Combinatorics of trivalent ribbon graphs with two faces.

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In this article we discuss combinatorics of trivalent ribbon graphs (*dessigne d'enfafnts*) with two faces. All the necessary definitions may be found in [2].

Consider the set $\mathcal{D}ess_{g,2}$ of trivalent genus g ribbon graphs D having the set of edges V, the set of vertices E, and the set of faces $F = \{X, Y\}$. Such graphs have $4g$ vertices and $6g$ edges. Consider also the set $F_{1,2}$ of $(1,2)$-flags, i.e. the set of pairs (e, C), where $e \in E$, $C \in F$, and the face C is incident to the edge e.

There are natural projections $F_{1,2}$ onto E and F:

$$\pi_1 \colon F_{1,2} \to E \qquad\qquad \pi_2 \colon F_{1,2} \to F \tag{1}$$

The set of edges E of a graph may be split into two parts

$$E_1 = \{e \in E, \qquad \text{such that} \qquad |\pi_1^{-1}(e)| = 1\} \tag{2}$$

— the set of edges incident to only one face, and

$$E_2 = \{e \in E, \qquad \text{such that} \qquad |\pi_1^{-1}(e)| = 2\} \tag{3}$$

— the set of edges separating one face from another. We shall call the edges from E_1 as *internal* edges, and the edges from E_2 — *separating* edges (See fig. 2). The set of internal edges in its turn splits into two parts

$$E_X = \{e \in E_1, \quad \pi_1^{-1}(e) = X\} \qquad \text{and} \qquad E_Y = \{e \in E_1, \quad \pi_1^{-1}(e) = Y\}, \tag{4}$$

denote

$$k = |E_2|, \qquad l_X = |E_X|, \qquad l_Y = |E_Y|, \tag{5}$$

so that $k + l_X + l_Y = 6g$. Denote

$$l = \min(l_X, l_Y), \qquad q = |l_X - l_Y|, \qquad 2M = 6g - 2l, \tag{6}$$

then evidently

$$q = 2M - k. \tag{7}$$

Let us describe the set U_g of pairs of integers (l, k) corresponding to ribbon graphs from $\mathcal{D}ess_{g,2}$.

Proposition 1 *The set U_g of ribbon graphs is defined on the plane (l, k), $k \geq 0$, $l \geq 0$ by the following inequalities:*

(1) $k \leq 4g$;

(2) $k \leq 6g - 2l$;

(3) $k \geq \kappa(l)$;

where $\kappa(l)$ is an integer function, periodic for $l > 0$ with period equal to 6, defined by the values:

$$\kappa(0) = 1;\ \kappa(1) = 4;\ \kappa(l) = 7 - l,\ l = 2, 3, 4, 5;\ \kappa(6) = 5. \tag{8}$$

Thus the domain corresponding to the ribbon graphs looks like a trapezium bounded from top by the line $k = 4g$, from left and right sides by the lines $l = 0$ and $k = 6g - 2l$, and from the bottom by the polygonal line $k = \kappa(l)$ (See fig. 1).

Let us proof the proposition 1. It is clear that in each vertex meets even number of separating edges, namely zero or two. Therefore the separating edges form a cycle (or several cycles). Hence the number of internal edges can not exceed the number of vertices which equals $4g$. This proves the necessity of the first inequality from the proposition 1. Necessity of the second inequality is also evident for in total there are k separating edges and at most $2l$ internal edges.

Next let us describe ribbon graphs corresponding to the lower border of the domain U_g i.e. to the points of the graph of the function κ. These ribbon graphs will be constructed from ribbon graphs with *half-edges*. Half-edge is an edge having one end incident to a vertex of the graph while the other

