Analytic cycles on real analytic manifolds

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1 Introduction

Let X be a compact real analytic manifold of dimension n. In this paper we study homology classes in $H_*(X, \mathbb{Z}/2)$ represented by analytic subsets of X. Here a subset V of X is called *analytic* if $V = f^{-1}(0)$, where $f : X \to \mathbb{R}$ is an analytic function. Thus analytic sets are zeros of global analytic functions.

Each analytic set $V \subseteq X$ of dimension k represents a homology class in $H_k(X, \mathbb{Z}/2)$, written $[V]_X$, that corresponds to the fundamental class of V (cf. [7]), and the set of all classes of this type forms a subgroup of $H_k(X, \mathbb{Z}/2)$, denoted $H_k^{an}(X, \mathbb{Z}/2)$. It is convenient to define and work with

$$H^{\ell}_{an}(X, \mathbb{Z}/2) = D_X^{-1}(H_k^{an}(X, \mathbb{Z}/2)),$$

where $D_X : H^{\ell}(X, \mathbb{Z}/2) \to H_k(X, \mathbb{Z}/2)$ is the Poincaré duality isomorphism and $k + \ell = n$. Any cohomology class in H^*_{an} is said to be *analytic* or *represented by an analytic set.*

It is well known that each homology class in $H_{n-1}(X, \mathbb{Z}/2)$ is represented by an analytic submanifold of X of codimension 1. In particular, one has always $H^1(X, \mathbb{Z}/2) = H^1_{an}(X, \mathbb{Z}/2)$. The situation is quite different for $H^2(X, \mathbb{Z}/2)$, where the analogous equality is not always satisfied.

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Surprisingly, the subgroup $H^2_{an}(X,\mathbb{Z}/2)$ admits a rather elegant and purely topological description. Let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of topological real vector bundles over X and define

$$W(X) = \{v \in H^2(X,\mathbb{Z}/2) |\, v = w_2(\xi) \text{ for some } \xi \text{ in } \mathrm{Vect}(X)\},$$

where $w_2(\xi)$ is the second Stiefel-Whitney class of ξ. One easily sees that $W(X)$ is a subgroup of $H^2(X,\mathbb{Z}/2)$, cf. for example [5, Proposition 1.4]. For X of dimension less than 6, one has $H^2(X,\mathbb{Z}/2) = W(X)$. However, for each $n \geq 6$, there exists a compact n-dimensional real analytic manifold X with $H^2(X,\mathbb{Z}/2) \neq W(X)$, cf. [19].

Theorem 1.1 *Let X be a compact real analytic manifold and let v be a cohomology class in $H^2(X,\mathbb{Z}/2)$. Then v is analytic if and only if v is in $W(X)$. Briefly,*

$$H^2_{an}(X,\mathbb{Z}/2) = W(X).$$

Corollary 1.2 *For each $n \geq 6$ there exists a compact n-dimensional real analytic manifold X with $H^2(X,\mathbb{Z}/2) \neq H^2_{an}(X,\mathbb{Z}/2)$.*

Similar problems for complex analytic manifolds and cohomology classes with coefficients in $\mathbb{Z}$ were investigated by Atiyah and Hirzebruch [2] by completely different methods.

The proof of Theorem 1.1 is given in Section 3.

2 The Grothendieck formula in real analytic setting

Let X be a paracompact real analytic manifold. By an *analytic subset* of X we mean a subset of the form $V = f^{-1}(0)$, where $f : X \to \mathbb{R}$ is an analytic function (the reader may refer to Whitney and Bruhat [21] for fundamental properties of such globally defined real analytic sets). Let $\mathcal{O}(X)$ (resp. $\mathcal{O}_x$) denote the ring of real analytic functions on X (resp.

