Some qualitative properties of equations of the type of slow, normal, and fast diffusion

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SOME QUALTITATIVE PROPERTIES OF EQUATIONS OF THE TYPE OF SLOW, NORMAL, AND FAST DIFFUSION

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1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 1$, $Q_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$, $\Gamma_T = S_T \cup [\bar{\Omega} \times (t=0)]$. Consider in Q_T equation

$$\mathcal{F}[u] \doteqdot \partial u/\partial t - div\, a(u, \nabla u) = 0 \tag{1.1}$$

where $a = (a^1, \dots, a^n)$, $\nabla u = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$ and functions $a^i(u,p)$, $i = 1, \dots, n$, are continuous in $\mathbb{R} \times \mathbb{R}^n$ and satisfy for all $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ inequalities

$$a(u,p) \cdot p \geq \nu_0 |u|^\ell |p|^m,\ \nu_0 > 0;\ |a(u,p)| \leq \mu_1 |u|^\ell |p|^{m-1},\ m > 1,\ \ell \geq 0. \tag{1.2}$$

Equations (1.1), (1.2) are special (in particular homogeneous) case of the, so-called, doubly nonlinear parabolic equations (DNPE). The prototype of these equations is

$$\partial u/\partial t - div[|u|^\ell | \nabla u|^{m-2} \nabla u] = 0,\ m > 1,\ \ell \geq 0. \tag{1.3}$$

In the case (1.3) we have $a(u,p) = |u|^\ell |p|^{m-2} p$ and for all $u \in \mathbb{R}$, $p \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$,

$$\frac{\partial a^i}{\partial p_j} \xi_i \xi_j \geq min\,(1, m-1) |u|^\ell |p|^{m-2} |\xi|^2. \tag{1.4}$$

From (1.4) it follows that equation (1.3) is parabolic at any point $(x,t) \in Q_T$ where u and ∇u do not equal zero. Equation (1.3) looks like an unify equation, but really it is an union of equations of three different types of PDE.

Definition 1.1. We say that equation (1.1), (1.2) is of the type of

$$\begin{aligned} \text{slow diffusion, } &\text{ if } \ m + \ell > 2, \\ \text{normal diffusion, } &\text{ if } \ m + \ell = 2, \\ \text{fast diffusion, } &\text{ if } \ m + \ell < 2. \end{aligned}$$

In this paper we study some qualitative properties of equations (1.1), (1.2). We show that equations of the type of slow, normal, and fast diffusion possess different properties.

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Definition 1.2. Any nonnegative bounded in Q_T function u is a weak solution of equation (1.1), (1.2) if

a) $u \in C([0,T]; L_2(\Omega))$, $\nabla u^{\sigma+1} \in L_m(Q_T)$, $\sigma = \frac{\ell}{m-1}$;

b) for any $\phi \in C^1(\bar{Q}_T)$, $\phi = 0$ on S_T, and any $t_1, t_2 \in [0,T]$

$$\int_\Omega u\phi\, dx|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_\Omega [-u\phi_t + a(u, u_x) \cdot \nabla\phi] dxdt = 0 \qquad (1.5)$$

where $u_x = (u_{x_1}, \ldots, u_{x_n})$ and u_{x_i}, $i = 1, \ldots, n$, are defined by

$$u_{x_i} = (1+\sigma)^{-1} u^{-\sigma} \partial u^{\sigma+1} / \partial x_i \quad \text{on} \quad [Q_T : u > 0], \; u_{x_i} = 0 \quad \text{on} \quad [Q_T : u = 0]. \qquad (1.6)$$

Consider Cauchy-Dirichlet problem

$$\mathcal{F}[u] = 0 \quad \text{in} \quad Q_T, \; u = \psi \quad \text{on} \quad \Gamma_T \qquad (\psi \geq 0, \; \psi \in W_1^1(Q_T)). \qquad (1.7)$$

Definition 1.3. Function u is a weak solution of Cauchy-Dirichlet problem (1.7) if u us a weak solution of equation (1.1) and $u = \psi$ on Γ_T.

Definition 1.4. We say that for equation (1.1), (1.2) there is a finite speed of propagation if any weak solution u of this equation possess the following property: if $u(x,t_0)$, $t_0 \in [0,T)$, has a compact support then the support of $u(x,t)$ is also compact for any $t \in (t_0, t_0 + \tau)$ with some $\tau \in (0, T - t_0]$.

Remark 1.1. In general τ depends on $supp\, u(x, t_0)$ and Ω.

Definition 1.5. We say that for equation (1.1), (1.2) there is a finite extinction time (or simple extinction) if there is $T_* \geq 0$ depending only on $n, m, \ell, \nu, \mu, |\Omega|$, and $\psi(x,0)$ such that any weak solution u of Cauchy-Dirichlet problem (1.7) with $\psi \in \overset{\circ}{W}{}_1^1(Q_T) \cap L_\infty(\Gamma_T)$ satisfies condition

$$u = 0 \quad a.e. \quad \text{in} \quad \Omega \quad \text{for any} \quad t \in [T_*, T]. \qquad (1.7)$$

This paper is dedicated to study some qualitative properties of equations (1.1), (1.2). The main results of the paper are propositions 5.1, 7.1 and 8.1 which are obtained as by-product of the proofs of Hölder estimates given in [1]-[3].

From these and other propositions we derive in particular that equations (1.1), (1.2) possess properties that can be reflected by the following table:

slow diffusion	normal diffusion	fast diffusion
finite speed of propagation	infinite speed of propagation	infinite speed of propagation
non-extinction	non-extinction	extinction

References to papers dedicated to investigation of properties mentioned in this table for the case of equation (1.3) can be found in [4].

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2. Existence of Hölder continuous weak solution

In this section we shall use also notion of strong solution of Cauchy-Dirichlet problem (1.6).

Definition 2.1. Let $inf(\psi, \Gamma_T) > 0$. We say that function u is a strong solution of Cauchy-Dirichlet problem (1.6) if u is a weak solution of this problem and moreover

$$inf(u, Q_T) > 0 \qquad \text{(and hence} \quad u \in W_m^{1,0}(Q_T)).$$

Consider Cauchy-Dirichlet problem with zero boundary condition

$$\mathcal{F}[u] = 0 \quad \text{in} \quad Q_T,\ u = 0 \quad \text{on} \quad S_T,\ u = u_0(x). \tag{2.1}$$

From results of paper [5] it follows in particular the following theorem.

Theorem 2.1. *Let the following conditions be fulfilled for equation (1.1):*

0) *functions* $u^{-\alpha}a^i(u, u^{-\alpha}p)$, $\alpha = \frac{\ell}{m}$, $m > 1$, $\ell \geq 0$, $i + 1, \dots, n$, *are continuous on* $\overline{\mathbb{R}_+} \times \mathbb{R}^n$;

1) *for any* $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ *inequalities (1.2) are satisfied;*

2) *there exists* $\nu_1 > 0$ *such that for any* $u \in \mathbb{R}$ *and* $p, q \in \mathbb{R}^n$

$$[a(u,p) - a(u,q)] \cdot (p - q) \geq \nu_1 |u|^\ell |p - q|^\kappa (|p|^m + |q|^m)^{1-\frac{\kappa}{m}},$$

where $\kappa = m$ *if* $m \geq 2$, $\kappa = 2$ *if* $m \in (1, 2)$;

3) *for any* $u, v \in [\epsilon, M]$, $\epsilon > 0$, $M > \epsilon$, *and any* $p \in \mathbb{R}^n$

$$|a(u,p) - a(v,p)| \leq \Lambda |u - v|(1 + |p|^{m-1}),\ \Lambda = \Lambda(\epsilon, M) \geq 0;$$

4) $\frac{\sigma+1}{\sigma+2} > \frac{1}{m} - \frac{1}{n}$, $\sigma = \frac{\ell}{m-1}$, $m > 1$, $\ell \geq 0$.