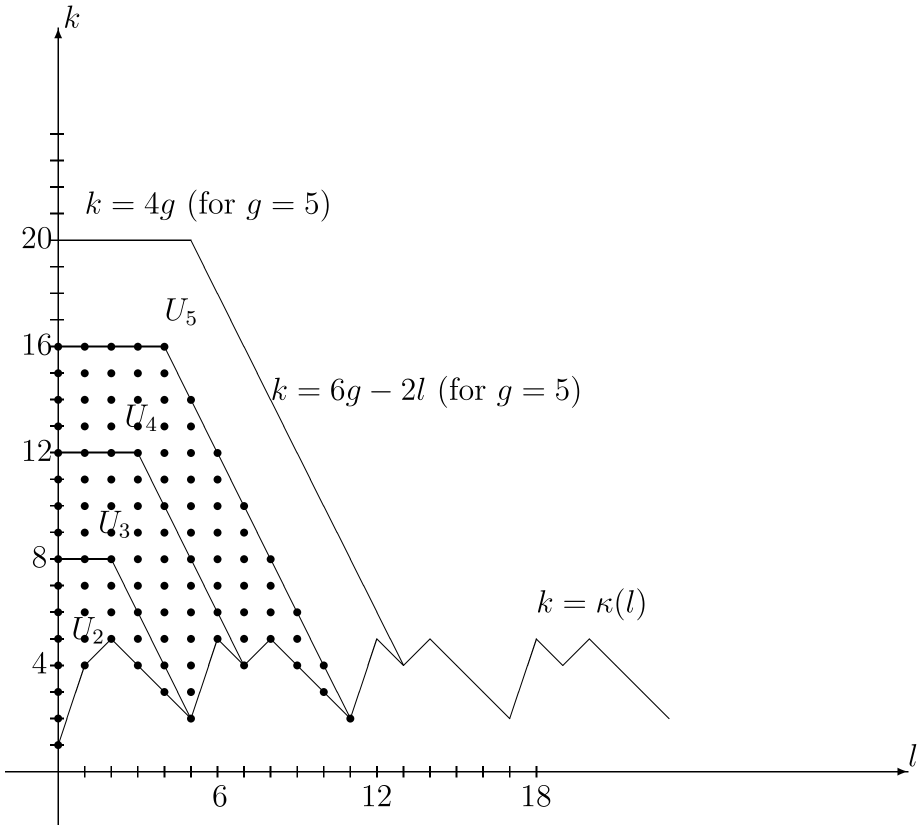


Figure 1: Domains U_g for $g = 2, 3, 4, 5$. On the picture we mark all the integral points of the domain U_4.

is free for further clutching together with another free end of some other half-edge (of the same or another ribbon graph with half-edges).[1].

[1]For completeness let us give a formal definition. A ribbon graph with half-edges Γ is defined by the following data $(V, \overrightarrow{E}, i, \overrightarrow{E}_-, s, \sigma)$, where:

(1) V — the set of *vertices* of Γ;

(2) $\overrightarrow{E}$ — the set of *oriented edges* of Γ;

(3) $i\colon \overrightarrow{E} \to \overrightarrow{E}$ — *the orientation change involution* (which is supposed to be fixed-point free);

(4) $\overrightarrow{E}_-$ — the set of *outgoing oriented edges* of Γ; it is supposed that $\overrightarrow{E} = \overrightarrow{E}_- \cup i(\overrightarrow{E}_-)$, (i. e. each edge is outgoing for at least one of its two possible orientations);

First let us consider two one-face trivalent ribbon graphs without half-edges D_1 and D_2 having respectively genus $h_i > 0$, $4h_i - 2$ vertices and $6h_i - 3$ edges, $i = 1, 2$. Let us also consider two-valent ribbon graph B_m with m vertices and m edges, (which is simply a circle with m vertices on it). Now let us transform all the vertices of B_m to trivalent by adding to each vertex a half-edge, so that m_1 half-edge would be directed to one face of B_m and $m_2 = m - m_1$ half-edge would be directed to the other face of B_m. Let us denote the constructed ribbon graph with half-edges by B_{m_1,m_2}. We shall perform the following operations on ribbon graphs D_1 and D_2:

(1) $p_1(D_i)$ — choose an arbitrary edge of D_i and insert a three-valent vertex with one half-edge into the middle of this edge;

(2) $p_2(D_i)$ — choose an arbitrary edge of D_i and cut it into two half-edges;

(3) $p_3(D_i)$ — remove an arbitrary vertex of D_i so that the three incident to this vertex edges become half-edges.

(4) $p_4(D_i)$ — choose an arbitrary edge of D_i and remove it so that the four adjacent edges become half-edges.

