CLASSIFICATION OF LOG DEL PEZZO SURFACES OF INDEX ≤ 2

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Abstract. This is an expanded version of our work [AN88], 1988, in Russian.

We classify del Pezzo surfaces over $\mathbb{C}$ with log terminal singularities (equivalently, quotient singularities) of index ≤ 2. By classification, we understand a description of the intersection graph of all exceptional curves on an appropriate (somewhat stronger than minimal) resolution of singularities together with the subgraph of the curves which are contracted to singular points.

The final results are similar to classical results about classification of non-singular del Pezzo surfaces and use the usual finite root systems. However, the intermediate considerations use the theory of K3 surfaces (especially of K3 surfaces with non-symplectic involutions) and the theory of reflection groups in hyperbolic spaces (especially of reflection groups of hyperbolic lattices).

As an "elementary" application, our results permit one to classify sextics in $\mathbb{P}^2$ with simple singularities and a component of geometric genus ≥ 2.

Contents

Introduction

This is an expanded version of our work [AN88], 1988, in Russian. See also [AN89], 1989, for a short exposition of the same results.

A complete algebraic surface Z over $\mathbb{C}$ with log terminal singularities is a del Pezzo surface if its anticanonical divisor $-K_Z$ is ample. A log terminal singularity in this case is analytically equivalent to a quotient singularity $\mathbb{C}^2/G$, where $G \subset \mathrm{GL}(2, \mathbb{C})$ is a finite subgroup. The index k of $z \in Z$ is the minimal positive integer for which the divisor kK_Z is a Cartier divisor in a neighborhood of z.

The aim of this work is to classify del Pezzo surfaces with log terminal singularities of index ≤ 2. We shortly call them *log del Pezzo surfaces of index* ≤ 2.

Classification of nonsingular del Pezzo surfaces is well known, and they are classical examples of rational surfaces (see, f.e. [Nag60, Man86, MT86]. Classification of log del Pezzo surfaces of index 1, i.e. Gorenstein del Pezzo surfaces, is the subject of a classical work of du Val [DV34] and papers of Nagata [Nag60], Demazure [Dem80], Urabe

[Ura83], Bindschadler-Brenton-Drucker [BBD84], Coray-Tsfasman [CT88]. See also survey [MT86].

By classification, we understand a description of the set of exceptional curves on an appropriate (see Section 4 for the precise definition) resolution of singularities $\sigma : Y \to Z$. Indeed, knowing the dual graph of exceptional curves on Y, we can describe all the ways to obtain Y and Z from the minimal nonsingular surfaces $\mathbb{P}^2$ and $\mathbb{F}_n$, $n = 0, 2, 3 \dots$. In particular, we obtain a complete list of 46 minimal, i.e. with $\operatorname{Pic} Z = \mathbb{Z}$, log del Pezzo surfaces of index ≤ 2 (see Section 4). Of those, 28 are Gorenstein and were previously known (see [BBD84]), and the other 18 are not Gorenstein. Note that minimal log del Pezzo surfaces play a special role in log Minimal Model Program, see f.e. [Sho85, Kaw84, Mor82, Rei83].

The main method for obtaining our classification is to reduce it to a classification of non-symplectic involutions on K3 surfaces (and to K3 surface theory). The main points of the latter are contained in [Nik80.1, Nik79, Nik83, Nik84.1, Nik87].

In Section 1, we show that for log del Pezzo surfaces Z of index ≤ 2 the linear system $|-2K_Z|$ contains a nonsingular curve, and that there exists an appropriate resolution of singularities $\sigma : Y \to Z$ for which the linear system $|-2K_Y|$ contains a nonsingular divisor X (i.e. Y is a "good" DPN surface) such that the component of C that belongs to $\sigma^*|-2K_Z|$ has genus ≥ 2 (i.e. the DPN surface Y is of elliptic type).

In Section 2, following [Nik80.1, Nik79, Nik83, Nik84.1, Nik87], we build a general theory of DPN surfaces Y. Here, we use the fact that the double cover X of Y branched along C is a K3 surface with a non-symplectic involution θ. In this way, the classification of DPN surfaces (Y, C) is equivalent to the classification of K3 surfaces with non-symplectic involution (X, θ). The switch to K3 surfaces is important because it is easy to describe exceptional curves on them and there are powerful tools available: the global Torelli theorem [PS-Sh71] and surjectivity of the period map [Kul77].

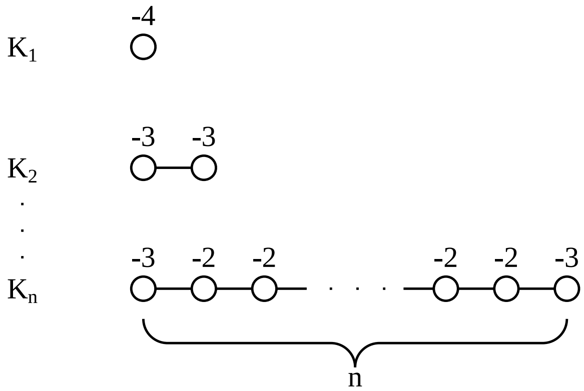
In Section 3, we extend this theory to a classification of DPN surfaces Y of elliptic type, i.e. when one of components of C has genus ≥ 2, by describing dual diagrams of exceptional curves on Y. In Section 3.6, we give an application of this classification to a classification of curves D of degree 6 on $\mathbb{P}^2$ with simple singularities in the case when one of components of D has geometric genus ≥ 2.

In obtaining results of Sections 2 and 3, a big role is played by the arithmetics of quadratic forms and reflection groups in hyperbolic spaces. From this point of view, the success of our classification hinges mainly on the fact that we explicitly describe subgroups generated by all reflections in automorphism groups of certain integral hyperbolic quadratic forms (even hyperbolic 2-elementary lattices of small rank, see Theorem 3.1). These computations are also important by themselves for the arithmetics of quadratic forms.

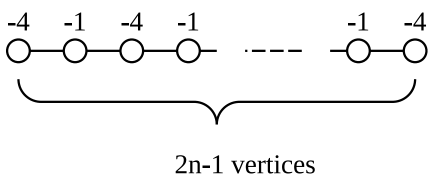
In Section 4, the results of Sections 1–3 are applied to the classification of log del Pezzo surfaces of index ≤ 2. Also, following [BBD84], we give an application of our classification to a description of rational compactifications of certain affine surfaces. In Section 4.3, we discuss some open questions. In particular, we give a formula for the dimension of moduli of log del Pezzo surfaces of index ≤ 2 and corresponding "good" DPN surfaces of elliptic type.

Below, we try to give an explicit and as elementary as possible exposition of our final results on classification of log del Pezzo surfaces of index ≤ 2.

Let Z be a log del Pezzo surface of index ≤ 2. Its singularities of index 1 are Du Val singularities classified by their minimal resolution of singularities. They are described by Dynkin diagrams A_k, D_m or E_n, with each vertex having weight -2. Singularities of Z of index 2 are singularities K_n which have minimal resolutions with dual graphs shown below:



To get the "good" resolution of singularities $\sigma : Y \to Z$, additionally one has to blow up all points of intersection of components in preimages of singular points K_n. Then the "good" resolution of a singular point K_n is described by the graph



In these graphs every vertex corresponds to an irreducible non-singular rational curve F_i with F_i^2 equal to the weight of the vertex. Two vertices are connected by an edge if $F_i \cdot F_j = 1$, and are not connected if $F_i \cdot F_j = 0$. Thus, the "good" resolution of singularities $\sigma : Y \to Z$ of a log del Pezzo surface of index ≤ 2 consists of minimal resolutions of singular points of index 1 and the "good" resolutions shown on the graphs above of singular points K_n of index 2.

Our classification of log del Pezzo surfaces Z of index ≤ 2 and the corresponding DPN surfaces of elliptic type which are "good" resolutions of singularities of Z is contained in Table 3 (see Sect. 3.5).

All cases of Table 3 are labeled by a number $1 \leq N \leq 50$. For $N = 7, 8, 9, 10, 20$ we add some letters and get cases: 7a,b, 8a–c, 9a–f, 10a–m, 20a–d. Thus, altogether, Table 3 contains

$$50 + (2-1) + (3-1) + (6-1) + (12-1) + (4-1) = 72$$

cases.

The labels $N = 1, \dots, 50$ enumerate the so called *main invariants* of log del Pezzo surfaces Z. They are triplets (r, a, δ) (equivalently $(k = (r-a)/2, g = (22-r-a)/2, \delta))$ where r, a, δ are integers: $r \geq 1$, $a \geq 0$, $\delta \in \{0, 1\}$, $g \geq 2$, $k \geq 0$. Thus, there exist exactly 50 possibilities for the main invariants (r, a, δ) (equivalently, (k, g, δ)) of log del Pezzo surfaces of index ≤ 2.

The main invariants have a very important geometric meaning. Any log del Pezzo surfaces Z of index ≤ 2 and its "good" resolution of singularities Y are rational. The number

$$r = \operatorname{rk} \operatorname{Pic} Y$$

is the Picard number of Y, i. e. $\operatorname{Pic} Y = \mathbb{Z}^r$. We prove that $|-2K_Z|$ contains a non-singular irreducible curve C_g of genus $g \geq 2$ which shows the geometric meaning of g. This is equivalent to saying that there is a curve

$$C = C_g + E_1 + \cdots + E_k \in |-2K_Y|,$$

where E_i are exceptional (i. e. irreducible with negative self-intersection) curves on Y with $(E_i)^2 = -4$. (The inequality $g \geq 2$ means that Y is of *elliptic type*). All these curves E_i come from the "good" resolution of singularities of Z described above. Thus, the invariant k equals the number of exceptional curves on Y with square -4. All of them are nonsingular and rational. E.g., if $k = 0$ then Z is Gorenstein and all of its singularities are Du Val. See Sect. 1.

Let us describe the invariant $\delta \in \{0, 1\}$. The components $C_g, E_1, \dots, E_k$ are disjoint. Since C is divisible by 2 in $\operatorname{Pic} Y$, it defines a double cover $\pi : X \to Y$ ramified in C. Let θ be the involution of the double cover. Then the set of fixed points $X^\theta = C$. Here, X is a K3 surface and

$$\delta = 0 \iff X^\theta \sim 0 \mod 2 \text{ in } H_2(X,\mathbb{Z}) \iff \tag{1}$$

there exist signs $(\pm)_i$ for which

$$\frac{1}{4}\sum_i \left[(\pm)_i cl(C^{(i)})\right] \in \operatorname{Pic} Y, \tag{2}$$

where $C^{(i)}$ are all irreducible components (i. e. C_g, $E_1, \dots, E_k$) of C.

The connection with K3 surfaces with non-symplectic involutions is the main tool of our classification.

As promised, our classification describes all intersection (or dual) graphs $\Gamma(Y)$ of exceptional curves on Y and also shows exceptional curves which must be contracted by $\sigma : Y \to Z$ to get the log del Pezzo surface Z of index ≤ 2 from Y. All these graphs can be obtained from graphs Γ in the right column of Table 3 and with the same main invariants (r, a, δ). Let describe this in more detail.

All exceptional curves E on Y are irreducible non-singular and rational. They are of three types:

(1) $E^2 = -4$, equivalently E is a component of genus 0 of $C \in |-2K_Y|$. In graphs of Table 3 these correspond to double transparent vertices;
(2) $E^2 = -2$. In graphs of Table 3 these correspond to black vertices;
(3) $E^2 = -1$ (the 1st kind). In graphs of Table 3 these correspond to simple transparent vertices.

All exceptional curves E_i with $(E_i)^2 = -4$, $i = 1, \dots k$, together with all exceptional curves F of the 1st kind such that there exist two different curves E_i, E_j, $i \neq j$, with $(E_i)^2 = (E_j)^2 = -4$ and $F \cdot E_i = F \cdot E_j = 1$ define the *logarithmic part* $\operatorname{Log}\Gamma(Y) \subset \Gamma(Y)$ *of* Y. Since $F \cdot (-2K_Y) = F \cdot C = 2$, the curves F are characterized by the property $C_g \cdot F = 0$. The logarithmic part $\operatorname{Log}\Gamma(Y)$ can be easily seen on graphs Γ of Table 3: curves E_i, $i = 1, \dots, k$, are shown as double transparent, the curves F of the first kind of $\operatorname{Log}\Gamma(Y)$ are shown as simple transparent vertices connected by two edges with (always two) double transparent vertices. This part of Γ is denoted by $\operatorname{Log}\Gamma$ and is also called *the logarithmic part of* Γ. Thus, we have:

$$\operatorname{Log}(\Gamma(Y)) = \operatorname{Log}\Gamma \tag{3}$$

(with the same main invariants (r, a, δ)). *The logarithmic part* $\operatorname{Log}\Gamma(Y)$ *gives precisely the preimage of singular points of* Z *of index two.*

All exceptional curves E on Y with $E^2 = -2$ define the *Du Val part* $\operatorname{Duv}\Gamma(Y) \subset \Gamma(Y)$ of $\Gamma(Y)$. Its connected components are Dynkin graphs A_n, D_n or E_n and they correspond to all Du Val singularities of Z. Thus *the Du Val part* $\operatorname{Duv}\Gamma(Y) \subset \Gamma(Y)$ *gives precisely the preimage of all Du Val (i. e. of index one) singular points of* Z. The Du Val part $\operatorname{Duv}\Gamma$ of a graphs Γ of Table 3 is defined by all its black vertices. We have:

$$D = \operatorname{Duv}\Gamma(Y) \subset \operatorname{Duv}\Gamma \tag{4}$$

(for the same main invariants (r, a, δ)). *Any subgraph* D *of* $\operatorname{Duv}\Gamma$ *can be taken.*

Let us describe the remaining part of $\Gamma(Y)$. Each graph Γ of Table 3 defines a lattice S in the usual way. It is

$$S = \left(\bigoplus_{v\in V(\Gamma)} \mathbb{Z}e_v\right) / \operatorname{Ker}$$

defined by the intersection pairing: $e_v^2 = -1$, if v is simple transparent, $e_v^2 = -2$, if v is black, $e_v^2 = -4$, if v is double transparent, $e_v \cdot e_{v'} = k$ if the vertices $v \ne v'$ are connected by k edges. "Ker" denotes the kernel of this pairing. We denote $E_v = e_v \mod \operatorname{Ker}$. In all cases except $N = 1$, 2, 11, the lattice S gives the Picard lattice $\operatorname{Pic} Y$ of Y. Thus, $\operatorname{Log}\Gamma(Y) = \operatorname{Log}\Gamma$ and $D = \operatorname{Duv}\Gamma(Y) \subset \operatorname{Duv}\Gamma$ define divisor classes E_v, $v \in V(\operatorname{Log}\Gamma(Y) \cup \operatorname{Duv}\Gamma(Y))$, of the corresponding exceptional curves on Y. Each exceptional curve E is evidently defined by its divisor class.

Black vertices $v \in V(\operatorname{Duv}\Gamma)$ define roots $E_v \in S$ with $E_v^2 = -2$ and define a finite Weyl group $W \subset O(S)$ generated by reflections in all these roots. The remaining part

$$\operatorname{Var}\Gamma(Y) = \Gamma(Y) - (\operatorname{Log}\Gamma(Y) \cup \operatorname{Duv}\Gamma(Y))$$

(it is called the *varying part of* $\Gamma(Y)$) is defined by the *varying part*

$$\operatorname{Var}\Gamma = \Gamma - (\operatorname{Duv}\Gamma \cup \operatorname{Log}\Gamma)$$

of the graph Γ of Table 3. Further, we identify exceptional curves $v \in V(\Gamma(Y))$ with their divisor classes $E_v \in S$. We have

$$V(\operatorname{Var}\Gamma(Y)) = \{E \in W(\{E_v \mid v \in V(\operatorname{Var}\Gamma)\}) \mid E_v \cdot D \ge 0\} \subset S. \tag{5}$$

The intersection pairing on S then defines the full graph $\Gamma(Y)$ of Y. This completes description of possible graphs $\Gamma(Y)$ of exceptional curves of log del Pezzo surfaces Z of index ≤ 2.

Thus, to find all possible graphs $\Gamma(Y)$ of exceptional curves of $\sigma :$ $Y \to Z$, one has to choose one of the graphs Γ of Table 3 (this also defines main invariants of Y), then one has to choose a subgraph $D = \operatorname{Duv}\Gamma(Y) \subset \operatorname{Duv}\Gamma$. Then $\Gamma(Y)$ consists of D, $\operatorname{Log}\Gamma(Y) = \operatorname{Log}\Gamma$ and the remaining part $\operatorname{Var}\Gamma(Y)$ defined by (5). See Theorems 3.16, 3.17 and 4.1.

We note two important opposite cases.

Extremal case. This is the case when $D = \operatorname{Duv}\Gamma(Y) = \operatorname{Duv}\Gamma$. Then $\Gamma(Y) = \Gamma$ is completely calculated in Table 3. This case is called *extremal* and gives log del Pezzo surfaces Z with Du Val singularities of the highest rank, respectively $\operatorname{rk}\operatorname{Pic} Z = r - \#V(\operatorname{Log}\Gamma(Y)) - \#V(\operatorname{Duv}\Gamma(Y))$ is minimal for the fixed main invariants. In particular, this case delivers all cases of minimal log del Pezzo surfaces of index ≤ 2 when $\operatorname{rk}\operatorname{Pic} Z = 1$. See Theorems 3.16, 4.2.

No Du Val singularities. This is the case when $D = \operatorname{Duv}\Gamma(Y) = \emptyset$. Equivalently, all singularities of Z have index 2, if they exist. Then $\Gamma(Y) = \operatorname{Log}\Gamma \cup \operatorname{Var}\Gamma(Y)$ where

$$V(\operatorname{Var}\Gamma(Y)) = W\left(\{E_v \mid v \in V(\operatorname{Var}\Gamma)\}\right).$$

Here, all the multiple cases 7a,b, 8a–c, 9a–f, 10a–m, 20a–d give the same surfaces, and one can always take cases 7a, 8a, 9a, 10a, 20a for the main invariants. This case is very similar to and includes the classical case of non-singular del Pezzo surfaces corresponding to the cases 1— 10. See Theorem 4.4 about this (without Du Val singularities) case. Log del Pezzo surfaces of this case are defined by their main invariants (r, a, δ) up to deformation. Du Val parts $\operatorname{Duv}\Gamma$ of graphs Γ of Table 3 can be considered as analogs to root systems (or Dynkin diagrams) which one usually corresponds to non-singular del Pezzo surfaces. In cases 7 — 10, 20, one should take graphs Γ of cases 7a — 10a, 20a.

The Root invariant. It is possible that two different subgraphs $D \subset \Gamma$, $D \subset \Gamma'$ of graphs of Table 3 (with the same main invariants (r, a, δ)) give isomorphic graphs $\Gamma(Y)$ and $\Gamma(Y')$ for the corresponding "good" resolutions, and then they give similar log del Pezzo surfaces Z and Z' of index ≤ 2, according to our classification. The *root invariant*

$$([D], \xi) \tag{6}$$

gives the necessary and sufficient condition when thisc happens.

To define the root invariant (6), we first remark that the main invariants (r, a, δ) define a unique hyperbolic (i. e. with one positive square) even 2-elementary lattice S with these invariants. Here $r = \operatorname{rk} S$, $S^*/S \cong (\mathbb{Z}/2)^a$, and $\delta = 0$, if and only if $(x^*)^2 \in \mathbb{Z}$ for any $x \in S^*$. In (6), the $[D]$ is the root lattice generated by D, and $\xi : [D]/2[D] \to S^*/S$ a homomorphism preserving finite forms $(x^2)/2 \mod 2$, $x \in [D]$, and $y^2 \mod 2$, $y \in S^*$. The construction of the root invariant (6) uses the double cover $\pi : X \to Y$ by K3 surface X (see above) with the non-symplectic involution θ of the double cover. Then $S = H^2(X,\mathbb{Z})^\theta$ is the sublattice where θ^* is identical. The root invariant (6) is considered up to automorphisms of S and the lattice D. See Sect. 2.4 about this construction and a very easy criterion (the kernel H of ξ is almost equivalent to ξ) about isomorphism of root invariants. The root invariant was first introduced and used in [Nik84.1] and [Nik87].

In practice, to calculate the root invariant for log del Pezzo surfaces of index ≤ 2, one should just go from the graphs Γ of Table 3 to the equivalent graphs $\Gamma(P(\mathcal{M}^{(2,4)}))$ or $\Gamma(P(X)_+)$ of Tables 1 or 2 of exceptional curves for the K3 pairs (X,θ) (see Sects 3.2, 3.5).

Thus, two Du Val subgraphs $D \subset \Gamma$, $D \subset \Gamma'$ of graphs of Table 3 give isomorphic full graphs $\Gamma(Y)$ and $\Gamma(Y')$ of their log del Pezzo surfaces, if and only if their root invariants (6) are isomorphic. Moreover, we constantly use the root invariant to prove existence of the corresponding K3 pairs (X,θ) and log del Pezzo surfaces Z. The main invariants (r,a,δ) and the root invariants (6) are the main our tools in classification. They are equivalent to the full graphs $\Gamma(Y)$ of exceptional curves on Y, but they are much more convenient to work with. For non-singular del Pezzo surfaces and log del Pezzo surfaces of index ≤ 2 without Du Val singularities the root invariant is zero. It is why, in this cases, we have so simple classification above.

Acknowledgment. We are grateful to V.A. Iskovskikh for useful discussions.

1. Log del Pezzo surfaces of index ≤ 2 and Smooth Divisor Theorem.

1.1. **Basic definitions and notations.** Let X be a normal algebraic surface over the field $\mathbb{C}$, and K_X be a canonical Weil divisor on it. The surface X is called $\mathbb{Q}$-Gorenstein if a certain positive multiple of K_X is Cartier, and $\mathbb{Q}$-factorial if this is true for any Weil divisor D. These properties are local: one has to require all singularities to be $\mathbb{Q}$-Gorenstein, resp. $\mathbb{Q}$-factorial.

Let us denote by $\mathrm{Div}(X)$, $Z^1(X)$ respectively the groups of Weil, resp. Cartier, divisors on X. Assume that X is $\mathbb{Q}$-factorial. Then the groups $\mathrm{Div}(X)\otimes\mathbb{Q}$, resp. $Z^1(X)\otimes\mathbb{Q}$ of $\mathbb{Q}$-Cartier divisors and $\mathbb{Q}$-Weil divisors coincide. The intersection form defines natural pairings

$$\mathrm{Div}(X)\otimes\mathbb{Q}\times\mathrm{Div}(X)\otimes\mathbb{Q}\to\mathbb{Q},$$

$$\mathrm{Div}(X)\otimes\mathbb{R}\times\mathrm{Div}(X)\otimes\mathbb{R}\to\mathbb{R}.$$

Quotient groups modulo kernels of these pairings are denoted $N_{\mathbb{Q}}(X)$ and $N_{\mathbb{R}}(X)$ respectively; if the surface X is projective, they are finite-dimensional linear spaces. The Mori cone is a convex cone $\overline{\mathrm{NE}}(X)$ in $N_{\mathbb{R}}(X)$, the closure of the cone generated by classes of effective curves.

Let D be a $\mathbb{Q}$-Cartier divisor on X. We will say that D is ample if some positive multiple is an ample Cartier divisor in the usual sense. By Kleiman's criterion, for this to hold it is necessary and sufficient that D, as a liner function on $N_{\mathbb{R}}(X)$ were strictly positive on $\overline{\mathrm{NE}}(X)-\{0\}$.

One say that the surface X has only log terminal singularities if it is $\mathbb{Q}$-Gorenstein and for a resolution of singularities $\pi : Y \to X$, in a natural formula $K_Y = \pi^* K_X + \sum \alpha_i F_i$, where F_i are simple divisors and $\alpha_i \in \mathbb{Q}$, one has $\alpha_i > -1$. The least common multiple of denominators of α_i is called the index of X.

It is known that two-dimensional log terminal singularities in characteristic zero are exactly the quotient singularities. They are listed in [Bri68]. They are all rational and $\mathbb{Q}$-factorial. We can now formulate the following

Definition 1.1. A normal complete surface X is called *a log del Pezzo surface of index* $\le k$ if it has only log terminal singularities of index $\le k$ and the anticanonical divisor $-K_X$ is ample. Equivalently, X is *a del Pezzo surface* (i. e. $-K_X$ is ample) *with log-terminal singularities of index* $\le k$.