the ring of germs of real analytic functions at x in X) and let

$$I(V) = \{f \in \mathcal{O}(X) |\ f|V = 0\}$$

$$J_{V,x} = \{f \in \mathcal{O}_x |\ f|V = 0\}$$

be the ideals of functions or germs of functions vanishing on V. We define the local dimension $\dim V_x$ of V at x to be the Krull dimension of the ring $\mathcal{O}_x/J_{V,x}$, and dimension $\dim V$ of V as

$$\dim V = \max\{\dim V_x |\ x \in V\}.$$

The set Reg$V$ of *globally regular* points of V is, by definition, the collection of all points x in V for which the local ring $\mathcal{O}_x/I(V)\mathcal{O}_x$ is regular of Krull dimension $\dim V$. We write "globally regular" to emphasize that the ideal $I(V)\mathcal{O}_x$, instead of $J_{V,x}$, is used in our definition. It follows from [21, Proposition 16] that $V \backslash \mathrm{Reg} V$ is an analytic subset of X (defined by global analytic functions on X) of dimension less than $\dim V$ ([21, Proposition 16] is applicable since the sheaf of ideals on X with stalk $I(V)\mathcal{O}_x$ at each point x in X is coherent, cf. [9, Corollaire (I,8)], and therefore by virtue of Cartan's Theorem A it is the largest coherent sheaf of ideals on X with zero locus containing V).

We shall use the following facts:

(i) If x is in Reg$V$, then $I(V)\mathcal{O}_x = J_{V,x}$ (cf. [1, p. 225, Proposition 4.4d]); in particular, there is a neighborhood U of x in X such that $V \cap U$ is an analytic submanifold of U of dimension $\dim V$ (however, a point of V having such a neighborhood is not necessarily globally regular).

(ii) For each x in X, the ring $\mathcal{O}_x$ is flat over $\mathcal{O}(X)_{m_x}$, the localization of the ring $\mathcal{O}(X)$ at the maximal ideal

$$m_x = \{f \in \mathcal{O}(X) |\ f(x) = 0\} \text{ (cf. [1, p. 226]).}$$

(iii) If X is compact, the ring $\mathcal{O}(X)$ is regular (that is, Noetherian whose localization at each maximal ideal, which is always of the form m_x for some x in X, is a regular local ring) of finite Krull dimension (cf. [9] and [1, p. 225, Proposition 4.4d]).

The following result is a real analytic version of the Grothendieck formula (for its real algebraic version, not needed here, the reader may consult [6]).

Theorem 2.1 *Let X be a compact real analytic manifold and let v be a cohomology class in $H^2_{an}(X, \mathbb{Z}/2)$. Then there exists an analytic vector bundle ξ over X such that $w_1(\xi) = 0$ and $w_2(\xi) = v$.*

Proof. Let V be an analytic subset of X of codimension 2 representing v, that is, $D_X(v) = [V]_X$. In the proof it is convenient to assume, which we can do without loss of generality, that X is connected and V is irreducible (in the class of analytic subsets of X defined by global real analytic function on X). Let $n = \dim X$.

Step 1. First we shall construct an auxiliary projective module of rank 2 over an appropriate localization of the ring $\mathcal{O}(X)$.

Let x_0 be a point in Reg$V$. Since $\mathcal{O}_{x_0}$ is a regular local ring, there exist (see [14, p. 168, Proposition 1.10]) two elements a_1 and a_2 in $I(V)$ such that

$$I(V)\mathcal{O}_{x_0} = (a_1, a_2)\mathcal{O}_{x_0}.$$

Hence, using flatness (property (ii) above), one gets

$$I(V)\mathcal{O}(X)_{m_{x_0}} = (a_1, a_2)\mathcal{O}(X)_{m_{x_0}}.$$

Since the ideal $I(V)$ is finitely generated and $V \backslash \mathrm{Reg}V$ is an analytic subset of V, we can choose λ in $\mathcal{O}(X)$ satisfying

(1)
$$\lambda I(V) \subseteq (a_1, a_2)\mathcal{O}(X) \text{ and } \lambda(x_0) \neq 0,\ V \backslash \mathrm{Reg}V \subseteq \lambda^{-1}(0).$$

Define

$$Y = X \backslash (V \cap \lambda^{-1}(0)) \text{ and } W = V \backslash (V \cap \lambda^{-1}(0)).$$

Irreducibility of V implies

$$\dim(X \backslash Y) = \dim(V \cap \lambda^{-1}(0)) < n - 2.$$

Clearly, W is an analytic subset of Y of codimension 2 and $W \subset \mathrm{Reg}V$.