Assume also that set Ω *and initial function* u_0 *satisfy correspondently conditions*

$$(\Omega) \exists \rho_0 > 0\, \exists \alpha_0 \in (0,1)\, \forall x_0 \in \partial\Omega\, \forall \rho \in (0, \rho_0) : |B_\rho(x_0) \cap \Omega| \leq (1 - \alpha_0)|B_\rho(x_0)|$$

and

$$(I)\, u_0 = u_0(x) \geq 0, u_0 \in C_\beta(\bar{\Omega}),\ \beta \in (0,1).$$

Then Cauchy-Dirichlet problem (2.1) has a Hölder continuous in $\bar{Q}_T$ weak solution u; moreover the regularized problems

$$\mathcal{F}[u_\epsilon] = 0 \quad in \quad Q_T,\ u_\epsilon = \epsilon \quad on \quad S_T,\ u_\epsilon = u_0(x) + \epsilon,\ \epsilon \in (0,1) \tag{2.2}$$

have Hölder continuous in $\bar{Q}_T$ strong solutions u_ϵ such that

$$inf(u_\epsilon, Q_T) \geq \epsilon,\ u_\epsilon \to u \quad in \quad C_{\alpha,\alpha/m}(\bar{Q}_T) \quad as \quad \epsilon \to 0 \tag{2.3}$$

where $\alpha \in (0,1)$ is independent of ϵ.

Remark 2.1. Conditions 0)-3) are fulfilled for equation (1.3) for any $m > 1$, $\ell \geq 0$.

Remark 2.2. Condition 4) defining admissible parameters m and ℓ can be rewritten as

$$(m,\ell) \in D\backslash\omega,\ D \doteqdot \{m > 1, \ell \geq 0\},\ \omega \doteqdot \{(m,\ell) \in D : \frac{\sigma+1}{\sigma+2} \leq \frac{1}{m} - \frac{1}{n},\ \sigma = \frac{\ell}{m-1}\}. \tag{2.4}$$

This condition means that point (m,ℓ) does not belong to the "bad set ω". It is easy to see that equation (1.1), (1.2) with $(m,\ell) \in \omega$ is equation of the type of fast diffusion. We constructed a counterexample ([6]) showing that it is impossible to establish local L_∞ -estimates and hence local hölderness for generalized solutions of equation (1.3) with $(m,\ell) \in \omega$.

3. Finite speed of propagation for equations of the type of slow diffusion

Proposition 3.1. For equations (1.1), (1.2) of the type of slow diffusion there is a finite speed of propagation.

This property is well-known at least for equation (1.3) in the case $m + \ell > 2$ (see survey [4] by A.S. Kalashnikov). Therefore we limit ourselves by illustration of this phenomenon with the aid of a simple modification of the Barenblatt explicit solution (such modification was given in [1]). Consider function

$$u(x,t) = t^{-\alpha\beta}\left[1 - c\left(\frac{|x|}{t^\alpha}\right)^\gamma\right]_+^\delta \tag{3.1}$$

where

$$\alpha^{-1} = n(m+\ell-2) + m,\ \beta = n,\ \gamma = \frac{m}{m-1},\ \delta = \frac{m-1}{m+\ell-2}. \tag{3.2}$$

It is easy to see that for appropriate constant $c > 0$ function (3.1), (3.2) is a weak solution of equation (1.3) in $\Omega \times (\epsilon, T]$, $\Omega \subset \mathbb{R}^n$, $\epsilon > 0$, $T > \epsilon$ if $m > 1$, $m + \ell > 2$. This function has a compact support for any $t > 0$.

The finite speed of propagation is one of the main properties of equations of the type of slow diffusion.

4. Extinction for equations of the type of fast diffusion

Proposition 4.1. *For equation (1.1) (1.2) of the type of fast diffusion with parameters (m, ℓ) satisfying condition (2.4) there is a finite extinction time.*

Proof. Let $\eta \in \overset{\circ}{W}{}_m^{1,0}(Q_T) \cap L_2(Q_T)$, $\eta \geq 0$, $0 < h < t_1 < t_2 < T - h$. Then for any weak solution u of Cauchy-Dirichlet problem (2.1) we have (see also [7])

$$\int_{t_1}^{t_2} \int_\Omega [u_{\bar{h}t}\eta + (a(u, u_x))_{\bar{h}} \cdot \nabla\eta] dxdt = 0, \int_{t_1}^{t_2} \int_\Omega [u_{ht}\eta + (a(u, u_x))_h \cdot \nabla\eta] dxdt = 0 \tag{4.1}$$

where $g_{\bar{h}t} \doteqdot \frac{1}{h} \int_{t-h}^{t} g(x, \tau)d\tau$, $d_h \doteqdot \frac{1}{h} \int_t^{t+h} g(x, \tau)d\tau$. Denote $\hat{u} \doteqdot u^{\sigma+1}$, $\sigma = \frac{\ell}{m-1}$. In view of (1.2), (1.6) we have

$$|a(u, u_x)| \leq c| \nabla \hat{u}|^{m-1} \tag{4.2}$$

and hence in view of (4.2) and Definition 1.3 integrals in (4.1) have sense. Set in (4.1) $\eta = \hat{u}$ (obviously such choice of test function η is admissble). Because function $u \to u^{\sigma+2}$ is concave we have

$$u_{\bar{h}t}u^{\sigma+1} \geq \frac{(u^{\sigma+2})\hat{h}t}{\sigma+2}, \; u_{ht}u^{\sigma+1} \leq \frac{(u^{\sigma+2})ht}{\sigma+2}. \tag{4.3}$$

Then letting $h \to 0$ in (4.1) (with $\eta = \hat{u}$) we obtain

$$\frac{1}{\sigma+2} \int_\Omega u^{\sigma+2} dx|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega a(u, u_x) \cdot \nabla\hat{u} \, dxdt = 0 \tag{4.4}$$

where $a(u, u_x) \cdot \nabla\hat{u} \in L_1(Q_T)$ in view of (4.2). Denote

$$\| u \|_{p,\Omega} \doteqdot \| u \|_{L_p(\Omega)}, \; p \geq 1.$$

Then from (4.4) it follows obviously that function $t \to \| u \|_{\sigma+2,\Omega}^{\sigma+2}$ has a derivative $\frac{d}{dt} \| u \|_{\sigma+2,\Omega}^{\sigma+2}$ a.e. on $[0, T]$; moreover $\frac{d}{dt} \| u \|_{\sigma+2,\Omega}^{\sigma+2} \in L_1([0, T])$. In view of (1.6) we have $u_x = (\sigma + 1)^{-1}u^{-\sigma} \nabla \hat{u}$ and hence from (1.2) it follows that

$$a(u, u_x) \cdot \nabla\hat{u} \geq \hat{\nu}| \nabla \hat{u}|^m, \; \hat{\nu} \doteqdot \nu(\sigma + 1)^{1-m}. \tag{4.5}$$

From (4.4), (4.5) we can derive obviously that for a.e. $t \in [0, T]$

$$\frac{1}{\sigma+2} \frac{d}{dt} \| u \|_{\sigma+2,\Omega}^{\sigma+2} + \hat{\nu} \| \nabla u^{\sigma+1} \|_{m,\Omega}^m \leq 0. \tag{4.6}$$