We will use the following notations. If D is a $\mathbb{Q}$-Weil divisor, $D = \sum c_i C_i$, $c_i \in \mathbb{Q}$, then $\ulcorner D \urcorner$ will denote the ceiling $\sum \ulcorner c_i \urcorner D_i$, and $\{D\} = \sum \{c_i\} C_i$ its fractional part. A divisor D is *nef* if for any curve C, $D \cdot C \ge 0$; D is *big* if $D^2 > 0$.

Below we will frequently use the following

Theorem 1.2 (Kawamata, Viehweg, [Kaw82, Vie82])**.** *Let Y be a smooth surface and let D be a $\mathbb{Q}$-divisor on Y such that*

(1) $\mathrm{supp}\{D\}$ *is a divisor with normal crossings;*
(2) D *is big and nef.*

Then $H^i(K_Y + \ulcorner D \urcorner) = 0$ for $i > 0$.

1.2. **Log terminal singularities of index 2.** Let (X, p) be a two-dimensional log terminal singularity of index ≤ 2, and $\pi : \widetilde{X} \to X$ be its minimal resolution. We have $K_{\widetilde{X}} = \pi^* K_X + \sum \alpha_i F_i$, where $-1 < \alpha_i \le 0$ and $F_i^2 \le -2$. Therefore, for each i one has $\alpha_i = -1/2$ or 0. One can rewrite the set of equations $K_{\widetilde{X}} \cdot F_i = -F_i^2 - 2$ in a matrix form:

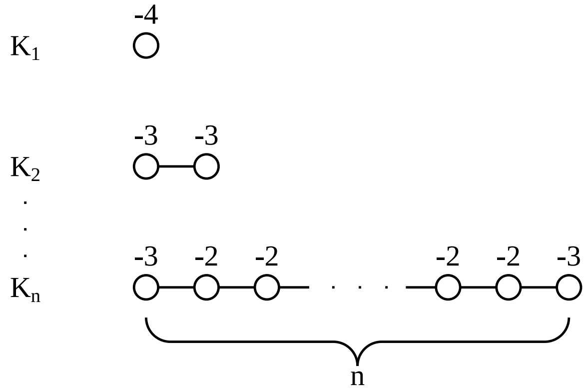
$$M \cdot (\alpha_1, \dots, \alpha_n)^t = (-F_1^2 - 2, \dots, -F_n^2 - 2)^t,$$

where $M = (F_i \cdot F_j)$ is the intersection matrix. M is nondegenerate and negative definite.

Let us give some easy consequences of this formula.

(1) If for some i_0, $\alpha_{i_0} = 0$ then all $\alpha_i = 0$, and the singularity (X, p) is Du Val, of type A_n, D_n, E_6, E_7 or E_8.

(2) If all $\alpha_i = -1/2$ then we get the following list of singularities:



In these graphs every curve F_i corresponds to a vertex with weight F_i^2, two vertices are connected by an edge if $F_i \cdot F_j = 1$ and are not connected if $F_i \cdot F_j = 0$.

1.3. **Basic facts about log del Pezzo surfaces.**

Lemma 1.3. *All log del Pezzo surfaces X are rational.*

Proof. Let $\pi : \widetilde{X} \to X$ be the minimal resolution of singularities, $K_{\widetilde{X}} = \pi^* K_X + \sum \alpha_i F_i$, $-1 < \alpha_i \leq 0$. Then

(7)
$$\begin{aligned} h^1(O_{\widetilde{X}}) =& h^1(K_{\widetilde{X}} + (-K_{\widetilde{X}})) = h^1(K_{\widetilde{X}} + \ulcorner -\pi^* K_X + \sum (-\alpha_i) F_i \urcorner) = \\ & h^1(K_{\widetilde{X}} + \ulcorner -\pi^* K_X \urcorner) = 0 \end{aligned}$$

by Kawamata-Viehweg theorem.

Also, $h^0(nK_{\widetilde{X}}) = 0$ for any positive integer n since $-\pi_* K_{\widetilde{X}} = -K_X$ is an effective nonzero $\mathbb{Q}$-Weil divisor. Therefore, by Castelnuovo criterion the surface $\widetilde{X}$, and hence also X, are rational. □

Lemma 1.4. *In the above notations, if $X \neq \mathbb{P}^2$ or $\mathbb{F}_n$ then the Mori cone of the surface $\widetilde{X}$ is generated by the curves F_i and exceptional curves of the 1st kind. The number of these curves is finite. There are no other curves with negative self-intersection number on $\widetilde{X}$.*

Proof. Let us show that on the surface $\widetilde{X}$ there exists a $\mathbb{Q}$-divisor Δ with $\Delta \geq 0$, $[\Delta] = 0$ and such that the divisor $-(K_{\widetilde{X}} + \Delta)$ is ample.

Choose $Z = \sum \beta_i F_i$ so that $Z \cdot F_i > 0$. Since the matrix $(F_i \cdot F_j)$ is negative definite and $F_i \cdot F_j \geq 0$ for $i \neq j$, one has $\beta_i < 0$. Let us show that for a small $0 < \epsilon \ll 1$ the divisor $T = -\pi^* K_X + \epsilon Z$ is ample. First of all $T^2 = K_X^2 + \epsilon^2 Z^2 > 0$ if $0 < \epsilon < \sqrt{(K_X^2)/(-Z^2)}$. We shall assume that. Let C be an irreducible curve on $\widetilde{X}$. Let us show that for small ϵ we have $C \cdot T > 0$.

If $C^2 \geq 0$, this follows from the fact that the intersection form on $N_{\mathbb{R}}(X)$ is hyperbolic.

If $C = F_i$ then $F_i \cdot T = \epsilon F_i \cdot Z = 0$.

If $C^2 < 0$ and $C \neq F_i$ then

$$C \cdot K_{\widetilde{X}} = C \cdot (\pi^* K_{\widetilde{X}} + \sum \alpha_i F_i) < 0.$$

On the other hand, $p_a(C) = \dfrac{C^2 + C \cdot K_{\widetilde{X}}}{2} + 1 \geq 0$. So, $C^2 < 0$ and $C \cdot K_{\widetilde{X}} < 0$ implies that $C^2 = -1$, $p_a(C) = 0$, i.e. C is an exceptional curve of the 1st kind. The conditions $nC \cdot \pi^* K_X \in \mathbb{Z}$ (where n is the index of X), $C^2 = -1$ and $0 < -C \cdot \pi^* K_X = 1 + \sum \alpha_i F_i \cdot C \leq 1$ imply that the number of exceptional curves of the 1st kind is finite. Hence, $C(-\pi^* K_X) > 0$ implies that for sufficiently small $0 < \epsilon \ll 1$ we have $T \cdot C = (-\pi^* K_X + \epsilon Z) \cdot C > 0$. Therefore the divisor $T = -\pi^* K_X + \epsilon Z$ is ample. But $-\pi^* K_X + \epsilon Z = -\left(K_{\widetilde{X}} + \sum(-\alpha_i - \epsilon\beta_i)F_i\right)$. Therefore, $\Delta \geq 0$, and for sufficiently small ϵ, we have $[\Delta] = 0$ because $\alpha_i > -1$.

Now, by [Kaw84, Thm.4.5], $NE(X) = \sum R_j$, where R_j are "good extremal rays". The curves F_i and exceptional curves of the 1st kind are obviously extremal rays. On the other hand, let R_j be a "good extremal ray", generated by an irreducible curve C. If $C \notin \{F_1, \dots, F_k\}$ then $C \cdot K_X = C \cdot (\pi^* K_X + \sum \alpha_i F_i) < 0$. Hence, by [Mor82] the curve C is an exceptional curve of the 1st kind, if X is different from $\mathbb{P}^2$ or $\mathbb{F}_n$. $\square$

1.4. **Smooth Divisor Theorem.**

Theorem 1.5. *Let X be a log del Pezzo surface of index ≤ 2. Then the linear system $|-2K_X|$ is nonempty, has no fixed components and contains a nonsingular element $D \in |-2K_X|$.*

Proof. Let $\pi : \widetilde{X} \to X$ be the minimal resolution of singularities. It is sufficient to prove the statement for the linear system $|-\pi^* 2K_X|$ on $\widetilde{X}$. We have $2K_{\widetilde{X}} = \pi^* 2K_X - \sum a_i F_i$, all $a_i = 0$ or 1 ($a_i = -2\alpha_i$).

1. *Nonemptiness.*

$$-\pi^* 2K_X = K_{\widetilde{X}} + (-3K_{\widetilde{X}} - \sum a_i F_i) = K_{\widetilde{X}} + \ulcorner D \urcorner,$$

where $D = \dfrac{3}{2}(-2K_{\widetilde{X}} - \sum a_i F_i)$ is big, nef, since $D = \dfrac{3}{2}(-\pi^* 2K_X)$. Hence, by Kawamata-Viehweg theorem1.2, $H^i(-\pi^* 2K_X) = 0$ for $i > 0$ and $h^0(-\pi^* 2K_X) = \chi(-\pi^* 2K_X) = 3K_X^2 + 1 > 0$.

2. *No fixed components.* Let E be the fixed part, so that $|-\pi^* 2K_X - E|$ is a movable linear system. Then

$$
\begin{aligned}
&h^0(-\pi^* 2K_X) = h^0(-\pi^* 2K_X - E),\\
&-\pi^* 2K_X - E = K_{\widetilde{X}} + (-3K_{\widetilde{X}} - \sum a_i F_i - E) = K_{\widetilde{X}} + \ulcorner D \urcorner,\\
&D = \frac{3}{2}(-2K_{\widetilde{X}} - \sum a_i F_i) - E = (-\pi^* 2K_X - E) + (-\pi^* K_X) =\\
&D_1 + D_2\,.
\end{aligned}
$$

Here, D_1 is movable and D_2 is big, nef, hence $D_1 + D_2$ is big, nef. By Kawamata-Viehweg Theorem 1.2,

$$h^i(-\pi^* 2K_X - E) = 0, \quad i > 0,$$

$$\chi(-\pi^* 2K_X) = \chi(-\pi^* 2K_X - E),$$

$$2\chi(-\pi^* 2K_X) - 2\chi(-\pi^* 2K_X - E) = E \cdot (-2\pi^* 2K_X - K_{\widetilde{X}} - E). \tag{8}$$

Let us show that this expression (8) is not equal to zero. Suppose

$$-\pi^* K_X \cdot (K_{\widetilde{X}} + E) = \pi^* K_X \cdot E - K_X^2 < 0.$$

Then the divisor $K_{\widetilde{X}} + E$ cannot be effective. Therefore,

$$\chi(-E) = h^0(-E) - h^1(-E) + h^0(K_{\widetilde{X}} + E) \leq 0.$$

Hence, $E \cdot (K_{\widetilde{X}} + E) = 2\chi(-E) - 2 < 0$, and the expression (8) is strictly positive. So, we can assume that $-\pi^* K_X \cdot E \geq K_X^2$. Let us write

$$E = \beta(-\pi^* K_X) + F, \quad F \in (\pi^* K_X)^{\perp}$$

in $N_{\mathbb{Q}}(\widetilde{X})$. One has $\beta \leq 2$ since $-\pi^* K_X \cdot (-\pi^* 2K_X - E) \geq 0$. Then

$$E(-2\pi^* 2K_X - K_{\widetilde{X}} - E) = (5 - \beta)\beta K_X^2 - F \cdot (\sum \alpha_i F_i + F).$$

The first term in this sum is ≥ 3 since $\beta K_X^2 = -\pi^* K_X \cdot E \geq K_X^2$ and $K_X^2 = \chi(\pi^* 2K_X) - 1$ is a positive integer. The second term achieves the minimum for $F = -\dfrac{1}{2}\sum \alpha_i F_i$ and equals $-\dfrac{m}{4}$, where m is the number of non-Du Val singularities. Therefore, all that remains to be shown is that surface X has less than 12 non-Du Val singularities. This will be obvious from the following Proposition.

Proposition 1.6. *Let X be a log del Pezzo surface of index ≤ 2 and $\pi : \widetilde{X} \to X$ be its minimal resolution of singularities. Then $K^2_{\widetilde{X}} \ge 0$.*

Proof. One has $K_{\widetilde{X}} = \pi^* K_X + \sum \alpha_i F_i$. Let us set $\overline{K} = \pi^* K_X - \sum \alpha_i F_i$. Let us show that $-\overline{K}$ is nef. By Lemma 1.4, one has to show that $-\overline{K} \cdot F_i \ge 0$ and $-\overline{K} \cdot C \ge 0$ if C is an exceptional curve of the 1st kind. Indeed, $-\overline{K} \cdot F_i = K_{\widetilde{X}} \cdot F_i = -F_i^2 - 2 \ge 0$ since resolution π is minimal. Next,

$$-\pi^* K_X \cdot C = -K_{\widetilde{X}} \cdot C + \sum \alpha_i F_i \cdot C = 1 + \sum \alpha_i F_i \cdot C > 0.$$

Since this number is half-integer,

$$-\overline{K} \cdot C = 1 + 2 \sum \alpha_i F_i \cdot C \ge 0.$$

So, $-\overline{K}$ is nef and $K^2_{\widetilde{X}} = \overline{K}^2 \ge 0$. Finally, if $\widetilde{X} = \mathbb{P}^2$ or $\mathbb{F}_n$ then $K^2_{\widetilde{X}} = 9$ or 8 respectively. □

By Lemma 1.3, surface $\widetilde{X}$ is rational. By Noether formula, $\operatorname{rk} \operatorname{Pic} \widetilde{X} \le 10$; hence in any case X has no more than 10 singular points.

3. *Existence of a smooth element.* Assume that all divisors in the linear system $|-\pi^* 2K_X|$ are singular. Then there exists a fixed pint P, and for a general element $D \in |-\pi^* 2K_X|$ the multiplicity of D at point P is $k \ge 2$. This point does not lie on F_i since $-\pi^* 2K_X \cdot F_i = 0$. Let $\epsilon : Y \to \widetilde{X}$ be the blowup at P, and $f = \pi\epsilon : Y \to X$, L be the exceptional divisor of ϵ. We have: $h^0(-f^* 2K_X) = h^0(-f^* 2K_X - L)$, the linear system $|-f^* 2K_X - kL|$ is movable, and

$$\begin{aligned}
&2K_Y = f^* 2K_X - \sum a_i F_i + 2L,\\
&-f^* 2K_X = K_Y + (-3K_Y - \sum a_i F_i + 2L) = K_Y + \ulcorner D \urcorner,\\
&D = \frac{3}{2}(-2K_Y - \sum a_i F_i + L) = \frac{3}{2}(-f^* 2K_X - L) =\\
&\frac{3}{2}[(-f^* 2K_X - kL) + (k-1)L].
\end{aligned}$$

The divisor D is nef since for any irreducible curve $C \ne L$, $C \cdot D \ge 0$, and also $D \cdot L = 1$. It is big since $(-f^* 2K_X - L)^2 = 4K_X^2 - 1 > 0$. Now,

$$\begin{aligned}
&-f^* 2K_X - L = K_Y + (-3K_Y - \sum a_i F_i + L) = K_Y + \ulcorner D \urcorner,\\
&D = \frac{3}{2}(-2K_Y - \sum a_i F_i + \frac{2}{3}L) = \frac{3}{2}[(-f^* 2K_X - kL) + (k - \frac{4}{3})L].
\end{aligned}$$

This divisor D is nef since for any irreducible curve $C \neq L$, $C \cdot D \geq 0$, and also $D \cdot L = 2$; D is big since

$$(-f^*2K_X - \frac{4}{3}L)^2 = 4K_X^2 - \frac{16}{9} > 0.$$

Now, again by Kawamata-Viehweg theorem, $h^i(-f^*2K_X) = h^i(-f^*2K_X - L) = 0$ for $i > 0$, and one must have $\chi(-f^*2K_X) = \chi(-f^*2K_X - L)$. But

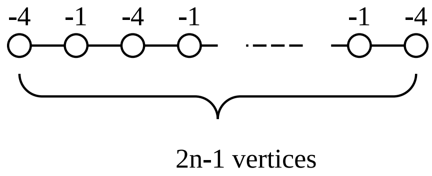
$$\begin{aligned} \chi(-f^*2K_X) - \chi(-f^*2K_X - L) &= \frac{1}{2}L \cdot (-2f^*2K_X - K_X - L) = \\ & \quad 1 + L \cdot (-f^*2K_X) > 0. \end{aligned}$$

The contradiction thus obtained completes the proof of the theorem. □

Remark 1.7. In the same way, parts 1 and 3 are proved for a log del Pezzo surface of arbitrary index n and the linear system $-\pi^*nK_X$. Part 2 is easy to prove under the assumption that $\pi(E)$ passes only through Du Val singularities.

1.5. **Reduction to DPN surfaces of elliptic type.** Let X be a log del Pezzo surface of index ≤ 2. Consider a resolution of singularities $f : Y \to X$ for which every Du Val singularity is resolved by inserting a usual tree of (-2)-curves, and the singularity K_n – by the following chain

(9)



The later resolution is obtained by blowing up all intersection points of exceptional curves on the minimal resolution of K_n points, see their diagrams in Section 1.2. In contrast to the minimal resolution, we will call this the *"good"* resolution of singularities. Consider a smooth element $C_g \in |-2K_X|$. It does not pass through singularities of the surface X. If we identify the curve C_g with its image under the morphism f, then it is easy to see from the formulas of Section 1.2 that $-f^*2K_X$ coincides with C_g, and $-2K_Y$ – with the disjoint union of C_g and curves in the above diagrams which have self-intersection -4. Moreover, it is easy to compute the genus of the curve C_g, and it equals

$g = K_X^2 + 1 \geq 2$. This shows that the surface Y is a "good" DPN surface of elliptic type in the sense of the next Section (see Sects 2.1 and 2.7).

Vice versa, the results of Sects 2 and 3 will imply (see Sect. 4) that a "good" DPN surface Y of elliptic type admits a unique contraction of exceptional curves $f : Y \to X$ to a log del Pezzo surface of index ≤ 2.

In this way, classification of log del Pezzo surfaces of index ≤ 2 is reduced to classification of "good" DPN surfaces of elliptic type.

2. General Theory of DPN surfaces

2.1. **General remarks.** As it was shown in Section 1, a description of log del Pezzo surfaces of index ≤ 2 is reduced to a description of rational surfaces Y containing a nonsingular curve $C \in |-2K_Y|$ and a certain configuration of exceptional curves. Such surfaces Y and exceptional curves on them were studied in the papers [Nik79, Nik83, Nik84.1, Nik87] of the second author. They are one of possible generalizations of del Pezzo surfaces.

Many other generalizations of del Pezzo surfaces were proposed, see f.e. [Dem80, Har85.1, Har85.2, Loo81], and most authors call their surfaces "generalized del Pezzo surfaces". Therefore, we decided following [Nik87] to call our generalization DPN surfaces.

Definition 2.1. A nonsingular projective algebraic surface Y over $\mathbb{C}$ is called a DPN surface if its irregularity $q(Y) = 0$, $K_Y \neq 0$ and there exists an effective divisor $C \in |-2K_Y|$ with only simple rational, i.e. A, D, E-singularities. Such a pair (Y, C) is called a DPN pair. A DPN surface Y is called *"good"* if there exists a nonsingular divisor $C \in |-2K_Y|$; in this case the pair (Y, C) is called "good" or nonsingular DPN pair.

The classification of surfaces implies that if $C = \emptyset$ then Y is an Enriques surface. If $C \neq \emptyset$ then Y is rational.

Using the well-known properties of blowups, the following results are easy to prove. Let (Y, C) be a DPN pair, $E \subset Y$ be an exceptional curve of the 1st kind on Y and $\sigma : Y \to Y'$ the contraction of E. Then $(Y', \sigma(C))$ is also a DPN pair. In this way, by contracting exceptional curves of the 1st kind, one can always arrive at a DPN pair (Y', C') where Y' is a relatively minimal rational surface. In this case, the only possibilities for Y' are $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$, since only for them $|-2K_{Y'}|$ contains a reduced divisor. If $Y' = \mathbb{P}^2$ then C' is a curve of degree 6; if $Y' = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ then C' is a curve of bidegree $(4, 4)$; if $Y' = \mathbb{F}_2$ then $C' \in |8f + 4s_2|$; if $Y' = \mathbb{F}_3$ then $D' = C_1 + s_3$, where $C_1 \in |10f + 3s_3|$; if $Y' = \mathbb{F}_4$ then $C' = D_1 + s_4$, where $D_1 \in |12f + 3s_4|$.

Here, the linear system $|f|$ is a pencil of rational curves on surface $\mathbb{F}_n$ with a section s_n, $s_n^2=-n$. Vice versa, if (Y',C') is a DPN pair, P is a singular point of C' and $\sigma : Y \to Y'$ is a blowup of P with an exceptional (-1)-curve E then (Y,C) is a DPN pair, where

$$C=\begin{cases}\sigma^{-1}(C') & \text{if } P \text{ has multiplicity 2 on } C'\\ \sigma^{-1}(C')+E & \text{if } P \text{ has multiplicity 3 on } C'\end{cases}$$

In this way, by blowups, from an arbitrary DPN pair (Y',C') one can always pass to a "good" DPN pair (Y,C), i.e. with a nonsingular C. A description of arbitrary DPN pairs and surfaces is thus reduced to a description of nonsingular DPN pairs and "good" DPN surfaces and exceptional curves on Y, where a curve $E\subset Y$ is called exceptional if E is irreducible and $E^2<0$.

Let (Y,C) be a nonsingular DPN pair. Since $C\in|-2K_Y|$, there exists a double cover $\pi : X\to Y$ defined by $-K_Y$, branched along C. From the classification of surfaces, it follows that X is a K3 surface, and the involution θ of the double cover π has the following property: $\theta^*(\omega_X)=-\omega_X$, where ω_X is a regular two-dimensional nowhere vanishing differential form on X. (Recall that $K_X=0$ and $q(X)=0$.) Such an involution θ on X is called *non-symplectic*. Vice versa, if (X,θ) is a K3 surface with a non-symplectic involution then the set X^θ of fixed points of the involution is nonsingular, and the pair $\big(Y=X/\{1,\theta\}, C=\pi(X^\theta)\big)$ is a "good" DPN pair where $\pi : X\to Y$ is the quotient morphism. Thus, a description of "good" DPN pairs (Y,C) is reduced to a description of K3 surfaces with a non-symplectic involution (X,θ).

2.2. **The lattice S, and the main invariants (r,a,δ) (equivalently (k,g,δ)) of pairs (X,θ) and (Y,C).** Let $S_X\subset H^2(X,\mathbb{Z})$ be the Picard lattice of a K3 surface X with a non-symplectic involution θ. Set

$$\begin{aligned}&(S_X)_\pm=\{x\in S_X\,|\,\theta^*(x)=\pm x\} \text{ and}\\ &H^2(X,\mathbb{Z})_\pm=\{x\in H^2(X,\mathbb{Z})\,|\,\theta^*(x)=\pm x\}.\end{aligned}$$

One has $(S_X)_+=H^2(X,\mathbb{Z})_+$. A fundamental invariant of the pair (X,θ) and the corresponding DPN pair (Y,C) is the lattice

$$S=(S_X)_+=H^2(X,\mathbb{Z})_+.$$

This is an even 2-elementary (see below) hyperbolic (it follows that X is algebraic) lattice. Up to an isomorphism, it is determined by a triple of invariants (r,a,δ). Here $r=\operatorname{rk} S$ is the rank of S. Here $a\in\mathbb{N}$ is defined by the condition $S^*/S\simeq(\mathbb{Z}/2\mathbb{Z})^a$ where $S^*=Hom(S,\mathbb{Z})$ is the dual lattice (such lattices S are called *2-elementary*). The invariant δ takes values in $\{0,1\}$. The $\delta=0 \iff (x^*)^2\in\mathbb{Z}$ for every $x^*\in S^* \iff$ *the discriminant quadratic form of* S

$$q_S:\mathfrak{A}=S^*/S\to\mathbb{Q}/2\mathbb{Z},\quad q_S(x^*+S)=(x^*)^2+2\mathbb{Z}$$

is even: it takes values in $(\mathbb{Z}/2\mathbb{Z})$. See [Nik80.2] about discriminant forms of lattices. The triple (r,a,δ) admits an interpretation in terms of $X^\theta=C$. If $(r,a,\delta)\neq(10,8,0)$ or $(10,10,0)$ then

$$X^\theta=C=C_g+E_1+\cdots+E_k,$$

where C_g is a nonsingular irreducible curve of genus g, and $E_1,\dots E_k$ are nonsingular irreducible rational curves, the curves are disjoint to each other,

$$g=(22-r-a)/2,\quad k=(r-a)/2$$

(we shall formally use the same formulae for g and k even in cases $(r,a,\delta)=(10,8,0)$ or $(10,10,0)$). If $(r,a,\delta)=(10,8,0)$ then $X^\theta\simeq C=C_1^{(1)}+C_1^{(2)}$, where $C_1^{(i)}$ are elliptic (genus 1) curves. If $(r,a,\delta)=(10,10,0)$ then $X^\theta\simeq C=\emptyset$, i.e. in this case Y is an Enriques surface. One has

(10)
$$\delta=0 \iff X^\theta\sim 0 \mod 2 \text{ in } S_X \text{ (equivalently in } H_2(X,\mathbb{Z})) \iff$$

there exist signs $(\pm)_i$ for which

$$\frac{1}{4}\sum_i\left[(\pm)_i cl(C^{(i)})\right]\in S_Y, \tag{11}$$

where $C^{(i)}$ go over all irreducible components of C. Signs $(\pm)_i$ for $\delta=0$ are defined uniquely up to a simultaneous change and define the other (with respect to the complex) orientation of components of C. All possibilities for (r,a,δ) (equivalently, $(k=(r-a)/2, g=(22-r-a)/2,\delta)$) are known and are shown on Figure 1.

