ABSOLUTE DETERMINANTS AND HILBERT SYMBOL

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ABSTRACT. A tensor category of "vector" spaces over "constants"(i.e., 0 and roots of unity) of a number field is constructed. We consider the "constants" as a monoid only and the tensor category has a very simple structure. Nevertheless, applying determinant theory (in the sense of the category considered) to number and local fields allows us to construct a theory of the classical Hilbert symbol.

INTRODUCTION

Many important constructions of algebraic geometry do not work in the scheme theory(and therefore in Arakelov geometry). For example, there is no two-dimensional space $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ and no sheaf of differentials, Ω^1, on $\operatorname{Spec}\mathbb{Z}$. The reason of difficulties seems to be that we have no properly defined "common" part of all the fields $\mathbb{F}_p$. Nevertheless, some objects defined over the non-existing "field with one element" are well-known(for example, $GL_n(\mathbb{F}_1)$ is the symmetric group S_n). Of course, all such kind constructions should be regarded only as a very rough approximation of "true" those, which are unknown in present time. So, our task on this level is to look for working constructions.

In [Sm] the natural "constants" in any number field (zero and all the roots of unity) were considered as a substitute of the constant field and as an extension of $\mathbb{F}_1$. (there a cosideration of $\mathbb{P}^1$ over the "fields" has led to several number-theoretical conjectures generalizing the well-known ABC-conjecture).

Recently, M.Kapranov wrote me that a tensor calculation over these "fields" (which he denotes by $\mathbb{F}_{1^n}$) lead to an interpretation of some classical symbols in the class field theory for number fields. More precisely, Kapranov pointed out that the n-power residue symbol $\left(\frac{a}{\mathfrak{p}}\right)_n \in \mu_n$ (here $\mathfrak{p}$ is a prime ideal in a ring of integers A of a number field) can be interpreted as an appropriately defined determinant of multiplication by a in $A/\mathfrak{p}$, considered as a "vector" space over $\mathbb{F}_{1^n}$:

$$\left(\frac{a}{\mathfrak{p}}\right)_n = \det(a : A/\mathfrak{p} \to A/\mathfrak{p}).$$

This strenghens the analogy between power residue symbols and resultants well known in the class field theory for functional fields. Namely, the resultant of two monic polinomials $f, g \in k[t]$ is

$$\operatorname{Res}(f, g) = \det(f : k[t]/(g) \to k[t]/(g)).$$

The reciprocity law is thus an analog of the symmetry property of the resultant.

In a talk on the conference on "Arithmetic geometry" on May 1994 in Lumini the author have mentioned this Kapranov's idea. After that A.N.Parshin told me about his interpratation of a Breen's construction for the tame symbol on algebraic curves (later and independently this construction were rediscovered in [ACK]) and suggested to look for an analogous construction for number fields. This is done in the present paper.

In the Sec.1 we describe a tensor category of "vector spaces over $\mathbb{F}_{1^n}$" and in the Sec.2 we study relations between this category, which is non-additive, and an abelian category of abelian groups with an $\mathbb{F}_{1^n}$- action(note Lemma 2.1.2, which plays a crucial role in what follows). In the Sec.3 we construct, following [ACK], a symbol $\{f,g\}$ of two commuting automorphisms of a universum V and in the Sec.4 we link this symbol with the tame Hilbert symbol(there V coincides with a finite extension K of $\mathbb{Q}_p$; $f, g \in K^*$). Besides, there a weak reciprocity law for global number fields is proved. In the Sec.5 we explaine the construction of the Sec.3 from a more invariant point of view: there we consider a category of line bundles and their meromorphic morphisms on a one-dimensional space instead of the universum V.

I would like to thank A.N.Parshin for his explanations and sugestion, and also the organizers of the mentioned conference for the invitation.

I thank also M.Kapranov, who wrote me about his ideas and later kindly consultated me concerning all my questions.

This paper is done during my stay at the Max-Planck Institute of Mathematics in framework of a special activity on the Arakelov theory and I thank MPIM and the organizers of this meeting for the invitation.

Notation

Let us fix notation, which will be used throughout the paper.

- μ is a finite abelian group (multiplicative);
- N is the number of elements of μ;
- $F = \{0\} \cup \mu$ is considered as a commutative monoid $(0 \cdot \varepsilon = 0)$;
- $\omega = \prod_{\varepsilon \in \mu} \varepsilon$ is the root sign of μ.

It is easy to see that

$$\omega = \begin{cases} \text{the element of order two in } \mu, & \text{if such an element is unique;} \\ 1, & \text{else.} \end{cases}$$

- $\tilde{V} = V \setminus \{\mathrm{pt}\}$ for any pointed set V.
- $\mathcal{M}$ is the category of F-spaces (see 1.1).
- $\mathcal{M}_{ab}$ is a subcategory of all the abelian groups of $\mathcal{M}$ (see 1.1).
- $\mathcal{P}$ is the category of free F-spaces (see 1.1).
- $\mathcal{P}_{ab} = \mathcal{P} \cap \mathcal{M}_{ab}$ (see 1.1).

1. Category of F-spaces

The constuctions of this section were discovered independently by M.Kapranov and the author. The later considered during long time the determinant $\det = \det_1$ (see 1.2.7) as bad one, and thought that a right determinant is $\det_{-1}$, which would be in K_1-theory of F, but not in F itself. A Kapranov's idea is that just $\det_1$ is interesting and just it arises in number theory.