Remark now that from condition (2.4) (or condition 4) of sect. 2) it follows in particular that $\overset{\circ}{W}{}_m^1(\Omega) \to L_{\frac{\sigma+2}{\sigma+1}}(\Omega)$ and hence

$$\| u^{\sigma+1} \|_{\frac{\sigma+2}{\sigma+1},\Omega} \leq \gamma_1 \| \nabla u^{\sigma+1} \|_{m,\Omega}, \; \gamma_1 = \gamma_1(|\Omega|, n, m)$$

or

$$\| u \|_{\sigma+2,\Omega}^{\sigma+1} \leq \gamma_1 \| \nabla u^{\sigma+1} \|_{m,\Omega} . \tag{4.7}$$

Then we derive fron (4.6), (4.7) that for a.e. $t \in [0,T]$

$$\frac{1}{\sigma+2}\frac{d}{dt} \| u \|_{\sigma+2,\Omega}^{\sigma+2} + \gamma \| u \|_{\sigma+2,\Omega}^{(\sigma+1)m} \leq 0 \tag{4.8}$$

where $\gamma = \hat{\nu}\gamma_1^{-m}$. In particular from (4.8) there follows that if $\| u \|_{\sigma+2,\Omega}= 0$ for some $t = t_0$ then $\| u \|_{\sigma+2,\Omega}= 0$ for any $t > t_0$. Denote $\tau = sup\{t \in \mathbb{R}_+ : \| u \|_{\sigma+2,\Omega}> 0\}$, assuming that $\| u_0 \|_{\sigma+2,\Omega}> 0$. Consider inequality (4.8) on $(0,\tau)$. Then we have

$$\frac{d}{dt} \| u \|_{\sigma+2,\Omega} + \gamma \| u \|_{\sigma+2,\Omega}^{(\sigma+1)(m-1)} \leq 0 \quad \text{on} \quad (0,\tau) \tag{4.9}$$

where $(\sigma+1)(m-1) = m+\ell-1$ and hence

$$\frac{1}{2-m-\ell}\frac{d}{dt} \| u \|_{\sigma+2,\Omega}^{2-m-\ell} \leq -\gamma \quad \text{on} \quad (0,\tau). \tag{4.10}$$

Integrating (4.10) over $(0,\tau)$ and using that $2-m-\ell > 0$ we obtain

$$0 \leq \frac{\| u \|_{\sigma+2,\Omega}}{2-m-\ell} \leq \frac{\| u_0 \|_{\sigma+2,\Omega}}{2-m-\ell} - \gamma\tau. \tag{4.11}$$

Obviously from (4.11) it follows that

$$\tau \leq T_* \doteqdot \frac{\| u_0 \|_{\sigma+2,\Omega}}{\gamma(2-m-\ell)}. \tag{4.12}$$

Proposition 4.1 is proved.

Remark 4.1. In the case $\ell = 0$ Proposition 4.1 is proved in [8]. Extinction for equation (1.3) in the case $m+\ell < 2$ is well-known (see about this in survey [4]).

5. Infinite speed of propagation for equation of the type of normal and fast diffusion

From the proof of lemmas 4.3 and 4.4 of paper [2] it is easy to derive the following result.

Lemma 5.1. *Let u be a weak solution of equation (1.1), (1.2) in Q_T of the type of fast or normal diffusion. Moreover assume that*

(i) $[a(u,p)-a(u,q)]\cdot(p-q) \geq 0$ *for any* $u \in \mathbb{R}$, $p,q \in \mathbb{R}^n$;
(j) $|a(u,p)-a(v,p)| \leq \Lambda|u-v|(1+|p|^{m-1})$ *for any* $u,v \in \mathbb{R}$, $p \in \mathbb{R}^n$;
(k) $u \in W_m^{1,0}(Q_T)$.

Let $\overline{B_\rho(x_0)} \times [t_0-\delta\rho^m, t_0+\delta\rho^m] \subset Q_T$ *and*

$$u(x,t) \geq u_0/4 \quad in \quad B_{\delta\rho}(x_0) \times [t_0-\delta\rho^m, t_0+\delta\rho^m] \tag{5.1}$$

for some $\delta \in (0,1)$ and $u_0 > 0$. Then for any $h_0 \in (0,1)$ there exists a number $\nu > 0$ depending on $u_0, \rho, \sigma, \delta$, and h_0, such that

$$u(x,t) \geq 1/2^{\nu} \quad in \quad B_\rho(x_0) \times [t_0 - (1-h_0)\delta\rho^m,\, t_0 + (1-h_0)\delta\rho^m].$$

In particular ν is independent of Λ and $\| \nabla u \|_{L_m(Q_T)}$ and

$$u(x,t_0) \geq 1/2^{\nu} \quad in \quad B_\rho(x_0). \tag{5.2}$$

Remark that from results of [2] it follows that function u from Lemma 5.1 is Hölder continuous in Q_T.

Proposition 5.1. *Let conditions 0)-4) and $(\Omega), (I)$ of Theorem 2.1 be fulfilled for equation (1.1) (1.2) of the type of fast or normal diffusion. Let u be a (Hölder continuous) weak solution of Cauchy-Dirichlet problem (2.1) Assume that for some $(x_0, t_0) \in Q_T$*

$$u(x_0,t_0) > 0. \tag{5.3}$$

Then for any ball $B_\rho(x_0)$ such that $\overline{B_\rho(x_0)} \subset \Omega$ we have

$$inf(u(x,t_0), B_\rho(x_0)) > 0. \tag{5.4}$$

Proof. From conditions $\overline{B_\rho(x_0)} \subset \Omega$ and $(x_0,t_0) \in Q_T$ it follows that $\overline{B_\rho(x_0)} \times [t_0 - \delta\rho^m,\, t_0 + \delta\rho^m] \subset Q_T$ and

$$u(x,t) \geq u_0/2 \quad \text{in} \quad B_{\delta\rho}(x_0) \times [t_0 - \delta\rho^m,\, t_0 + \delta\rho^m] \tag{5.5}$$

for some $\delta \in (0,1)$ and $u_0 = u(x_0,t_0) > 0$. In view of Theorem 2.1 there exists strong solutions u_ϵ of regularized problems (2.2) satisfying condition (2.3). Using Hölder equicontinuity of u_ϵ (see the second condition in (2.3)) we derive from (5.5) inequalities

$$u_\epsilon(x,t) \geq u_0/4 \quad \text{in} \quad B_{\delta\rho}(x_0) \times [t_0 - \delta\rho^m,\, t_0 + \delta\rho^m]. \tag{5.6}$$

Because assumption (i) follows from 2), assumption j) follows from 3) and the first condition in (2.3), while definition of strong solution implies assumption (k) we can apply Lemma 5.1 to solutions u_ϵ of regularized problems (2.3). Then in view of (5.2) we have

$$u_\epsilon(x,t_0) \geq 1/2^{\nu} \quad \text{in} \quad B_\rho(x_0) \tag{5.7}$$

where number $\nu > 0$ is independent of ϵ. Using again the second condition in (2.3) we obatin (5.4). Proposition 5.1 is proved.

Corollary 5.1. *For equations (1.1), (1.2) of the type of fast or normal diffusion satisfying conditions 0), 2)-4) (in particular for equation (1.3) with $m+\ell \leq 2$, $(m,\ell) \in D \setminus \omega$) there is an infinite speed of propagation.*

Proof. Let u be a Hölder continuous weak solution of Cauchy-Dirichlet problem (2.1) for equation (1.1), (1.2) in the case $m+\ell \leq 2$, $\Omega = B_R(0)$, $R > 0$, with non-negative Hölder continuous in Ω function u_0 having a compact support containing the origin (so that $u(0,0)=0$). In view of continuity of $u(x,t)$ we have $u(0,t)>0$ for all sufficient small $t>0$. Hence in view of Proposition 5.1

$$u(x,t)>0 \quad \text{in} \quad \Omega\times(0,\tau)$$

for some $\tau>0$. Corollary 5.1 is proved.