The invariants (r,a,δ) (equivalently (k,g,δ)) of K3 surfaces with non-symplectic involutions, and the corresponding DPN pairs and DPN surfaces play a crucial role in our classification, and henceforth we call them *The Main Invariants.* Pairs (X,θ) and DPN pairs (Y,C) with given main invariants (r,a,δ) form connected smooth families $\mathcal{M}_{(r,a,\delta)}$ of surfaces without polarization. Here, $\dim\mathcal{M}_{(r,a,\delta)}=20-r$. For a general point $m\in\mathcal{M}_{(r,a,\delta)}$ the corresponding surface $X=X_m$ has $S=(S_X)_+=S_X$, involution θ is canonically defined by the surface X itself as an automorphism $\theta\in\operatorname{Aut}X$ which acts identically on S_X and as multiplication by (-1) on ω_X (or $T_X=(S_X)^\perp_{H^2(X,\mathbb{Z})}$).

The results described in this section were obtained in papers [Nik80.1, Nik80.2, Nik83] (see also [Nik87]).

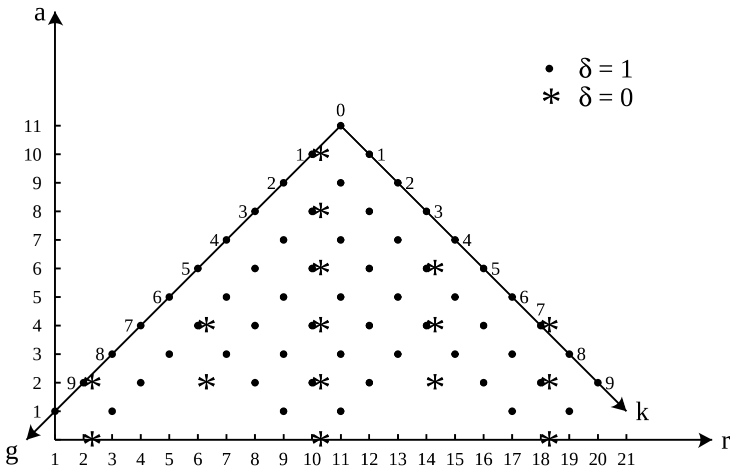


FIGURE 1. All possible main invariants (r, a, δ)

2.3. **Exceptional curves on** (X, θ) **and** Y**.** A description of exceptional curves on Y can also be reduced to a description of K3 surfaces X with a non-symplectic involution θ considered above.

Let X be any algebraic K3 surface and $F \subset X$ be an exceptional curve on X, i.e. $F^2 < 0$ and F is irreducible. Then $F^2 = -2$ and F is a nonsingular rational curve on X. The subset $P(X) \subset S_X$ of divisor classes of all exceptional curves F on X has the following description. Let

$$V(X) = \{x \in S_X \otimes \mathbb{R} \,|\, x^2 > 0\}.$$

Since the lattice S_X is hyperbolic (i.e. has signature $(1, t)$), the cone $V(X)$ has two convex halves. One of these halves $V^+(X)$ is distinguished by the fact that it contains the ray $\mathbb{R}^+ h$ of a polarization h on X. It defines the hyperbolic space

$$\mathcal{L}(X) = V^+(X)/\mathbb{R}^+.$$

The elements of this space are rays $\mathbb{R}^+ x$, where $x \in S_X \otimes \mathbb{R}$, $x^2 > 0$ and $x \cdot h > 0$. Each element $\beta \in S_X \otimes \mathbb{R}$ with square $\beta^2 < 0$ defines a half-space

$$\mathcal{H}^+_\beta = \{\mathbb{R}^+ x \in \mathcal{L}(X) \,|\, \beta \cdot x \geq 0\},$$

so that β is perpendicular to the bounding hyperplane

$$\mathcal{H}_\beta = \{\mathbb{R}^+ x \in \mathcal{L}(X) \,|\, \beta \cdot x = 0\},$$

and faces outward. The set

$$\mathcal{M}(X) = \bigcap_{\substack{f\in S_X, f^2=-2\\ f \text{ is effective}}} \mathcal{H}_\delta^+ = \bigcap_{f\in P(X)} \mathcal{H}_\delta^+$$

is a locally finite convex polytope in $\mathcal{L}(X)$. The set $P(\mathcal{M}(X))$ of vectors with square -2, perpendicular to the facets of $\mathcal{M}(X)$ and directed outward, is exactly the set $P(X)$ of divisor classes of exceptional curves on X. Moreover, $\mathcal{M}(X)$ admits a description in terms of groups. Let $O'(S_X)$ be the subgroup of index two of the full automorphism group $O(S_X)$ of the lattice S_X which consists of automorphisms which preserve the half-cone $V^+(S_X)$. Let $W^{(2)}(S_X) \subset O'(S_X)$ be the subgroup of $O'(S_X)$ generated by reflections s_f with respect to all elements $f \in S_X$ with square (-2):

$$S_f(x) = x + (x\cdot f)f, \qquad x \in S_X.$$

The action of the group $W^{(2)}(S_X)$, as well as $O'(S_X)$, on $\mathcal{L}(X)$ is discrete. $W^{(2)}(S_X)$ is the group generated by reflections in hyperplanes $\mathcal{H}_f$, $f \in S_X$ and $f^2=-2$. The set $\mathcal{M}(X)$ is a chamber for this group, i.e. $W^2(S_X)(\mathcal{M}(X))$ defines a decomposition of $\mathcal{L}(X)$ into polytopes which are congruent to $\mathcal{M}(X)$, and $W^2(S_X)$ acts transitively on this decomposition (cf. [PS-Sh71, Vin85]). The fundamental polytope $\mathcal{M}(X)$ is distinguished from other polytopes by the fact that it contains the ray $\mathbb{R}^+h$ of polarization, since

$$\mathcal{M}(X) = \{x \in V^+(X) \,|\, x\cdot f \ge 0, \forall f \in P(X)\}.$$

Now let (X,θ) be a K3 surface with a nonsymplectic involution and $(Y = X/\{1,\theta\}, C = \pi(X^\theta))$ be the corresponding DPN pair. If $E \subset Y$ is an exceptional curve then the curve $F = \pi^{-1}(E)$ is either an irreducible curve on X with negative square, or $F = F_1 + \theta(F_1)$, where

$$F_1^2 + (\theta(F_1))^2 + 2F_1\cdot\theta(F_1) = 2E^2 < 0.$$

The curves F_1 and $\theta(F_1)$ are irreducible and have negative square (i.e. equal to (-2)). Using this, in an obvious way we get the following four possibilities for E and F (see Fig. 2)

- I $E^2=-4$, E is a component of C; resp. F is a component of X^θ, and $F^2=-2$.
- IIa $E^2=-1$, $E\cdot C=2$ and E intersects C transversally at two points; resp. F is irreducible and $F^2=-2$.
- IIb $E^2=-1$, $E\cdot C=2$ and E is tangent to C at one point; resp. $F=F_1+\theta(F_1)$, where $(F_1)^2=-2$ and $F_1\cdot\theta(F_1)=1$.
- III $E^2=-2$, $E\cap C=\emptyset$; resp. $F=F_1+\theta(F_1)$, where $(F_1)^2=-2$ and $F_1\cdot\theta(F_1)=0$.

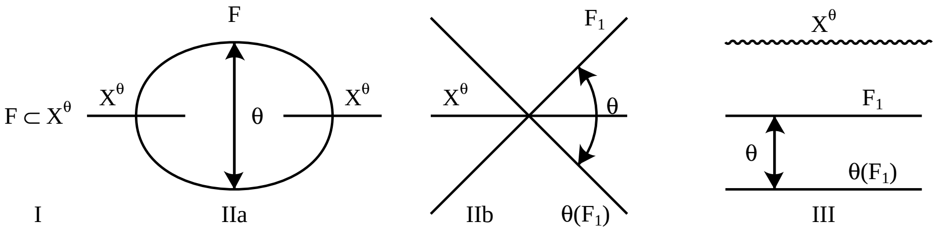


FIGURE 2. Pictures of exceptional curves.

Let $P(Y) \subset S_Y$ be the subset of divisor classes of all exceptional curves E on Y (here S_Y is the Picard lattice of Y modulo torsion, if Y is Enriques surface), and $P(X)_+ \subset S = (S_X)_+$ be the set of divisor classes of all $F = \pi^{-1}(E)$. We call them *exceptional classes of the pair* (X,θ). By what we said above, $P(Y)$ and $P(X)_+$ are divided into subsets:

$$\begin{aligned} P(Y) &= P(Y)_I \coprod P(Y)_{IIa} \coprod P(Y)_{IIb} \coprod P(Y)_{III}. \\ P(X)_+ &= P(X)_{+I} \coprod P(X)_{+IIa} \coprod P(X)_{+IIb} \coprod P(X)_{+III}. \end{aligned}$$

It turns out that $P(X)_+$ and, respectively $P(Y)$ admit a description analogous to the description of $P(X)$ for the set of exceptional curves on the surface X. Since the lattice $S = (S_X)_+$ is hyperbolic, and $S \subset S_X$, we have embedding of cones

$$V(S) \subset V(S_X), \qquad V^+(S) \subset V^+(S_X)$$

and hyperbolic spaces $\mathcal{L}(S) \subset \mathcal{L}(X)$.

The set $\mathcal{L}(S)$ contains the polarization ray $\mathbb{R}^+(h+\theta^* h)$ of surface X, therefore, $\mathcal{M}(S)_+^{(2,4)} = \mathcal{M}(X) \cap \mathcal{L}(S)$ is non-degenerate (i.e. containing a nonempty open subset of $\mathcal{L}(S)$) convex locally finite polytope in $\mathcal{L}(S)$. Since S is even, it is easy to see that $P(X)_+$ is precisely the set of primitive elements of S, perpendicular to facets of $\mathcal{M}(X)_+^{(2,4)}$ and directed outward. One has

$$\begin{aligned} P^{(2)}(X)_+ &\overset{\text{Def}}{=} \{f \in P(X)_+ \mid f^2 = -2\} \\ &= P(X)_{+I} \coprod P(X)_{+IIa} \coprod P(X)_{+IIb}, \\ P^{(4)}(X)_+ &\overset{\text{Def}}{=} \{f \in P(X)_+ \mid f^2 = -4\} = P(X)_{+III}. \end{aligned}$$

Moreover, $\mathcal{M}(X)_+$, as well as $\mathcal{M}(X)$ admits a description in terms of groups.

Let

$$\begin{aligned}
\Delta^{(4)}_{\pm} &= \{f_{\pm}\in (S_X)_{\pm}\,|\, f_{\pm}^2=-4,\ \text{and}\ \exists f_{\mp}\in (S_X)_{\mp},\\
&\qquad \text{for which}\ f_{\mp}^2=-4\ \text{and}\ (f_{\pm}+f_{\mp})/2\in S_X\};\\
\Delta^{(2)}_{+} &= \Delta^{(2)}(S)=\{f_+\in S\,|\, f_+^2=-2\};\\
\Delta^{(2)}_{+t} &= \{f_+\in \Delta^{(2)}(S)\,|\,\exists f_-\in (S_X)_-,\\
&\qquad \text{for which}\ f_-^2=-6\ \text{and}\ (f_++f_-)/2\in S_X\};\\
\Delta^{(6)}_{-} &= \{f_-\in (S_X)_-\,|\, f_-^2=-6\ \text{and}\ \exists f_+\in\Delta^{(2)}_{+t},\\
&\qquad \text{for which}\ (f_++f_-)/2\in S_X\}.
\end{aligned}$$

If $f_{\pm}\in\Delta^{(4)}_{\pm}$, then $f_{\pm}\cdot(S_X)_{\pm}\equiv 0 \mod 2$. Hence, $f_{\pm}\in\Delta^{(4)}_{\pm}$ are roots of $(S_X)_{\pm}$, and there exists a reflection $s_{f_{\pm}}\in O'((S_X)_{\pm})$ with respect to element $f_{\pm}$,

$$s_{f_{\pm}}(x)=x+\frac{(x\cdot f_{\pm})}{2}f_{\pm},\quad x\in (S_X)_{\pm},$$

and one has the very important property

$$s_{f_{\pm}}(\Delta^{(2)}_{\pm}\cup\Delta^{(4)}_{\pm})=\Delta^{(2)}_{\pm}\cup\Delta^{(4)}_{\pm}\quad \forall f_{\pm}\in\Delta^{(2)}_{\pm}\cup\Delta^{(4)}_{\pm} \tag{12}$$

where we formally put $\Delta^{(2)}_{-}=\emptyset$ because the lattice $(S_X)_-$ has no elements with square (-2). It follows that reflections with respect to all elements $\Delta^{(2)}(S)\cup\Delta^{(4)}_{+}(S)=\Delta^{(2,4)}_{+}(S)$ generate a group $W^{(2,4)}_{+}\subset O'(S)$ which geometrically is a group generated by reflections in hyperplanes of $\mathcal{L}(S)$ which are orthogonal to $\Delta^{(2,4)}_{+}(S)$, any reflection in hyperplane of $\mathcal{L}(S)$ from this group is reflection in elements of $\Delta^{(2)}_{\pm}\cup\Delta^{(4)}_{\pm}$. It easily follows that $\mathcal{M}(S)^{(2,4)}_{+}$ is a chamber for $W^{(2,4)}_{+}$ (see [Dol84, Nik84.1, Nam85]). In this way,

$$P(X)_+=P(\mathcal{M}(S)^{(2,4)}_{+})$$

is the set of primitive elements of S, which are orthogonal to facets of $\mathcal{M}(S)^{(2,4)}_{+}$ and directed outward. Denote by

$$A(X,\theta)=\{\phi\in O'(S)\,|\,\phi(\mathcal{M}_+)=\mathcal{M}_+\}$$

the group of automorphisms of $\mathcal{M}_+$ in $O'(S)$ and by $\mathrm{Aut}(X,\theta)$ – the normalizer of the involution θ in $\mathrm{Aut}\,X$. The action of $\mathrm{Aut}(X,\theta)$ on S_X and S defines a contravariant representation $\mathrm{Aut}(X,\theta)\to A(X,\theta)$ with a finite kernel. Global Torelli theorem for K3 surfaces [PS-Sh71] implies [Dol84, Nik84.1, Nam85] that over the base field $\mathbb{C}$ the image has finite index in $A(X,\theta)$. Therefore, there is an isomorphism up to finite groups $\mathrm{Aut}(X,\theta)\approx A(X,\theta)$.

2.3.1. *Computing $P(X)_+$.* First, we consider the calculation of the fundamental chamber $\mathcal{M} \subset \mathcal{L}(S)$ of $W^{(2)}(S)$.

For that, it is important to consider a larger group $W^{(2,4)}(S)$ generated by reflections in all elements $\Delta^{(2)}(S)$ with square (-2) of the lattice S and all elements of the set

$$\Delta^{(4)}(S) = \{f \in S \mid f^2 = -4 \text{ and } f \cdot S \equiv 0 \mod 2\}$$

of all roots with square (-4) of the lattice S. Both of these sets $\Delta^{(2)}(S)$ and $\Delta^{(4)}(S)$ are invariant with respect to $W^{(2,4)}(S)$. It follows that every reflection from $W^{(2,4)}(S)$ gives a hyperplane $\mathcal{H}_f$ where $f \in \Delta^{(2)}(S) \cup \Delta^{(4)}(S)$. The subgroup $W^{(2)}(S) \triangleleft W^{(2,4)}(S)$ is normal, and any reflection from $W^{(2)}(S)$ is reflection in an element of $\Delta^{(2)}(S)$. Similarly, the subgroup $W^{(4)}(S) \triangleleft W^{(2,4)}(S)$, generated by reflections in $\Delta^{(4)}(S)$, is normal and any reflection from $W^{(4)}(S)$ is reflection in an element of $\Delta^{(4)}(S)$.

This implies the following description of a fundamental chamber $\mathcal{M}^{(2)}$ of $W^{(2)}(S)$. Let $\mathcal{M}^{(2,4)} \subset \mathcal{L}(S)$ be a fundamental chamber of $W^{(2,4)}(S)$. Let $P^{(2)}(\mathcal{M}^{(2,4)})$ and $P^{(4)}(\mathcal{M}^{(2,4)})$ are directed outwards and orthogonal to $\mathcal{M}^{(2,4)}$ (i. e. to facets of $\mathcal{M}^{(2,4)}$) elements from $\Delta^{(2)}(S)$ and $\Delta^{(4)}(S)$ respectively.

Proposition 2.2. *Let $W^{(4)}(\mathcal{M}^{(2)})$ be the group generated by reflections in all elements of $P^{(4)}(\mathcal{M}^{(2,4)})$.*

Then the fundamental chamber $\mathcal{M}^{(2)}$ of $W^{(2)}(S)$ containing $\mathcal{M}^{(2,4)}$ is equal to

$$\mathcal{M}^{(2)} = W^{(4)}(\mathcal{M}^{(2)})(\mathcal{M}^{(2,4)}),$$
$$P(\mathcal{M}^{(2)}) = W^{(4)}(\mathcal{M}^{(2)})(P^{(2)}(\mathcal{M}^{(2,4)})).$$

Moreover,

$$W^{(4)}(\mathcal{M}^{(2)}) = \{w \in W^{(4)}(S) \mid w(\mathcal{M}^{(2)}) = \mathcal{M}^{(2)}\}.$$

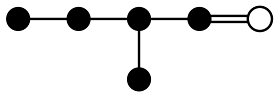
Reflections which are contained in $W^{(4)}(\mathcal{M}^{(2)})$ are exactly the reflections in elements

$$\Delta^{(4)}(\mathcal{M}^{(2)}) = \{f \in \Delta^{(4)}(S) \mid \mathcal{H}_f \textit{ intersects the interior of } \mathcal{M}^{(2)}\}.$$

Proof. This easily follows from the facts that $W^{(2)}(S) \triangleleft W^{(2,4)}(S)$ and $W^{(4)}(S) \triangleleft W^{(2,4)}(S)$ are normal subgroups, and $\Delta^{(2)}(S) \cap \Delta^{(4)}(S) = \emptyset$. We leave details to the reader. $\square$

Example 1. Let us consider the hyperbolic 2-elementary lattice $S = \langle 2 \rangle \oplus 5\langle -2 \rangle$ with the invariants $(r, a, \delta) = (6, 6, 1)$. Here and in what follows we denote by $\langle A\rangle$ the integral lattice given by an integral symmetric matrix A in some its basis. We denote by $\oplus$ the orthogonal sum of lattices.

The Dynkin diagram of $W^{(2,4)}(S)$ (which is equivalent to the Gram matrix of all elements of $P(\mathcal{M}^{(2,4)})$) is (see [Vin85])



Here black vertices correspond to elements from $P^{(4)}(\mathcal{M}^{(2,4)})$ and white vertices correspond to elements from $P^{(2)}(\mathcal{M}^{(2,4)})$ (see Sect. 3.1 below about edges). From the diagram, one can see that $W^{(4)}(\mathcal{M}^{(2)})$ is the Weyl group of the root system D_5, the $\Delta^{(4)}(\mathcal{M}^{(2)})$ is the root system D_5, the set $P^{(2)}(\mathcal{M}^{(2)})$ is the orbit of the Weyl group of D_5 applied to the unique element of $P^{(2)}(\mathcal{M}^{(2,4)})$ which corresponds to the white vertex. Calculations show that the set $P(\mathcal{M}^{(2)})$ consists of 16 elements, and it is not easy to draw their Dynkin (or Gram) diagram, but it is completely defined by the diagram above which has only 6 vertices.

Now let us consider a subset $\Delta^{(4)}_+\subset\Delta^{(4)}(S)$ and the subgroup $W^{(2,4)}_+$ of reflections generated by this subset and by the set $\Delta^{(2)}(S)$. As in our case (12), we shall assume that the set $\Delta^{(4)}_+$ is $W^{(2,4)}_+$-invariant. Then each reflection from $W^{(2,4)}_+$ is a reflection in a hyperplane $\mathcal{H}_f$, $f\in\Delta^{(2)}(S)\cup\Delta^{(4)}_+$. As before, $W^{(2)}(S)\lhd W^{(2,4)}_+$ is a normal subgroup. We denote by $W^{(4)}_+$ the group generated by reflections in $\Delta^{(4)}_+$, it is normal in $W^{(2,4)}_+$ as well. Thus, for a fundamental chamber $\mathcal{M}^{(2,4)}_+\subset\mathcal{M}^{(2)}$ of $W^{(2,4)}_+$ we can similarly define $P^{(4)}(\mathcal{M}^{(2,4)}_+)$, $W^{(4)}_+(\mathcal{M}^{(2)})$, $\Delta^{(4)}_+(\mathcal{M}^{(2)})$ and prove similar statements to Proposition 2.2. We shall do this below.

First, let us show how one can calculate a fundamental chamber $\mathcal{M}^{(2,4)}_+$ of $W^{(2,4)}_+$ contained in the fixed fundamental chamber $\mathcal{M}^{(2)}$ of $W^{(2)}(S)$. Let

$$W^{(4)}_+(\mathcal{M}^{(2)})=\{w\in W^{(4)}_+\mid w(\mathcal{M}^{(2)})=\mathcal{M}^{(2)}\}$$

and

$$\Delta^{(4)}_+(\mathcal{M}^{(2)})=\{f\in\Delta^{(4)}_+\mid \mathcal{H}_f \text{ intersects the interior of } \mathcal{M}^{(2)}\}.$$

By the analogous statement of Proposition 2.2 for $W^{(2,4)}_+$, the group $W^{(4)}_+(\mathcal{M}^{(2)})$ is generated by reflections in the elements of $\Delta^{(4)}_+(\mathcal{M}^{(2)})$, and every reflection from $W^{(4)}_+(\mathcal{M}^{(2)})$ is a reflection in a hyperplane $\mathcal{H}_f$, $f\in\Delta^{(4)}(\mathcal{M}^{(2)})$.

Obviously, the fundamental chamber $\mathcal{M}_+^{(2,4)}$ for $W_+^{(2,4)}$ contained in $\mathcal{M}^{(2)}$ is the fundamental chamber of the group $W_+^{(4)}(\mathcal{M}^{(2)})$ considered as a group acting on $\mathcal{M}^{(2)}$. Let $P^{(4)}(\mathcal{M}_+^{(2,4)})$ and $P^{(2)}(\mathcal{M}_+^{(2,4)})$ are the sets of all elements in $\Delta_+^{(4)}$ and $\Delta^{(2)}(S)$ respectively which are orthogonal to facets of $\mathcal{M}_+^{(2,4)}$.