1.1 Objects and morphisms.

Definition 1.1.1. An *F-space* is a pointed set V (the point pt$\in V$ will always be denoted by the symbol 0) with an action of F such that for any $\varepsilon \in F$ and any $x \in V$ we have $0 \cdot x = 0$ and $\varepsilon \cdot 0 = 0$.

A morphism of F-spaces is an F-equivariant map of pointed sets. We denote the category of F-spaces by $\mathcal{M}$ (or $\mathcal{M}_F$). Consider in $\mathcal{M}$ a subcategory $\mathcal{M}_{ab}$ of abelian groups and their homomorphisms (the action of F must agree with the structure of the abelian group).

Definition 1.1.2. Let $V \in \mathcal{M}$. We say that V is *free*, if μ acts freely on $\tilde{V}$.

The free objects of $\mathcal{M}$ generate a full subcategory, which will be denoted by $\mathcal{P}$ and the free objects of $\mathcal{M}_{ab}$ generate a full subcategory, which will be denoted by $\mathcal{P}_{ab}$. For $V \in \mathcal{P}$ put

$$\dim V = \mathrm{Card}(\tilde{V}/\mu).$$

For $V \in \mathcal{P}$ define a basis of V as a collection of elements $x_1, x_2, ..., x_d \in \tilde{V}$, such that in each orbit of the action of μ on $\tilde{V}$ lies exactly one of the x_i's.

1.2 Tensor operations. In this subsection by symbols V, W, etc. we denote free F-spaces.

1.2.1 Duality. The space V^* coincides as a set with V. We denote the identity map $V \to V^*$ by $x \mapsto x^*$. The action of F on V^* is defined as follows: if $\varepsilon \in \mu$, then $\varepsilon \cdot x^* = (\varepsilon^{-1} \cdot x)^*$.

1.2.3 Direct sum. Put $V \oplus W = \{0\} \cup \tilde{V} \cup \tilde{W}$ with the obvious action of μ.

1.2.4 Tensor product. Consider the antidiagonal action of μ on $\tilde{V} \times \tilde{W}$ defined by $\varepsilon \cdot (x \times y) = \varepsilon x \times \varepsilon^{-1} y$.

Put $V \otimes W = \{0\} \cup \widetilde{V \otimes W}$, where $\widetilde{V \otimes W}$ is the factor of $\tilde{V} \times \tilde{W}$ by the antidiagonal action of μ.

Let $x \otimes y$ denote the image of the couple (x, y) under the natural projection. The action of F on V induces an action of F on $V \otimes W$ and it is clear, that $(\varepsilon x) \otimes y = x \otimes \varepsilon y$.

1.2.5 Exterior powers. There are two natural actions of the symmetric group S_k on $V^{\otimes k}$:

(a) $\sigma(x_1 \otimes ... \otimes x_k) = x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$;

(b) $\sigma(x_1 \otimes ... \otimes x_k) = \omega^{\text{parity}(\sigma)} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$.

Accordingly, there are two theories of exterior powers: $\Lambda_1^k V$ and $\Lambda_\omega^k V$ (we write often $\Lambda^k V$ instead of $\Lambda_1^k V$). To define these spaces, consider the following subset of $V^{\otimes k}$:

$$Z = \{x_1 \otimes ... \otimes x_k | \text{there exist indices } i,\ j, \text{ and } \lambda \in F, \text{ such that } x_j = \lambda \cdot x_i\}.$$

Put

$$\Lambda_\varepsilon^k V = \{0\} \cup (V^{\otimes k} \setminus Z)/S_k,$$

where ε denotes 1 or ω and the action of S_k is as in (a) for $\varepsilon = 1$, and as in (b) for $\varepsilon = \omega$.

There is a natural projection $V^{\otimes k} \to \Lambda_\varepsilon^k V$ (by which $Z \to \{0\}$). The image of $x_1 \otimes ... \otimes x_k$ under this projection is denoted by $x_1 \wedge ... \wedge x_k$.

1.2.6 *Remark.* In the algebra $\Lambda_1^\bullet V$, we have, as usual, $x \wedge x = 0$, but $x \wedge y = y \wedge x$. The existence of the commutative exterior power is a special phenomenon of our situation.

1.2.7 Determinants. Let $\dim V = d < \infty$. We set

$$\det_\varepsilon V = \Lambda_\varepsilon^d V.$$

This is a one-dimensional space.

For any isomorphism $\phi : V \to W$ we obtain a natural morphism

$$\det_\varepsilon \phi : \det_\varepsilon V \to \det_\varepsilon W; \quad y_1 \wedge \cdots \wedge y_d \to \phi(y_1) \wedge \cdots \wedge \phi(y_d).$$

In the case $V = W$, the morphism $\det_\varepsilon \phi$ is given by the multiplication by an element of F (since $\dim(\det_\varepsilon V) = 1$), which is denoted by the same symbol $\det_\varepsilon \phi$.