Corollary 5.2. *Let all conditions of Theorem 2.1 are fulfilled for equation (1.1), (1.2) of the type of fast diffusion in the case $\Omega = B_R(0)$ and $T \geq T_*$ where T_* is defined by formulae (4.12). Let u be a (Hölder continuous) weak solution of Cauchy-Dirichlet problem (2.1). Let $u_0(x)\not\equiv 0$ in Ω. Then there exists $\tau\in(0,T_*]$ such that*

$$u(x,t)>0 \quad in \quad \Omega\times(0,T),\ u(x,\tau)=0 \quad in \quad \Omega. \tag{5.8}$$

Proof. The result of Corollary 5.2 follows directly from Proposition 4.1 and Proposition 5.1.

6. Some auxiliary propositions

In this sections we state auxiliary propositions that will be used in the next solutions.

From the proofs of Theorem 4.3 of paper [1] and Lemma 2.1 of paper [2] we can derive that in the case of (homogeneous) equations of the type (1.1), (1.2) the following proposition holds.

Lemma 6.1. *Let u be a weak solution of equation (1.1), (1.2) with any $m>1$, $\ell\geq 0$ in Q_T and let $\hat u = u^{\sigma+1}$, $\sigma=\frac{\ell}{m-1}$. Then for any $Q=Q_{R;t_1,t_2}=B_R(x_0)\times[t_1,t_2]$, $\bar Q\subset Q_T$, we have :*
(B) If $sup((u-\kappa)^-,Q)\leq H^-$, $(u-\kappa)^-=(\kappa-u)^+=sup(\kappa-u,0), \kappa\in\mathbb{R}_+$, then function

$$g=g(H^-,(u-\kappa)^-,\gamma)\doteqdot \ell n_+[H^-/(H^- -(u-\kappa)^-+\gamma)] \tag{6.1}$$

satisfies inequality

$$\sup_{t\in[t_1,t_2]}\int_{B_R(x_0)} g^2\xi^2dx \leq \int_{B_R(x_0)} g^2\xi^2dx|^{t=t_1}+\mu\iint_{Q_{R;t_1,t_2}} u^\ell g|g'|^{2-m}|\nabla\xi|^m dxdt \tag{6.2}$$

where $\mu = const\geq 0$, $\xi=\xi(x)$ be a piecewise smooth function defined in the ball $B_R(x_0)$ such that $0\leq\xi\leq 1$ and $\xi=0$ on the boundary of $B_R(x_0)$.

(C) For any $\hat\kappa\in\mathbb{R}_+$ function $\hat u=u^{\sigma+1}$, $\sigma=\frac{\ell}{m-1}$, satisfies inequality

$$\sup_{t\in[t_1,t_2]}\int_{B_R(x_0)}\mathcal{F}^-((\hat u-\hat\kappa)^-)\zeta^m dx+\nu\iint_{Q_{R;t_1,t_2}}|\nabla(\hat u-\hat\kappa)^-|^m\zeta^m dxdt\leq$$

$$\leq\int_{B_R(x_0)}\mathcal{F}^-((\hat u-\hat\kappa)^-)\zeta^m dx|^{t=t_1}+\iint_{Q_{R;t_1,t_2}}[\mathcal{F}^-((\hat u-\hat\kappa)^-)(\zeta^m)_t+\mu|(\hat u-\hat\kappa)^-|^m|\nabla\zeta|^m]dxdt \tag{6.3}$$

where $\nu = const > 0$, $\mu = const \geq 0$, $\zeta = \zeta(x,t)$, *be a piecewise smooth function defined in the cylinder* $Q_{R;t_1,t_2}$ *such that* $0 \leq \zeta \leq 1$ *and* $\zeta = 0$ *on the lateral surface of* $Q_{R;t_1,t_2}$ and

$$\mathcal{F}^-((\hat{u}-\hat{\kappa})^-) = \frac{1}{\sigma+1}\int^{(\hat{u}-\hat{\kappa})^-} (\hat{\kappa}-\xi)^{\frac{1}{\sigma+1}-1}\xi\, d\xi,\ \sigma = \frac{\ell}{m-1}. \tag{6.4}$$

The following important lemma is a slight generalization of Lemma 6.2 of paper [1] (see also Lemma 2.2 in [2]).

Lemma 6.2. *Let* $w \in L_\infty((0,T); L_m(B_1)) \cap W_m^{1,0}(Q_1)$, $m > 1$, *where* $B_1 = B_1(0)$, $Q_1 = B_1 \times [-1,0]$. *Let* $\mu > 0$ *and* $\sup(w, Q_1) \leq \mu$. *Let* $\delta \in (0,1)$. *Assume that for all* $\kappa', \kappa \in [0,1]$, $\kappa' < \kappa$, *and all* $\xi = \xi(x)$, $\xi \in C_0^1(B_1(0))$, $0 \leq \xi \leq 1$, *and some* $\kappa > 0$

$$\sup_{t\in[-1,0]}\int_{B_1} |(w-\kappa)^+|^m \xi^m dx + \iint_{Q_1} |\nabla (w-\kappa)^+|^m \xi^m dxdt \leq$$
$$\leq c_0 \max_{B_1}(1 + |\nabla \xi|^m)[1 + (\frac{\mu}{\kappa-\kappa'})^\kappa]\iint_{Q_{1,\xi}} |(w-\kappa')^+|^m dxdt \tag{6.5}$$

where $Q_{1,\xi} \doteqdot \{Q_1 : \xi(x) > 0\}$. *Then there exists a constant* $\epsilon_0 > 0$ *depending only on* n, m, δ, c_0 *and* κ *such that from inequality*

$$\iint_{Q_1} |w^+|^m dxdt \leq \epsilon_0 |Q_1| \mu^m \tag{6.6}$$

it follows that

$$sup(w, B_{1/2}(0)) \times [-1,0]) \leq \delta\mu. \tag{6.7}$$

7. Non-extinction for equations of the type of slow and normal diffusion

We proved in section 5 that weak solutions of equation (1.1), (1.2) of the type of the fast diffusion satisfy conditions (5.8) with some $\tau \leq T_* \leq T$ (see (4.12)). In this section we show that such property characterizes equation of the type of fast diffusion because for equations of the type of slow or normal diffusion we have the following

Proposition 7.1. *Let* u *be a weak solution of equation (1.1), (1.2) in* Q_T *of the type of slow or normal diffusion. Assume that*

$$u(x,t) > 0 \quad in \quad \Omega \times (t_0 - \epsilon, t_0) \quad for\ some \quad t_0 \in (0,T] \quad and \quad \epsilon > 0. \tag{7.1}$$

Then

$$u(x,t_0) > 0 \quad in \quad \Omega \times [t = t_0]. \tag{7.2}$$

Remark 7.1. From results of [1] and [3] it follows that any weak solution of equation (1.1), (1.2) is Hölder continuous in Q_T.