Proposition 2.3. *We have:*

$$P^{(2)}(\mathcal{M}^{(2)}) = W_+^{(4)}(\mathcal{M}^{(2)})(P^{(2)}(\mathcal{M}_+^{(2,4)}))$$

and

$$P^{(2)}(\mathcal{M}_+^{(2,4)}) = \{f \in P^{(2)}(\mathcal{M}^{(2)}) \mid f \cdot P^{(4)}(\mathcal{M}_+^{(2,4)}) \geq 0\}.$$

Proof. The first statement is analogous to Proposition 2.2. We denote the right hand side of the proving second equality as $P^{(2)}$. Since $P^{(2)} \subset P(\mathcal{M}^{(2)})$ and $\mathcal{M}^{(2)}$ has acute angles, $f \cdot f' \geq 0$ for any two different elements $f,\ f' \in P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})$. It follows that $P(\mathcal{M}_+^{(2,4)}) \subset P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})$ because the fundamental chamber $\mathcal{M}_+^{(2,4)}$ must have acute angles. Then

$$\bigcap_{f \in P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})} \mathcal{H}_f^+ \subset \mathcal{M}_+^{(2,4)}$$

where the left hand side is not empty. Indeed, it contains the non-empty subset

$$\bigcap_{f \in P^{(2)}(\mathcal{M}^{(2)}) \cup P^{(4)}(\mathcal{M}_+^{(2,4)})} \mathcal{H}_f^+ = \mathcal{M}^{(2)} \bigcap \left(\bigcap_{f \in P^{(4)}(\mathcal{M}_+^{(2,4)})} \mathcal{H}_f^+ \right) \supset \mathcal{M}_+^{(2,4)}.$$

It follows (see Proposition 3.1 in [Vin85]) that

$$P(\mathcal{M}_+^{(2,4)}) = P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})$$

because for all $f \neq f' \in P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})$ we have $f \cdot f' \geq 0$. □

Propositions 2.2 and 2.3 imply the result. It will be very important in further considerations.

Theorem 2.4. *Let $\mathcal{M}^{(2,4)}$ be a fundamental chamber of $W^{(2,4)}(S)$ in $\mathcal{L}(S)$, and $W^{(4)}(\mathcal{M}^{(2)})$ the group generated by reflections in all elements of $P^{(4)}(\mathcal{M}^{(2,4)})$, and $\Delta^{(4)}(\mathcal{M}^{(2)}) = W^{(4)}(\mathcal{M}^{(2)})(P^{(4)}(\mathcal{M}^{(2,4)}))$.*
Then

(1) Subsets $\Delta_+^{(4)} \subset \Delta^{(4)}(S)$ *which are invariant for the group* $W_+^{(2,4)}$ *generated by reflections in all elements of* $\Delta^{(2)}(S) \cup \Delta_+^{(4)}$ *are in one-to-one correspondence with subsets* $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ *which are invariant for the group* $W_+^{(4)}(\mathcal{M}^{(2)})$ *generated by reflections in all elements of* $\Delta_+^{(4)}(\mathcal{M}^{(2)})$*. This correspondence is given by*

$$\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta_+^{(4)} \cap \Delta^{(4)}(\mathcal{M}^{(2)});\quad \Delta_+^{(4)} = W^{(2)}(S)(\Delta_+^{(4)}(\mathcal{M}^{(2)})).$$

(2) The fundamental chamber $\mathcal{M}^{(2)}$ *of* $W^{(2)}(S)$ *containing* $\mathcal{M}_+^{(2,4)}$ *is* $\mathcal{M}^{(2)} = W^{(4)}(\mathcal{M}^{(2)})(\mathcal{M}^{(2,4)})$.

(3) Under the one-to-one correspondence in (1), any fundamental chamber $\mathcal{M}_+^{(2,4)} \subset \mathcal{M}^{(2)}$ *of* $W_+^{(2,4)}$ *can be obtained as follows: Let* $\mathcal{M}_+^{(4)}(\mathcal{M}^{(2)})$ *be a fundamental chamber for* $W_+^{(4)}(\mathcal{M}^{(2)})$*. Then*

$$\mathcal{M}_+^{(2,4)} = \mathcal{M}^{(2)} \cap \mathcal{M}_+^{(4)}(\mathcal{M}^{(2)}) \text{ and } P^{(4)}(\mathcal{M}_+^{(2,4)}) = P^{(4)}(\mathcal{M}_+^{(4)}(\mathcal{M}^{(2)})),$$

$$P^{(2)}(\mathcal{M}_+^{(2,4)}) = \{f \in W^{(4)}(\mathcal{M}^{(2)})(P^{(2)}(\mathcal{M}^{(2,4)})) \mid f \cdot P^{(4)}(\mathcal{M}_+^{(2,4)}) \geq 0\}.$$

Proof. Only the statement (1) requires some clarification. Assume that $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ is invariant with respect to the subgroup $W_+^{(4)}(\mathcal{M}^{(2)})$ generated by reflections in $\Delta_+^{(4)}(\mathcal{M}^{(2)})$. Note that $\mathcal{M}^{(2)}$ is invariant with respect to $W_+^{(4)}(\mathcal{M}^{(2)})$. It follows that the fundamental chamber $\mathcal{M}_+^{(2,4)}$ for $W_+^{(4)}(\mathcal{M}^{(2)})$ acting in $\mathcal{M}^{(2)}$ will be the fundamental chamber for the group $W_+^{(2,4)}$ generated by reflections in all elements of $\Delta^{(2)}(S)$ and $\Delta_+^{(4)}(\mathcal{M}^{(2)})$. It follows that $\Delta_+^{(4)} = W^{(2)}(S)(\Delta_+^{(4)}(\mathcal{M}^{(2)}))$ is invariant with respect to $W_+^{(2,4)}$. It follows that $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta_+^{(4)} \cap \Delta^{(4)}(\mathcal{M}^{(2)})$.

The remaining statements are obvious. ∎

In Sect. 3 we apply this theorem to describe DPN surfaces of elliptic type.

2.4. **Root invariants of pairs** (X,θ)**.** To describe $W_+^{(2,4)}$ and sets $P(X)_{+III}$, and $P(X)_{+IIa}$, $P(X)_{+IIb}$, one should add to the main invariants (r,a,δ) (equivalently (k,g,δ)) of (X,θ)) the so called *root invariants*. We describe them below. For the first time, root invariants for DPN surfaces had been considered and used in [Nik84.1] and [Nik87].

Let M be a lattice. Following [Nik80.2], $M(k)$ denotes a lattice obtained from M by multiplication of the form of M by $k \in \mathbb{Q}$.

Let $K(2)$ be the sublattice of $(S_X)_-$ generated by $\Delta_-^{(4)} \subset (S_X)_-$. Since $\Delta_-^{(4)} \cdot (S_X)_- \equiv 0 \mod 2$, the lattice $K = K(2)(1/2)$ is integral and is generated by its subset $\Delta_-^{(4)} \subset K$ of elements with square (-2) defining in K a root system, since reflections with respect to elements of $\Delta_-^{(4)}$ send $\Delta_-^{(4)}$ to itself. If follows that the lattice K is isomorphic to an orthogonal sum of root lattices A_n, D_m and E_k corresponding to the root systems $\mathbb{A}_n$, $\mathbb{D}_m$, $\mathbb{E}_k$ (or their Dynkin diagrams), and $\Delta_-^{(4)} = \Delta^{(2)}(K)$ is the set of all elements of K with square (-2). Equivalently, $\Delta_-^{(4)} = \Delta^{(4)}(K(2))$ is the set of all elements with square (-4) of $K(2)$. Moreover, we have a natural homomorphism of groups

$$\xi : Q = \frac{1}{2}K(2)/K(2) \to \mathfrak{A}_S = S^*/S \tag{13}$$

such that $\xi(\frac{1}{2}f_- + K(2)) = \frac{1}{2}f_+ + S$, if $f_\mp \in \Delta_\mp^{(4)}$ and $(f_- + f_+)/2 \in S_X$. This defines a homomorphism of finite quadratic forms $\xi : q_{K(2)}|Q \to -q_S$ with values in $\frac{1}{2}\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}$. (Here $q_M : \mathfrak{A}_M = M^*/M \to \mathbb{Q}/2\mathbb{Z}$ denotes the discriminant quadratic form of an even lattice M.) The homomorphism ξ is equivalent to the homomorphism (which we denote by the same letter ξ) of the finite quadratic forms

$$\xi : K \mod 2 \to -q_S, \tag{14}$$

by the natural isomorphism $\frac{1}{2}K(2)/K(2) \cong K/2K$, where $q_{K(2)}|Q$ is replaced by the finite quadratic form $\frac{1}{2}x^2 \mod 2$ for $x \in K$. The triplet

$$R(X,\theta) = (K(2), \Delta_-^{(4)}, \xi) \cong (K(2), K(2)^{(4)}, \xi) \cong (K, \Delta^{(2)}(K), \xi), \tag{15}$$

considered up to isomorphisms of lattices K and automorphisms of the lattice S is called *the root invariant of the pair* (X,θ) or the corresponding DPN pair (Y,C). (Clearly, similarly we can introduce *abstract root invariants,* without any relation to K3 surfaces with involutions and DPN pairs; we leave to a reader the corresponding definition).

We have the following statement from [Nik80.1]

Lemma 2.5. *Let S be an even hyperbolic 2-elementary lattice.*
Then the natural homomorphism $O(S) \to O(q_S)$ is surjective.

Proof. We remind the proof from [Nik80.1]. If $\operatorname{rk} S \geq 3$, this follows from the general theorem 1.14.2 in [Nik80.2] (see also [Nik80.1]). If $\operatorname{rk} S = 2$, then $S \cong U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $U(2)$, $\langle 2\rangle \oplus \langle -2\rangle$. If $\operatorname{rk} S = 1$, then $S = \langle 2\rangle$. For all these lattice one can check the statement directly. $\square$

Lemma 2.6. *Let S be an even hyperbolic 2-elementary lattice and $\operatorname{rk} S \geq 2$. Then every $x \in \mathfrak{A}_S$ with $q_S(x) = n/2 \mod 2$, $n \in \mathbb{Z}$, can be represented as $x = u/2 \mod S$ where $u \in S$ and $u^2 = 2n$.*

Proof. If $\operatorname{rk} S = 2$, then $S \equiv U, U(2), \langle 2\rangle \oplus \langle -2\rangle$, and the statement can be checked directly. Assume that the statement is valid for $\operatorname{rk} S \le k$ where $k \ge 2$. Let $\operatorname{rk} S = k+1$. By Theorem 1.12.2 from [Nik80.2] about existence of an even lattice with a given discriminant quadratic form, we get that $S = S' \oplus T$ where S' is a hyperbolic 2-elementary lattice of the rank $\operatorname{rk} S' \ge 2$, and T is a negative definite 2-elementary lattice of the $\operatorname{rk} T \ge 1$. Let $x = y \oplus z, x \in \mathfrak{A}_S$, $y \in \mathfrak{A}_T$, and assume $z = u/2 \mod T$ where $u \in T$ and $u^2 = 2m$, $m \in \mathbb{Z}$. By the induction assumption, there exists $v \in S'$ with $y = v/2 \mod S'$ and $v^2 = 2n - 2m$ since $q_{S'}(y) = q(x) - q(z) = (n-m)/2 \mod 2$. $\square$

Lemma 2.7. *Let $q : A \to \mathbb{Q}/2\mathbb{Z}$ be a non-degenerate quadratic form on a finite 2-elementary group A and $\phi : H_1 \cong H_2$ be an isomorphism of two subgroups in A which preserves $q|H_1$ and $q|H_2$. Assume that the characteristic element a_q of q on A either does not belongs to both subgroups H_1 and H_2 or belongs to both of them. In the second case we additionally assume that $\phi(a_q) = a_q$.*

Then ϕ can be extended to an authomorphism of q.

Proof. See Proposition 1.9.1 in [Nik84.2]. We recall that $a_q \in A$ is the *characteristic element of q*, if $q(x) \equiv b_q(x, a_q) \mod 1$ for any $x \in A$. Here b_q is the bilinear form of q. This defines the characteristic element a_q uniquely. $\square$

Lemmas 2.5 — 2.7 imply

Proposition 2.8. *The root invariant $R(X,\theta)$ of (X,θ) (or (Y,C)) is equivalent to the triplet*

$$R(X,\theta) = (K(2); H; \alpha, \overline{a}) \cong (K; H; \alpha, \overline{a}).$$

Here $H = Ker\ \xi$ is an isotropic for $q_{K(2)}$ subgroup in Q (equivalently in $K \mod 2$); $\alpha = 0$, if $\xi(Q) = \xi(K \mod 2)$ contains the characteristic element a_{q_S} of the quadratic form q_S, and $\alpha = 1$ otherwise; if $\alpha = 0$, the element $\overline{a} = \xi^{-1}(a_{q_S}) + H \in Q/H$; if $\alpha = 1$, the element $\overline{a}$ is not defined.

The root invariant $R(X,\theta)$ is important because it defines

$$\Delta_+^{(4)} = \{f_+ \in S \mid\ f_+^2 = -4,\ f_+/2 \mod S \in$$
$$\xi(\frac{1}{2}\Delta^{(4)}(K(2)) \mod K(2)) = \xi(\Delta^{(2)}(K) \mod 2K)\},$$

and $W_+^{(2,4)}$, $\mathcal{M}(X)_+^{(2,4)}$, $P(\mathcal{M}(X)_+^{(2,4)}) = P(X)_+$. Moreover, for $f_+ \in P(X)_+^{(4)}$, the root invariant defines the decomposition $f_+ = f + \theta^*(f)$, $f \in P(X)$, up to the action of $O(S)$. More precisely, we have the following. Let $f_- \in \Delta^{(4)}(K(2))$ and $\xi(f_-/2 \mod K(2)) = f_+/2 \mod S$.

Then $f=(f_+\pm f_-)/2$, $\theta(f)=(f_+\mp f_-)/2$. Indeed, if $f_-{}'\in \Delta^{(4)}(K(2))$ satisfies the same conditions, then $(f_- + f_-{}')/2\in K(2)$. In $K(2)$, if $f_-{}'\neq \pm f_-$ then either $f_-\cdot f_-{}'=0$ or $f_-\cdot f_-'=\pm 2$. If $f_-\cdot f_-{}'=0$ then $((f_-+f_-{}')/2)^2=-2$, and we get a contradiction since $(S_X)_-$ does not have elements with the square (-2). If $f_-\cdot f_-{}'=\pm 2$ then $f_-\cdot(f_-+f_-{}')/2=-2\pm 1$, and we get a contradiction since $f_-\cdot (S_X)_-\equiv 0 \mod 2$. Thus, $f_-{}'=\pm f_-$, and the pair of elements f and $\theta^\ast(f)$ is defined uniquely.

Similarly one can define a *generalized root invariant*

$$R_{gen}(X,\theta)=(K_{gen}(2),\ \Delta_-^{(4)}\cup\Delta_-^{(6)},\ \xi_{gen})\cong$$

$$(K_{gen},\Delta_-^{(2)}\cup\Delta_-^{(3)},\xi_{gen}),$$

where for $f_-\in\Delta_-^{(6)}$ one has

$$\xi_{gen}(f_-/2 \mod K_{gen}(2))=f_+/2 \mod S$$

where $f_+\in\Delta_{+t}^{(2)}(S)$ and $(f_++f_-)/2\in S_X$. Here $K_{gen}(2)\subset (S_X)_-$ is generated by $\Delta_-^{(4)}\cup\Delta_-^{(6)}$.

Using Lemmas 2.5 — 2.7, one can similarly prove that it is equivalent to the tuple

$$R_{gen}(X,\theta)=(K_{gen}(2),\Delta_-^{(4)}\cup\Delta_-^{(6)};\ H_{gen};\ \alpha_{gen},\ \overline{a}_{gen})\cong$$

$$(K_{gen},\Delta_-^{(2)}\cup\Delta_-^{(3)};\ H_{gen};\ \alpha_{gen},\ \overline{a}_{gen}).$$

It is defined similarly to the root invariant.

Importance of the generalized root invariant is that it contains the root invariant $R(X,\theta)$. Thus, it defines $W_+^{(2,4)}$, $\mathcal{M}(X)_+^{(2,4)}$ and also $P(\mathcal{M}(X)_+^{(2,4)})=P(X)_+$ up to the action of $O(S)$. But, it also defines

$$\Delta_{+t}^{(2)}=\{f_+\in S\ |\ (f_+)^2=-2,\ f_+/2 \mod S\in$$

$$\xi_{gen}(\frac{1}{2}\Delta_-^{(6)} \mod K_{gen}(2)=\xi(\Delta_-^{(3)} \mod 2K_{gen})\},$$

and then it defines

$$P(X)_{+IIb}=P^{(2)}(X)_{+t}=\{f_+\in P(X)_+\mid f_+\in\Delta_{+t}^{(2)}\}.$$

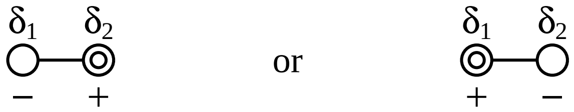
Thus, using root invariants, we know how to find $P(X)_{+III}=P^{(4)}(X)_+$, $P(X)_{+IIb}=P^{(2)}(X)_{+t}$, and hence, we know $P(X)_{+I}\cup P(X)_{+IIa}=P^{(2)}(X)_+-P^{(2)}(X)_{+t}$. To distinguish $P(X)_{+I}$ and $P(X)_{+IIa}$, it is sufficient to know how one can find $P(X)_{+I}$.

2.5. **Finding the locus X^θ.** Here we show how one can find $P(X)_{+I}$. This is based on the following considerations (similar to [Nik83]):

1) Since $W^{(2)}(S) \lhd W_+^{(2,4)}$ is a normal subgroup, the fundamental polyhedron $\mathcal{M}_+^{(2,4)}$ is contained in one fundamental chamber $\mathcal{M}^{(2)}$ of $W^{(2)}(S)$; we have $\mathcal{M}_+^{(2,4)} \subset \mathcal{M}^{(2)}$. One can consider replacing $\mathcal{M}_+^{(2,4)}$ by $\mathcal{M}^{(2)}$ as a deformation of a pair (X,θ) to a general pair $(\widetilde{X},\widetilde{\theta})$ having $S_{\widetilde{X}} = S$, $\mathcal{M}(\widetilde{X}) = \mathcal{M}^{(2)}$ and $P(\widetilde{X}) = P(\mathcal{M}^{(2)})$.

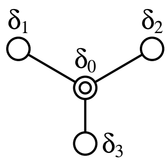
The divisor classes of fixed points of the involution do not change under this deformation, thus $P(X)_{+I} = P(\widetilde{X})_{+I}$. In particular, $P(X)_{+I}$ does not change when a root invariant changes (with fixed main invariants (r,a,δ) equivalent to the lattice S).

2) Let δ_1, δ_2 belong to $P^{(2)}(X)_+$ and $\delta_1\cdot\delta_2 = 1$, i. e. the corresponding to them curves D_1, D_2 intersect transversally. Then one of δ_1, δ_2 belongs to $P(X)_{+I}$, and another to $P(X)_{+II} = P(X)_{+IIa} \cup P(X)_{+IIb} = P(X)_{+IIa}$ for the general case we consider. See the diagrams below where an element of $P(X)_{+I}$ is denoted by a double transparent vertex, and an element of $P(X)_{+II}$ by a single transparent vertex.



Indeed, the intersection point $D_1 \cap D_2$ is a fixed point of θ, tangent directions of D_1 and D_2 at this point are eigenvectors of θ_*. We know that they have eigenvalues $+1$ and -1.

3) If the Gram diagram of elements δ_0, δ_1, δ_2 and δ_3 from $P^{(2)}(X)_+$ has the form as shown



then $\delta_0 \in P(X)_{+I}$, and δ_1, δ_2, $\delta_3 \in P(X)_{+II} = P(X)_{+IIa}$ (for the general case we consider). Indeed, the rational curve corresponding to δ_0 has three different fixed points of θ, and hence belongs to X^θ.

4) If $\delta \in P(X)_{+III} = P^{(4)}(X)_+$ and $\delta_1 \in P(X)_{+I}$, then $\delta_1 \cdot \delta = 0$. This is obvious from the definition of $P(X)_{+III}$.

Considering all possible lattices S, it is not difficult to see that statements 1) — 3) are sufficient for finding $P(X)_{+I}$ and the divisor class of the irreducible component C_g of the curve X^θ of fixed points. The statement 4) simplifies these considerations, if some elements of $P^{(4)}(X)_+$ are known.

2.6. **Conditions for the existence of root invariants.** Assume that the main invariants (r, a, δ) (equivalently (k, g, δ)) of (X, θ) are known and fixed. Here we want to give conditions which are necessary and sufficient for the existence of a pair (X, θ) with a given root or a generalized root invariant. We consider the case of a root invariant. Similarly, one can consider the case of a generalized root invariant.

For the existence of a root invariant, it is necessary

Condition 1. *The lattice*

$$K_H = [K; x/2 \text{ where } x + 2K \in H]$$

does not have elements with the square (-1)*. Equivalently, the lattice*

$$K_H(2) = [K(2); x/2 \text{where } x/2 + K(2) \in H]$$

does not have elements with the square (-2)*.*

Indeed, the lattice $K_H(2) \subset (S_X)_-$, but the lattice $(S_X)_-$ does not have elements with the square (-2). If $f \in (S_X)_-$ has $f^2 = -2$ then $\pm f$ is effective (by Riemann-Roch Theorem applied to the K3 surface X). On the other hand, $\theta^*(\pm f) = \mp f$ is then also effective which is impossible.

Condition 2. *If* $k = \mathbb{C}$*, then* $\operatorname{rk} S + \operatorname{rk} K = r + \operatorname{rk} K \leq 20$.

Indeed, $S \oplus K \subset S_X$ and $\operatorname{rk} S_X \leq 20$.

A pair (X, θ) (or the corresponding pair (Y, C))is called *standard,* if $K_H(2)$ is a primitive sublattice of $(S_X)_-$, and the primitive sublattice $[S \oplus K(2)]_{\mathrm{pr}}$ in S_X generated by $S \oplus K(2)$ is defined by the homomorphism ξ, i. e. it is equal to

$$\begin{aligned} M =&[S \oplus K(2); \{a + b \,|\, \forall a \in S^*,\ \forall b \in K(2)/2,\\ &\text{such that } \xi(b + K(2)) = a + S\}]. \end{aligned} \tag{16}$$

Clearly, $M \subset [S \oplus K(2)]_{\mathrm{pr}}$ is always a sublattice of finite index.

Let $l(\mathfrak{A})$ *be the minimal number of generators of a finite Abelian group* $\mathfrak{A}$. Let $\mathfrak{A}_M = M^*/M$ *be the discriminant group of a lattice* M*.*

Proposition 2.9. *Assume that* $k = \mathbb{C}$*.*

Then a standard pair (X, θ) *with a given root invariant* $R(X, \theta)$ *satisfying Conditions 1 and 2 does exist, if additionally*

$$r + a + 2l(H) < 22 \tag{17}$$

and

$$r + \operatorname{rk} K + l(\mathfrak{A}_{K_p}) < 22 \tag{18}$$

for any prime $p > 2$.

Proof. By Global Torelli Theorem [PS-Sh71] and Surjectivity of Torelli Map [Kul77] for K3, the pair (X, θ) does exist, if there exists a primitive embedding of the lattice M described in (16) into an even unimodular lattice $L_{K3} \cong H^2(X, \mathbb{Z})$ of the signature $(3, 19)$ (see the proof of Proposition 2.10 below). By Corollary 1.12.3 in [Nik80.2], such a primitive embedding does exist, if $\operatorname{rk} M + l(\mathfrak{A}_{M_p}) < 22$ for all prime $p \geq 2$.

If $p > 2$, then $\operatorname{rk} M + l(\mathfrak{A}_{M_p}) = r + rkK + l(\mathfrak{A}_{K_p}) < 22$ by (18) (here we remember that the lattice S is 2-elementary).