Consider the map $F : X \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,t) = F_t(x) = (a_1(x) + t_1 b(x)^2,\ a_2(x) + t_2 b(x)^2),$$

where x is in X, $t = (t_1, t_2)$ is in $\mathbb{R}^2$, and b is an element of $\mathcal{O}(X)$ with $b^{-1}(0) = V$. Observe that $F|Y \times \mathbb{R}^2$ is transverse to 0 in $\mathbb{R}^2$. Indeed, let (x,t) be a point in $Y \times \mathbb{R}^2$ and $F(x,t) = 0$. If x is not in W, then transversality at (x,t) is obvious. If x is in W, then $\lambda(x) \neq 0$ and hence (1) implies $I(V)\mathcal{O}_x = (a_1, a_2)\mathcal{O}_x$. Since x belongs to $\mathrm{Reg}V$, it follows from (i) that also $J_{V,x} = (a_1, a_2)\mathcal{O}_x$, which implies transversality of $(a_1, a_2) : X \to \mathbb{R}^2$ at x to 0 in $\mathbb{R}^2$, and therefore also transversality of F_t at x and F at (x,t) to 0 in $\mathbb{R}^2$.

By a standard transversality theorem [12, p. 79, Theorem 2.7], we can choose t in $\mathbb{R}^2$ such that for

$$F_t = (f_1, f_2) : X \to \mathbb{R}^2,$$

the restriction $F_t|Y : Y \to \mathbb{R}^2$ is transverse to 0 in $\mathbb{R}^2$. It follows that for each point x in W, one has $J_{V,x} = (f_1, f_2)\mathcal{O}_x$ and hence in view of (i),

(2)
$$I(V)\mathcal{O}_x = (f_1, f_2)\mathcal{O}_x.$$

Using again property (ii) (flatness), one deduces

(3)
$$I(V)\mathcal{O}(X)_{m_x} = (f_1, f_2)\mathcal{O}(X)_{m_x}.$$

for all x in W. Condition (2) implies

(4)
$$f_1^{-1}(0) \cap f_2^{-1}(0) \cap Y = W \cup W' \text{ with } W \cap W' = \emptyset.$$

The intersection V' of all analytic subsets of X containing W' is an analytic subset of X (cf. [21, p. 154]) satisfying

(5)
$$V' \subseteq f_1^{-1}(0) \cap f_2^{-1}(0).$$

Claim. $W' = V' \cap Y$.

The Claim is proved as follows. By (3), for each point x in W, there exists an analytic function μ_x in $\mathcal{O}(X)$ such that

$$\mu_x I(V) \subseteq (f_1, f_2)\mathcal{O}(X) \text{ and } \mu_x(x) \neq 0.$$

Let $\mu_1, \ldots, \mu_p$ be a finite system of generators of the ideal of $\mathcal{O}(X)$ generated by all μ_x with x in W, and let $\mu = \mu_1^2 + \cdots + \mu_p^2$. Then