If $x_1, \ldots, x_d$ is a basis of V and $\phi(x_i) = \varepsilon_i x_{\sigma(i)}$, then

$$det_\varepsilon \phi = \varepsilon^{\text{parity}(\sigma)} \cdot \prod_{i=1}^{d} \varepsilon_i.$$

We write often $\det V$ instead of $\det_1 V$ and $\det \phi$ instead of $\det_1 \phi$.

1.2.8 *Remark.* Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{1, \omega\}$. It is easy to see that if $\varepsilon_1\varepsilon_2 = \varepsilon_3\varepsilon_4$, then there is a canonical isomorphism

$$\det_{\varepsilon_1} V \otimes \det_{\varepsilon_2} W \cong \det_{\varepsilon_3} V \otimes \det_{\varepsilon_4} W.$$

In particular, if we put $[\omega]_V = \det_\omega V \otimes (\det_1 V)^*$, then for any V and W the spaces $[\omega]_V$ and $[\omega]_W$ are canonically isomorphic.

Besides, there is the canonical isomorphism:

$$(\det_1 V)^{\otimes 2} \cong (\det_\omega V)^{\otimes 2}.$$

1.2.9 Agreement about signs. Below we adhere to the following convention:

(a) $\det V = \det_1 V$, $\det \phi = \det_1 \phi$, $\Lambda^k V = \Lambda_1^k V$;

(b) $V \otimes W \cong W \otimes V$, $x \otimes y \to y \otimes x$;

(c) $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$, $(v \otimes w) \otimes u \to v \otimes (w \otimes u)$;

(d) $V^{**} \cong V$, $x^{**} \to x$.

2. Interaction between $\mathcal{M}$ and $\mathcal{M}_{ab}$

2.1 Restriction of scalars (inclusion $\mathcal{M}_{ab} \subset \mathcal{M}$).

Lemma 2.1.1. *(i) Let A be a finite object of $\mathcal{M}_{ab}$. Then A is free, iff for each $\varepsilon \in (\mu \setminus \{1\})$ the operator $(1-\varepsilon)$ is invertible on A. It is enough to verify this only for a system of generators of μ.*

(ii) Let $0 \to A \to B \to C \to 0$ be an exact sequence of finite objects of $\mathcal{M}_{ab}$. Then B is free, iff A and C are free.

Proof. (i) is obvious in view of the finiteness of A; (ii) follows from (i).

Lemma 2.1.2. *Let $E : 0 \to A \to B \to C \to 0$ be an exact sequence of finite objects of $\mathcal{P}_{ab}$. Then*

(i) $dim B = dim A + dim C \pmod N$;

(ii) there is a canonical isomorphism

$$\phi_E : \det A \otimes \det C \tilde{\to} \det B.$$

Proof. (i) This is obvious ($\mathrm{Card}\, A = 1 + N \dim A$, etc.).

(ii) First we construct ϕ_E in terms of any two bases of A and C and then check that it does not depend on the choice of the bases. Let $x_1, \dots, x_n$ be a basis of A and $y_1, \dots, y_m$ be a basis of C. Define ϕ_E by

$$\phi_E(x_1 \wedge \dots \wedge x_n \otimes y_1 \wedge \dots \wedge y_m) = \bigwedge_{i=1}^{n} x_i \wedge \bigwedge_{j=1}^{m} \bigwedge_{\tilde{y} \mapsto y_j} \tilde{y}.$$

Here $\tilde{y}_j$ runs over all the preimages of y_j in B. (We remind the reader that our exterior product is commutative, see Remark 1.2.6, and thus the exterior products do not depend on the order of factors.) For new bases $x'_1, \dots, x'_n$, $y'_1, \dots, y'_m$, the same formula yields a map ϕ'_E. Let $x'_i = \varepsilon_i x_i$, $y'_j = \omega_j y_j$. It is easily seen that

$$\frac{\phi_E(*)}{\phi'_E(*)} = \frac{\prod_{i=1}^{n} \varepsilon_i \prod_{j=1}^{m} (\omega_j)^{\mathrm{Card}\, A}}{\prod_{i=1}^{n} \varepsilon_i \prod_{j=1}^{m} \omega_j},$$

Obviously, the right hand side is equal to unity, and so $\phi'_E = \phi_E$.

Remark 2.1.3. A striking feature of our definition of ϕ_E is that it is obviously nonsymmetric with respect to A and C. Actually, besides ϕ_E we can define another canonical isomorphism ψ_E. To do this choose any liftings $\tilde{y}_1, \dots, \tilde{y}_m$ in B of the elements $y_1, \dots, y_m$ and set

$$\psi_E(x_1 \wedge \dots \wedge x_n \otimes y_1 \wedge \dots \wedge y_m) = \bigwedge_{i=1}^{n} x_i \wedge \bigwedge_{j=1}^{m} \tilde{y}_j \wedge \bigwedge_{\substack{i=1,\dots,n \\ j=1,\dots,m \\ \varepsilon \in \mu}} (x_i + \varepsilon \tilde{y}_j).$$

We easily see that $\psi_E = \omega^{mn} \phi_E$, where ω is the root sign of μ.

2.2 Extension of scalars. Let us consider a F-field K (I mean that μ lies in the multiplicative group of K). In this situation we have the following functor:

$$V \to K \otimes V; \quad \mathcal{P} \to \mathrm{Vect}\, K.$$

It is easy to see that the following lemma holds.