Assume that $p = 2$. Let Γ_ξ be the graph of ξ. Then $\mathfrak{A}_M = (\Gamma_\xi)^\perp/\Gamma_\xi$ for the discriminant form $q_S \oplus q_{k(2)}$. Let $Q = H \oplus Q'$ (see (13)) where Q' is a complimentary subgroup, and $\xi' = \xi|Q'$. Then $\Gamma_{\xi'} \subset \Gamma_\xi$, moreover $\Gamma_{\xi'} \subset \mathfrak{A}_S \oplus \mathfrak{A}_{K(2)} = \mathfrak{A}$ is a 2-elementary subgroup, and $\Gamma_{\xi'} \cap 2\,\mathfrak{A} = \{0\}$ since $2\,\mathfrak{A} = \{0\} \oplus 2\,\mathfrak{A}_{K(2)}$ and ξ' is injective. Let $\mathfrak{A}^{(2)}$ be the kernel of multiplication by 2 in $\mathfrak{A}$, and $q = (q_S \oplus q_{K(2)})|\,\mathfrak{A}^{(2)}$. It is easy to see that the kernel $\operatorname{Ker} q = \mathfrak{A}^{(2)} \cap 2\,\mathfrak{A}$. Since $\Gamma_{\xi'} \cap 2\,\mathfrak{A} = \{0\}$, then $\Gamma_{\xi'} \cap \operatorname{Ker} q = \{0\}$. Let $\mathfrak{A}_1^{(2)}$ be a complementary to $\operatorname{Ker} q$ subgroup in $\mathfrak{A}^{(2)}$ which contains $\Gamma_{\xi'}$. Then $q_S \oplus q_{K(2)} = q_1 \oplus q_2$ where $q_1 = q_S \oplus q_{K(2)}|\,\mathfrak{A}_1^{(2)}$ and q_2 is the orthogonal complement to q_1 (since q_1 is non-degenerate). The subgroup $\Gamma_{\xi'}$ is isotropic for the non-degenerate 2-elementary form q_1 and has rank $\operatorname{rk} K - \operatorname{rk} H$. It follows that

$$l((\Gamma_{\xi'})^\perp_{q_1}/\Gamma_{\xi'}) = l(\mathfrak{A}_1^{(2)}) - 2l(\Gamma_{\xi'}),$$

and then

$$l(\mathfrak{A}_{M_2}) \leq l(\mathfrak{A}) - 2l(\Gamma_{\xi'}) = a + \operatorname{rk} K - 2(\operatorname{rk} K - l(H)).$$

This implies that

$$\operatorname{rk} M + l(\mathfrak{A}_{M_2}) \leq r + a + 2l(H) < 22$$

by (17). $\square$

Finally, in general, by Global Torelli Theorem [PS-Sh71] and Surjectivity of Torelli Map [Kul77] for K3 (actually, The Global Torelli Theorem and local surjectivity of Torelli Map for K3 are sufficient for the proof, see [Nik80.1]), we have the following necessary and sufficient conditions of existence of a pair (X, θ) with a root invariant $(K(2), \xi)$. It even takes under consideration the more delicate invariant which is the isomorphism class of embedding of lattices $M \subset H^2(X, \mathbb{Z}) \cong L_{K3}$.

Proposition 2.10. *(over $\mathbb{C}$) There exists a K3 pair (X,θ) with a root invariant $(K(2),\xi)$ and the isomorphism class of embedding $\phi: M\subset L_{K3}$ of lattices (see (16)), if and only if*

1) $\phi(S)\subset L_{K3}$ is a primitive sublattice;

2) the primitive sublattice $\phi(K(2))_{\mathrm{pr}}\subset L_{K3}$ generated by $\phi(K(2))$ in L_{K3} does not have elements with square (-2);

3) we have:

(19)
$$\begin{aligned}\phi(\Delta^{(4)}(K(2))) = &\{f_-\in \phi(K(2))_{\mathrm{pr}}\mid f_-^2=-4,\ \text{and } \exists f_+\in S\\ &\quad \text{such that } f_+^2=-4 \text{ and } (\phi(f_+)+f_-)/2\in L_{K3}\}.\end{aligned}$$

We remind that here $\Delta^{(4)}(K(2))$ is the set of all elements in $K(2)$ with the square (-4).

Proof. Let us consider an *abstract root invariant* $(K(2),\xi)$. This means that K is a negative definite lattice generated by its elements with square (-2), and $K(2)$ is obtained by multiplication of the form of K by 2. The map

$$\xi: q_{K(2)}|Q\to -q_S,\quad Q=\frac{1}{2}K(2)/K(2),$$

is a homomorphism of finite quadratic forms. We assume that for each $f_-\in\Delta^{(4)}(K(2))$ there exists $f_+\in\Delta^{(4)}(S)$ such that $\xi(f_-/2+K(2))=f_+/2+S$ (by Lemma 2.4.2, this condition satisfies always).

Using (16), we construct the even lattice M which contains $K(2)\oplus S\subset M$ as a sublattice of finite index. It contains $S\subset M$ as a primitive sublattice, and it's primitive sublattice generated by $K(2)$ is $K(2)_H$ where $H=\operatorname{Ker}\xi$.

Let $\phi: M\to L_{K3}$ be an embedding of lattices. If ϕ corresponds to a K3 pair (X,θ), with the root invariant $(K(2),\xi)$, then conditions 1), 2) and 3) must satisfy. Now we assume that they are valid for the abstract embedding $\phi: M\to L_{K3}$ of lattices we consider.

Then $\phi(S)\subset L_{K3}$ is a primitive sublattice. To simplify notations, we identify $S=\phi(S)\subset L_{K3}$ and $K(2)=\phi(K(2))$. Since S is 2-elementary, there exists an involution α on L_{K3} with $(L_{K3})_+=S$ and $(L_{K3})_-=S^\perp$. Then $\alpha=-id$ on $K(2)$. We denote by $\widetilde{M}$ the primitive sublattice in L_{K3} generated by $\phi(M)=M$.

Assume that $f\in\widetilde{M}$ satisfies $f^2=-2$, $f=f_-^*+f_+^*$ where $f_-^*\in(K(2)_{\mathrm{pr}})^*$, $f_+^*\in S^*$ and $f_+^2<0$. Since $2f_-^*=f_-=f-\alpha(f)\in K(2)_{\mathrm{pr}}$, $2f_+^*=f_+=f+\alpha(f)\in S$, $K(2)$ is negative definite and satisfies 2), it follows that either $f=(f_-+f_+)/2$ where $f_-=0$ and $f=f_+/2\in$

$\Delta^{(2)}(S)$, or $f = (f_- + f_+)/2$ where $f_- \in K(2)^{(4)}$, $f_+ \in \Delta^{(4)}(S)$, or $f = (f_- + f_+)/2$ where $(f_-)^2 = -6$ and $f_+ \in \Delta^{(2)}(S)$.

It follows that there exists $h_+ \in S$ with $(h_+)^2 > 0$ such that $h_+ \cdot f \neq 0$ for any $f \in \Delta^{(2)}(\widetilde{M})$.

By surjectivity of Torelli map for K3 surfaces [Kul77], we can assume that there exists a K3 surface X with $H^2(X,\mathbb{Z}) = L_{K3}$, $S_X = \widetilde{M}$ and a polarization h_+. The involution α preserves periods of X. By Global Torelli Theorem for K3 [PS-Sh71], $\alpha = \theta^*$ corresponds to an automorphism θ of X. The automorphism θ is symplectic because $H^2(X,\mathbb{Z})_+ = (S_X)_+ = \widetilde{M}_+ = S$ is hyperbolic. By 3), the root invariant of (X,θ) is $(K(2),\xi)$. $\square$

This implies the important for us result.

Corollary 2.11. *Let $(K(2),\xi)$ be the root invariant of a pair (X,θ) and $K'(2) \subset K(2)$ a primitive sublattice of $K(2)$ generated by its elements $\Delta^{(4)}(K'(2))$ with the square (-4).*

Then the pair $(K'(2), \xi' = \xi | Q' = \frac{1}{2}K'(2)/K'(2))$ is also the root invariant of some pair (X',θ').

Corollary 2.11 shows that to describe all possible root invariants of pairs (X,θ), it is enough to describe all possible root invariants of extremal pairs. Here a *pair* (X',θ') is called *extremal,* if its root invariant $R(X',\theta') = (K'(2),\xi')$ cannot be obtained using Corollary 2.11 from the root invariant $R(X,\theta) = (K(2),\xi)$ of any other pair (X,θ') with $\operatorname{rk} K(2) > \operatorname{rk} K'(2)$.

2.7. **Three types of non-symplectic involutions of K3 surfaces and DPN surfaces.** It is natural to divide non-symplectic involutions (X,θ) of K3 and the corresponding DPN surfaces in three types:

Elliptic type: $X^\theta \cong C \cong C_g + E_1 + \cdots + E_k$ where C_g is an irreducible curve of genus $g \geq 2$ (equivalently, $(C_g)^2 > 0$), and $E_1, \dots, E_k$ are non-singular irreducible rational curves. By Sect. 2.2, this is equivalent to $r + a \leq 18$ and $(r,a,\delta) \neq (10,8,0)$. Then $\operatorname{Aut}(X,\theta)$ is finite because $(C_g)^2 > 0$.

Parabolic type: Either $X^\theta \cong C \cong C_1 + E_1 + \cdots + E_k$ (using the same notations), or $X^\theta \cong C \cong C_1^{(1)} + C_1^{(2)}$ is sum of two elliptic (i. e. of genus 1) curves. By Sect. 2.2, this is equivalent to either $r + a = 20$ and $(r,a,\delta) \neq (10,10,0)$, or $(r,a,\delta) = (10,8,0)$. Then $\operatorname{Aut}(X,\theta)$ is Abelian up to finite index and usually non-finite. Here $(C_1)^2 = 0$.

Hyperbolic type: Either $X^\theta \cong C \cong E_0 + \cdots + E_k$ is sum of non-singular irreducible rational curves, or $X^\theta = \emptyset$. By Sect. 2.2, this is equivalent to either $r+a=22$, or $(r,a,\delta)=(10,10,0)$. Then $\mathrm{Aut}(X,\theta)$ is usually non-Abelian up to finite index.

Thus, pairs (X,θ) of elliptic type are the most simple, and we describe them completely in Sect. 3. On the other hand, for classification of log del Pezzo surfaces of index ≤ 2 we need only these pairs.

3. Classification of DPN surfaces of elliptic type

3.1. **Fundamental chambers of $W^{(2,4)}(S)$ for elliptic type.** The most important property of the lattices S for elliptic type is that the subgroup $W^{(2)}(S)\subset O(S)$ has finite index. We remark that it is parallel to Lemma 1.4, and it is an important step to prove that log del Pezzo surfaces of index ≤ 2 are equivalent to DPN surfaces of elliptic type.

This finiteness was first observed and used for classification of all hyperbolic lattices M with finite index $[O(M):W^{(2)}(M)]$ in [Nik83]. We repeat arguments of [Nik83]. Let us take a general pair (X,θ) with $(S_X)_+=S$. Then $S_X=S$, and the involution θ of X is unique by the condition that it is identical on $S_X=S$ and is -1 on the orthogonal complement to S_X in $H^2(X,\mathbb{Z})$. Thus, $\mathrm{Aut}\,X=\mathrm{Aut}(X,\theta)$. By Global Torelli Theorem for K3 (see [PS-Sh71]), the action of $\mathrm{Aut}\,X$ on S_X gives that $\mathrm{Aut}\,X$ and $O(S_X)/W^{(2)}(S_X)$ are isomorphic up to finite groups. In particular, they are finite simultaneously. Thus, $[O(S):W^{(2)}(S)]$ is finite, if and only if $\mathrm{Aut}(X,\theta)$ is finite. If (X,θ) has elliptic type, then $\mathrm{Aut}(X,\theta)$ preserves X^θ and its component C_g with $(C_g)^2>0$. Since S_X is hyperbolic, it follows that the action of $\mathrm{Aut}(X,\theta)$ in S_X is finite. But it is known for K3 (see [PS-Sh71]) that the kernel of this action is also finite. It follows that $\mathrm{Aut}(X,\theta)$ and $[O(S):W^{(2)}(S)]$ are finite.

Since $O(S)$ is arithmetic, $W^{(2)}(S)$ has a fundamental chamber $\mathcal{M}^{(2)}$ in $\mathcal{L}(S)$ of finite volume and with a finite number of faces (e. g. see [Vin85]). Since $W^{(2)}(S)\subset W^{(2,4)}(S)\subset O(S)$, the same is valid for $W^{(2,4)}(S)$.

Let $\mathcal{M}^{(2,4)}\subset\mathcal{L}(S)$ be a fundamental chamber of $W^{(2,4)}(S)$, and $\Gamma(P(\mathcal{M}^{(2,4)}))$ its Dynkin diagram (see [Vin85]). Vertices corresponding to different elements $f_1,\ f_2\in P(\mathcal{M}^{(2,4)})$ *are not connected by any edge,* if $f_1\cdot f_2=0$. They are connected by a *simple edge of the weight* k (equivalently, by $k-2$ *simple edges,* if $k>2$ is small), if

$$\frac{2\,f_1\cdot f_2}{\sqrt{f_1^2f_2^2}}=2\cos\frac{\pi}{k},\quad k\in\mathbb{N}.$$

They are connected by a *thick edge,* if

$$\frac{2\,f_1\cdot f_2}{\sqrt{f_1^2 f_2^2}}=2.$$

They are connected by a *punctured edge of the weight* t*,* if

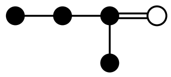
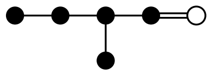
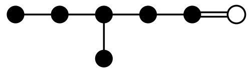
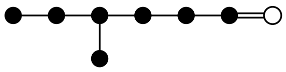
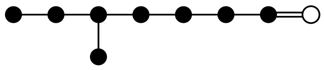
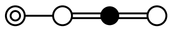
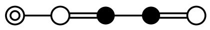
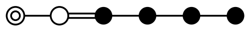
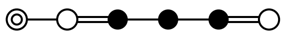
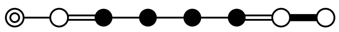
$$\frac{2\,f_1\cdot f_2}{\sqrt{f_1^2 f_2^2}}=t>2.$$

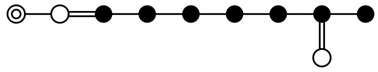
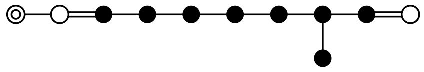
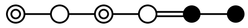
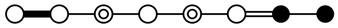
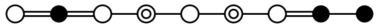
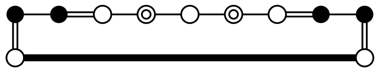
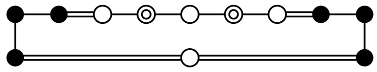
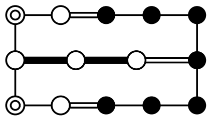
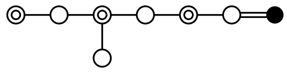
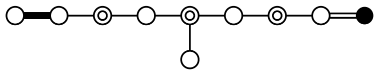
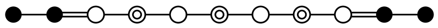
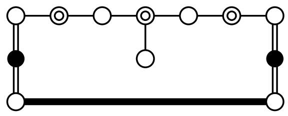
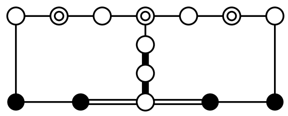
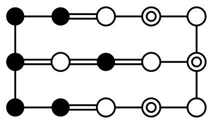
Moreover, a vertex corresponding to $f\in P^{(4)}(\mathcal{M}^{(2,4)})$ is *black.* It is *transparent,* if $f\in P^{(2)}(\mathcal{M}^{(2,4)})$. It is *transparent and doubled,* if $f\in P(X)_{+I}$ (i. e. it corresponds to the class of a rational component of X^θ), otherwise, it is *simple transparent.* Of course, here we assume that $\mathcal{M}^{(2,4)}\subset\mathcal{M}(X)_+^{(2,4)}$ for a K3 surface with involution (X,θ) and $(S_X)_+=S$.

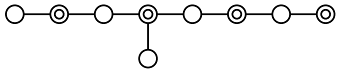
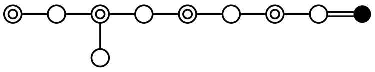
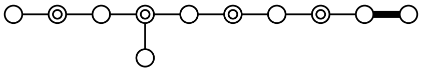
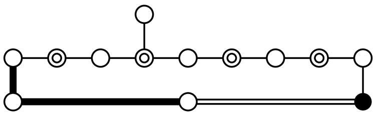
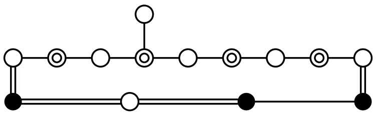
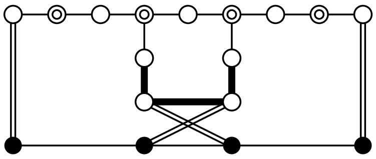
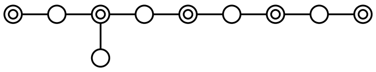
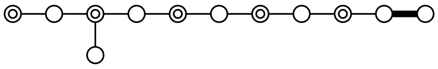
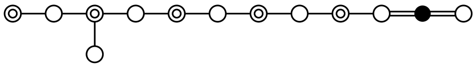
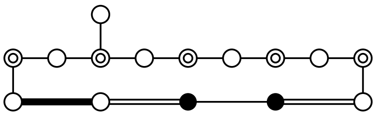
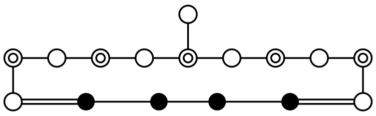
Classification of DPN surfaces of elliptic type is based on the purely arithmetic calculations of the fundamental chambers $\mathcal{M}^{(2,4)}$ (equivalently, of the graphs $\Gamma(P(\mathcal{M}^{(2,4)}))$ of the reflection groups $W^{(2,4)}(S)$ of the lattices S of elliptic type. Since S is 2-elementary and even, $W^{(2,4)}(S)=W(S)$ is the full reflection group of the lattice S, any root $f\in S$ has $f^2=-2$ or -4. We have

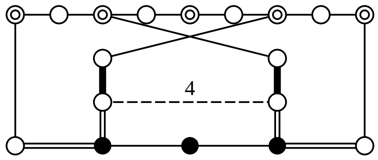
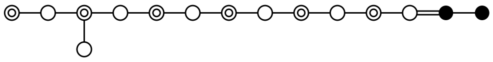
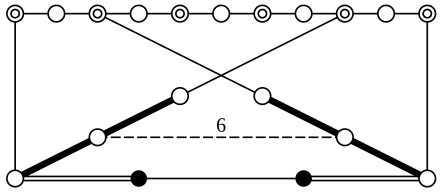
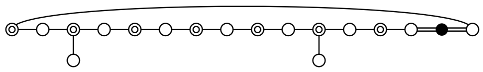
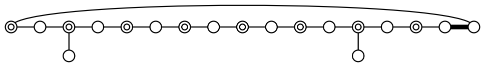
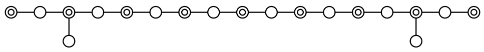
Theorem 3.1. *2-elementary even hyperbolic lattices* S *of elliptic type have fundamental chambers* $\mathcal{M}^{(2,4)}$ *for their reflection groups* $W^{(2,4)}(S)$ *(it is the full reflection group of* S*), equivalently the corresponding Dynkin diagrams* $\Gamma(P(\mathcal{M}^{(2,4)}))$*, which are given in Table 1 below, where the lattice* S *is defined by its invariants* (r,a,δ)*, see Sect. 2.2.*

Table 1. *Fundamental chambers $\mathcal{M}^{(2,4)}$ of reflection groups $W^{(2,4)}(S)$ for 2-elementary even hyperbolic lattices S of elliptic type.*

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(\mathcal{M}^{(2,4)}))$
1	1	1	1	0	10	0	$\Gamma=\varnothing$
2	2	2	0	0	9	0	
3	2	2	1	0	9	0	
4	3	3	1	0	8	0	
5	4	4	1	0	7	0	
6	5	5	1	0	6	0	
7	6	6	1	0	5	0	
8	7	7	1	0	4	0	
9	8	8	1	0	3	0	
10	9	9	1	0	2	0	
11	2	0	0	1	10	0	
12	3	1	1	1	9	0	
13	4	2	1	1	8	0	
14	5	3	1	1	7	0	
15	6	4	0	1	6	0	
16	6	4	1	1	6	0	
17	7	5	1	1	5	0	
18	8	6	1	1	4	1	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(\mathcal{M}^{(2,4)}))$
19	9	7	1	1	3	1	
20	10	8	1	1	2	1	
21	6	2	0	2	7	0	
22	7	3	1	2	6	0	
23	8	4	1	2	5	0	
24	9	5	1	2	4	0	
25	10	6	0	2	3	1	
26	10	6	1	2	3	1	
27	11	7	1	2	2	1	
28	8	2	1	3	6	0	
29	9	3	1	3	5	0	
30	10	4	0	3	4	0	
31	10	4	1	3	4	0	
32	11	5	1	3	3	0	
33	12	6	1	3	2	1	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(\mathcal{M}^{(2,4)}))$
34	9	1	1	4	6	0	
35	10	2	0	4	5	0	
36	10	2	1	4	5	0	
37	11	3	1	4	4	0	
38	12	4	1	4	3	0	
39	13	5	1	4	2	0	
40	10	0	0	5	6	0	
41	11	1	1	5	5	0	
42	12	2	1	5	4	0	
43	13	3	1	5	3	0	
44	14	4	0	5	2	0	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(\mathcal{M}^{(2,4)}))$
45	14	4	1	5	2	0	
46	14	2	0	6	3	0	
47	15	3	1	6	2	0	
48	16	2	1	7	2	0	
49	17	1	1	8	2	0	
50	18	0	0	9	2	0	

Proof. When S is unimodular (i.e. $a = 0$) or $r = a$ (then $S(1/2)$ is unimodular), i. e. for cases 1—11, 40, 50, these calculations were done by Vinberg. In all other cases they can be done using Vinberg's algorithm for calculation of the fundamental chamber of a hyperbolic reflection group. See [Vin72], [Vin85]. For lattices $S \cong U \oplus D_{2k} \oplus mA_1$, $k \geq 3$, we used a computer.

To describe elements of $P(X)_{+I}$ (i. e. double transparent vertices), we use the results of Sect. 2.5 and the fact that that their number is known, by results of Sect. 2.2. $\square$

Remark 3.2. Using diagrams of Theorem 3.1, one can easily find the class in S of the component C_g, $g \geq 2$, of X^θ as an element $C_g \in S$ such that $C_g \cdot x = 0$, if x corresponds to a black or a double transparent vertex, and $C_g \cdot x = 2 - s$ if x corresponds to a simple transparent vertex which has s edges to double transparent vertices.

3.2. **Root invariants, and subsystems of roots in $\Delta^{(4)}(\mathcal{M}^{(2)})$ for elliptic case.** We use the notations and results of Sect. 2.3.1. Let $\mathcal{M}^{(2)} \supset \mathcal{M}^{(2,4)}$ be the fundamental chamber of $W^{(2)}(S)$ containing $\mathcal{M}^{(2,4)}$. Dynkin diagram of $P^{(4)}(\mathcal{M}^{(2,4)})$ (i. e. black vertices) consists of components of types $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ (see Table 1). Thus, the group $W^{(4)}(\mathcal{M}^{(2)})$ generated by reflections in all elements of $P^{(4)}(\mathcal{M}^{(2,4)})$ is a finite Weyl group. It has to because $W^{(4)}(\mathcal{M}^{(2)})(\mathcal{M}^{(2,4)}) = \mathcal{M}^{(2)}$ has finite volume, and $\mathcal{M}^{(2,4)}$ is the fundamental chamber for the action of $W^{(4)}(\mathcal{M}^{(2)})$ in $\mathcal{M}^{(2)}$. Thus,

$$\Delta^{(4)}(\mathcal{M}^{(2)}) = W^{(4)}(\mathcal{M}^{(2)})P^{(4)}(\mathcal{M}^{(2,4)})$$

is a finite root system of the corresponding type in a negative definite sublattice

$$R(2) = [P^{(4)}(\mathcal{M}^{(2,4)})] \subset S.$$

Let (X,θ) be a K3 surface with a non-symplectic involution, and $(S_X)_+ = S$. Let $\Delta_+^{(4)} \subset \Delta^{(4)}(S)$ be the defined by (X,θ) subset which is invariant with respect to $W_+^{(2,4)}$. By Theorem 2.4, $\Delta_+^{(4)} = W^{(2)}(S)\Delta_+^{(4)}(\mathcal{M}^{(2)})$ where $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta_+^{(4)} \cap \Delta^{(4)}(\mathcal{M}^{(2)})$ is a root subsystem in $\Delta^{(4)}(\mathcal{M}^{(2)})$. Let

$$K^+(2) = [\Delta_+^{(4)}(\mathcal{M}^{(2)})] \subset R(2) \subset S \tag{20}$$

be the corresponding negative definite sublattice in S, and

$$Q = \frac{1}{2}K^+(2)/K^+(2), \quad \xi^+ : q_{K^+(2)}|Q \to q_S \tag{21}$$

a homomorphism such that $\xi^+(x/2 + K^+(2)) = x/2 + S$, $x \in K^+(2)$. We obtained a pair $(K^+(2), \xi^+)$ which is similar to a root invariant, and it is equivalent to the root invariant for elliptic type.