(6)
$$W \subset \bigcup_{x\in W}(X \backslash \mu_x^{-1}(0)) = X \backslash \bigcap_{x \in W} \mu_x^{-1}(0) = X\backslash \mu^{-1}(0),$$

and $\mu I(V) \subseteq (f_1, f_2)\mathcal{O}(X)$. The last inclusion implies

(7)
$$f_1^{-1}(0) \cap f_2^{-1}(0) \subseteq \mu^{-1}(0) \cup V.$$

By (4) and (5),

(8)
$$V' \cap Y \subseteq W \cup W'.$$

Since in view of (4) and (7),

$$W \cup W' \subseteq (\mu^{-1}(0) \cup V) \cap Y = (\mu^{-1}(0) \cap Y) \cup (V \cap Y) = (\mu^{-1}(0) \cap Y) \cup W,$$

one gets $W' \subseteq \mu^{-1}(0) \cap Y \subseteq \mu^{-1}(0)$, the sets W and W' being disjoint. Hence also

(9)
$$V' \subseteq \mu^{-1}(0).$$

Conditions (8), (9), and (6) imply

$$V' \cap Y \subseteq (W \cup W') \cap \mu^{-1}(0) = W' \cap \mu^{-1}(0) \subseteq W'$$

and the Claim follows, the inclusion $W' \subseteq V' \cap Y$ being trivial.

Choose ψ_1 and ψ_2 in $\mathcal{O}(X)$ with $\psi_1^{-1}(0) = V$ and $\psi_2^{-1}(0) = V'$. Then $\psi_1^{-1}(0) \cap Y = W$ and, by the Claim, $\psi_2^{-1}(0) \cap Y = W'$. In view of (4) and the fact that $(f_1, f_2)|Y : Y \to \mathbb{R}^2$

is transverse to 0 in $\mathbb{R}^2$, for each point x in Y, the germ of $\psi_1\psi_2$ at x belongs to the ideal $(f_1, f_2)\mathcal{O}_x$. Hence making use of (ii), one gets

(10)
$$\psi_1\psi_2 \in (f_1, f_2)\mathcal{O}(X)_{m_x} \text{ for all } x \text{ in } Y = X\backslash(V \cap \lambda^{-1}(0)).$$

Define a multiplicatively closed subset S of the ring $\mathcal{O}(X)$ by

$$S = \{f \in \mathcal{O}(X) |\ f^{-1}(0) \subset V \cap \lambda^{-1}(0)\}$$

and consider the localization $A = S^{-1}\mathcal{O}(X)$ of $\mathcal{O}(X)$ at S. Each maximal ideal of A is of the form $m_x A$, where m_x is the maximal ideal of $\mathcal{O}(X)$ corresponding to a point x in Y. Hence (10) implies that $\psi_1\psi_2$ belongs to $(f_1, f_2)A$, that is,

$$\psi_1\psi_2 = (h_1 f_1 + h_2 f_2)/c,$$

where h_1, h_2 are in $\mathcal{O}(X)$ and c is in S. It follows from the last equation that the matrices

$$g_{12} = \begin{pmatrix} h_1/c\psi_2^2 & h_2/c\psi_2^2 \\ -f_2\psi_1/\psi_2 & f_1\psi_1/\psi_2 \end{pmatrix}, \quad g_{21} = \begin{pmatrix} f_1\psi_2/\psi_1 & -h_2/c\psi_1^2 \\ f_2\psi_2/\psi_1 & h_1/c\psi_1^2 \end{pmatrix},$$

with entries in the field $\mathcal{O}(X)_{(0)}$ of fractions of $\mathcal{O}(X)$, are invertible and satisfy $g_{21} = g_{12}^{-1}$. The entries of g_{21} and g_{12} are analytic on $U_1 = Y\backslash W$ and $U_2 = Y\backslash W'$, respectively, and $\{U_1, U_2\}$ is an open cover of Y. Regarding A as a subring of $\mathcal{O}(X)_{(0)}$, set

$$P = \{(\alpha, \beta) \in A^2 \times A^2 |\ \beta = g_{21}\alpha\},$$

where elements of A^2 are viewed as column vectors. Since the ring $\mathcal{O}(X)$ is Noetherian (recall (iii)), so is the localization A of $\mathcal{O}(X)$, and hence P is a finitely generated submodule of the A-module $A^2 \times A^2$.