Because in the case of equations of the type of slow or normal diffusion with $m \geq 2$, $\ell \geq 0$ we prove in the next section Proposition 8.1 from which Proposition 7.1 follows as a particular case, we shall assume in this section hat

$$m \in (1,2),\ \ell \geq 0,\ m + \ell \geq 2. \tag{7.3}$$

We shall say that some constant c depends only on the data if c depends on $n, m, \ell, \delta, \nu_0, \mu_1$, and $sup(u, Q_T)$.
Without loss of generality we can and shall assume in the remainder of this paper that

$$sup(u, Q_T) \leq 1. \tag{7.4}$$

In view of Remark 7.1 it is easy to see that Proposition 7.1 is a consequence of the following

Proposition 7.2. *Let u be a weak solution of equation (1.1), (1.2) in Q_T with parameters m, ℓ satisfying conditions (7.3). There exists a number $\hat{\nu} > 0$ depending only on the data such that if*

$$\overline{B_{\hat{\theta}_\rho}(x_0)} \times [t_0 - \rho^m, t_0] \subset Q_T,\ \hat{\theta} = 2^{\frac{2-m}{m}\hat{\nu}},\ \rho > 0, \tag{7.5}$$

and

$$u \geq 2^{-\delta} \quad on \quad B_{\hat{\theta}_\rho}(x_0) \times [t = t_0 - \rho^m] \tag{7.6}$$

for some $\delta > 0$ then

$$u \geq 2^{-(\delta+\nu+1)} \quad on \quad B_{\theta\rho/4}(x_0) \times [t_0 - \rho^m, t_0] \tag{7.7}$$

where $\nu = \hat{\nu}^{\frac{1}{\sigma+1}}$.

For establishing Proposition 7.2 we prove two lemmas.

Lemma 7.1. *Let u be a weak solution of equation (1.1), (1.2) in Q_T with parameters m, ℓ satisfying conditions (7.3) and let $\hat{u} = u^{\sigma+1}$, $\sigma = \frac{\ell}{m-1}$. Assume that for some $r > 0$, $\hat{\nu} \geq 0$, $c_1 > 0$, $\hat{\delta} \geq 0$*

$$Q(r) \doteqdot B_{\hat{\theta}_r}(x_0) \times [t_0 - c_1 r^m, t_0],\ \hat{\theta} = 2^{\frac{2-m}{m}\hat{\nu}}, \overline{Q(r)} \subset Q_T \tag{7.8}$$

and

$$\hat{u} \geq 2^{-(\hat{\delta}+\hat{\nu})} \quad on \quad B_{\hat{\theta}_r}(x_0) \times [t = t_0 - c_1 r^m]. \tag{7.9}$$

Then there exists a number $\hat{\alpha}_0 \in (0,1)$ depending only on the data and c_1 such that if

$$|\{Q(r) : \hat{u} < 2^{-(\hat{\delta}+\hat{\nu})}\}| \le \hat{\alpha}_0 |Q(r)| \tag{7.10}$$

then

$$\hat{u} \ge 2^{-(\hat{\delta}+\hat{\nu}+1)} \quad in \quad B_{\hat{\theta}r/2}(x_0) \times [t_0 - c_1 r^m, t_0]. \tag{7.11}$$

Proof. Consider inequality (6.3) for $\hat{\kappa} \in [2^{-(\hat{\delta}+\hat{\nu}+1)}, 2^{-(\hat{\delta}+\hat{\nu})}]$, $R = \hat{\theta} r$, $t_1 = t_0 - c_1 r^m$, $t_2 = t_0, \zeta = \xi(x) \in C_0^1(B_{\hat{\theta}r}(x_0))$. In view of (7.9) we have $(\hat{u} - \hat{\kappa})^- = 0$ on $B_{\hat{\theta}r}(x_0) \times [t = t_0 - c_1 r^m]$ and hence (see (6.4))

$$\mathcal{F}^-((\hat{u} - \hat{\kappa})^-) = 0 \quad \text{on} \quad B_{\hat{\theta}r}(x_0) \times [t = t_0 - c_1 r^m]. \tag{7.12}$$

Using that for $\hat{\alpha} \doteqdot \frac{\sigma}{\sigma+1}$ we have

$$\mathcal{F}^-((\hat{u} - \hat{\kappa})^-) \ge \frac{\kappa^{-\hat{\alpha}}}{2(\sigma+1)} |(\hat{u} - \hat{\kappa})^-|^2 \ge c\, 2^{(\hat{\delta}+\hat{\nu})\hat{\alpha}} |(\hat{u} - \hat{\kappa})^-|^2 \tag{7.13}$$

we derive from (6.3) that

$$2^{(\hat{\delta}+\hat{\nu})\hat{\alpha}} \sup_{t \in [t_1, t_2]} \int_{B_{\hat{\theta}r}(x_0)} |(\hat{u} - \hat{\kappa})^-|^2 \xi^m dx + \iint_{Q(r)} |\nabla (\hat{u} - \hat{\kappa})^-|^m \xi^m dx dt \le$$
$$\le \iint_{Q(r)} |(\hat{u} - \hat{\kappa})^-|^m |\nabla \xi|^m dx dt. \tag{7.14}$$

Let $\hat{\kappa}_1, \hat{\kappa} \in [2^{-(\hat{\delta}+\hat{\nu}+1)}, 2^{-(\hat{\delta}+\hat{\nu})}]$, $\hat{\kappa}_1 < \hat{\kappa}$. Obviously in view of (7.3)

$$\int_{B_{\hat{\theta}r}(x_0)} |(\hat{u} - \hat{\kappa})^-|^2 \xi^m dx \ge (\hat{\kappa} - \hat{\kappa}_1)^{2-m} \int_{B_{\hat{\theta}r}(x_0)} |(\hat{u} - \hat{\kappa}_1)^-|^m \xi^m dx. \tag{7.15}$$

Denote

$$v = 2^{\hat{\delta}+\hat{\nu}} \hat{u},\ \tilde{\kappa}_1 = 2^{\hat{\delta}+\hat{\nu}} \hat{\kappa}_1,\ \tilde{\kappa} = 2^{\hat{\delta}+\hat{\nu}} \hat{\kappa} \qquad (\tilde{\kappa}_1 < \tilde{\kappa}) \tag{7.16}$$

and introduce new variables

$$\tilde{x} = \frac{x - x_0}{\hat{\theta} r},\ \tilde{t} = \frac{t - t_0}{c_1 r^m} \qquad (\hat{\theta} = 2^{\frac{2-m}{m}\hat{\nu}}). \tag{7.16}$$

Change (7.16) transforms $B_{\hat{\theta}r}(x_0)$ and $Q(r)$ into $B_1(0)$ and $Q_1 = B_1(0) \times [-1, 0]$ correspondently. Obviously also that

$$\partial/\partial x_i = 2^{\frac{m-2}{m}\hat{\nu}} r^{-1} \partial/\partial \tilde{x}_i,\ dt = c_1 r^m\, d\tilde{t}. \tag{7.17}$$

Using (7.15)-(7.17) we can derive from (7.14) inequality

$$2^{\hat{\delta}(m+\hat{\alpha}-2)}2^{\hat{\alpha}\hat{\nu}}2^{(m-2)\hat{\nu}}(\tilde{\kappa}-\tilde{\kappa}_1)^{2-m}\sup_{\tilde{t}\in[-1,0]}\int\limits_{B_1(0)}|(v-\tilde{\kappa}_1)^-|^m\xi^m dx+$$
$$+2^{(m-2)\hat{\nu}}\iint\limits_{Q-1}|\tilde{\nabla}(v-\tilde{\kappa}_1)^-|^m\xi^m d\tilde{x}d\tilde{t}\leq$$
$$\leq c(c_1)2^{(m-2)\hat{\nu}}\iint\limits_{Q_1}|(v-\tilde{\kappa})^-|\tilde{\nabla}\xi|^m d\tilde{x}d\tilde{t}. \tag{7.18}$$