Lemma 2.2.1. *(i) The functor $V \to K \otimes V$ is a functor of tesor categories;*

(ii) if $\omega = -1$, then

$$K \otimes \Lambda_\omega^k V = \Lambda^k (K \otimes V).$$

.

Remark 2.2.2. (a) Let $E : 0 \to A \to B \to C \to 0$ be an exact sequence of finite objects of $\mathcal{P}_{ab}$. The same sequence E , considered over F, is not exact. One can see that, after $\otimes$ -multiplication by K, the sequence of K-vector spaces $K \otimes E$ has a homology $H_K(E)$ in the middle. Although a homology $H(E)$ of E itself is not defined, Lemma 2 shows, that $\det H(E) = 1$ and $\dim H(E) = 0 \pmod{N}$. It is interesting to introduce some structure on the vector space $H_K(E)$, which would reflect a "triviality" of the space. It seems that the structure should arise from special bases of $K \otimes A, K \otimes B, K \otimes C$, related to the additive structure of A, B, C. I mean bases, related to characters of those groups in the multiplicative group of K.

(b) Each $V \in \mathcal{P}$ gives a functor $K \to K \otimes V$. We can extend the category $\mathcal{P}$, considering functors with appropriative properties as new F-spaces. The functor $K \to H_K(E)$ shows that there exsist really new F-spaces.

3. Abstract symbols

3.1 Symbols $(A|B)$ and $[A|B]$. Having Lemma 2.1.2 we can follow the paper [ACK].

Let us fix $V \in \mathcal{M}_{ab}$. Below we use the termin "subgroup of V" instead of "subobject of V".

Definition 3.1.1. Let A and B be subgroups of V. We say that A and B are commensurable and write $A \sim B$, iff there exists a subgroup $C \subset (A \cap B)$, such that A/C and B/C are finite and free.

Lemma 3.1.2. *(i) If $A \sim B$, then one can take $A \cap B$ itself instead of C in the definition* 3.1.1;

(ii) the relation $A \sim B$ is an equivalence.

Proof. That follows easily from Lemma 2.1.1,(ii).

Definition 3.1.3. Let $A \sim B$.

(i) Put $[A|B] = \dim(A/(A \cap B)) - \dim(B/(A \cap B)) \pmod N$.

(ii) Put $(A|B) = \det(A/(A \cap B)) \otimes \det(B/(A \cap B))^*$.

Lemma 3.1.4. *Let C be a subgroup of $(A \cap B)$, $A \sim B \sim C$. Then there is a canonical isomorphism*

$$(A|B) \cong \det(A/C) \otimes \det(B/C)^*$$

Proof. Cosider the following exact sequences:

$$E_1 : 0 \to (A \cap B)/C \to A/C \to A/(A \cap B) \to 0,$$

and

$$E_2 : 0 \to (A \cap B)/C \to B/C \to B/(A \cap B) \to 0.$$

Lemma 2.1.2,(ii) gives the canonical isomorphisms:

$$\phi_{E_1} : \det((A \cap B)/C) \otimes \det(A/(A \cap B)) \tilde{\to} \det(A/C)$$

$$\phi_{E_2} : \det((A \cap B)/C) \otimes \det(B/(A \cap B)) \tilde{\to} \det(B/C)$$

So we have the isomorphism $\phi_{E_1} \otimes \phi^*_{E_2}$:

$$det((A \cap B)/C) \otimes \det(A/(A \cap B)) \otimes \det((A \cap B)/C)^* \otimes \det(B/(A \cap B))^* \tilde{\to} \det(A/C) \otimes \det(B/C)^*.$$

After a reduction of the left hand side we have the isomorphism to be constructed.

3.2 Properties of the symbols $(A|B)$ and $[A|B]$. Below we write $l_1 \cdot l_2$ instead of $l_1 \otimes l_2$ and l^{-1} instead of l^*, where l_1, l_2, l are one-dimensional F-spaces. By 1 we denote the standart one-dimensional spase, namely F itself.

Lemma 3.2.1. *Let A, B, C be subgroups of V; $A \sim B \sim C$. Then there are the canonical isomorphisms:*

(a) $(A|A) \cong 1$;

(b) $(A|B)(B|C) \cong (A|C)$;

(c) $(A|B)^{-1} \cong (B|A)$;

(d) *$i_f : (A|B) \tilde{\to} (fA|fB)$, where f is an automorphism of V. Besides, $i_f = 1$ in the case $A = B$, if we identify $(A|A)$ and $(fA|fA)$ with 1.*

Proof. (a) is obvious, (c) follows from (a) and (b).

(b) To construct the isomorphism of (b) put $D = A \cap B \cap C$. Using Lemma 3.1.4, we can write

$$(A|B)(B|C) \cong \det(A/D)(\det(B/D)^{-1})\det(B/D)(\det(C/D))^{-1}$$

After a reduction of the right hand side we have , by the Lemma 3.1.4, just $(A|C)$.

(d) Put $D = A \cap B$. The automorphism f gives the following isomorphisms: $\det(A/D) \cong \det(fA/fD)$ and $\det(B/D) \cong \det(fB/fD)$, which give the obvious isomorphism i_f.

By the same way we obtain the following lemma.