We assert that P is a projective A-module of rank 2. To establish this we just have to prove that the localization P_m of P at each maximal ideal m of A is a free A_m-module of rank 2 [14, p. 113, Corollary 3.6]. We proceed as follows. Define

$$\gamma_1 = c\psi_1^2(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, g_{21}\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = (\begin{pmatrix} c\psi_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} cf_1\psi_1\psi_2 \\ f_2\psi_2 \end{pmatrix}),$$

$$\gamma_2 = c\psi_1^2(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, g_{21}\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = (\begin{pmatrix} 0 \\ c\psi_1^2 \end{pmatrix}, \begin{pmatrix} -h_2 \\ h_1 \end{pmatrix}),$$

$$\eta_1 = c\psi_2^2(g_{12}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = (\begin{pmatrix} h_1 \\ -cf_2\psi_1\psi_2 \end{pmatrix}, \begin{pmatrix} c\psi_2^2 \\ 0 \end{pmatrix}),$$

$$\eta_2 = c\psi_2^2(g_{12}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = (\begin{pmatrix} h_2 \\ cf_1\psi_1\psi_2 \end{pmatrix}, \begin{pmatrix} 0 \\ c\psi_2^2 \end{pmatrix}).$$

Each of these elements is in P. Given any maximal ideal $m = m_x A$ of A, with x in U_1 (resp. x in U_2), the element $c\psi_1^2$ (resp. $c\psi_2^2$) is a unit in A_m, and hence γ_1, γ_2 (resp. η_1, η_2) generate freely P_m over A_m. The assertion is proved.

Step 2. Construction of a topological vector bundle ζ over X with $w_2(\zeta) = v$.

Consider the finitely generated projective A-module P of rank 2 constructed in Step 1. Since the ring $\mathcal{O}(X)$ is regular of finite Krull dimension (recall (iii)), its projective dimension is also finite [14, p. 208, Proposition 2.4]. Hence, by [3, p. 453, Proposition 2.1, p. 492, Proposition 6.1], there exists a finitely generated projective $\mathcal{O}(X)$-module Q such that the A-modules $A \otimes_{\mathcal{O}(X)} Q$ and P are stably isomorphic.

Each element of A determines an analytic function in $\mathcal{O}(Y)$. This gives rise to a canonical homomorphism $A \to \mathcal{C}(Y)$ from A into the ring $\mathcal{C}(Y)$ of all continuous functions from Y into $\mathbb{R}$. The $\mathcal{C}(Y)$-modules $\mathcal{C}(Y) \otimes_A (A \otimes_{\mathcal{O}(X)} Q)$ and $\mathcal{C}(Y) \otimes_A P$ are stably isomorphic. Making use of the restriction homomorphisms $\mathcal{O}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}(X) \to \mathcal{C}(Y)$, we obtain the following isomorphisms of $\mathcal{C}(Y)$-modules:

$$\mathcal{C}(Y) \otimes_A (A \otimes_{\mathcal{O}(X)} Q) \cong \mathcal{C}(Y) \otimes_{\mathcal{O}(X)} Q \cong \mathcal{C}(Y) \otimes_{\mathcal{C}(X)} (\mathcal{C}(X) \otimes_{\mathcal{O}(X)} Q).$$

Hence the finitely generated projective $\mathcal{C}(Y)$-modules $\mathcal{C}(Y) \otimes_{\mathcal{C}(X)} (\mathcal{C}(X) \otimes_{\mathcal{O}(X)} Q)$ and $\mathcal{C}(Y) \otimes_A P$ are stably isomorphic.

Let ζ (resp. η) be the topological real vector bundle over X (resp. Y) corresponding to $\mathcal{C}(X) \otimes_{\mathcal{O}(X)} Q$ (resp. $\mathcal{C}(Y) \otimes_A P$), cf. [18]. It follows that $\zeta|Y$ and η are stably isomorphic.

