{1}{2}\int_{C\cup C^*} log(\lambda-\eta)\rho(\eta)d\eta,$$

where $C\cup C^*$ is oriented counterclockwise and $\rho(\eta)d\eta$ is a non-positive measure on C which is strictly negative in a non-trivial subset of C and is extended to C^* by the condition

$$\rho(\eta^*)=(\rho(\eta))^*.$$

The actual choice of logarithm branch is irrelevant for our purposes.

Differentiating, one gets

$$\frac{dg(\lambda)}{d\lambda}=\frac{1}{2}\int_{C\cup C^*}(\lambda-\eta)^{-1}\rho(\eta)d\eta,$$

and at $\lambda=0$,

$$\frac{dg(\lambda)}{d\lambda}=\frac{-1}{2}\int_{C\cup C^*}\frac{\rho(\eta)}{\eta}d\eta.$$

(The integral converges because of the square root behavior of ρ near 0.) Making use of the symmetry with respect to complex reflection and remembering that the orientation is counterclockwise for both C and C^* one gets

$$[\frac{dg(\lambda)}{d\lambda}](\lambda=0) = 2i\int_C \frac{\rho(\eta)Im(\eta)}{|\eta|^2}d\eta.$$

Hence $[\frac{dg(\lambda)}{d\lambda}](\lambda=0)-ix$ is imaginary with strictly negative imaginary part (remember $x \geq 0$). It follows that at $\lambda=0$, $\frac{d\zeta}{d\lambda}<0$.

So, as claimed earlier, the point $\lambda=0$ indeed is a common end-point of two intervals J_l^- and J_{l+1}^-.

Consider first the intervals J_l^+. By the Cauchy-Riemann relations we have $d(Im\zeta)/d(Im\lambda) > 0$, for $\lambda \in J_l^+$, in the positive direction perpendicular to the real line. This means that in an area of the upper half-plane, close to J_l^+, the real part of $i\zeta$ is negative, so $exp(i\zeta/h)$ is exponentially decaying. Similarly, in an area of the upper half-plane, close to J_l^-, the real part of $i\zeta$ is positive, so $exp(-i\zeta/h)$ is exponentially decaying.

Let us now introduce the following lens-like contours. For each interval J_l^+ consider small deformations upwards, say $J_l^{+,up}$, keeping the end points fixed, but otherwise lying entirely in the upper half-plane. Similarly, $J_l^{+,dn}$, keeping the end points fixed, but otherwise lying entirely in the lower half-plane. For each interval J_l^- consider small deformations upwards, say $J_l^{-,up}$, keeping the end points fixed, but otherwise lying entirely in the upper half-plane. Similarly, $J_l^{-,dn}$, keeping the end points fixed, but otherwise lying entirely in the lower half-plane. All orientations are compatible with $Re(\lambda)$ increasing. We also make sure that $J_l^{+,up}, J_l^{+,dn}, J_l^{-,up}, J_l^{-,dn}$ cut the real line at angles $\neq 0, \pi/2$ at the point $\lambda=0$ and that they do not cut through C, C^*.

Let the open regions bounded by J_l^+ and $J_l^{+,up}$ be denoted by $D_l^{+,up}$ and the open regions bounded by J_l^+ and $J_l^{+,dn}$ be denoted by $D_l^{+,dn}$. Similarly, let the open regions bounded by J_l^- and $J_l^{-,up}$ be denoted by $D_l^{-,up}$ and the open regions bounded by J_l^- and $J_l^{-,dn}$ be denoted by $D_l^{-,dn}$. We make sure that $Re(i\zeta)<0$, in $D_l^{+,up} \cup J_l^{+,up} \cup D_l^{-,dn} \cup J_l^{-,dn}$. Similarly, we make sure that $Re(i\zeta)>0$, in

$D_l^{-,up} \cup J_l^{-,up} \cup D_l^{+,dn} \cup J_l^{+,dn}$.