Taking into account (7.3) we have $m+\hat{\alpha}-2=\frac{m+\ell-2}{\sigma+1}\geq 0$, $\hat{\alpha}\geq 0$ and hence

$$2^{\hat{\delta}(m+\hat{\alpha}-2)}2^{\hat{\alpha}\hat{\nu}}\geq 1. \tag{7.19}$$

Denote

$$w=1-v,\ \kappa=1-\tilde{\kappa}_1,\ \kappa'=1-\tilde{\kappa}. \tag{7.20}$$

Obviously that from (7.16) and (7.20) we can derive that

$$\kappa',\kappa\in[0,1/2],\ \kappa'<\kappa,\ sup(w,Q_1)\leq 1 \tag{7.21}$$

and

$$(v-\tilde{\kappa})^-=(w-\kappa')^+,\ (v-\tilde{\kappa}_1)^-=(w-\kappa)^+,\ \tilde{\kappa}-\tilde{\kappa}_1=\kappa-\kappa'. \tag{7.22}$$

Then from (7.18)-(7.22) it follows that

$$\sup_{\tilde{t}\in[-1,0]}\int\limits_{B_1(0)}|(w-\kappa)^+|^m\xi^m d\tilde{x}+\iint\limits_{Q_1}|\tilde{\nabla}(w-\kappa)^+|^m\xi^m d\tilde{x}\,d\tilde{t}\leq$$
$$\leq c(c_1)\max_{B_1(0)}(1+|\tilde{\nabla}\xi|^m)(\frac{1}{\kappa-\kappa'})^{2-m}\iint\limits_{Q_{1,\xi}}|(w-\kappa')^+|^m d\tilde{x}\,d\tilde{t} \tag{7.23}$$

where $Q_{1,\xi}=\{Q_1:\xi(x)>0\}$. Using Lemma 6.2 in the case $\mu=1$, $\delta=1/2$, $\kappa=2-m$ we derive that there exists a constant $\epsilon_0>0$ depending only on the data and c_1 such that from inequality (6.6) it follows that inequality (6.7) holds. Using that conditions (6.6) and (7.10) coincide for $\hat{\alpha}_0=\epsilon_0$ and that inequalities (6.7) with $\delta=1/2$ and (7.11) are equivalent we can conclude that Lemma 7.1 is proved.

Remark 7.1. It is important that number $\hat{\alpha}_0$ from Lemma 7.1 is independent of $\hat{\nu}$ and $\hat{\delta}$.

Lemma 7.2. *Let u be a weak solution of equation (1.1), (1.2) in Q_T with parameters m, ℓ satisfying conditions (7.3) and let $\hat{u} = u^{\sigma+1}$, $\sigma = \frac{\ell}{m-1}$. Let $\hat{\alpha}_1 \in (0,1)$, $r > 0$, $c_1 > 0$, $\hat{\delta} \geq 0$ are fixed. There exists a number $\hat{\nu}$ depending only on the data, c_1, and $\hat{\alpha}_1$ such that if conditions (7.8) and*

$$\hat{u} \geq 2^{-\hat{\delta}} \quad on \quad B_{\hat{\theta}r}(x_0) \times [t = t_0 - c_1 r^m] \tag{7.24}$$

hold then for every $t \in [t_0 - c_1 r^m, t_0]$ we have

$$|\{B_{\hat{\theta}r/2}(x_0) : \hat{u} < 2^{-(\hat{\delta}+\hat{\nu})}\}| \leq \hat{\alpha}_1 |B_{\hat{\theta}r/2}(x_0)|. \tag{7.25}$$

Proof. Let conditions (7.8) and (7.24) hold for some $\hat{\nu}$ which will be fixed later. Denote $\delta = \hat{\delta}^{1/(\sigma+1)}$. From (7.24) it follows that

$$u > 2^{-\delta} \quad \text{on} \quad B_{\hat{\theta}r}(x_0) \times [t = t_0 - c_1 r^m]. \tag{7.26}$$

Consider inequality (6.2) in the case $\kappa = 2^{-\delta}$, $R = \hat{\theta}r$, $t_1 = t_0 - c_1 r^m$, $t_2 = t_0$, $\xi \in C_0^1(B_{\hat{\theta}r}(x_0))$, $0 \leq \xi \leq 1$, $\xi = 1$ in $B_{\hat{\theta}r/2}(x_0)$, $| \nabla \xi| \leq c_0 2^{\frac{m-2}{m}\hat{\nu}} r^{-1}$, $H^- \doteqdot \sup((u - \kappa)^-, Q(r)) = 2^{-\delta} - inf(u, Q(r))$, $\mu = 2^{-(\delta+\nu)}$, where $\nu > 2$ will be chosen below. Without loss of generality we can and shall assume that

$$H^- > 2^{-(\delta+1)} \tag{7.27}$$

because otherwise $H^- = 2^{-\delta} - inf(u, Q(r)) \leq 2^{-(\delta+1)}$ and hence $inf(u, Q(r)) \geq 2^{-(\delta+1)}$. But then $\inf(\hat{u}, Q(r)) \geq 2^{-(\delta+1)(\sigma+1)}$ and (7.25) are trivially fulfilled with $\hat{\nu} = \sigma + 1$.
From (7.26) it follows that

$$g(H^-, (u-\kappa)^-, \gamma) = 0 \quad \text{on} \quad B_{\hat{\theta}r}(x_0) \times [t = t_0 - c_1 r^m]. \tag{7.28}$$

Taking into account that $\mu = 2^{-(\delta+\nu)}$, $H^- \leq 2^{-\delta}$ we derive (see (6.1)) that

$$g(H, (u-\kappa)^-, \gamma) \leq \ell n(H^-/\gamma) \leq \nu\, \ell n\, 2, \tag{7.29}$$

$$|g'(H^-, (u-\kappa)^-, \gamma))|^{2-m} \leq 2^{(\delta+\nu)(2-m)} \chi(\{u < 2^{-\delta}\}). \tag{7.30}$$

Then from (6.2), (7.28)-(7.30) and estimate for $| \nabla \xi|$ it follows that for any $t \in [t_0 - c_1 r^m, t_0]$

$$\int\limits_{B_{\hat{\theta}r/2}(x_0)} g^2(H^-, (-\kappa)^-, \gamma) dx \leq c(c_1) \nu 2^{\delta(2-m-\ell)} 2^{\nu(2-m)} 2^{(m-2)\hat{\nu}} |B_{\hat{\theta}r/2}(x_0)|. \tag{7.31}$$

Taking into account that $m + \ell - 2 \geq 0$ and choosing

$$\hat{\nu} = \nu(\sigma + 1) \tag{7.32}$$

we obtain from (7.31), (7.32) that

$$\int\limits_{B_{\hat\theta r/2}(x_0)} g^2(H^-,(u-\kappa)^-,\gamma)dx \le c\nu|B_{\hat\theta r/2}(x_0)|,\ \forall t\in[t_0-c_1r^m,t_0], \tag{7.33}$$

where $c = c(c_1)$. Now we estimate the left-hand side of (7.33) from below. It is obvious that on the set $\{B_{\hat\theta r/2}(x_0) : u(x,t) < 2^{-(\delta+\nu)}\}$ we have

$$H^- - (u-\kappa)^- + \gamma \le 2^{-(\delta+\nu-1)}. \tag{7.34}$$

Then from (7.27) and (7.34) we derive that for any $t\in[t_0-c_1r^m,t_0]$

$$\int\limits_{B_{\hat\theta r/2}(x_0)} g^2(H^-,(u-\kappa)^-,\gamma)dx \ge (\nu-2)^2\ell n^2 2|\{B_{\hat\theta r/2}(x_0) : u < 2^{-(\delta+\nu)}\}|. \tag{7.35}$$

Taking into account that $\{B_{\hat\theta r/2}(x_0) : \hat u < 2^{-(\hat\delta+\hat\nu)}\} = \{B_{\hat\theta r/2}(x_0) : u < 2^{-(\delta+\nu)}\}$ we can derive from (7.33) and (7.35) that for any $t\in[t_0-c_1r^m,\ t_0]$

$$|\{B_{\hat\theta r/2}(x_0) : \hat u < 2^{-(\hat\delta+\hat\nu)}\}| \le c\frac{\nu}{(\nu-2)^2}|B_{\hat\theta r/2}(x_0)|. \tag{7.36}$$

Choose ν so large that $c\nu/(\nu-2)^2 \le \hat\alpha_1$. Then (7.25) follows from (7.36). Lemma 7.2 is proved.