Lemma 3.2.2. *Let A, B, C be subgroups of V; $A \sim B \sim C$. Then*

(a) $[A|A] = 0 \pmod N$;

(b) $[A|B] + [B|C] = [A|C] \pmod N$;

(c) $[A|B] = -[B|A] \pmod N$;

(d) $[A|B] = [fA|fB] \pmod N$, *where f is an automorphism of V.*

3.3 Symbol $\{f, g\}$. Let us fix a couple $A \subset V$ of objects in $\mathcal{M}_{ab}$.

We denote by $GL(V, A)$ a set of all the automorphisms f of V, s.t., $A \sim fA$.

Let f and g be two commutating elements in $GL(V, A)$. Consider the following diagram $\mathcal{D}_A(f, g)$ (or $\mathcal{D}_A^V(f, g)$):

$$\begin{array}{ccc}
(A|fA)(fA|fgA)(A|gA)^{-1}(gA|fgA)^{-1} & \xleftarrow[\alpha]{1\otimes i_f\otimes 1\otimes i_g^*} & (A|fA)(A|gA)(A|gA)^{-1}(A|fA)^{-1} \\
\downarrow \psi & & \varphi \downarrow \\
1 & & 1
\end{array}$$

Here ψ and φ are the compositions of certain natural convolutions of the types (b) and (c) from Lemma 3.2.1.

Definition 3.3.1. Let $f, g \in GL(V, A)$. We set

$$\{f, g\} = \psi \circ \alpha \circ \varphi^{-1}.$$

Here $\{f, g\}$ is the automorphism of the one-dimensional space 1, i.e., an element of F. If necessary, we use the notation $\{f, g\}_A$ or $\{f, g\}_A^V$ to emphasize the dependence of $\{f, g\}$ on A and V.

Remark 3.3.2. A.N.Parshin explained to the author the paper $[ACK]$ using following drawings (maybe, they will help to a reader):

$$\begin{array}{ccc}
fA & \xrightarrow{(fA|gfA)} & fgA \\
(A|fA) \uparrow & & \downarrow (gA|fgA)^{-1} \\
A & \xleftarrow[(A|gA)^{-1}]{} & gA
\end{array}$$

. The diagram is "commutativ", because the both triangles are "commutative" (Lemma 3.2.1 (b)), and the cycle gives just the upper left hand corner of the preceding diagram. Two following arrows give an automorphism of the cycle:

$$\begin{array}{ccc} * & & * \\ {\scriptstyle (A|fA)}\Big\uparrow & g \quad \Longrightarrow & \Big\uparrow{\scriptstyle (gA|gfA)} \\ * & & * \end{array}$$

, and

$$\begin{array}{c} * \xrightarrow{(fA|fgA)} * \\ \Uparrow f \\ * \xrightarrow[(A|gA)]{} * \end{array}$$

. This automorphism coincides just with the symbol $\{f, g\}$.

Lemma 3.3.3. *Let h be an automorphism of V, such that $fh = hf, gh = hg, B \sim A$, where $B = hA$. Then*

$$\{f, g\}_A = \{f, g\}_B.$$

Proof. It is enough to construct an isomorphism of the diagrams $\mathcal{D}_A(f, g)$ and $\mathcal{D}_B(f, g)$. This isomorphism is given by $i_h \otimes i_h \otimes i_h^* \otimes i_h^*$.

Proposition 3.3.4. *Let f, g, h be automorphisms of V, s.t., $fg = gf, gh = hg, hf = fh$ and $A \sim fA \sim gA \sim hA$. Then*
$(i)\{f, g\} = \{g, f\}^{-1}$;
$(ii)\{fg, h\} = \{f, h\}\{g, h\}$.

Proof. (i) follows immediatly from the definition of the symbols.

(ii) Consider a diagram $\mathcal{D} = \mathcal{D}_A(f, h) \otimes \mathcal{D}_{fA}(g, h)$. I mean, that we take the tensor product of the objects and the morphisms on corresponding plases of the diagrams. It is enough to demonstrate that $\mathcal{D} \cong \mathcal{D}_A(fg, h)$(then Lemma 3.3.3 will give the needed assertion).

Let us point out here only the natural isomorphism of the upper left hand corners of the diagrams.

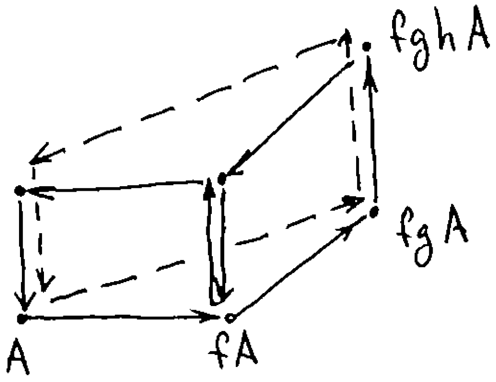
By a certain sequence of natural convolutions we obtain the following isomorphism:

$$(A|fgA)(fgA|fghA)(fghA|hA)(hA|A) \cong$$

$$(A|fA)\overbrace{(fA|fgA)(fgA|fghA)(fghA|fhA)(fhA|fA)}(fA|fhA)(fhA|hA)(hA|A).$$

The product on the left hand side gives the upper left hand corner of the diaram $\mathcal{D}_A(fg, h)$. The product on the right hand side being under the bracket gives the upper left hand corner of the diaram $\mathcal{D}_{fA}(g,h)$. The product of the four others symbols gives the upper left hand corner of the diaram $\mathcal{D}_A(f,h)$.