Next, we can make use of the factorizations given in (15+), (15-). Set

$$
\begin{aligned}
Z(\lambda) &= Q(\lambda),\ for\ \lambda \in \mathbb{C} \setminus \cup_l [\bar{D}_l^{+,up} \cup \bar{D}_l^{+,dn} \cup \bar{D}_l^{-,up} \cup \bar{D}_l^{-,dn}],\\
Z(\lambda) &= Q(\lambda)U^{-1}(\lambda),\ for\ \lambda \in \cup_l D_l^{+,up},\\
Z(\lambda) &= Q(\lambda)L(\lambda),\ for\ \lambda \in \cup_l D_l^{+,dn},\\
Z(\lambda) &= Q(\lambda)T^{-1}(\lambda),\ for\ \lambda \in \cup_l D_l^{-,up},\\
Z(\lambda) &= Q(\lambda)S(\lambda),\ for\ \lambda \in \cup_l D_l^{-,dn},\\
Z(\lambda) &= Q(\lambda)\ otherwise.
\end{aligned} \tag{19}
$$

The Riemann-Hilbert problem becomes $Z_+(\lambda) = Z_-(\lambda)v_Z(\lambda)$, where

(20)

$$v_Z(\lambda) = \begin{pmatrix} 1 & -r(\lambda)e^{\frac{2g-2i\lambda x-2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^{+,up},$$

$$v_Z(\lambda) = \begin{pmatrix} 1 & 0 \\ r^*(\lambda^*)e^{\frac{-2g+2i\lambda x+2i\lambda^2 t}{h}} & 1 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^{+,dn},$$

$$v_Z(\lambda) = \begin{pmatrix} 1 & 0 \\ \frac{r^*(\lambda^*)}{1+|r(\lambda)|^2}e^{\frac{-2g+2i\lambda x+2i\lambda^2 t}{h}} & 1 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^{-,up},$$

$$v_Z(\lambda) = \begin{pmatrix} 1 & \frac{r(\lambda)}{1+|r(\lambda)|^2}e^{\frac{2g-2i\lambda x-2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^{-,dn},$$

$$v_Z(\lambda) = \begin{pmatrix} e^{\frac{g_+-g_-}{h}} & -i\ exp(\frac{1}{h}\int log(\lambda-\eta)d\mu(\eta))exp(-\frac{1}{h}(-g_+-g_-+2i\lambda x+2i\lambda^2 t - X(\lambda))) \\ 0 & e^{\frac{g_--g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C,$$

$$v_Z(\lambda) = \begin{pmatrix} e^{\frac{g_+-g_-}{h}} & 0 \\ i\ exp(\frac{-1}{h}\int log(\lambda-\eta)d\mu(\eta))exp(\frac{1}{h}(-g_+-g_-+2i\lambda x+2i\lambda^2 t + X^*(\lambda^*))) & e^{\frac{g_--g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C^*,$$

$$v_Z(\lambda) = \begin{pmatrix} (1+|r(\lambda)|^2)^{-1} & 0 \\ 0 & 1+|r(\lambda)|^2 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^-.$$

Also $lim_{\lambda\to\infty} Z(\lambda) = I$.

At this point, we still have a Riemann-Hilbert that is equivalent to the original one (11)-(12). Now, we can finally start taking into account the fact that $h \to 0$. Indeed, all terms involving $r(\lambda)$ and not supported on the real line can be neglected, not because r itself is small (it is not, see (16)-(16*)) but it always appears multiplied by something exponentially small.

In general jump matrices of the form $I +$ *exponentially small* can be neglected asymptotically as long as it is proved that the solution is uniformly (in x, t) bounded in h. We shall see that this is eventually the case.