We shall clutch the half-edges of the ribbon graph D_1 with the half edges of the ribbon graph B_{m_1,m_2} directed to one of its faces and the half-edges of the ribbon graph D_1 with the half edges of the ribbon graph B_{m_1,m_2} directed to its other face; the result we shall denote by the sign "+".

Now we can describe the ribbon graphs from the lower border of the domain U_g. The points $(l, \kappa(l))$ corresponding to the following ribbon graphs ($0 < l \leq l_{max}$, $h_1 = \left[\frac{l}{6}\right] + 1$):

(1) $p_1(p_1(D_1)) + (B_{1,1} \sqcup B_{1,1}) + p_1(p_1(D_2))$ corresponding to the value $l = 6h_1+1$, $\kappa(l) = 4$ ($h_2 = g - h_1 - 1$; for $l = 1$ instead of adding D_1 we simply clutch together the pair of half-edges of the two ribbon graphs $B_{1,1}$);

(2) $p_4(D_1) + B_{4,1} + p_1(D_2)$ corresponding to the value $l = 6h_1 - 4$, $\kappa(l) = 5$ ($h_2 = g - h_1$);

(5) $s\colon \overrightarrow{E}_- \to V$ — a surjective mapping attaching to every outgoing edge its source;

(6) σ — a cyclic order on the set $s^{-1}(v)$ of oriented edges directed from each vertex v.

(3) $p_3(D_1) + B_{3,1} + p_1(D_2)$ corresponding to the value $l = 6h_1 - 3$, $\kappa(l) = 4$ $(h_2 = g - h_1)$;

(4) $p_2(D_1) + B_{2,1} + p_1(D_2)$ corresponding to the value $l = 6h_1 - 2$, $\kappa(l) = 3$ $(h_2 = g - h_1)$;

(5) $p_1(D_1) + B_{1,1} + p_1(D_2)$ corresponding to the value $l = 6h_1 - 1$, $\kappa(l) = 2$ $(h_2 = g - h_1)$;

(6) $p_2(p_1(D_1)) + (B_{2,1} \sqcup B_{1,1}) + p_1(p_1(D_2))$ corresponding to the value $l = 6h_1$, $\kappa(l) = 5$ $(h_2 = g - h_1 - 1)$.

The value $l = 0$ is special: for $\kappa(0) = 1$ we take $B_{0,1} + p_1(D_2)$ with $h_2 = g$.

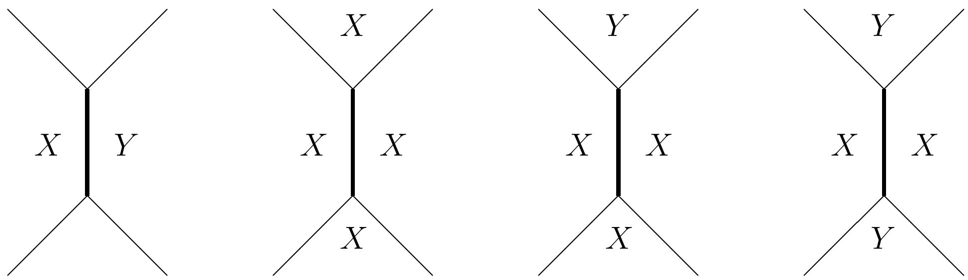
Necessity of the condition $k \geq \kappa(l)$ is proved by ordinary enumeration of possibilities. Cutting the corresponding Riemann surface along all the separating edges provides two Riemann surfaces of genus $h_1 > 0$ and $h_2 > 0$ respectively each having $s > 0$ holes, so that $h_1 + h_2 + s - 1 = g$. But since $k \geq 2s$ and $\kappa(l) \leq 5$ it is sufficient to study the cases $s = 1$ and $s = 2$. It is easy to see that all such cases are contained in our list (1)-(6).

It is left to verify that all the other points of the domain U_g also correspond to some ribbon graphs. We shall construct these ribbon graphs applying *flips* to the already constructed ribbon graphs. Recall that flip at the edge e of the trivalent ribbon graph D is the following operation. We contract the edge e and then insert a new edge e' to divide the obtained four-valent vertex into two tree-valent in a different way.