Proposition 3.3. *Let (X,θ) be a K3 surface with a non-symplectic involution of elliptic type, and $S = (S_X)_+$.*

In this case, the root invariant $R(X,\theta)$ is equivalent to the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$, considered up to the action of $O(S)$ (i. e. two root subsystems $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ and $\Delta_+^{(4)}(\mathcal{M}^{(2)})' \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ are equivalent, if $\Delta_+^{(4)}(\mathcal{M}^{(2)})' = \phi(\Delta_+^{(4)}(\mathcal{M}^{(2)}))$ for some $\phi \in O(S)$):

The root invariant $R(X,\theta) \cong (K^+(2), \xi^+)$ is defined by (20) *and* (21).

The fundamental chamber $\mathcal{M}(X)_+^{(2,4)}$ is defined by the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ (up to above equivalence), by Theorem 2.4. Moreover, $P^{(4)}(\mathcal{M}(X)_+^{(2,4)})$ coincides with a basis of the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})$.

Proof. Let E_i, $i \in I$, are all non-singular rational curves on X such that $E_i \cdot \theta(E_i) = 0$, i. e.

$$\mathrm{cl}(E) + \mathrm{cl}(\theta(E)) = \delta \in P^{(4)}(\mathcal{M}(X)_+^{(2,4)}) = P^{(4)}(X)_+ = P(X)_{+III}.$$

Since $E_i \cdot C_g = 0$ and $C_g^2 = 2g - 2 > 0$, the curves E_i, $i \in I$, generate in S_X a negative definite sublattice. Thus, their components define a Dynkin diagram Γ which consists of several connected components $\mathbb{A}_n$, $\mathbb{D}_m$ and $\mathbb{E}_k$. The involution θ acts on these diagrams and corresponding curves without fixed points. Thus it necessarily changes connected components of Γ. Let $\Gamma = \Gamma_1 \coprod \Gamma_2$ where $\theta(\Gamma_1) = \Gamma_2$, and $I = I_1 \coprod I_2$ the corresponding subdivision of vertices of Γ. Then

$$\delta_i^+ = \mathrm{cl}(E_i) + \mathrm{cl}(\theta(E_i)), \quad i \in I_1,$$

and

$$\delta_i^- = \mathrm{cl}(E_i) - \mathrm{cl}(\theta(E_i)), \quad i \in I_1$$

give bases of root systems $\Delta_+^{(4)}(\mathcal{M}^2)$ and $\Delta_-^{(4)} = \Delta^{(4)}(K(2))$ respectively. The map

$$\delta_i^- = \mathrm{cl}(E_i) - \mathrm{cl}(\theta(E_i)) \mapsto \delta_+ = \mathrm{cl}(E_i) + \mathrm{cl}(\theta(E_i)),\ i \in I_1,$$

defines an isomorphism $\Delta_-^{(4)} \cong \Delta_+^{(4)}(\mathcal{M}^{(2)})$ of root systems, since it evidently preserves the intersection pairing. The homomorphism ξ of the root invariant $R(X,\theta) = (K(2),\xi)$ of the pair (X,θ) then goes to ξ^+.

In the opposite direction, the root invariant $R(X,\theta)$ defines $\Delta_+^{(4)}$ and $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta^{(4)}(\mathcal{M}^{(2)}) \cap \Delta_+^{(4)}$.

The last statement follows from Sect. 2.3.1. $\square$

By Proposition 3.3 in the elliptic case instead of root invariants one can consider root subsystems $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ (in $\Delta^{(4)}(\mathcal{M}^{(2)})$). We say that a *root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ "is contained" (respectively "is primitively contained") in a root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$, if $\phi(\Delta_+^{(4)}(\mathcal{M}^{(2)})) \subset \Delta_+^{(4)}(\mathcal{M}^{(2)})'$ (respectively $[\phi(\Delta_+^{(4)}(\mathcal{M}^{(2)}))] \subset [\Delta_+^{(4)}(\mathcal{M}^{(2)})']$ is a primitive embedding of lattices) for some $\phi \in O(S)$.* By Corollary 2.11, we obtain

Proposition 3.4. *If a root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ in $\Delta^{(4)}(\mathcal{M}^{(2)})$ corresponds to a K3 surface with non-symplectic involution (X,θ), then any primitive root subsystem in $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ corresponds to a K3 surface with non-symplectic involution. Thus, it is enough to describe extremal pairs (X,θ) such that their root subsystems $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ in $\Delta^{(4)}(\mathcal{M}^{(2)})$ are not contained as primitive root subsystems of strictly smaller rank in a root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$ in $\Delta^{(4)}(\mathcal{M}^{(2)})$ corresponding to another pair (X',θ').*

3.3. **Classification of non-symplectic involutions (X,θ) of elliptic type of K3 surfaces.** We have

Theorem 3.5. *Let (X,θ) and (X',θ') be two non-symplectic involutions of elliptic type of K3 surfaces.*

Then the following three conditions are equivalent:

(i) Their main invariants (r,a,δ) (equivalently, (k,g,δ)) coincide, and their root invariants are isomorphic.

(ii) Their main invariants (r,a,δ) coincide, and the root subsystems $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ are equivalent.

(iii) Dynkin diagrams $\Gamma(P(X)_+)$ and $\Gamma(P(X')_+)$ of their exceptional curves are isomorphic, and additionally the genera g are equal, if these diagrams are empty. The diagram $\Gamma(P(X)_+)$ is empty if and only if either $(r,a,\delta)=(1,1,1)$ (then $g=10$), or $(r,a,\delta)=(2,2,0)$ (then $g=9$) and the root invariant is zero. The corresponding DPN surfaces are $\mathbb{P}^2$ or $\mathbb{F}_0$ respectively.

Proof. By Sections 3.2 and 2.4, the conditions (i) and (ii) are equivalent, and they imply (iii).

Let us show that (iii) implies (i).

Assume that $r=\operatorname{rk} S\ge 3$.

First, let us show that S is generated by $\Delta^{(2)}(S)$, if $r=\operatorname{rk} S\ge 3$. If $r\ge a+2$, then it is easy to see that either $S\cong U\oplus T$ or $S\cong U(2)\oplus T$ where T is orthogonal sum of A_1, D_{2m}, E_7, E_8 (one can get all possible invariants (r,a,δ) of S taking these orthogonal sums). We have $U=[c_1,c_2]$ where $c_1^2=c_2^2=0$ and $c_1\cdot c_2=1$ (the same for $U(2)$, only $c_1\cdot c_2=2$). Then S is generated by elements with square -2 which are

$$\Delta^{(2)}(T)\cup(c_1\oplus\Delta^{(2)}(T))\cup(c_2\oplus\Delta^{(2)}(T)).$$

If $r=a$ then $S\cong\langle 2\rangle\oplus tA_1$. Let $h,e_1,\dots,e_t$ be the corresponding orthogonal basis of S where $h^2=2$ and $e_i^2=-2$, $i=1,\dots,t$. Then S is generated by elements with square (-2) which are $e_1,\dots,e_t$ and $h-e_1-e_2$.

Let us show that $P(X)_+$ generates S. Indeed, every element of $\Delta^{(2)}(S)\cup\Delta_+^{(4)}$ can be obtained by composition of reflections in elements of $P(X)_+$ from some element of $P(X)_+$. It follows, that it is an integral linear combination of elements of $P(X)_+$. Since we can get in this way all elements of $\Delta^{(2)}(S)$ and they generate S, it follows that $P(X)_+$ generates S.

It follows that the lattice S with its elements $P(X)_+$ is defined by the Dynkin diagram $\Gamma(P(X)_+)$. From S, we can find invariants (r,a,δ) of S, and they define invariants (k,g,δ).

Let $K^+(2) \subset S$ be a sublattice generated by $P^{(4)}(X)_+$ (i. e. by the black vertices), and $\xi^+ : Q = (1/2)K^+(2)/K^+(2) \to q_S$ the homomorphism with $\xi^+(x/2 + K^+(2)) = x/2 + S$. By Proposition 3.3, the pair $(K^+(2), \xi^+)$ coincides with the root invariant $R(X, \theta)$.

Now assume that $r = \operatorname{rk} S = 1$, 2 for the pair (X, θ). Then $S \cong \langle 2\rangle$, $U(2)$, U or $\langle 2\rangle \oplus \langle -2\rangle$.

In the first two cases $\Delta^{(2)}(S) = \emptyset$ and then $P^{(2)}(X)_+ = \emptyset$. In the last two cases $\Delta^{(2)}(S)$ and $P^{(2)}(X)_+$ are not empty.

Thus, only the first two cases give an empty diagram $P^{(2)}(X)_+$. This distinguishes these two cases from all others. In the case $S = \langle 2\rangle$, the invariant $g = 10$, and the root invariant is always zero because S has no elements with square -4. Thus, in this case, the diagram $P(X)_+$ is always empty. This case gives $Y = X/\{1, \theta\} \cong \mathbb{P}^2$. In the case $S = U(2)$, the diagram $P^{(2)}(X)_+$ is empty, but $P^{(4)}(X)_+ = \emptyset$, if the root invariant is zero, and $P^{(4)}(X)_+$ consists of one black vertex, if the root invariant is not zero (see Table 1 for this case). First case gives $Y = \mathbb{F}_0$. Second case gives $Y = \mathbb{F}_2$. In both these cases $g = 9$. Thus difference between two cases when the diagram is empty ($\mathbb{P}^2$ or $\mathbb{F}_1$) is in genus: $g = 10$ for the first case, and $g = 9$ for the second.

The difference of $S = U(2)$ with a non-empty diagram $\Gamma(P(X)_+)$ from all other cases is that this diagram consists of only one black vertex. All cases with $\operatorname{rk} S \geq 3$ must have at least 3 different vertices to generate S. In cases $S = U$ and $S = \langle 2\rangle \oplus \langle -2\rangle$, the diagram $\Gamma(P(X)_+)$ also consists of one vertex, but it is respectively double transparent and simple transparent (see Table 1). Moreover, this consideration also shows the difference between cases $S = U$ and $S = \langle 2\rangle \oplus \langle -2\rangle$ and with all other cases. $\square$

Theorem 3.5 shows that to classify pairs (X, θ) of elliptic type, we can use any of the following invariants: either the root invariant, or the root subsystem (together with the main invariants (k, g, δ) or (r, a, δ)), or the Dynkin diagram of exceptional curves.

It seems that the most natural and geometric is the classification by the Dynkin diagram. Using this diagram, on the one hand, it easy to calculate all other invariants. On the other hand, considering the corresponding DPN surface, we get the Gram diagram of all exceptional curves on it and all possibilities to get the DPN surface by blow-ups from relatively minimal rational surfaces.

However, the statements (i) and (ii) of Theorem 3.5 are also very important since they give a simple way to find out if two pairs (X, θ)

and (X',θ') (equivalently, the corresponding DPN surfaces) have isomorphic Dynkin diagrams of exceptional curves. Moreover, the classification in terms of root invariants and root subsystems is much more compact, since the full Gram diagram of exceptional curves can be very large (e. g. recall the classical non-singular del Pezzo surface corresponding to $\mathbb{E}_8$).

We have the following

Theorem 3.6 (Classification Theorem in the extremal case of elliptic type)**.** *Over $\mathbb{C}$, a K3 surface with a non-symplectic involution (X,θ) of elliptic type is* **extremal,** *if and only if the number of its exceptional curves with the square (-4), i. e. $\#P^{(4)}(X)_+$, is equal to $\#P^{(4)}(\mathcal{M}^{(2,4)})$ (see Theorem 3.1) where $\mathcal{M}^{(2,4)}$ is a fundamental chamber of $W^{(2,4)}(S)$, $S=(S_X)_+$. Equivalently, numbers of black vertices of Dynkin diagrams $\Gamma(P(X)_+)$ and $\Gamma(P(\mathcal{M}^{(2,4)}))$ with the same invariants (r,a,δ) are equal.*

Moreover, the diagram $\Gamma(P(X)_+)$ is isomorphic to (i. e. coincides with) $\Gamma(P(\mathcal{M}^{(2,4)}))$ (see Table 1) in all cases of Theorem 3.1 except cases 7, 8, 9, 10 and 20 of Table 1. In the last five cases, all possible diagrams $\Gamma(P(X)_+)$ are given in Table 2. All diagrams of Tables 1 and 2 correspond to some extremal K3 (X,θ) over $\mathbb{C}$.

Proof. See Sect. 3.4 below. $\square$

Now let us consider a description of non-extremal pairs (X,θ). The worst way to describe them is using full diagrams $\Gamma(P(X)_+)$, since the number of non-extremal pairs (X,θ) is very large and diagrams $\Gamma(P(X)_+)$ can be huge. It is better to describe them using Proposition 3.4 and Theorem 3.5. It is better to describe them by primitive root subsystems $\Delta^{(4)}_+(\mathcal{M}^{(2)})'$ in the root subsystems $\Delta^{(4)}_+(\mathcal{M}^{(2)})\subset \Delta^{(4)}(\mathcal{M}^{(2)})$ of extremal pairs $(\widetilde{X},\widetilde{\theta})$.

Let us choose $\mathcal{M}^{(2)}$ in such a way that $\mathcal{M}^{(2)}\supset \mathcal{M}(\widetilde{X})^{(2,4)}_+$. By Sect. 2.3.1, then $\Delta^{(4)}_+(\mathcal{M}^{(2)})=\Delta^{(4)}([P^{(4)}(\widetilde{X})_+])$ is the subsystem of roots with the basis $P^{(4)}(\widetilde{X})_+$, i. e. $\Delta^{(4)}_+(\mathcal{M}^{(2)})=\Delta^{(4)}([P^{(4)}(\widetilde{X})_+])$ is the set of all elements with the square (-4) in the sublattice $[P^{(4)}(\widetilde{X})_+]$ generated by $P^{(4)}(\widetilde{X})_+$ in $S=(S_{\widetilde{X}})_+$. Equivalently, $\Delta^{(4)}([P^{(4)}(\widetilde{X})_+])= W^{(4)}_+(\widetilde{X})(P^{(4)}(\widetilde{X})_+)$, where $W^{(4)}_+(\widetilde{X})$ is the finite Weyl group generated by reflections in all elements of $P^{(4)}(\widetilde{X})_+$.

Replacing a primitive root subsystem $\Delta^{(4)}_+(\mathcal{M}^{(2)})'\subset\Delta^{(4)}([P^{(4)}(\widetilde{X})_+])$ for a non-extremal pair (X,θ) by an equivalent root subsystem $\phi(\Delta^{(4)}_+(\mathcal{M}^{(2)})')$, $\phi\in W^{(4)}_+(\widetilde{X})$, we can assume (by primitivity) that a basis of $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$ is a part of the basis $P^{(4)}(\widetilde{X})_+$ of the root system $\Delta^{(4)}([P^{(4)}(\widetilde{X})_+])$. Thus, we can assume that the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$ is defined by a subdiagram

$$D \subset \Gamma(P^{(4)}(\widetilde{X})_+)$$

where $\Gamma(P^{(4)}(\widetilde{X})_+)$ is the subdiagram of the full diagram $\Gamma(P(\widetilde{X})_+)$ generated by all its black vertices. The D is a basis of $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$.

By Propositions 2.2, 2.3 and Theorem 2.4, the subdiagram $D \subset \Gamma(P^{(4)}(\widetilde{X})_+)$ defines the full Dynkin diagram $\Gamma(P(X)_+)$ of the pair (X, θ) with the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$: We have

$$P^{(2)}(X)_+ = \{f \in W_+^{(4)}(\widetilde{X})(P^{(2)}(\widetilde{X})_+) \mid f \cdot D \geq 0\}. \tag{22}$$

The subdiagram of $\Gamma(P(X)_+)$ defined by all its black vertices coincides with D. It is called *Du Val's part of* $\Gamma(P(X)_+)$, and it is denoted by $\operatorname{Duv}\Gamma(P(X)_+)$. Thus,

$$\operatorname{Duv}\Gamma(P(X)_+) = D \subset \operatorname{Duv}\Gamma(P(\widetilde{X})_+).$$

Double transparent vertices of $\Gamma(P(X)_+)$ identify with double transparent vertices of $\Gamma(P(\widetilde{X})_+)$ (see Sect. 2.5), and single transparent vertices of $P(X)_+$ which are connected by two edges with double transparent vertices of $\Gamma(P(X)_+)$ identify with such vertices of $\Gamma(P(\widetilde{X})_+)$. Indeed, they are orthogonal to the set $P^{(4)}(\widetilde{X})_+$ which defines the reflection group $W_+^{(4)}(\widetilde{X})$ as the group generated by reflections in all elements of $P^{(4)}(\widetilde{X})_+$. Thus, the group $W_+^{(4)}(\widetilde{X})$ acts identically on all these vertices, and all of them satisfy (22). All double transparent vertices and all single transparent vertices connected by two edges with double transparent vertices of $\Gamma(P(X)_+)$ define the *logarithmic part of* $\Gamma(P(X)_+)$, and it is denoted by $\operatorname{Log}\Gamma(P(X)_+)$. Thus, we have

$$\operatorname{Log}\Gamma(P(X)_+) = \operatorname{Log}\Gamma(P(\widetilde{X})_+),$$

logarithmic parts of X and $\widetilde{X}$ identify. Moreover, the Du Val part $\operatorname{Duv}\Gamma(P(X)_+)$ and the logarithmic part $\operatorname{Log}\Gamma(P(X)_+)$ are *disjoint in* $\Gamma(P(X)_+)$ because they are orthogonal to each other. Thus, the logarithmic part of $\Gamma(P(X)_+)$ is stable, it is the same for all pairs (X, θ) with the same main invariants (r, a, δ). On the Du Val part of $\Gamma(P(X)_+)$ we have only a restriction: it is a subdiagram of Du Val part of one of extremal pairs $(\widetilde{X}, \widetilde{\theta})$ described in Theorems 3.1 and 3.6 (with the same main invariants (r, a, δ)).

All vertices of $\Gamma(P(X)_+)$ which do not belong to $\operatorname{Duv}\Gamma(P(X)_+) \cup \operatorname{Log}\Gamma(P(X)_+)$ define a subdiagram $\operatorname{Var}\Gamma(P(X)_+)$ which is called the *varying part of* $\Gamma(P(X)_+)$. By (22), we have

$$\operatorname{Var} P(X)_+ = \{f \in W_+^{(4)}(\widetilde{X})(\operatorname{Var} P(\widetilde{X})_+) \mid f \cdot D \ge 0\}$$

(we skip Γ when we consider only vertices). It describes $\operatorname{Var}\Gamma(P(X)_+)$ by the intersection pairing in S.

Of course, different Dynkin subdiagrams $D \subset \Gamma(P^{(4)}(\widetilde{X})_+)$ and $D \subset \Gamma(P^{(4)}(\widetilde{X}')_+)$ of extremal pairs $(\widetilde{X}, \widetilde{\theta})$ and $(\widetilde{X}', \widetilde{\theta}')$ (with the same main invariants) can give isomorphic Dynkin diagrams $\Gamma(P(X)_+)$ and $\Gamma(P(X')_+)$ for defining by them K3 pairs (X, θ) and (X', θ'). To have that, it is necessary and sufficient that defining by them root invariants $([D], \xi^+)$ and $([D], (\xi')^+)$ are isomorphic. We remind that they can be obtained by restriction on $[D]$ of the root invariants of pairs $(\widetilde{X}, \widetilde{\theta})$ and $(\widetilde{X}', \widetilde{\theta}')$ respectively, and they can be easily computed. We remind that to have $([D], \xi^+)$ and $([D], (\xi')^+)$ isomorphic, there must exist an automorphism $\gamma : D \to D$ of diagrams and an automorphism $\overline{\phi} \in O(q_S)$ of the discriminant quadratic form of the lattice S which send ξ^+ for $(\xi')^+$. Sect. 2.4 gives the very simple and effective method for that. Thus, we have a very simple and effective method to find out when different subdiagrams D above give isomorphic K3 pairs.

Note that we have used all equivalent conditions (i), (ii) and (iii) of Theorem 3.5 which shows their importance. Finally, we get

Theorem 3.7 (Classification Theorem in the non-extremal, i. e. general, case of elliptic type)**.** *Dynkin diagrams* $\Gamma(P(X)_+)$ *of exceptional curves of non-extremal (i. e. general) non-symplectic involutions* (X, θ) *of elliptic type of K3 surfaces are described by arbitrary (without restrictions) Dynkin subdiagrams* $D \subset \operatorname{Duv}\Gamma(P(\widetilde{X})_+)$ *of extremal pairs* $(\widetilde{X}, \widetilde{\theta})$ *(see Theorem 3.6) with the same main invariants* (r, a, δ) *(equivalently* (k, g, δ)*). Moreover,*

$$\operatorname{Duv}\Gamma(P(X)_+) = D, \quad \operatorname{Log}\Gamma(P(X)_+) = \operatorname{Log}\Gamma(P(\widetilde{X})_+),$$

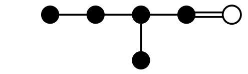
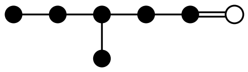
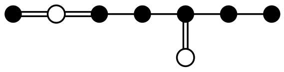
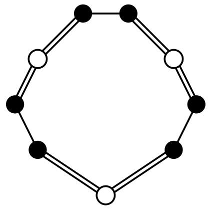
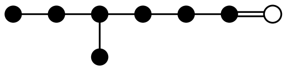
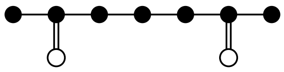
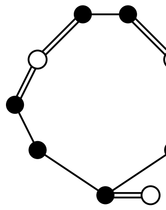
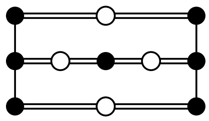
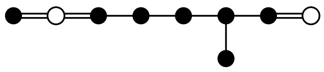
and they are disjoint to each other,

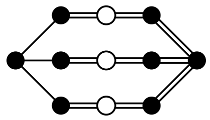
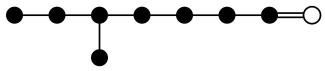
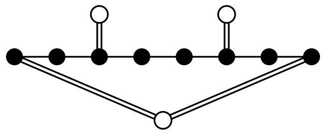
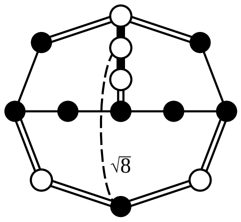
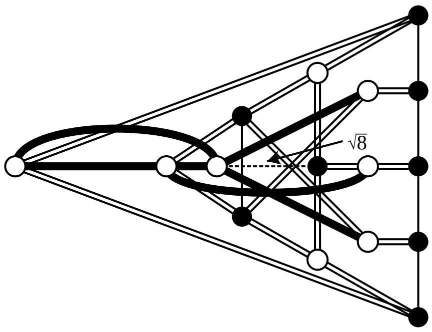
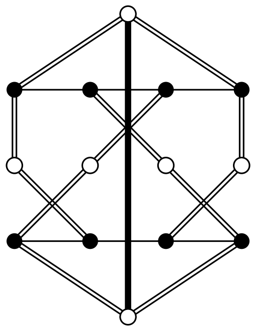
$$\operatorname{Var} P(X)_+ = \{f \in W_+^{(4)}(\widetilde{X})(\operatorname{Var} P(\widetilde{X})_+) \mid f \cdot D \ge 0\}$$

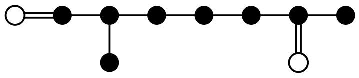
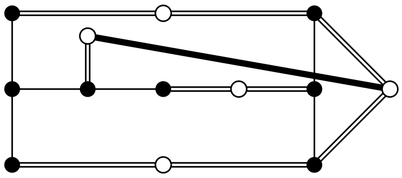
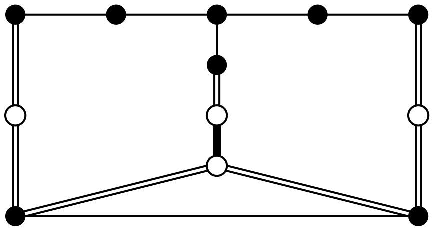
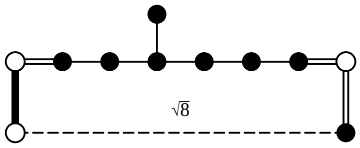
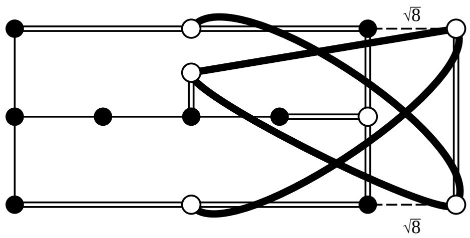
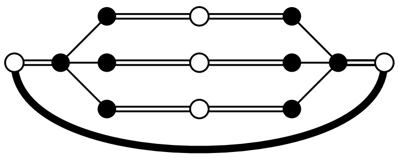
where $W_+^{(4)}(\widetilde{X})$ *is generated by reflections in all elements of* $\operatorname{Duv} P(\widetilde{X})_+ = P^{(4)}(\widetilde{X})_+$.