3.3.5 *Remark.* Maybe, the following drawing(like those in Remark 3.3.2) will be useful to understand the preciding proof.



The cycle $A \to fgA \to fghA \to hA \to A$ from the diagram $\mathcal{D}_A(fg,h)$ splits in the composition of two cycles $A \to fA \to fhA \to hA \to A$ and $fA \to fgA \to fghA \to fhA \to fA$ from the diagrams $\mathcal{D}_A(f,h)$ and $\mathcal{D}_{fA}(g,h)$, correspondingly.

3.4 Symbols $\mathrm{sign}(f,g)$ **and** $< f,g >$. Let f,g,A be as in 3.3. We set

$$\mathrm{sign}(f,g) = \omega^{[A|fA][A|gA]};$$
$$< f,g >= \mathrm{sign}(f,g)\cdot\{f,g\}.$$

If it is nesassary to indicate A explicitly, we write $\mathrm{sign}(f,g)_A$ and $< f,g >_A$.

Proposition 3.4.1. *Let* $f,g,h \in GL(V,A)$, $fg = gf, gh = hg, hf = fh$. *Then*

$(i)\, \mathrm{sign}(f,g) = \mathrm{sign}(g,f)^{-1}$;
$\mathrm{sign}(fg,h) = \mathrm{sign}(f,h)\,\mathrm{sign}(g,h)$;
$(ii)\, < f,g >=< g,f >^{-1}$;
$< fg,h >=< f,h >< g,h >$.

Proof. This follows immediatly from Lemma 3.2.2 and Proposition 3.3.4.

Remark 3.4.2. It easy to see that the symbols $\{*,*\}$, $\mathrm{sign}(*,*)$ and $< *,* >$ do not depend on V. This means that for a space W, s.t.,$V \subset W$, all the symbols constructed by the couple (V,A) coincide with those constructed by (W,A).

3.5 Inclusion - exclusion property.

Let A and B be two subgroups of V; $f,g \in GL(V,A)\cap GL(V,B)$, $fg = gf$.

Proposition 3.5.1.

$$< f,g >_A \cdot < f,g >_B = < f,g >_{A+B} \cdot < f,g >_{A\cap B}$$

Proof. The proposition is proving by a direct calculation, exactly in $[ACK]$. One should permanently use Lemma 2.1.1, Lemma 2.1.2 and take into account that the isomorphism $\det A \otimes \det B \cong \det(A \oplus B)$ from Lemma 2.1.2(ii) depends on the order of the couple (A,B)(see Remark 2.1.3).

4. Symbols of class field theory

4.1 Hilbert symbol. Let K be a finite extension of $\mathbb{Q}_p$, $v : K^* \to \mathbb{Z}$ the corresponding valuation, $\mathcal{O}$ is the ring of the integers in K, $\mathfrak{M}$ is the maximal ideal of $\mathcal{O}$, $\mathbb{F}_q$ is the residue field.

Let $\mu \subset K^*$; $(N,p) = 1$, where N is the order of μ. Let $d = \dim \mathbb{F}_q = (q-1)/N$.

Let us take $V = K, A = \mathcal{O}$. Let $f, g \in K^*$. The multiplications by f and g give operators on V. The condition $(N,p) = 1$ implies that $A \sim fA$. So we have defined the symbols $\{f,g\}, \operatorname{sign}(f,g)$ and (f,g).

Theorem 4.1.1.
(*i*) $[A|fA] = v(f) \cdot d, \quad \operatorname{sign}(f,g) = \omega^{v(f)v(g)d^2}$;
(*ii*) $\{f,g\} = (f^{v(g)}/g^{v(f)})^d \pmod{\mathfrak{M}}$;
(*iii*) $< f,g >$ *is the classical tame Hilbert symbol.*

Proof.
(*i*) is easy,(*iii*) follows from (*i*),(*ii*) and the stadart formula for the Hilbert symbol(for example see [Se]).

(*ii*) In view of the Proposition 3.4.3 it is enough to consider only the cases:
(a) f and g are units in $\mathcal{O}$;
(b) $f = \eta$ is an unit in $\mathcal{O}$, $g = \pi$ is an uniformizing element of $\mathcal{O}$.

The case (a) is obvious. To proof the case (b) consider the diagramm $\mathcal{D}(\eta,\pi)$ from 3.4 in our case.

$$\begin{array}{ccc} 1 \otimes \det \mathbb{F}_q \otimes (\det \mathbb{F}_q)^{-1} \otimes 1 & \xleftarrow[\alpha]{1\otimes i_\eta \otimes 1 \otimes i_\pi^*} & 1 \otimes \det \mathbb{F}_q \otimes (\det \mathbb{F}_q)^{-1} \otimes 1 \\ \psi \downarrow & & \downarrow \varphi \\ 1 & & 1 \end{array}$$

The operator i_π is equal to 1(see 3.2.1,(d)). Let $x_1, ..., x_d$ be a bases of $\mathbb{F}_q$, $\bar{\eta}$ be the residue of $\eta \pmod{I}$. Let us write the diagram in terms of the bases.

$$\begin{array}{ccc} 1 \otimes (\bar{\eta}x_1 \wedge ... \wedge \bar{\eta}x_d) \otimes (x_1 \wedge ... \wedge x_d)^{-1} \otimes 1 & \xleftarrow[\alpha]{1\otimes i_\eta \otimes 1 \otimes i_\pi^*} & 1 \otimes (x_1 \wedge ... \wedge x_d) \otimes (x_1 \wedge ... \wedge x_d)^{-1} \otimes 1 \\ \psi \downarrow & & \downarrow \varphi \\ \bar{\eta}^d & & 1 \end{array}$$

By the definition 3.3.1 we have: $\{\eta, \pi\} = \bar{\eta}^d$.