Proof of Proposition 7.2. Let $\hat\alpha\in(0,1)$ be defined by Lemma 7.1 corresponding to the case $c_1 = 2^m$. Let $\hat\nu > 0$ be defined by Lemma 7.2 corresponding to the case $c_1 = 1$, $\hat\alpha_1 = \hat\alpha_0$ with such $\hat\alpha_0$. Applying Lemma 7.2 in the case $c_1, \hat\alpha_1$ chosen and for $r = \rho$ we obtain for any $t\in[t_0-\rho^m,t_0]$

$$|\{B_{\hat\theta\rho/2}(x_0) : \hat u < 2^{-(\hat\delta+\hat\nu)}\}| \le \hat\alpha_0|B_{\hat\theta\rho/2}(x_0)| \tag{7.37}$$

because (7.24)with $r=\rho$, $c_1 = 1$, $\hat\delta = \delta(\sigma+1)$ follows from (7.6). But from (7.37) it follows that condition (7.10) with $r = \rho/2$, $c_1 = 2^m$ is fulfilled. Then using Lemma 7.1 in the case $r=\rho/2$, $c_1 = 2^m$, $\hat\delta = \delta(\sigma+1)$ and $\hat\nu$ chosen we obtain (7.11) (with $r = \rho/2$, $c_1 = 2^m$) and hence inequality (7.7) is established. Proposition 7.2 is proved.

8. Remaining of positivity for weak solutions of equations (1.1), (1.2) with $m \ge 2, \ell \ge 0$

In this section we assume that

$$m \ge 2,\ \ell \ge 0. \tag{8.1}$$

Proposition 8.1. *Let u be a weak solution of equation (1.1), (1,2) in Q_T with parameters m, ℓ satisfying condition (8.1). Assume that*

$$u(x_0, t_0) > 0 \quad \text{for some} \quad (x_0, t_0) \in \Omega \times [0, T). \tag{8.2}$$

Then

$$u(x_0, t) > 0 \quad \text{for any} \quad t \in (t_0, T]. \tag{8.3}$$

Remark 8.1. In view of results of [1] it follows that any weak solution of equation (1.1), (1.2) with $m \geq 2$, $\ell \geq 0$ is Hölder continuous in Q_T. Obviously that Proposition 7.1 follows from Proposition 8.1.

We prove Proposition 8.1 as a consequence of the forthcoming propositions 8.2 and 8.3.

Proposition 8.2. *Let u be a weak solution of equation (1.1), (1.2) in Q_T with parameters m, ℓ satisfying conditions (8.1). Assume that*

$$\overline{B_\rho(x_0)} \times [t_0 - \rho^m, t_0] \subset Q_T, \ \rho > 0, \tag{8.4}$$

and

$$u \geq 2^{-s} \quad \text{on} \quad B_\rho(x_0) \times [t = t_0 - \rho^m] \tag{8.5}$$

for some $s > 0$. Then there exists a number $\nu > 0$ depending only on the data such that

$$u \geq 2^{-(s+\nu+1)} \quad \text{on} \quad B_{\rho/4}(x_0) \times [t_0 - \rho^m, t_0]. \tag{8.6}$$

For establishing Proposition 8.2 we prove two lemmas which are similar to lemmas 7.1 and 7.2.

Lemma 8.1. *Let u be a weak solution of equation (1.1), (1.2), (8.1) in Q_T and let $\hat{u} = u^{\sigma+1}$, $\sigma = \frac{\ell}{m-1}$. Assume that for some $r > 0$, $c_1 > 0$, $\hat{\delta} \geq 0$*

$$Q(r) \doteqdot B_r(x_0) \times [t_0 - c_1 r^m, t_0], \ \overline{Q(r)} \subset Q_T \tag{8.7}$$

and

$$\hat{u} \geq 2^{-\hat{\delta}} \quad \text{on} \quad B_r(x_0) \times [t = t_0 - c_1 r^m]. \tag{8.8}$$

Then there exists a number $\hat{\alpha}_0 \in (0,1)$ depending only on the data and c_1 such that if

$$|\{Q(r) : \hat{u} < 2^{-\hat{\delta}}\}| \leq \hat{\alpha}_0 |Q(r)| \tag{8.9}$$

then

$$\hat{u} \geq 2^{-(\hat{\delta}+1)} \quad \text{in} \quad B_{r/2}(x_0) \times [t_0 - c_1 r^m, t_0]. \tag{8.10}$$

Proof. The proof of Lemma 8.1 is similar to the one of Lemma 7.1 in the case $\hat{\nu} = 0$. In particular we have inequality (7.14) with $\hat{\nu} = 0$. Instead of (7.15) we estimate

$$\int\limits_{B_r(x_0)} |(\hat{u} - \hat{\kappa})^-|^2 \xi^m dx \geq 2^{\hat{\delta}(m-2)} \int\limits_{B_r(x_0)} |(\hat{u} - \hat{\kappa})^-|^m \xi^m dx \tag{8.11}$$

and introduce the new variables (7.16) with $\hat{\theta} = 1$. Then instead of (7.18) we obtain

$$2^{\hat{\delta}(m+\hat{\alpha}-2)} \sup_{\tilde{t}\in[-1,0]} \int\limits_{B_1(0)} |(v - \tilde{\kappa})^-|^m \xi^m d\tilde{x} + \iint\limits_{Q_1} |\tilde{\nabla}(v - \tilde{\kappa})^-|^m \xi^m d\tilde{x} d\tilde{t} \leq$$
$$\leq c(c_1) \iint\limits_{Q_1} |(v - \tilde{\kappa})^-|^m |\tilde{\nabla}\xi|^m d\tilde{x} d\tilde{t} \tag{8.12}$$

where $v = 2^{\hat{\delta}}\hat{u}$, $\tilde{\kappa} = 2^{\delta}\kappa$. Denote $w = 1 - v$, $\kappa = 1 - \tilde{\kappa}$. Then $\kappa \in [0, 1/2]$, $\sup(w, Q_1) \leq 1$ and $(v - \tilde{\kappa})^- = (w - \kappa)^+$. Moreover using that $m + \hat{\alpha} - 2 \geq 0$ we derive from (8.12) that

$$\sup_{\tilde{t}\in[-1,0]} \int\limits_{B_1(0)} |(w - \kappa)^+|^m \xi^m d\tilde{x} + \iint\limits_{Q_1} |\tilde{\nabla}(w - \kappa)^+|^m d\tilde{x} d\tilde{t} \leq$$
$$\leq c(c_1) \max_{B_1(0)} (1 + |\tilde{\nabla}\xi|^m) \iint\limits_{Q_{1,\xi}} |(w - \kappa)^+|^m d\tilde{x} d\tilde{t}. \tag{8.13}$$

where $Q_{1,\xi} = \{Q_1 : \xi(x) > 0\}$. Obviously we can apply Lemma 6.2 in the case $\mu = 1$, $\delta = 1/2$, $\kappa = 0$. The remainder of the proof is the same as in the case of Lemma 7.1. Lemma 8.1 is proved.