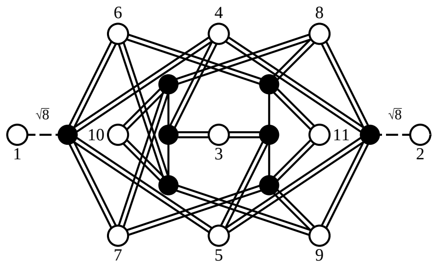
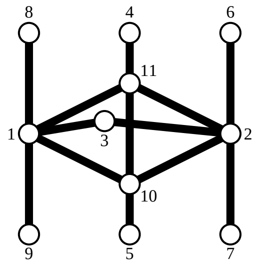
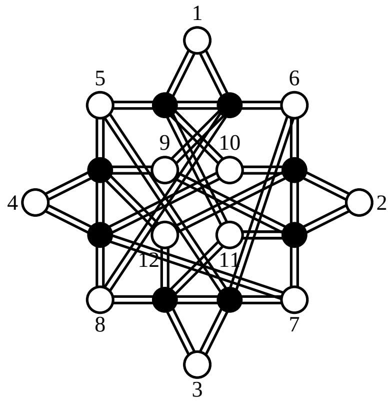
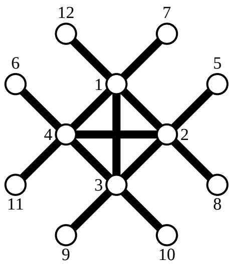
Dynkin subdiagrams $D \subset \operatorname{Duv}\Gamma(P(\widetilde{X})_+)$ *give pairs* (X, θ) *with isomorphic Dynkin diagrams* $\Gamma(P(X)_+)$*, if and only if the defining by them root invariants* $([D], \xi^+)$ *are isomorphic.*

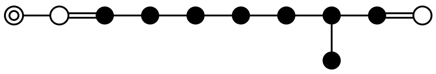
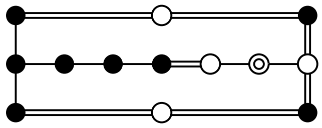
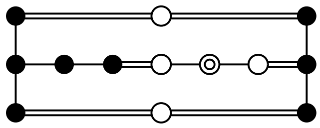
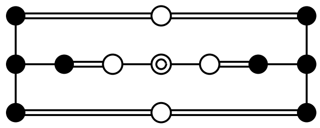
Table 2. *Diagrams $\Gamma(P(X)_+)$ of extremal (X,θ) of elliptic type which are different from Table 1 (in (a) we repeat the corresponding case of Table 1)*

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
7	6	6	1	0	5		
a						0	
b						1	
8	7	7	1	0	4		
a						0	
b						1	
c						0	
9	8	8	1	0	3		
a						0	
b						1	
c						1	
d						1	
e						1	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
9	8	8	1	0	3		
f						1	
10	9	9	1	0	2		
a						0	
b						0	
c						1	
d						1	
e						0	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
10	9	9	1	0	2		
f						1	
g						1	
h						0	
i						1	
j						2	
k						2	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
10	9	9	1	0	2		
l						2	 
m						0	 

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
20	10	8	1	1	2		
a						1	
b						2	
c						1	
d						2	

3.4. **Proof of Theorem 3.6.** Let (X, θ) be a non-symplectic involution of elliptic type of a K3 surface over $\mathbb{C}$ with the main invariants (r, a, δ), and (X, θ) is an extremal pair.

By Theorem 3.5, the $\Gamma(P(X)_+)$ is defined by the corresponding to (X, θ) root subsystem $\Delta^{(4)}_+(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ where $\mathcal{M}^{(2)}$ is a fundamental chamber of $W^{(2)}(S)$, and $S = (S_X)_+$ has the invariants (r, a, δ). We can assume that $\mathcal{M}^{(2)} \supset \mathcal{M}(X)^{(2,4)}_+ \supset \mathcal{M}^{(2,4)}$ where $\mathcal{M}^{(2,4)}$ is a fundamental chamber of $W^{(2,4)}(S)$ defined by a choice of a basis $P^{(4)}(\mathcal{M}^{(2,4)})$ of the root system $\Delta^{(4)}(\mathcal{M}^{(2)})$ (see Sect. 2.3.1).

Let $\Gamma(P^{(4)}(\mathcal{M}^{(2,4)}))$ be the Dynkin diagram of the root system $\Delta^{(4)}(\mathcal{M}^{(2)})$ and $W^{(4)}(\mathcal{M}^{(2)})$ the Weyl group of the root system $\Delta^{(4)}(\mathcal{M}^{(2)})$. We use the following description of a root subsystem $\Delta^{(4)}_+(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$.

3.4.1. Let $T \subset R$ be a root subsystem of a root system R and all components of R have types $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$. We consider two particular cases of root subsystems.

Let $T \subset R$ be a primitive root subsystem. Then T can be replaced by an equivalent root subsystem $\phi(T)$, $\phi \in W(R)$, such that there exists a basis B of R such that a part of the basis B gives a basis of T (see [Bour68]). Thus (up to equivalence defined by the Weyl group $W(R)$), primitive root subsystems $T \subset R$ can be described by Dynkin subdiagrams $\Gamma \subset \Gamma(B)$.

Now let $T \subset R$ be a root subsystem of a finite index. Let R_i be a component of R. Let r_j, $j \in J$, be a basis of R_i. Let $r_{\max} = \sum_{j\in J} k_j r_j$ be the maximal root of R_i corresponding to this basis. Dynkin diagram of the set of roots

$$\{r_j \,|\, j \in J\} \cup \{-r_{\max}\}$$

is an extended Dynkin diagram of the Dynkin diagram $\Gamma(\{r_j \,|\, j \in J\})$. Let us replace the component R_i of the root system R by the root subsystem $R_i' \subset R_i$ of finite index having by its basis the set $(\{r_j \,|\, j \in J\} \cup \{-r_{\max}\}) - \{r_t\}$ where $t \in J$ is some fixed element. We get a subsystem $R' \subset R$ or the finite index equals to k_t. It can be shown [Dyn] that iterations of this procedure give any root subsystem of finite index of R up to the action of $W(R)$.

Description of an arbitrary root subsystem $T \subset R$ can be reduced to these two particular cases, moreover it can be done in two ways.

Firstly, any root subsystem $T \subset R$ is a subsystem of finite index $T \subset T_{\mathrm{pr}}$ where $T_{\mathrm{pr}} \subset R$ is a primitive root subsystem generated by T.

Secondly, any root subsystem $T \subset R$ can be considered as a primitive root subsystem $T \subset R_1$ where $R_1 \subset R$ is root subsystem of finite index. One can take R_1 generated by T and by any $u = \operatorname{rk} R - \operatorname{rk} T$ roots $r_1, \dots, r_u$ such that $\operatorname{rk}[T, r_1, \dots, r_u] = \operatorname{rk} R$.

3.4.2. Here we show that the root subsystems $\Delta_+(\mathcal{M}^{(2)})$ which coincide with the full root system $\Delta^{(4)}(\mathcal{M}^{(2)})$ can be realized by K3 pairs (X, θ). Obviously, they are extremal, for them $\mathcal{M}(X)_+^{(2,4)} = \mathcal{M}^{(2,4)}$, and the Dynkin diagrams $\Gamma(P(X)_+) = \Gamma(P(\mathcal{M}^{(2,4)}))$ coincide. All these diagrams are described in Table 1 of Theorem 3.1. It is natural to call such pairs (X, θ) as super-extremal. Thus, a non-symplectic involution (X, θ) of elliptic type of K3 (equivalently, the corresponding DPN pair (Y, C) or DPN surface) is called *super-extremal,* if for the corresponding root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ we have $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta^{(4)}(\mathcal{M}^{(2)})$ (equivalently, $\Delta_+^{(4)} = \Delta^{(4)}(S)$). We have

Proposition 3.8. *For any possible elliptic triplet of main invariants (r, a, δ) there exists a super-extremal, i. e.*

$$\Gamma(P(X)_+) = \Gamma(P(\mathcal{M}^{(2,4)})),$$

and standard (see Sect. 2.6) K3 pair (X, θ).

See the description of their graphs $\Gamma(P(X)_+) = \Gamma(P(\mathcal{M}^{(2,4)}))$ in Table 1 of Theorem 3.1).

Proof. Let us consider an elliptic triplet of main invariants (r, a, δ) and the corresponding Dynkin diagram $\Gamma(P(\mathcal{M}^{(2,4)}))$ which is described in

Theorem 3.1. Denote $K^+(2) = [P^{(4)}(\mathcal{M}^{(2,4)})]$, i. e. it is the sublattice generated by all black vertices of $\Gamma(P(\mathcal{M}^{(2,4)}))$. Consider the corresponding root invariant $(K^+(2), \xi^+)$, see (20) and (21). Consider $H = \operatorname{Ker} \xi^+$. By Propositions 3.3 and 2.9, there exists a super-extremal standard pair (X, θ), if the inequalities

$$r + \operatorname{rk} K^+ + l(\mathfrak{A}_{(K^+)_p}) < 22 \text{ for all prime } p > 2,$$

$$r + a + 2l(H) < 22$$

are valid together with Conditions 1 and 2 from Sect. 2.6.

By trivial inspection of all cases in Table 1, we can see that first inequality is valid. To prove second inequality, it is enough to show that $l(H) \le 1$ since $r+a \le 18$ in elliptic case. The inequality $l(H) \le 1$ can be proved by direct calculation of $l(H)$ in all cases of Table 1 of Theorem 3.1.1. These calculations simplify by the general statement.

Lemma 3.9. *In elliptic super-extremal case,*

$$l(H) = \#P^{(4)}(\mathcal{M}^{(2,4)}) - l(\mathfrak{A}_S^{(1)})$$

where $\mathfrak{A}_S^{(1)} \subset \mathfrak{A}_S$ *is the subgroup generated by all elements* $x \in \mathfrak{A}_S$ *such that* $q_S(x) = 1 \mod 2$. *Moreover, we have:*

If $\delta = 0$ *then* $l(\mathfrak{A}_S^{(1)}) = a$ *except (*$a = 2$ *and* $\operatorname{sign} S = 2 - r \equiv 0 \mod 8$*). In the last case* $l(\mathfrak{A}_S^{(1)}) = a - 1$.

If $\delta = 1$, *then* $l(\mathfrak{A}_S^{(1)}) = a - 1$ *except cases (*$a = 2$ *and* $\operatorname{sign} S \equiv 0 \mod 8$*), (*$a = 3$ *and* $\operatorname{sign} S \equiv \pm 1 \mod 8$*), and (*$a = 4$ *and* $\operatorname{sign} S \equiv 0 \mod 8$*). In these cases* $l(\mathfrak{A}_S^{(1)}) = a - 2$.

Proof. We know (see Sect. 2.3.1) that $\Delta^{(4)}(S) = W^{(2)}(S)(\Delta^{(4)}(\mathcal{M}^{(2,4)}))$. The group $W^{(2)}(S)$ acts identically on $\mathfrak{A}_S$. Therefore,

$$\operatorname{Im} \xi^+ = [\{\xi^+(f/2 + K^+(2)) \mid f \in \Delta^{(4)}(\mathcal{M}^{(2,4)})\}] =$$

$$[\{f/2 + S \mid f \in \Delta^{(4)}(S)\} = \mathfrak{A}_S^{(1)}.$$

In the last equality, we use Lemma 2.6. For $Q = (K^+(2)/2)/K^+(2)$, we have $l(Q) = \operatorname{rk} K^+ = \#P^{(4)}(\mathcal{M}^{(2,4)})$. Thus, $l(H) = l(Q) - l(\mathfrak{A}_S^{(1)}) = \#P^{(4)}(\mathcal{M}^{(2,4)}) - l(\mathfrak{A}_S^{(1)})$.

The rest statements of Lemma can be proved by direct calculations using a decomposition of a 2-elementary non-degenerate finite quadratic form as sum of elementary ones: $q_{\pm 1}^{(2)}(2)$, $u_+^{(2)}(2)$ and $v_+^{(2)}(2)$ (in notations of [Nik80.2]). ☐

One can easily check Condition 2 of Sect. 2.6.

To check Condition 1 of Sect. 2.6, note that if the lattice $K_H^+(2)$ has elements with the square (-2), than the sublattice $[P^{(4)}(\mathcal{M}^{(2,4)})]_{\mathrm{pr}}$ of S also has elements with the square (-2). Let us show that this is not the case.

Let us consider the subspace

$$\gamma = \bigcap_{f\in P^{(4)}(\mathcal{M}^{(2,4)})} \mathcal{H}_f$$

of $\mathcal{L}(S)$ which is orthogonal to $[P^{(4)}(\mathcal{M}^{(2,4)})]$ (equivalently, we consider the corresponding face $\gamma \cap \mathcal{M}^{(2,4)}$ of $\mathcal{M}^{(2,4)}$). If the sublattice $[P^{(4)}(\mathcal{M}^{(2,4)})]_{\mathrm{pr}} \subset S$ has elements with square (-2), then some hyperplanes $\mathcal{H}_e$, $e \in \Delta^{(2)}(S)$, also contain γ and give reflections from $W^{(2,4)}(S)$. On the other hand (e. g. see [Vin85]), all hyperplanes of reflections from $W^{(2,4)}(S)$ containing γ must be obtained from the hyperplanes $\mathcal{H}_f$, $f \in P^{(4)}(\mathcal{M}^{(2,4)})$, by the group generated by reflections in $P^{(4)}(\mathcal{M}^{(2,4)})$. All these hyperplanes are then also orthogonal to elements with square (-4) from S. They cannot be orthogonal to elements with square (-2) from S too.

This finishes the proof of Proposition 3.8.

3.4.3. Let us prove Theorem 3.6 in all cases except 7 — 10 and 20 of Table 1. These cases (i. e. different from 7 — 10 and 20 of Table 1) characterize by the property that Dynkin diagram $\Gamma(P^{(4)}(\mathcal{M}^{(2,4)}))$ consists of components of type $\mathbb{A}$ only. By Sect. 3.4.1, any primitive root subsystem $\Delta^{(4)}_+(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ is then primitive. In particular, any root subsystem $\Delta^{(4)}_+(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ of finite index is $\Delta^{(4)}_+(\mathcal{M}^{(2)}) = \Delta^{(4)}(\mathcal{M}^{(2)})$. By Proposition 3.8, we then obtain

Proposition 3.10. *For any elliptic triplet (r, a, δ) of main invariants which is different from (6,6,1), (7,7,1), (8,8,1), (9,9,1) and (10,8,1), any extremal K3 pair (X, θ) is super-extremal, i. e. $\Gamma(P(X)_+) = \Gamma(P(\mathcal{M}^{(2,4)}))$ (see their description in Table 1 of Theorem 3.1).*

3.4.4. Now let us consider cases 7—10 and 20 of Table 1. In these cases, the root system $R = \Delta^{(4)}(\mathcal{M}^{(2)})$ is $\mathbb{D}_5$ in the case 7, $\mathbb{E}_6$ in the case 8, $\mathbb{E}_7$ in the case 9, $\mathbb{E}_8$ in the case 10, and $\mathbb{D}_8$ in the case 20.

We have

Lemma 3.11. *If R is one of root systems of types $\mathbb{D}_5$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ or $\mathbb{D}_8$, then its root subsystem $T \subset R$ of finite index is defined by the isomorphism type of the root system T itself, up to the action of $W(R)$. Moreover, the type of T can be the following and only the following* *which is given in Table of Lemma 3.11 below (we identify the type with the isomorphism class of the corresponding root lattice).*

Table of Lemma 3.11.

N	R	T
7	D_5	a) D_5, b) $A_3 \oplus 2A_1$
8	E_6	a) E_6, b) $A_5 \oplus A_1$, c) $3A_2$
9	E_7	a) E_7, b) A_7, c) $A_5 \oplus A_2$, d) $2A_3 \oplus A_1$, e) $D_6 \oplus A_1$, f) $D_4 \oplus 3A_1$, g) $7A_1$
10	E_8	a) E_8, b) A_8, c) $A_7 \oplus A_1$, d) $A_5 \oplus A_2 \oplus A_1$, e) $2A_4$, f) D_8, g) $D_5 \oplus A_3$, h) $E_6 \oplus A_2$, i) $E_7 \oplus A_1$, j) $D_6 \oplus 2A_1$, k) $2D_4$, l) $2A_3 \oplus 2A_1$, m) $4A_2$, n) $D_4 \oplus 4A_1$, o) $8A_1$
20	D_8	a) D_8, b) $D_6 \oplus 2A_1$, c) $D_5 \oplus A_3$, d) $2D_4$, e) $2A_3 \oplus 2A_1$, f) $D_4 \oplus 4A_1$, g) $8A_1$

Proof. It is done by inspection using description of root subsystems of finite index in Sect. 3.4.1. □

Consider a root subsystem $T \subset R$ of Lemma 3.11. By Theorem 2.4, the root subsystem $T \subset R$ defines a subset $\Delta^{(4)}_+(S) \subset \Delta^{(4)}(S)$, the corresponding reflection group $W^{(2,4)}_+$, and Dynkin diagram $\Gamma(P(\mathcal{M}^{(2,4)}_+))$ of its fundamental chamber $\mathcal{M}^{(2,4)}_+$. Direct calculation of these diagrams using Theorem 2.4 gives diagrams of Table 2 of Theorem 3.6 (where $\Gamma(P(\mathcal{M}^{(2,4)}_+))$ is replaced by $\Gamma(P(X)_+))$ in all cases 7a, b; 8a — c; 9a — f; 10a — m; 20a — d. In the rest cases 9g; 10n, o; 20e — g we get diagrams $\Gamma(P(\mathcal{M}^{(2,4)}_+))$ which we describe below.

In the *Case 9g,* it is better to describe $\Gamma(P(\mathcal{M}^{(2,4)}_+))$ indirectly. Its transparent vertices correspond to all points of $\mathbb{P}^2(F_2)$ which is the projective plane over the field F_2 with two elements. Its black vertices correspond to all lines in $\mathbb{P}^2(F_2)$. Both sets have seven elements. Transparent vertices are disjoint; black vertices are also disjoint; a transparent vertex is connected with a black vertex by the double edge, if the corresponding point belongs to the corresponding line, otherwise, they are disjoint.

In the *Case 10n,* the diagram $\Gamma(P(\mathcal{M}^{(2,4)}_+))$ is given below in Figure 3. Since it is quite complicated, we divide it in three subdiagrams shown. The first one shows all its edges connected black and transparent vertices. The second one shows the edge connected the transparent vertices numerated by 1 and 2. The third one shows edges connected transparent vertices 3 — 12. Each edge of $\Gamma(P(\mathcal{M}^{(2,4)}_+))$ is shown in one of these diagrams. (All other our similar descriptions of diagrams

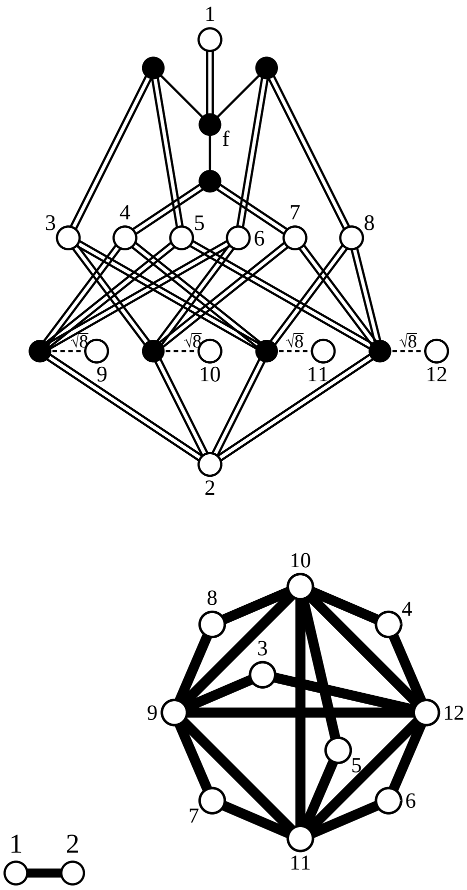


FIGURE 3. The diagram 10n

as unions of their subdiagrams have the same meaning. In particular, we have used it in some diagrams of Table 2.

In the *Case 10o,* we describe the diagram $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ indirectly. Its black vertices f_v, $v \in V$, correspond to all points of a three-dimensional affine space V over F_2. Its transparent vertices are of two types. Vertices e_v of the first type also correspond to all points $v \in V$. Vertices e_π of the second type correspond to all (affine) planes $\pi \subset V$ (there are 14 of them). Black vertices f_v are disjoint. A black vertex f_v is connected with a transparent vertex $e_{v'}$, if and only if $v = v'$; the edge has the weight $\sqrt{8}$. A black vertex f_v is connected with a transparent vertex e_π, if and only if $v \in \pi$; the edge is double. Transparent vertices e_v, $e_{v'}$ are connected by a thick edge. A transparent vertex e_v is connected with a transparent vertex e_π, if and only if $v \notin \pi$; the edge is thick.

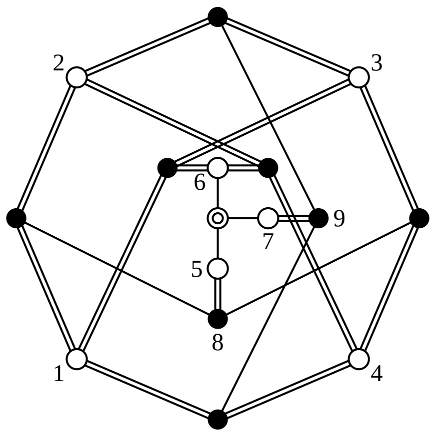


FIGURE 4. The diagram 20e

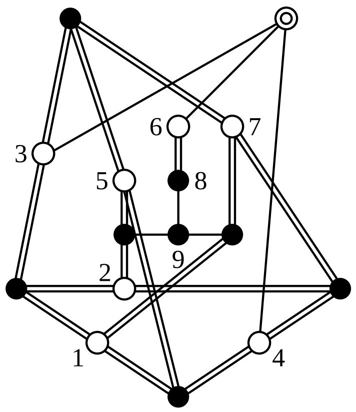


FIGURE 5. The diagram 20f

Transparent vertices e_π, $e_{\pi'}$ are connected by edge, if and only if $\pi\|\pi'$; the edge is thick.

In *Cases 20e, 20f and 20g* diagrams $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ are shown in figures 4—6 below.

We remark that a calculation of $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ in cases 7a,b, 8a—c, 9a—g and 10a—o can be obtained from results of [BBD84] where (in our notations) the dual diagram of all exceptional curves on the quotient $Y = X/\{1,\theta\}$ is calculated using completely different method (under the assumption that Y does exist). By Sect. 2.4, both diagrams can be easily obtained from one another (compare with Sect. 3.5 below). Therefore, we explain our method of calculation of $\Gamma(P(\mathcal{M}_+^{(2,4)}))$

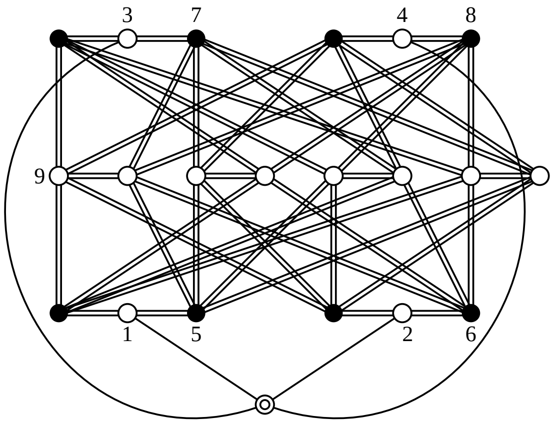


FIGURE 6. The diagram 20g

in more details than it has done in Sect. 2.3.1 only in the Case 20 (i. e. cases 20a—g).