Let us call an internal edge of our ribbon graph *m-internal*, $m = 0, 1, 2$, if m of its ends are in the border of the opposite face. (See fig. 2)

It is not hard to see that after a flip at a 1-internal edge e incident to the face X the edge e' becomes separating, therefore such a flip increases k by 1 and correspondingly decreases l_X. To the contrary, after a flip at a 2-internal edge e of the face X the new edge e' becomes the 2-internal edge e of the second face Y. Therefore such flip does not change k but increases l_Y by 1 and correspondingly decreases l_X.

Pick some ribbon graph D corresponding to the point $(l, \kappa(l))$, assume that $l_X = l$. Performing consequently flips at 1-internal edges of the face Y we shall each time obtain ribbon graphs with the same value of l and consequently increasing by one value of k, at least until $l_X < l_Y$. If at certain step we achieve $l_X = l_Y$ the that would mean that $k + 2l = 6g$ and therefore we have reached the right slanting border of the domain U_g. If after a certain



Separating edge. 0-internal edge. 1-internal edge. 2-internal edge.

Figure 2: Separating edge and internal edges.

flip we shall still have $l_X < l_Y$ but no 1-internal edges of the face Y left this would mean that all the internal edges of the face Y are 2-internal. (It is not hard to see that if a face has 0-internal edges then it also has 1-internal edges.) In the latter case we shall apply flips at 2-internal edges of the face Y which as we have seen do not change the value of k. After $l_Y - l$ such flips the two faces would change their roles: now we would have $l_X > l_Y = l$ and all the internal edges of the face Y would be 2-internal. In case we still have some 1-internal edges of the face X we shall apply the discussed above procedure of increasing k by flips at 1-internal edges of the face X until all the internal edges of the face X would also become 2-internal. Thus we would obtain a ribbon graph all whose internal edges are 2-internal, which means that $k = 4g$ and therefore the corresponding point is located on the upper border of U_g. Proposition 1 is proved.

Using the described stratification of the set $\mathcal{D}ess_{g,2}$ it is not hard to see that for ribbon graphs with two faces there exist many identities analogous to the Kontsevich identity ([1]). To recall the Kontsevich identity let us fix the followung notations. For a genus g ribbon graph $D \in \mathcal{D}ess_{g,\gamma}$ we denote by V the set of its vertices, by E the set of its edges and by F the set of its faces, $\gamma = |F|$. Consider the rational function in variables $\lambda_1, \lambda_2, \ldots \lambda_\gamma$

$$K(D) = S\left(\prod_{e \in E} \frac{1}{\lambda_i + \lambda_j}\right), \tag{9}$$

where the variables λ_i and λ_j correspond to the faces incident to the edge e ($i = j$ is possible), and S means symmetrization in variables $\lambda_1, \lambda_2, \dots \lambda_\gamma$. Then the Kontsevich identity claims that the function

$$\mathcal{K}_{g,\gamma} = \sum_{D \in \mathcal{D}ess_{g,\gamma}} C(D) \cdot K(D), \tag{10}$$

(where $C(D) = \frac{1}{|\operatorname{Aut} D|}$) is a linear combination of monomials in $\frac{1}{\lambda_i}$ i. e.

$$\mathcal{K}_{g,\gamma} = \sum A_{n_1,,n_\gamma} \frac{1}{\lambda_1^{n_1} \dots \lambda_\gamma^{n_\gamma}}. \tag{11}$$

In the discussed case $\gamma = 2$ it is not hard to verify that the same statement holds for many choices of positive rational numbers $C(D)$. More precisely, such collections of $C(D)$ is a convex rational cone.

Proposition 2 *For trivalent ribbon graphs with two faces the set of collections of the coefficients $C(D)$ satisfying (11) is a convex rational cone of dimension at least $g(4g - 7)$.*

Let us use the new variables $x = \frac{1}{\lambda_1}$ and $y = \frac{1}{\lambda_2}$, corresponding to the faces X and Y. (9) In these notations the formula (9) looks as follows:

$$K(D) = \frac{1}{2^{6g-k}} \cdot (xy)^l \cdot \frac{(xy)^k}{(x+y)^k} \cdot (x^{2M-k} + y^{2M-k}) = \frac{1}{2^{6g-k}} \cdot R_{l,k}(x,y), \tag{12}$$

where the numbers k, l, M are defined in (5) and (6), and

$$R_{l,k}(x,y) = \frac{(xy)^{k+l}}{(x+y)^k} \cdot (x^{2M-k} + y^{2M-k}). \tag{13}$$

It is not hard to verify the following properties of the functions $R_{l,k}$.