Remark 8.2. It is important that number $\hat{\alpha}_0$ from Lemma 8.1 is independent of $\hat{\delta}$.

Lemma 8.2. *Let u be a weak solution of equation (1.1), (1.2), (8.1) in Q_T and let $\hat{u} = u^{\sigma+1}$, $\sigma = \frac{\ell}{m-1}$. Let $\hat{\alpha}_1 \in (0, 1)$, $r > 0$, $c_1 > 0$, $\hat{s} \geq 0$ are fixed. There exists a number $\hat{\nu}$ depending only on the data, c_1, and $\hat{\alpha}_1$ such that if conditions (8.7) and*

$$\hat{u} \geq 2^{-\hat{s}} \quad on \quad B_r(x_0) \times [t = t_0 - c_1 r^m] \tag{8.14}$$

hold then for every $t \in [t_0 - c_1 r^m, t_0]$ we have

$$|\{B_{r/2}(x_0) : \hat{u} > 2^{-(\hat{s}+\hat{\nu})}\}| \leq \hat{\alpha}_1 |B_{r/2}(x_0)|. \tag{8.15}$$

Proof. The proof of Lemma 8.2 is similar to the one of Lemma 7.2 in the case $\hat{\theta} = 1$, $\hat{\delta} = \hat{s}$. But because now $|\tilde{\nabla}\xi| \leq c_0 r^{-1}$ we have instead of (7.31) inequality

$$\int\limits_{B_{r/2}(x_0)} g^2(H^-, (u - \kappa)^-, \gamma) dx \leq c(c_1)\nu 2^{(2-m-\ell)} 2^{\nu(2-m)} |B_{r/2}(x_0)|. \tag{8.16}$$

Using that $m+\ell-2 \geq 0$, $m \geq 2$ we derive from (8.16) that (7.33) with $\hat{\theta}=1$ holds. The remainder of the proof is the same as in the case of Lemma 7.2. Lemma 8.2 is proved.

Proof of Proposition 8.2. Let $\hat{\alpha}_0 \in (0,1)$ is defined by Lemma 8.1 corresponding to the case $c_1 = 2^m$. Let $\hat{\nu} > 0$ is defined by Lemma 8.2 corresponding to the case $c_1 = 1$, $\hat{\alpha}_1 = \hat{\alpha}_0$ with $\hat{\alpha}_0$ chosen. Applying Lemma 8.2 in the case c_1, $\hat{\alpha}_1$ chosen and for $r=\rho$, $\hat{s} = s(\sigma+1)$ we obtain for any $t \in [t_0 - \rho^m, t_0]$

$$|\{B_{\rho/2}(x_0) : \hat{u} < 2^{-(\hat{s}+\hat{\nu})}\}| \leq \hat{\alpha}_0 |B_{\rho/2}(x_0)|. \tag{8.17}$$

Obviously from (8.17) it follows that condition (8.8) with $r=\rho/2$, $c_1 = 2^m$ and $\hat{\delta} = \hat{s}+\hat{\nu}$ is fulfilled. Then using Lemma 8.1 in the case $r=\rho/2$, $c_1=2^m$, $\hat{\delta} = \hat{s}+\hat{\nu}$, $\hat{s}=s(\sigma+1)$ and $\hat{\nu}$ chosen we obtain (8.9) (with $r=\rho/2$, $c_1=2^m$, $\hat{\delta}=\hat{s}+\hat{\nu}$) and hence inequality (8.6) is established. Proposition 8.2 is proved.

Proposition 8.3. *Let u be a weak solution of equation (1.1), (1.2) in Q_T with parameters m, ℓ satisfying conditions (8.1). Assume that conditions (8.4) and (8.5) are fulfilled with some $s>0$. Let $\beta \in (0,1)$ be fixed. Then there exists a number $\nu > 0$ depending only on the data and $\beta \in (0,1)$ such that*

$$u \geq 2^{-(s+\nu)} \quad on \quad B_{\beta\rho}(x_0) \times [t_0 - \rho^m, t_0]. \tag{8.18}$$

Proof. Proposition 8.3 can be proved absolutely in the same way as in the case $\beta = 1/4$ (see the proof of Proposition 8.2).

Proof of Proposition 8.1. Without loss of generality we can and shall assume that $x_0 = 0$. Assume that

$$u(0,t) > 0 \quad \text{for some} \quad t_0 \in [0,T). \tag{8.19}$$

In view of continuity of function u (see Remark 8.1) we can assume that

$$u(x,t_0) > 2^{-\delta_0} \quad \text{on} \quad B_\rho \times [t=t_0] \quad (B_\rho \doteqdot B_\rho(0))$$

for some $\rho > 0$ and $\delta_0 > 0$. Using Proposition 8.2 with some $\beta_1 \in (0,1)$ we obtain

$$u(x,t) > 2^{-\delta_1} \quad \text{on} \quad B_{\beta_1\rho} \times [t_0,t_1],\ t_1 = t_0 + \rho^m \tag{8.20}$$

for some $\delta_1 > 0$. In particular

$$u(x,t_1) > 2^{-\delta_1} \quad \text{on} \quad B_{\beta_1\rho} \times [t=t_1].$$

Repeating this argumentation we obtain for some $\beta_2, \dots, \beta_\kappa$ and $\delta_2, \dots, \delta_\kappa$

$$u(x,t) > 2^{-\delta_2} \quad \text{on} \quad B_{\beta_1\beta_2\rho} \times [t_1,t_2],\ t_2 = t_1 + (\beta_1\rho)^m, \tag{8.21}$$

....

$$u(x,t) > 2^{-\delta_\kappa} \quad \text{on} \quad B_{\beta_1,\beta_2\dots\beta_\kappa\rho} \times [t_{\kappa-1},t_\kappa], t_\kappa = t_{\kappa-1} + (\beta_1 \dots \beta_{\kappa-1}\rho)^m. \tag{8.22}$$

We can choose sequence $\{\beta_i\}$ so that

$$(\beta_1 \dots \beta_{\kappa-1})^m = \frac{1}{\kappa}, \; \kappa = 2, 3, \dots$$

Without loss of generality we can count that $\{\delta_i\}$ is increasing. Then from (8.20) -(8.22) it follows that

$$u(x,t) > 2^{-\delta_\kappa} \quad \text{on} \quad B_{\frac{\rho}{(\kappa+1)^m}} \times [t_0, t_0 + (1 + \dots \frac{1}{\kappa})\rho^m].$$

In particular

$$u(0,t) > 2^{-\delta_\kappa} \quad \text{for} \quad t \in [t_0, t_0 + (1 + \dots \frac{1}{\kappa})\rho^m]. \tag{8.23}$$

Obviously that result of Proposition 8.1, i.e., inequality (8.3) follows from (8.23). Proposition 8.1 is proved.

Notes added in proof.

1) In view of the weak maximum principle for homogeneous equations (1.1), (1.2) Theorem 2.1 remains to be true if instead of condition 4) we assume only that $m > 1$, $\ell \geq 0$.
2) Uniqueness of solution of Cauchy-Dirichlet problem (2.1) established by Theorem 2.1 can be derived from paper [9].

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