4.2 Weak reciprocity law. Let K be a number field, $\mathcal{O}$ be the ring of the integers, $\mu \subset K$.

Theorem 4.2.1. *Let $f,g \in K^*$. If f and g are prime to N(this means that $v_{\mathfrak{p}}(f) = v_{\mathfrak{p}}(g) = 0$ for each prime divisor $\mathfrak{p}|N$), then*

$$\prod_{\mathfrak{p}} < f,g >_{\mathfrak{p}} = < f,g >_{\mathcal{O}},$$

where $\mathfrak{p}$ runs over all the prime ideals of $\mathcal{O}$. Here $< f,g >_{\mathfrak{p}}$ denotes $< f,g >_{\mathcal{O}_{\mathfrak{p}}}^{K_{\mathfrak{p}}}$, where $K_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic completation of K and $\mathcal{O}_{\mathfrak{p}}$ is its ring of integers; the symbol $< f,g >_{\mathcal{O}}$ denotes $< f,g >_{\mathcal{O}}^{K}$.

Proof. Let V be the space of all the finite adels of K, $A = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$ be the space of all the integer adels. Let us to apply Proposition 3.5.1 to the couple A and $B = K$(the condition of the primity of f and g to N does all the symbols correctly defined). We have:

$$< f,g >_A^V \cdot < f,g >_K^V = < f,g >_{A\cap K}^V \cdot < f,g >_{A+K}^V.$$

Remark 3.4.2 shows that $< f,g >_K^V = < f,g >_K^K$, and the latest, obviously, is equal to 1. The same remark shows that $< f,g >_{A\cap K}^V = < f,g >_{\mathcal{O}}^K$, because $A \cap K = \mathcal{O}$. Besides, $V = A + K$(for example, see [L]), and , consequently, $< f,g >_{A+K}^V = 1$.

To complete the proof we should verify, that

$$< f,g >_A^V = \prod_{\mathfrak{p}} < f,g >_{\mathfrak{p}}.$$

Let S be the support of the divisors of f and g. Let us write

$$A = \prod_{\mathfrak{p}\in S} \mathcal{O}_{\mathfrak{p}} \times A',$$

where A' is the ring of all the integer adels without the S-component. Applying Proposition 3.5.1 several times, we see that:

$$< f,g >_A^V = \prod_{\mathfrak{p}\in S} < f,g >_{\mathcal{O}_{\mathfrak{p}}}^V \cdot < f,g >_{A'}^V.$$

By Remark 3.4.2 $< f,g >_{A'}^V = < f,g >_{A'}^{A'}$, and, this is equal to 1. By the same remark $< f,g >_{\mathcal{O}_{\mathfrak{p}}}^V = < f,g >_{\mathcal{O}_{\mathfrak{p}}}^{K_{\mathfrak{p}}}$.

Remark 4.2.2. (a) The symbols $< f,g >_{\mathfrak{p}}$ for $\mathfrak{p}|N$ are correctly defined in the product of Theorem 4.2.1, because the $\mathfrak{p}$-components of all the finite groups, involved into the constructions of the symbols and the proof of the theorem, are trivial(f and g are invertible $\pmod N$)). By the sme reason the symbol $< f,g >_{\mathfrak{p}} = 1$ for $\mathfrak{p}|N$ and does not coincide with the Hilbert symbol.

(b) An absense of a good theory at divisors of N and at archimedian points does not allow us to prove the classical reciprocity law by the same way as it is done in

[ACK] for the geometrical case. In fact, to prove the classical reciprocity law we should verify that

$$\prod_{\mathfrak{p}|N}(f,g)_{\mathfrak{p}} =< f,g >_{\mathcal{O}}^{-1},$$

where $(f,g)_{\mathfrak{p}}$ is the Hilbert symbol. For example, in the case $K = \mathbb{Q}$ we should verify that for two odd integer m and n

$$(-1)^{(m-1)/2\cdot(n-1)/2} =< m,n >_{\mathbf{Z}} .$$

For an arbitrary K similar formulas are much more complicated (see [FV]).

5. Categorical approach

Here we sketch in brief a more categorial approach to the symbols introduced in Sec. 3.

5.1 Category $\mathcal{L}$. We consider instead of the universum V from 3.1 and 3.2 a category $\mathcal{L}$, whose objects we name as line bundles on an one-dimensional spase X and whose morphisms we name as meromorphic morphisms . Let F be a constant "field". Namely, we deal with one of the following situations:

(a) Geometric case: here X is a connected riemannien surface, the words meromorphic morphisms should be understood directly, $F = \mathbb{C}$ is a usual field.