Proposition 3 *(1)* $R_{l,k} + R_{l+1,k} + R_{l,k-1} = R_{l+1,k-2}$

(2) $R_{l,k} = R_{6g-k-l,k}$

(3) $R_{l,k}$ *is polynomial for* $k \leq 1$

Iterating the equality (1) for successive pairs of values of k we obtain analogous identities for arbitrary $2m$ successive values of k.

Corollary 1

$$R_{l+1,2m} + \sum_{k=1}^{k=2m} R_{l,k} = R_{l+1,0} \tag{14}$$

To prove proposition 2 let us denote the identity (1) of the proposition 3 by I_{lk} and define the *support* of the identity I_{lk} by

$$\operatorname{supp} I_{lk} = \{(l,k),(l+1,k),(l,k-1),(l+1,k-2)\}.$$

Thus any identity I_{lk} such that $\operatorname{supp} I_{lk} \subset U_g$ defines a linear variation of the possible coefficients $C(D)$. Next let us prove that any finite collection of the identities I_{lk} are linearly independent. Suppose that we have some non-trivial linear combination of such identities that is equal to zero. Let us mark all the points (l,k) of the supports all the involved identities. Choose any left bottom marked point (i. e. such a marked point (l,k) that any other point (l',k') with $l' \le l$ and $k' \le k$ is not marked). Then such a point can be involved in only one identity I_{lk} and therefore this identity ought to have zero coefficient. Contradiction. The estimate in the proposition 2 is simply the size of such a rectangle that it is contained in U_g and $\operatorname{supp} I_{lk} \subset U_g$ for any point (l,k) of this rectangle.

For completeness let us give an independent proof of existence of the discussed coefficients. Let us prove that there exist such natural numbers $C_{l,k}$ that the function

$$\sum_{(l,k)\in U_g} C_{l,k} R_{l,k} \tag{15}$$

is a polynomial.

First let us note that any function $R_{l,k}$, $0 \le l \le l_{max}$, $1 \le k < \kappa(l)$, can be linearly expressed by $R_{l,k}$ for $(l,k) \in U_g$ with non-negative integer coefficients. The proof it by induction. This is of course true for $l = 0$. To pass from l to $l+1$ we express the functions $R_{l+1,k}$ one after another using the formula (1) of the proposition 3 starting from $k = \kappa(l) - 1$ till $k = 1$. Note that in the last row ($l = l_{max}$) to get $R_{l_{max},\kappa(l_{max})-1}$ we shall need the function $R_{l_{max},\kappa(l_{max})+1}$ which does not correspond to a point of U_g but in this case the identity (2) of the proposition 3 provides $R_{l_{max},\kappa(l_{max})+1} = R_{l_{max}-1,\kappa(l_{max})+1}$, where $(l_{max}-1, \kappa(l_{max})+1) \in U_g$.

Therefore it is sufficient to prove our statement for the domain $\bar{U}_g$, circumscribed by the lines $k = 4g$ on the top, $k = 1$ at the bottom, $l = 0$ on the left and $l = l_{max}$ and $k = 6g - 2l$ on the right.

Now it is left for each $l \leq l_{max}$ to apply the formula (14) to the column (l, k), $k = 1, \ldots, \min(4g, 6g-2l)$. The only problem is that for $l \geq g$ (i. e. on the slanting side of $\bar{U}_g$) the point $(l+1, 6g-2l)$ is not in the domain $\bar{U}_g$, but for this case the identity (2) of the proposition 3 provides $R_{l+1,6g-2l} = R_{l-1,6g-2l}$, where $(l-1, 6g-2l) \in \bar{U}_g$.

Note that it is possible to trace the coefficients $C_{l,k}$ for $k > 6$ (which where not used to express the functions $R_{l,k}$ corresponding to points which are lower than the graph of the function κ): $C_{g-1,4g} = 3$ (for $g > 2$); $C_{l,4g} = 2$ for $1 < l < g-1$, $l = g$ and $C_{l-1,6g-2l} = 2$ for $g < l < l_{max} - 3$; all the other $C_{l,k} = 1$ $(k > 6)$.

References

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