Assuming that, we can simply delete the non-real part of the contour, at least away from the endpoints of the intervals J_k. But eventually (see section 4 later) we can also delete the remaining small crosses centered at such points. We end up with a matrix valued function W, such that $W \sim Z$ near infinity, and $W_+(\lambda) = W_-(\lambda) v_W(\lambda)$, where

(21)
$$v_W(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & -i\ exp(\frac{1}{h}\int log(\lambda-\eta)d\mu(\eta)) exp(-\frac{1}{h}(-g_+ - g_- + 2i\lambda x + 2i\lambda^2 t - X(\lambda))) \\ 0 & e^{\frac{g_- - g_+}{h}} \end{pmatrix}, \quad for\ \lambda \in C,$$

$$v_W(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & 0 \\ i\ exp(\frac{-1}{h}\int log(\lambda-\eta)d\mu(\eta)) exp(\frac{1}{h}(-g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*))) & e^{\frac{g_- - g_+}{h}} \end{pmatrix}, \quad for\ \lambda \in C^*,$$

$$v_W(\lambda) = \begin{pmatrix} (1+|r(\lambda)|^2)^{-1} & 0 \\ 0 & 1+|r(\lambda)|^2 \end{pmatrix}, \quad for\ \lambda \in \cup_l J_l^-.$$

Also $lim_{\lambda\to\infty} W(\lambda) = I$.

We will eventually see that the solution of (21) exists and is uniformly (in x, t) bounded (in the L^2 sense) as $h \to 0$. This will justify neglecting the exponentially small terms in (20) according to a basic principle for deformations of Riemann-Hilbert problems (see e.g. [DZ]). The passage from (20) to (21) will then be a posteriori justified, again at least except for the remaining small crosses centered at a finite number of real points. But again, for the crosses see section 4 (Remark 5) later.

The next transformation will get rid of the jump across the real line. Define $\tilde{d}$ as follows. Let $\tilde{d}$ be analytic in $\mathbb{C} \setminus \cup J_l^-$ and such that

$$\tilde{d}_+(\lambda) = \tilde{d}_-(\lambda)(1+|r(\lambda)|^2), \ \lambda \in \cup J_l^-,$$

$$lim_{\lambda\to\infty} d(\lambda) = 1.$$

One can even give an explicit formula for $\tilde{d}$. For our purposes, all we need is that $|\tilde{d}|, |\tilde{d}^{-1}| \leq const. < \infty$.

Let

(22)
$$Y(\lambda) = W(\lambda)\Delta(\lambda),$$
$$where$$
$$\Delta(\lambda) = \begin{pmatrix} \tilde{d}(\lambda) & 0 \\ 0 & \tilde{d}^{-1}(\lambda) \end{pmatrix}.$$

Then $Y_+ = Y_- v_Y$, where

(23)
$$v_Y(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & -i\ \tilde{d}^{-2}\ exp[\frac{1}{h}(\int log(\lambda-\eta)d\mu(\eta) + g_- - g_- - 2i\lambda x - 2i\lambda^2 t + X(\lambda))] \\ 0 & e^{\frac{g_- - g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C,$$
$$v_Y(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & 0 \\ i\ \tilde{d}^2\ exp[\frac{1}{h}(-\int log(\lambda-\eta)d\mu(\eta) - g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*))] & e^{\frac{g_- - g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C^*.$$

Also $lim_{\lambda\to\infty} W(\lambda) = I$.

This is a problem that can be treated along the lines of Chapter 4 of [KMM].

4. THE RESULT

We have reduced the problem with barrier data to a problem very similar to the one studied in [KMM] for the real analytic data case. So, we will not pursue the detailed analysis. Instead, we first state the result, and then give a sketch of the proof, faithfully following Chapter 4 of [KMM], and commenting on any possible new features.

We assume that the finite genus ansatz holds for the semiclassical asymptotics, an assumption that we will prove in a later paper. This means that the x,t-plane can be divided into (possibly empty) open regions $R_G, G = 0, 2, 4, ...$ together with their boundaries, such that within each region the asymptotics of the solution of (1) with barrier data as in (2) can be given as follows.