(b) Algebraic case: here X is a smooth unreducible algebraic curve over a usual field F, meromorphic morphisms are the rational those.

(c) Arithmetic case: here $X = \mathrm{Spec}(\mathcal{O})$, where $\mathcal{O}$ is a localisation of the ring of the integers in a number field ; F is the union of $\{0\}$ and the group μ of all the roots of unity in $\mathcal{O}$(in this case F is not a field). We assume that $N = \mathrm{Card}\,\mu$ is invertible in $\mathcal{O}$.

In all these cases we can consider instead of X also formal subschemas of X and non-connected spaces. It is useful, if we deal with coverings of X. Next constructions can be transfer to this case by an obvious way, and we will not discuss these cases here.

5.2 Divisors. In all the cases we have a theory of divisors. Namely, we have the notion of a divisor; for each morphism $f : L_1 \to L_2$ we have a divisor $\mathrm{div}\, f$; for each divisor D and each line bundle L we have the line bundle $L(D)$:

$$\Gamma(U, L(D)) = \{s : \mathcal{O}|_U \to L|_U \quad | \operatorname{div} s + D \geqslant 0\}$$

and the meromorphic morphism $1_{L,D} : L \to L(D), s \to s$.

5.3 Symbols Det **and** Ind. Let us consider a $\mathcal{L}$-morphism

$$f : L_1 \to L_2.$$

Let $\operatorname{div} f = D^+ - D^-$, where D^+ and D^- are effective divisors with disjoint supports. Then we have the following sequence:

$$L_1(D^+) \xrightarrow{g} L_1 \xrightarrow{f} L_2,$$

where $g = (1_{L_1,D^+})^{-1}$. It is easy to see that, in fact, the morphisms g and $f \circ g$ are holomorphic. So we can put

$$\operatorname{Det} f = \det \operatorname{coker}(g \circ f) \otimes (\det \operatorname{coker}(g))^*;$$

$$\operatorname{Ind} f = \dim \operatorname{coker}(g \circ f) - \dim \operatorname{coker}(g).$$

Let us link the symbols $\operatorname{Ind} f$ and $\operatorname{Det} f$ with the symbols of Sec.3.

We take the ring of adeles or the algebra of meromorphic functions on X as V. Let L_1 and L_2 be two line bundles with trivialisations in all the general point of X. In this situation there are two fractional ideal A and B in V, corresponding to L_1 and L_2. Also the quotion of the two trivialisations gives the meromorphic map $f : L_1 \to L_2$. We have:

$$[A|B] = \operatorname{Ind} f \quad (\text{in the case (c) only} \quad (\operatorname{mod} N));$$

$$(A|B) = \operatorname{Det} f,$$

where the left hand side symbols are defined in [ACK] for the cases (a) and (b) and in 3.1 above for the case (c).

5.4 Symbol $< f, g >$.

It is clear that each holomorphic isomorphism $f : L_1 \to L_2$ gives a certain isomorphism $i_f : \operatorname{Det} 1_{L_1,D} \to \operatorname{Det} 1_{L_2,D}$.

Let us fix two meromorphic functions f, g and a line bundle L. Let $D_1 = -\operatorname{div} f$, $D_2 = -\operatorname{div} g$. It is easy to see that the morphisms of the multiplicstion by f and g give the holomorphic isomorphisms:

$$f : L \to L(D_1) \quad \text{and} \quad g : L \to L(D_2).$$

Exactly as in 3.3 we consider the following diagram:

$$\begin{array}{ccc} l_1 \otimes l_2 \otimes l_3^{-1} \otimes l_4^{-1} & \xleftarrow[\alpha]{1 \otimes i_f \otimes 1 \otimes i_g^*} & l_1 \otimes l_3 \otimes l_3^{-1} \otimes l_1^{-1} \\ \downarrow \psi & & \varphi \downarrow \\ 1 & & 1 \end{array}$$

where $l_1 = \operatorname{Det} 1_{L,D_1}$; $l_2 = \operatorname{Det} 1_{L(D_1),D_2}$; $l_3 = \operatorname{Det} 1_{L,D_2}$; $l_4 = \operatorname{Det} 1_{L(D_2),D_1}$.

Set: $\{f, g\}_X^L = \psi \circ \alpha \circ \phi^{-1}$ and $< f, g >_X^L = \omega^{ab}\{f, g\}_X^L$, where $a = \operatorname{Ind} 1_{L,D_1}$; $b = \operatorname{Ind} 1_{L,D_2}$; $\omega = -1$ for the cases $(a), (b)$ and ω is the same as in Sec.3 for the case (c).

It is easy to see that the symbol does not depend on L and coincides with $< f, g >_{\Gamma(X,\mathcal{O})}$ from Sec.3.

5.5 Mayer-Vietoris property. Let U, V be two open subsets of X. The inclusion-exclusion property from 3.5 becomes

$$< f,g >_{U\cup V} = < f,g >_U \cdot < f,g >_V \cdot < f,g >_{U\cap V}^{-1}$$

.

The weak reciprocity law from 4.2 becomes

$$\prod_x < f,g >_x = < f,g >_X,$$

where x runs over all the points of X; $< f,g >_x = < f,g >_{U_x}$, U_x is the formal neighbourhood of x in X.

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