For our purposes we need to describe the vector bundle η more explicitly. Define

$$E = \{(y, u_1, u_2) \in Y \times \mathbb{R}^2 \times \mathbb{R}^2 \,|\; u_2 = g_{21}(y)u_1 \text{ for } y \text{ in } U_1 \text{ and}$$
$$u_1 = g_{12}(y)u_2 \text{ for } y \text{ in } U_2\},$$

$$\pi : E \to Y, \;\; \pi(y, u_1, u_2) = y.$$

Since $\{U_1, U_2\}$ is an open cover of Y, the set E is a closed analytic submanifold of $Y \times \mathbb{R}^2 \times \mathbb{R}^2$ and π is an analytic map. For each point y in Y, the fiber $E_y = \pi^{-1}(y)$ is a vector subspace of $\{y\} \times \mathbb{R}^2 \times \mathbb{R}^2$. Furthermore, the map

$$U_k \times \mathbb{R}^2 \to \pi^{-1}(U_k), \;\; (y, u) \to (y, g_{1k}(y)u, g_{2k}(y)u)$$

is an analytic diffeomorphism for $k = 1, 2$, where $g_{kk}(y)$ denotes the identity matrix. Thus $\eta' = (E, \pi, Y)$ is an analytic real vector bundle of rank 2 over Y.

We claim that η and η' are isomorphic as topological vector bundles. In order to demonstrate this claim it suffices to show that the $\mathcal{C}(Y)$-module Γ of all continuous sections of η' is isomorphic to $\mathcal{C}(Y) \otimes_A P$ (cf. [18]). Note that

$$\Gamma = \{(\varphi, \psi) \in \mathcal{C}(Y)^2 \times \mathcal{C}(Y)^2 |\; \psi(y) = g_{21}(y)\varphi(y) \text{ for } y \text{ in } U_1 \text{ and}$$
$$\varphi(y) = g_{12}(y)\psi(y) \text{ for } y \text{ in } U_2\}.$$

On the other hand, since P is a projective submodule of the A-module $A^2 \times A^2$, we obtain that $\mathcal{C}(Y) \otimes_A P$ is isomorphic to the submodule $\mathcal{C}(Y)P$ of the $\mathcal{C}(Y)$-module $\mathcal{C}(Y)^2 \times \mathcal{C}(Y)^2$, where $\mathcal{C}(Y)P$ is the submodule generated by P over $\mathcal{C}(Y)$ (this follows from the fact that any projective module is flat). One readily check $\mathcal{C}(Y)P = \Gamma$, and hence the vector bundles η and η' are topologically isomorphic, as claimed.

Henceforth we shall identify η with η' and regard η as an analytic vector bundle over Y. The analytic map

$$s : Y \to E, \;\; s(y) = (y, \begin{pmatrix} \psi_1(y) \\ 0 \end{pmatrix}, \begin{pmatrix} f_1(y)\psi_2(y) \\ f_2(y)\psi_2(y) \end{pmatrix})$$

is a section of $\eta = \eta' = (E, \pi, Y)$ with $s^{-1}(0_E) = W$, where 0_E is the image of the zero section of η. On U_2 the section s is represented by $(f_1, f_2) : U_2 \to \mathbb{R}^2$, and therefore s is transverse to 0_E.

Let $[W]_Y$ be the Borel-Moore homology class in $H^{\mathrm{BM}}_{n-2}(Y, \mathbb{Z}/2)$ represented by W (since Y and possibly W are not compact, the use of the Borel-Moore homology is necessary [7]). Since $W = s^{-1}(0_E)$ and s is transverse to 0_E, it follows that

(11)
$$D_Y(w_2(\eta)) = [W]_Y,$$

where $D_Y : H^*(Y, \mathbb{Z}/2) \to H^{\mathrm{BM}}_*(Y, \mathbb{Z}/2)$ is the Poincaré duality isomorphism.