THEOREM 2. Let x_0, t_0 lie in region R_G. The solution $u^h(x,t)$ is asymptotically described (locally) as a slowly modulated $G+1$ phase wavetrain. Setting $x = x_0 + h\hat{x}$ and $t = t_0 + h\hat{t}$, so that x_0, t_0 are "slow" variables while $\hat{x}, \hat{t}$ are "fast" variables, there exist parameters

$a, U = (U_0, U_1, ..., U_G)^T$, $k = (k_0, k_1, ..., k_G)^T$, $w = (w_0, w_1, ..., w_G)^T$, $Y = (Y_0, Y_1, ..., Y_G)^T$, $Z = (Z_0, Z_1, ..., Z_G)^T$ depending on the slow variables x_0 and t_0 but not h (or $\hat{x}, \hat{t}$) such that

$$u^h(x,t) = u^h(x_0 + h\hat{x}, t_0 + h\hat{t}) \sim a(x_0,t_0) e^{iU_0(x_0,t_0)/h} e^{i(k_0(x_0,t_0)\hat{x} - w_0(x_0,t_0)\hat{t})} \cdot \tilde{d}^2(\lambda_0) \frac{\Theta(Y(x_0,t_0) + iU(x_0,t_0)/h + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}{\Theta(Z(x_0,t_0) + iU(x_0,t_0)/h + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}. \tag{24}$$

All parameters can be defined in terms of an underlying Riemann surface X. The moduli of X are given by $\lambda_j,\ j = 0, ..., G$ and their complex conjugates $\lambda_j^*,\ j = 0, ..., G$. The genus of X is G. The moduli of X vary slowly with x, t, i.e. they depend on x_0, t_0 but not $\hat{x}, \hat{t}$. For the exact formulae for the parameters as well as the definition of the theta functions we present the following construction.

We define the analytic arcs $I_j, I_j^*, j = 1, ..., G/2$ as follows (they come in conjugate pairs). Let the points $\lambda_j,\ j = 0, ..., G$, in the open uper half-plane be the branch points of the function g. (The fact that there are $G+1$ of them is a consequence of the definition of g, according to the "finite genus ansatz".) All such points lie on the contour C and we order them as $\lambda_0, \lambda_1, ..., \lambda_G$, according to the direction given to C. The points $\lambda_0^*, \lambda_1^*, ..., \lambda_G^*$ are their complex conjugates. Then let $I_0 = [0, \lambda_0]$ be the subarc of C joining points 0 and λ_0. Similarly, $I_j = [\lambda_{2j-1}, \lambda_{2j}],\ j = 1, ..., G/2$. The points $\lambda_j,\ j = 0, ..., G$ lie in the open uper half-plane and they are determined by a set of transcendental equations (see Remark after Lemma 5.1.5 of [KMM]).

The Riemann surface X is constructed by cutting two copies of the complex sphere along the slits $I_0 \cup I_0^*, I_j, I_j^*, j = 1...., G$, and pasting the "top" copy to the "bottom" copy along these very slits.

We define the homology cycles $a_j, b_j,\ j = 1, ..., G$ as follows. Cycle a_1 goes around the slit $I_0 \cup I_0^*$ joining λ_0 to λ_0^*, remaining on the top sheet, oriented counterclockwise, a_2 goes through the slits I_{-1} and I_1 starting from the top sheet, also oriented counterclockwise, a_3 goes around the slits $I_{-1}, I_0 \cup I_0^*, I_1$ remaining on the top sheet, oriented counterclockwise, etc. Cycle b_1 goes through I_0 and I_1 oriented counterclockwise, cycle b_2 goes through I_{-1} and I_1, also oriented counterclockwise, cycle b_3 goes through I_{-1} and I_2, and around the slits $I_{-1}, I_0 \cup I_0^*, I_1$, oriented counterclockwise, etc.