As $\zeta|Y$ and η are stably isomorphic, one has

(12)
$$j^*(w_2(\zeta)) = w_2(\zeta|Y) = w_2(\eta),$$

where $j : Y \hookrightarrow X$ is the inclusion map and $j^* : H^*(X, \mathbb{Z}/2) \to H^*(Y, \mathbb{Z}/2)$ the induced homomorphism.

On the other hand, commutativity of the diagram

$$\begin{array}{ccc} H^2(X, \mathbb{Z}/2) & \xrightarrow{j^*} & H^2(Y, \mathbb{Z}/2) \\ {\scriptstyle D_X}\big\downarrow & & \big\downarrow{\scriptstyle D_Y} \\ H_{n-2}(X, \mathbb{Z}/2) & \xrightarrow{\rho} & H^{\mathrm{BM}}_{n-2}(Y, \mathbb{Z}/2), \end{array}$$

where ρ is the restriction homomorphism (which is defined for each open subset Y of X, cf. [7]), implies

$$j^*(v) = j^*(D_X^{-1}([V]_X)) = D_Y^{-1}(\rho([V]_X)) = D_Y^{-1}([W]_Y).$$

Hence, according to (11),

(13)
$$j^*(v) = w_2(\eta).$$

The homomorphism j^* is injective, as we can see by examining the exact sequence

$$H^2(X, Y, \mathbb{Z}/2) \to H^2(X, \mathbb{Z}/2) \xrightarrow{j^*} H^2(Y, \mathbb{Z}/2).$$

Indeed, since $H^2(X, Y, \mathbb{Z}/2)$ is isomorphic to $H_{n-2}(X\backslash Y, \mathbb{Z}/2)$ (cf. [17, p. 351, Lemma 14]) and $\dim(X\backslash Y) < n-2$, we have $H^2(X, Y, \mathbb{Z}/2) = 0$ and j^* is injective. Hence comparing (12) and (13), one gets $w_2(\zeta) = v$. The construction in Step 2 is accomplished.

Step 3. Construction of an analytic real vector bundle ξ over X with $w_1(\xi) = 0$ and $w_2(\xi) = v$.

Let r be the rank of ζ and $\Theta = \Lambda^r\zeta$. Since $w_1(\Theta) = w_1(\zeta)$ (cf. [13, p. 246]), the topological vector bundle $\xi = \zeta \oplus \Theta \oplus \Theta \oplus \Theta$ over X satisfies $w_1(\xi) = 0$ and $w_2(\xi) = v$.

In order to complete the proof it remains to recall that each topological vector bundle over X is isomorphic to an analytic one. This is true since any vector bundle over X is a pullback of the universal vector bundle over an appropriate Grassmannian and the following approximation theorem holds: For any compact real analytic manifolds Z and Z', the set of analytic maps from Z into Z' is dense in the space of all smooth (of class $\mathcal{C}^\infty$) maps from Z into Z', cf. [16, Theorem 1.6.5] and [11, Theorem 3]. $\square$

3 Analytic cycles

Proof of Theorem 1.1. Theorem 2.1 implies $H^2_{an}(X,\mathbb{Z}/2) \subseteq W(X)$. To prove the inclusion in the opposite direction, let u be in $W(X)$ and choose a topological real vector bundle ξ of rank p over X with $w_2(\xi) = u$. Let $f : X \to G_{m,p}(\mathbb{R})$ be a continuous map into the Grassmannian $G_{m,p}(\mathbb{R})$ for some m, such that ξ is isomorphic to the pullback $f^*\gamma$ of the universal vector bundle γ over $G_{m,p}(\mathbb{R})$. Since $H^*(G_{m,p}(\mathbb{R}),\mathbb{Z}/2) = H^*_{an}(G_{m,p}(\mathbb{R}),\mathbb{Z}/2)$ (in fact, each cohomology class in $H^*(G_{m,p}(\mathbb{R}),\mathbb{Z}/2)$ is represented by an algebraic subset [4, Proposition 11.3.3]), Proposition 3.1 below implies that the cohomology class $u = w_2(\xi) = w_2(f^*\gamma) = f^*(w_2(\gamma))$ is in $H^2_{an}(X,\mathbb{Z}/2)$. This shows $W(X) \subseteq H^2_{an}(X,\mathbb{Z}/2)$, which completes the proof. □