On X there is a complex G-dimensional linear space of holomorphic differentials, with basis elements $\nu_k(P)$ for $k = 1, \ldots, G$ that can be written in the form

$$\nu_k(P) = \frac{\sum_{j=0}^{G-1} c_{kj}\lambda(P)^j}{R_X(P)}\, d\lambda(P)\,,$$

where $R_X(P)$ is a "lifting" of the function $R(\lambda)$ from the cut plane to X: if P is on the first sheet of X then $R_X(P) = R(\lambda(P))$ and if P is on the second sheet of X then $R_X(P) = -R(\lambda(P))$. The coefficients c_{kj} are uniquely determined by the constraint that the differentials satisfy the normalization conditions:

$$\oint_{a_j} \nu_k(P) = 2\pi i \delta_{jk}.$$

From the normalized differentials, one defines a $G \times G$ matrix H (the period matrix) by the formula:

$$H_{jk} = \oint_{b_j} \nu_k(P).$$

It is a consequence of the standard theory of Riemann surfaces that H is a symmetric matrix whose real part is negative definite.

In particular, we can define the theta function

$$\Theta(w) := \sum_{n \in \mathbb{Z}^G} \exp(\frac{1}{2} n^T H n + n^T w),$$

where H is the period matrix associated to X. Since the real part of H is negative definite, the series converges.

We arbitrarily fix a base point P_0 on X. The Abel map $A : X \to Jac(X)$ is then defined componentwise as follows:

$$A_k(P; P_0) := \int_{P_0}^{P} \nu_k(P'),\ k = 1, \ldots, G,$$

where P' is an integration variable.

A particularly important element of the Jacobian is the Riemann constant vector K which is defined, modulo the lattice Λ, componentwise by

$$K_k := \pi i + \frac{H_{kk}}{2} - \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq k}}^{G} \oint_{a_j} \left(\nu_j(P) \int_{P_0}^{P} \nu_k(P') \right),$$

where the index k varies between 1 and G.

Next, we will need to define a certain meromorphic differential on X. Let $\Omega(P)$ be holomorphic away from the points ∞_1 and ∞_2, where it has the behavior

$$\Omega(P) = dp(\lambda(P)) + \left(\frac{d\lambda(P)}{\lambda(P)^2}\right),\ P \to \infty_1,$$
$$\Omega(P) = -dp(\lambda(P)) + O\left(\frac{d\lambda(P)}{\lambda(P)^2}\right),\ P \to \infty_2,$$

and made unique by the normalization conditions

$$\oint_{a_j} \Omega(P) = 0, j = 1, \ldots, G.$$

Let the vector $U \in \mathbb{C}^G$ be defined componentwise by

$$U_j := \oint_{b_j} \Omega(P).$$

Note that $\Omega(P)$ has no residues.

Let the vectors V_1, V_2 be defined componentwise by

$$V_{1,k} = (A_k(\lambda_{1+}^*) + A_k(\lambda_{2+}) + A_k(\lambda_{3+}^*) + \cdots + A_k(\lambda_{G+})) + A_k(\infty) + \pi i + \frac{H_{kk}}{2},$$
$$V_{2,k} = (A_k(\lambda_{1+}^*) + A_k(\lambda_{2+}) + A_k(\lambda_{3+}^*) + \cdots + A_k(\lambda_{G+})) - A_k(\infty) + \pi i + \frac{H_{kk}}{2},$$

where $k = 1, ..., G$, and the $+$ index means that the integral for A is to be taken on the first sheet of X, with base point λ_+^0.

Finally, let

$$a = \frac{\Theta(Z)}{\Theta(Y)} \sum_{k=0}^{G} (-1)^k \Im(\lambda_k),$$
$$k_n = \partial_x U_n,\ w_n = -\partial_t U_n,\ n = 0, \ldots, G,$$

where

$$Y = -A(\infty) - V_1,\ Z = A(\infty) - V_1,$$

and U_0 depends explicitly on the initial data (see [KMM]).

Now, the parameters appearing in formula (24) are completely described.

We simply note here that the U_i and hence the k_i and w_i are real. We also note that the denominator in (24) never vanishes (for any $x_0, t_0, \hat{x}, \hat{t}$).

REMARKS AND SKETCH OF PROOF.

1. Formula (24) is locally a so-called finite gap expression. It describes violent oscillations of bounded amplitude but high frequency $O(1/h)$.