Proposition 3.1 *Let $f : X \to Y$ be a continuous map between compact real analytic manifolds. Then the induced homomorphism $f^* : H^*(Y,\mathbb{Z}/2) \to H^*(X,\mathbb{Z}/2)$ satisfies*

$$f^*(H^k_{an}(Y,\mathbb{Z}/2)) \subseteq H^k_{an}(X,\mathbb{Z}/2)$$

for each $k \geq 0$.

Proof. Let w be a cohomology class in $H^k_{an}(Y,\mathbb{Z}/2)$. We shall show that $f^*(w)$ is in $H^k_{an}(X,\mathbb{Z}/2)$.

Choose an analytic subset W of Y such that $w = D_Y^{-1}([W]_Y)$. Let $\{W_1,\dots,W_s\}$ be a stratification of W satisfying Whitney's condition (a) (for the definition see [10] or [4, p. 242], for the existence see [15]). The set of smooth maps from X into Y transverse to each of the submanifolds $W_1,\dots,W_s$ is open and dense in the space of all smooth maps [8, Proposition 3.6] (it is Whitney's condition (a) that guarantees openness). Since the set of analytic maps from X into Y is dense in the space of all smooth maps, there exists an analytic map $g : X \to Y$ homotopic to f and transverse to W_i for $1 \leq i \leq s$. The inverse image $V = g^{-1}(W)$ is an analytic subset of X and, by [7, Proposition 2.15], the cohomology class $v = D_X^{-1}([V]_X)$ satisfies $v = g^*(w)$. The proof is complete since v is in $H^k_{an}(X, \mathbb{Z}/2)$ and $v = g^*(w) = f^*(w)$. $\square$

As a corollary to Proposition 3.1 we immediately obtain the following result.

Proposition 3.2 *For any compact real analytic manifold X, the group*

$$H^*_{an}(X, \mathbb{Z}/2) = \bigoplus_{k \geq 0} H^k_{an}(X, \mathbb{Z}/2)$$

is a subring of the cohomology ring $H^*(X, \mathbb{Z}/2)$.

Proof. Let u and v be cohomology classes in $H^k_{an}(X, \mathbb{Z}/2)$ and $H^\ell_{an}(X, \mathbb{Z}/2)$, respectively. It readily follows that the cross product $u \times v$ belong to $H^{k+\ell}_{an}(X \times X, \mathbb{Z}/2)$. Recall that for the cup product we have $u \cup v = \delta^*(u \times v)$, where $\delta : X \to X \times X$ is defined by $\delta(x) = (x, x)$ for all x in X. Hence $u \cup v$ is in $H^{k+\ell}_{an}(X, \mathbb{Z}/2)$ by virtue of Proposition 3.1. $\square$

In contrast to Proposition 3.1, analytic homology classes to not behave in a functorial manner.

Remark 3.3. With notation as in Proposition 3.1, it is *not* true, in general, that

$$(*) \qquad f_*(H_*^{an}(X, \mathbb{Z}/2)) \subseteq H_*^{an}(Y, \mathbb{Z}/2).$$

Indeed, let Y be a compact real analytic manifold having a homology class z which is not in $H_*^{an}(Y, \mathbb{Z}/2)$. By [20], there are a compact real analytic manifold X and an analytic map $f : X \to Y$ such that $f_*([X]) = z$. Hence the validity of $(*)$ would lead to a contradiction.

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