2. As in [KMM] weak limits of densities exist. $\rho = lim_{h\to 0}|u^h|^2$ and $\mu = lim_{h\to 0}\frac{-ih}{2}(\bar{u}^h u^h_x - u^h \bar{u}^h_x)$. These limits are actually strong in the genus zero region.

3. Naturally, since the initial data is discontinuous and the limiting Euler system (1a) is elliptic (see [KMM]) one expects the break-time of the limiting system to be zero. This is indeed the case, as numerical experiments by H.Ceniceros and F.Tian have shown [T], or as can be shown analytically by considering the limiting Euler system directly. Indeed, from the second equation of (1a) it is obvious that μ_t is infinite at $x = \pm\frac{1}{2}, t = 0$. Of course a genus zero region still exists but it touches the $t = 0$ axis of the x, t-plane at $x = \pm 1/2$.

4. The proof of the results in [KMM] makes use of the assumption that the eigenvalue density can be analytically extended in the upper half-plane with the spike where eigenvalues accumulate deleted. This *is* the case here, see formula (9'). The branch root singularity at iA does not play any role, because it is integrable.

5. A priori, neglecting the exponentially terms in (20) only allows us to "delete" arcs $J_l^{+,up}, J_l^{+,dn}, J_l^{-,up}, J_l^{-,dn}$ only away from the real endpoints of J_l^+, J_l^-. For entirely rigorous justifications of the deletion of the small crosses centered at each such point remaining after the deletion of the bulk of the arcs $J_l^{+,up}, J_l^{+,dn}, J_l^{-,up}, J_l^{-,dn}$, one must construct local parametrices of the Riemann-Hilbert problem and make sure they match with the solution of (21) away from the endpoints. This is a procedure that is by now standard in the literature, so we omit the cumbersome but uninspiring details. In fact, essentially the same situation has appeared in [DZ]. The "local" Riemann-Hilbert problem can be solved via parabolic cylinder functions. (In [DZ], of course, the contribution was not negligible, since one was trying to evaluate the term of order $O(t^{-1/2})$ of the asymptotics.)

However, the point 0 is somewhat exceptional, because the contours C, C^* also run through it. So there are actually *eight* branches of the contour emanating from 0. In fact, once one follows the steps of [KMM], section 4.4.3, one ends with *fourteen* branches. But it is still easy to prove the existence of the appropriate local parametrix, using the Fredholm theory described in [KMM], and then match it with the solution of the "outer problem". (The "cyclic" relation, that the product of limits of jumps at 0 is the identity, still holds.)

6. The Bohr-Sommerfeld condition (8) is not quite the same as the condition postulated in [KMM] defining the so-called soliton ensemble, which translates as

$$(A^2 + \lambda_k^2)^{1/2} = hk\pi$$

for our present problem (1)-(2). Now, one can notice that the difference between the two conditions gives rise to a uniform error of higher order $O(h^2)$. The analysis in [KMM] (Chapter 3) of estimates needed for the passage from a "discrete" Riemann-Hilbert problem to a "continuum" Riemann-Hilbert problem is not altered by this innocent modification.

7. From formulae (6)-(7) it is obvious that $\alpha(0) = 0$ if Ah is an odd multiple of $\pi/2$. So, for a particular sequence of h going to zero we have a *spectral singularity* at the real point 0. The Riemann-Hilbert problem jump becomes singular at 0. By assumption we have excluded such values of h. We plan to study the effect of a real spectral singularity on the semiclassical behavior of the focusing NLS in a later publication.

8. The proof now follows the discussion of Chapter 4 of [KMM]. One minor change is the extra factor $\tilde{d}^{\pm 2}$ appearing in the off-diagonal terms of (23). This factor can be taking care of by the auxiliary scalar Riemann-Hilbert problem (4.38) in section 4.3.1 of [KMM]. The proof goes through with only a minor change: α_k has to be substituted by $\alpha_k + 2ih\ log\tilde{d}$. Of course $\tilde{d}$ has to appear in the final formula (24).

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