In the *Case 20,* the lattice S has invariants $(r, a, \delta) = (10, 8, 1)$, and we can take in $S \otimes \mathbb{Q}$ an orthogonal basis h, a, $v_1, \dots, v_8$ with $h^2 = 2$, $a^2 = v_1^2 = \cdots = v_8^2 = -2$. As $P(\mathcal{M}^{(2,4)})$, we can take

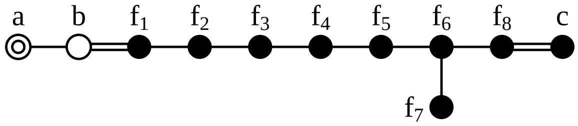
(23)

$$\begin{aligned} P^{(4)}(\mathcal{M}^{(2,4)}) =& \{f_1 = v_1 - v_2, f_2 = v_2 - v_3, f_3 = v_3 - v_4, f_4 = v_4 - v_5, \\ & f_5 = v_5 - v_6, f_6 = v_6 - v_7, f_7 = v_7 - v_8, f_8 = v_7 + v_8, \} \end{aligned}$$

and

$$P^{(2)}(\mathcal{M}^{(2,4)}) = \{a,\, b = \frac{h}{2} - \frac{a}{2} - v_1,\, c = h - \frac{1}{2}(v_1 + v_2 + \cdots + v_8)\}.$$

These elements have Dynkin diagram



of the case 20a, and they generate and define S.

By Sect. 2.3.1, the set $P^{(2)}(\mathcal{M}^{(2)})$, where $\mathcal{M}^{(2)} \supset \mathcal{M}^{(2,4)}$, is

$$W(\mathcal{M}^{(2)})(\{a, b, c\})$$

where $W(\mathcal{M}^{(2)})$ is generated by reflections in $f_1, \dots, f_8$. It follows that

$$P(\mathcal{M}^{(2)}) = P^{(2)}(\mathcal{M}^{(2)}) = \{a; b_{\pm i}; c_{i_1 \dots i_k}\}$$

where

$$b_{\pm i} = \frac{h}{2} - \frac{a}{2} \pm v_i,\ \ i = 1, 2, \dots, 8;$$

$$c_{i_1 \dots i_k} = h + \frac{1}{2}(v_1 + v_2 + \dots + v_8) - v_{i_1} - \dots - v_{i_k},$$

where $1 \le i_1 < i_2 < \dots < i_k \le 8$ and $k \equiv 0 \mod 2$. Here all $b_{\pm i}$ give the $W(\mathcal{M}^{(2)})$-orbit of b, and all $c_{i_1 \dots i_k}$ give the $W(\mathcal{M}^{(2)})$-orbit of c.

Elements $f_1, \dots, f_8$ give a basis of the root system R of type $\mathbb{D}_8$. If $T \subset R$ is its subsystem of rank m, and $t_1, \dots, t_m$ a basis of T, then defining by T and by its basis $t_1, \dots, t_m$ fundamental chamber $\mathcal{M}_+^{(2,4)} \subset \mathcal{M}^{(2)}$ has $P(\mathcal{M}_+^{(2,4)}) = P^{(4)}(\mathcal{M}_+^{(2,4)}) \cup P^{(2)}(\mathcal{M}_+^{(2,4)})$ where

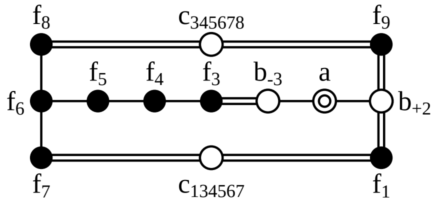
$$\begin{aligned} P^{(4)}(\mathcal{M}_+^{(2,4)}) =& \{t_1, \dots, t_m\}, \\ P^{(2)}(\mathcal{M}_+^{(2,4)}) =& \{a\} \cup \{b_{\pm i} \mid b_{\pm i} \cdot t_s \ge 0,\ 1 \le s \le m\} \\ & \cup \{c_{i_1 \dots i_k} \mid c_{i_1 \dots i_k} \cdot t_s \ge 0,\ 1 \le s \le m\}. \end{aligned} \tag{24}$$

This describes $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ completely.

For example, assume that $T \subset R$ has the type $2A_1 \oplus D_6$ with the basis $f_1,\ f_9 = -v_1 - v_2,\ f_3,\ \dots,\ f_8$. Then we get (after simple calculations)

$$P^{(2)}(\mathcal{M}_+^{(2,4)}) = \{a, b_2, b_{-3}, c_{345678}, c_{134567}\},$$

and $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ is



This gives Case 20b of Table 2.

Exactly the same calculations of $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ can be done in all cases 20a—g, and cases 7a,b — 10a—o of Table of Lemma 3.11 as well. Details of these calculations require too much space, and we leave them to a reader. The final results have been presented.

3.4.5. Here we prove

Proposition 3.12. *Cases 9g, 10n,o and 20 e — g of root subsystems $T \subset R$ of Lemma 3.11 do not correspond to non-symplectic involutions (X, θ) of K3 (in characteristic 0 and even in characteristic ≥ 3).*

Proof. Assume that a root subsystem $T \subset R$ corresponds to a K3 pair (X, θ). Then the corresponding Dynkin diagram $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ given in Sect. 3.4.4 coincides with Dynkin diagram $\Gamma(P(X)_+)$ of exceptional curves of the pair (X, θ). It follows the dual diagram of exceptional curves $\Gamma(P(Y))$ on the corresponding DPN surface $Y = X/\{1, \theta\}$ (see Sect. 2.3). Using this diagram, it is easy to find a sequence of exceptional curves $E_1, \dots, E_k$ on Y where $k = r - 1$ such that their contraction gives a morphism $\sigma : Y \to \mathbb{P}^2$. Then other (different from $E_1, \dots, E_k$) exceptional curves on Y corresponding to Du Val and logarithmic part of $\Gamma(P(Y))$ give a configuration of rational curves on $\mathbb{P}^2$ which cannot exist in characteristic 0 and even in characteristic ≥ 3 (but it exists in characteristic 2). In cases 9g; 10n,o; 20e,f we get Fano's configuration of seven lines of the finite projective plane over F_2 which can exist only in characteristic 2. In the case 9g one should contract exceptional curves corresponding to all transparent vertices. In the case 10n — corresponding to vertices 1, f, 3 — 8. In the case 10o — corresponding to vertices e_π where π contains a fixed point $0 \in V$ and e_0; then curves corresponding to f_v, $v \ne 0$, give Fano's configuration. In cases 20e,f — corresponding to vertices 1 — 9. In the case 20g — corresponding to vertices 1 — 9, then we get a conic (corresponding to the double transparent vertex) and four its tangent lines (corresponding to black vertices different from 5 — 8) passing through one point. It is possible only in characteristic 2. $\square$

Another purely arithmetic proof of Proposition 3.12 can be obtained using Proposition 2.10. This proof is more complicated, but it can also be done. Here we preferred more short (if diagrams have calculated) the geometric considerations.

$\square$

3.4.6. Here we prove

Proposition 3.13. *Cases 7a,b; 8a—c, 9a—f; 10a—m and 20a—d of Table 2 of Theorem 3.6 correspond to standard extremal non-symplectic K3 involutions* (X, θ) *over* $\mathbb{C}$.

Proof. Let us calculate corresponding to these cases root invariants $(K^+(2), \xi^+)$.

Consider the sequence of embeddings of lattices

$$K^+(2) = [T] \subset [R] \subset S.$$

It defines the homomorphism

$$\xi^+ : Q = \frac{1}{2}K^+(2)/K^+(2) \to S^*/S \subset \frac{1}{2}S/S$$

with the kernel H. It can be decomposed as

$$\xi^+ : Q \xrightarrow{\widetilde{\xi}^+} \frac{1}{2}[R]/[R] \xrightarrow{\xi^+_R} S^*/S \subset \frac{1}{2}S/S. \tag{25}$$

Let $H_R = \operatorname{Ker} \xi^+_R$. Then $H = (\widetilde{\xi}^+)^{-1}(H_R)$. As we know (from our considerations in the super-extremal case), $H_R = 0$ in cases 7, 8, 9, 10. In the case 20, the $H_R = \mathbb{Z}/2\mathbb{Z}$ is

$$H_R = [\frac{1}{2}(f_1 + f_3 + f_5 + f_7) + R]/[R]$$

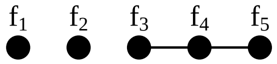
(see Sect. 3.4.4 about this case). Thus, H can be identified with $H = (\frac{1}{2}[T] \cap [R])/[T]$ in cases 7, 8, 9, 10, and with

$$H = (\frac{1}{2}[T] \cap [\frac{1}{2}(f_1 + f_3 + f_5 + f_7) + R])/[T]$$

in the case 20. Using Lemma 3.11 and the described in Sect. 3.4.1 method of construction of root subsystems $T \subset R$, the subgroups H can be easily calculated.

In Case 7a the $H = 0$.

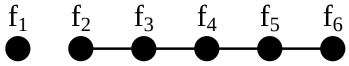
In Case 7b with the basis (in T)



the $H = [(f_1 + f_2 + f_3 + f_5)/2] \mod [T]$.

In Case 8a the $H = 0$.

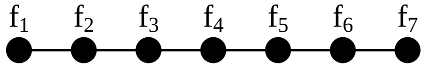
In Case 8b with the basis



the $H = [(f_1 + f_2 + f_4 + f_6)/2] \mod [T]$.

In Cases 8c and 9a the $H = 0$.

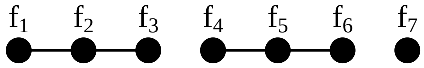
In Case 9b with the basis



the $H = [(f_1 + f_3 + f_5 + f_7)/2] \mod [T]$.

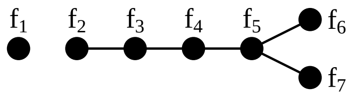
In Case 9c the $H = 0$.

In Case 9d with the basis



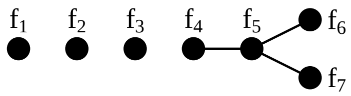
the $H = [(f_1 + f_3 + f_4 + f_6)/2] \mod [T]$.

In Case 9e with the basis



the $H = [(f_1 + f_2 + f_4 + f_6)/2] \mod [T]$.

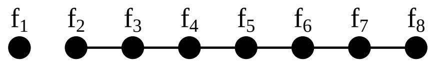
In Case 9f with the basis



the $H = [(f_1 + f_2 + f_4 + f_6)/2] \mod [T]$.

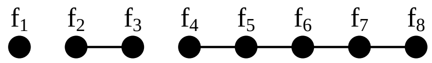
In Cases 10a and 10b the $H = 0$.

In Case 10c with the basis



the $H = [(f_2 + f_4 + f_6 + f_8)/2] \mod [T]$.

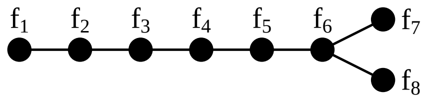
In Case 10d with the basis



the $H = [(f_1 + f_4 + f_6 + f_8)/2] \mod [T]$.

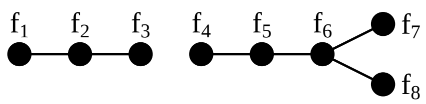
In Case 10e the $H = 0$.

In Case 10f with the basis



the $H = [(f_1 + f_3 + f_5 + f_7)/2] \mod [T]$.

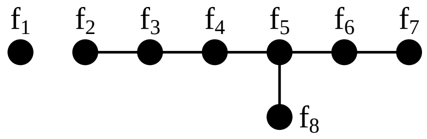
In Case 10g with the basis



the $H = [(f_1 + f_3 + f_7 + f_8)/2] \mod [T]$.

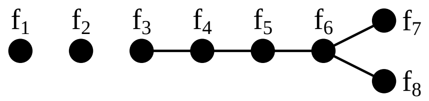
In Case 10h the $H = 0$.

In Case 10i with the basis



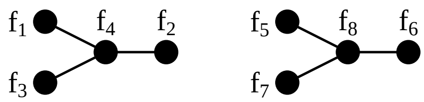
the $H = [(f_1 + f_2 + f_4 + f_8)/2] \mod [T]$.

In Case 10j with the basis



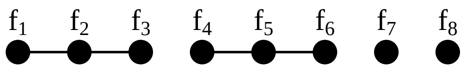
the $H = [(f_1 + f_3 + f_5 + f_7)/2,\ (f_2 + f_3 + f_5 + f_8)/2] \mod [T]$.

In Case 10k with the basis



the $H = [(f_1 + f_2 + f_5 + f_6)/2,\ (f_2 + f_3 + f_6 + f_7)/2] \mod [T]$.

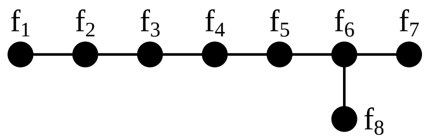
In Case 10l with the basis



the $H = [(f_1 + f_3 + f_7 + f_8)/2,\ (f_4 + f_6 + f_7 + f_8)/2] \mod [T]$.

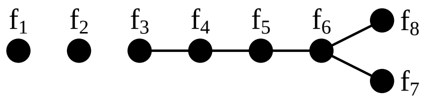
In Case 10m the $H = 0$.

In Case 20a with some basis



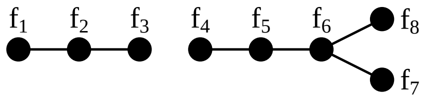
the $H = [(f_1 + f_3 + f_5 + f_7)/2] \mod [T]$.

In Case 20b with the basis



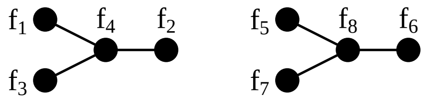
the $H = [(f_1 + f_3 + f_5 + f_7)/2,\ (f_2 + f_3 + f_5 + f_8)/2] \mod [T]$.

In Case 20c with the basis



the $H = [(f_1 + f_3 + f_7 + f_8)/2] \mod [T]$.

In Case 20d with the basis



the $H = [(f_1 + f_2 + f_5 + f_6)/2,\ (f_2 + f_3 + f_6 + f_7)/2] \mod [T]$.

¿From these calculations, we get values of $l(H)$ given in Table 2 of Theorem 3.6.

As in Sect. 3.4.2, using Proposition 2.9, one can prove that all these cases when

$$r + a + 2l(H) < 22 \tag{26}$$

correspond to standard extremal non-symplectic K3 involutions (X, θ) (always over $\mathbb{C}$). Thus, we only need to consider cases when the inequality (26) fails. There are exactly five such cases: 10j,k,l and 20b,d. Further we consider these cases only.

In cases 10j,k,l the discriminant form of S is $q_S = q_1^{(2)}(2) \oplus 4u_+^{(2)}(2)$ (in notations of [Nik80.2]). Here, the generator of the first summand $q_1^{(2)}(2)$ gives the characteristic element a_{q_S} of the q_S, and the second summand $4u_+^{(2)}(2)$ gives the image of ξ_R^+ from (25), by Lemma 3.9. Thus, the image of ξ^+ belongs to $4u_+^{(2)}(2)$. The discriminant form of the lattice M (from 2.6) is obtained as follows (see [Nik80.2] about this technique). Let

$$\Gamma_{\xi^+} \subset Q \oplus \mathfrak{A}_S \subset \mathfrak{A}_{K^+(2)} \oplus \mathfrak{A}_S$$

be the graph of the homomorphism ξ^+ in $\mathfrak{A}_{K^+(2)} \oplus \mathfrak{A}_S$. Then

$$q_M = (q_{K^+(2)} \oplus q_S \,|\, (\Gamma_{\xi^+})^{\perp}_{q_{K^+(2)} \oplus q_S})/\Gamma_{\xi^+} \tag{27}$$

(here Γ_{ξ^+} is an isotropic subgroup).Therefore, $q_M \cong q_1^{(2)}(2) \oplus q'$ since the image of ξ^+ belongs to the orthogonal complement of the summand $q_1^{(2)}(2)$. Considerations in the proof of Proposition 2.9 show that

$$\operatorname{rk} M + l(\mathfrak{A}_{M_2}) \leq 22 \tag{28}$$

since $r + a + 2l(H) = 22$ in cases 10j,k,l. It is easy to see that

$$\operatorname{rk} M + l(\mathfrak{A}_{M_p}) < 22$$

for all prime $p>2$. Then, by Theorem 1.12.2 in [Nik80.2], there exists a primitive embedding $M\subset L_{K3}$ when either the inequality (28) is strict or $q_{M_2}\equiv q^{(2)}_{\pm 1}(2)\oplus q'$, if it gives the equality. Thus, it always does exist. It follows that all cases 10j,k,l correspond to standard extremal non-symplectic K3 involutions (X,θ) over $\mathbb{C}$ by Proposition 2.10 (where we have used fundamental Global Torelli Theorem [PS-Sh71] and surjectivity of Torelli map [Kul77] for K3).

In cases 20b,d, the proof is exactly the same, but it is more difficult to prove that $q_{M_2}\cong q^{(2)}_{\theta}(2)\oplus q'$ where $\theta=\pm1$. In these cases

$$q_S=3u^{(2)}_{+}(2)\oplus q^{(2)}_{1}(2)\oplus q^{(2)}_{-1}(2).$$

If α_1 and α_2 are generators of the summands $q^{(2)}_1(2)$ and $q^{(2)}_{-1}(2)$ respectively, then $\alpha_{q_S}=\alpha_1+\alpha_2$ is the characteristic element of q_S, and the image of ξ^+ belongs to $3u^{(2)}_{+}(2)\oplus[\alpha_{q_S}]$. In these cases, the lattice $K^+_H(2)$ (see Sect. 2.6) is isomorphic to $E_8(2)$. For example, this is valid because the subgroups H are the same in cases 10j and 20b, and in cases 10k and 20d, besides, in cases 10j and 10 k we have $E_8/K^+\cong H$. It follows that

$$q_{K^+_H(2)}=(q_{K^+(2)}\mid(H)^{\perp}_{q_{K^+(2)}})/H\cong q_{E_8(2)}\cong 4u^{(2)}_{+}(2).$$

We set $\overline{\Gamma}_{\xi^+}=\Gamma_{\xi^+}/H$. By (27)

$$q_M=(q_{K^+_H(2)}\oplus q_S\mid(\overline{\Gamma}_{\xi^+})^{\perp}_{q_{K^+_H(2)}\oplus q_S})/\overline{\Gamma}_{\xi^+}.$$

We have $q_{K^+_H(2)}\oplus q_S=7u^{(2)}_{+}(2)\oplus q^{(2)}_{1}(2)\oplus q^{(2)}_{-1}(2)$. Since $u^{(2)}_{+}(2)$ takes values in $\mathbb{Z}/2\mathbb{Z}$, the element α_{q_S} (more exactly, $0\oplus\alpha_{q_S}$) is the characteristic element of $q_{K^+_H(2)}\oplus q_S$ again. Moreover, $\alpha_{q_S}\notin\overline{\Gamma}_{\xi^+}$ since Γ_{ξ^+} is the graph of a homomorphism with the kernel H. Therefore $(\overline{\Gamma}_{\xi^+})^{\perp}_{q_{K^+_H(2)}\oplus q_S}$ contains v which is not orthogonal to α_{q_S}. Then

$$(q_{K^+_H(2)}\oplus q_S)(v)=\pm\frac{1}{2}\mod 2$$

and

$$[v\mod\overline{\Gamma}_{\xi^+}]\cong q^{(2)}_{\theta}(2),\ \theta=\pm1,$$

is the orthogonal summand of q_{M_2} we were looking for. □

Remark 3.14. We can give another proof of Proposition 3.13 which uses Theorem 1.5 and considerations which are inverse to the proof of the previous Proposition 3.12. Indeed, by Theorem 1.5, it is enough to prove existence of rational surfaces with Picard number r and configuration of rational curves defined by Dynkin diagram of Table 2 of Theorem 3.6 (assuming that these Dynkin diagrams correspond to

K3 pairs (X,θ) and considering the quotient by θ). One can prove existence of these rational surfaces considering appropriate sequences of blow-ups of appropriate relatively minimal rational surfaces $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{F}_3$ or $\mathbb{F}_4$ with appropriate configurations of rations curves defined by Dynkin diagrams of Table 2 of Theorem 3.6 (see the proof of Proposition 3.12). This proof does not use Global Torelli Theorem and surjectivity of Torelli map for K3. This gives a hope that results of Sect. 2 and Sect. 3 can be generalized to characteristic $p > 0$. Unfortunately, we have proved Theorem 1.5 in characteristic 0 only. Thus, we preferred the proof of Proposition 3.13 which is independent on the results of Sect. 1.

3.4.7. To finish the proof of Theorem 3.6, we need to prove only

Proposition 3.15. *Let (X,θ) be a non-symplectic involution of K3 which corresponds to one of cases 7 — 10 or 20 of Table 1 of Theorem 3.1 and a root subsystem $T \subset R = \Delta^{(4)}(\mathcal{M}^{(2)})$.*

If (X,θ) is extremal, then $\operatorname{rk} T = \operatorname{rk} R$

Proof. We can assume (see Sect. 3.4.1) that T has a basis which gives a part of a basis of a root subsystem $\widetilde{T} \subset R = \Delta^{(4)}(\mathcal{M}^{(2)})$ of the same rank $\operatorname{rk}\widetilde{T} = \operatorname{rk} R$. Then $\widetilde{T} \subset R$ is one of root subsystems of Lemma 3.11. If the root subsystem $\widetilde{T} \subset R$ corresponds to a non-symplectic involution of K3, i. e. $\widetilde{T}$ gives cases 7a—b, 8a—c, 9a—f, 10a—m and 20a—d, then T is extremal, only if $T = \widetilde{T}$ (by definition). Then $\operatorname{rk} T = \operatorname{rk}\widetilde{T} = \operatorname{rk} R$ as we want. Thus, it is enough to consider $\widetilde{T}$ of cases 9g, 10n—o, 20e—g and $T \subset \widetilde{T}$ to be a primitive root subsystem of a strictly smaller rank. Let us consider all these cases.

Case 9g. Then $\widetilde{T} = 7A_1$, and $T = kA_1$, $k \leq 6$. The root subsystem $\widetilde{T} = 7A_1$ is obtained from the root system $D_4 \oplus 3A_1$ by the procedure described in Sect. 3.4.1. One can see that then T is a primitive root subsystem in $D_4 \oplus 3A_1$. Since $D_4 \oplus 3A_1$ corresponds to K3, the T is not extremal.

Case 10n. Then $\widetilde{T} = D_4 \oplus 4A_1$. It is obtained from the root subsystems $D_6 \oplus 2A_1$ or $D_4 \oplus D_4$ by the procedure described in Sect. 3.4.1. It follows that T is a primitive root subsystem in $D_6 \oplus 2A_1$, $D_4 \oplus D_4$ or $2A_3 \oplus 2A_1$ (and it is not then extremal) in all cases except when $T = 7A_1$.

Let us consider the last case $T = 7A_1$ and show (as in Sect. 3.4.5) that it does not correspond to K3. As in Sect.3.4.4, one can calculate Dynkin diagram $\Gamma = \Gamma(P(\mathcal{M}_+^{(2,4)}))$. It is similar to the case 10o (see Sect. 3.4.4), but it is more complicated. We describe it indirectly. One can relate with this diagram a 3-dimensional linear vector space V over F_2.

Black vertices f_v of Γ correspond to $v \in V - \{0\}$ (there are seven of them). Its transparent vertices (all of them are simple) are
e_v, $v \in V - \{0\}$; $e_0^{(+)}$, $e_0^{(-)}$;
e_π, $\pi \subset V$ is any hypersurface in V which does not contain 0;
$e_\pi^{(+)}$, $e_\pi^{(-)}$, $\pi \subset V$ is any hyperspace ($0 \in \pi$) of V.

Edges which connect f_v, e_v, $e_0^{(+)}$, e_π, $e_\pi^{(+)}$ are the same as for the diagram 10o (forget about $(+)$). The same is valid for f_v, e_v, $e_0^{(-)}$, e_π, $e_\pi^{(-)}$ (forget about $(-)$). Vertices $e_0^{(+)}$ and $e_0^{(-)}$ are connected by the punctured edge of the weight 6. Vertices $e_0^{(+)}$ and $e_\pi^{(-)}$ (and $e_0^{(-)}$, $e_\pi^{(+)}$ as well) are connected by the punctured edge of the weight 4. This gives all edges of Γ.

Assume that Γ corresponds to a K3 pair (X, θ). Consider the corresponding DPN surface and contract exceptional curves corresponding to $e_\pi^{(+)}$ and $e_0^{(+)}$. Then exceptional curves of f_v, $v \in V - \{0\}$, give Fano's configuration on $\mathbb{P}^2$ which exists only in characteristic 2. We get a contradiction.

Case 20e. Then $\widetilde{T} = 2A_3 \oplus 2A_1$. It is obtained from the root subsystems $D_6 \oplus 2A_1$ or $D_5 \oplus A_3$ by the procedure described in Sect. 3.4.1. It follows that T is a primitive root subsystem in $D_6 \oplus 2A_1$ or $D_5 \oplus A_3$ (and it is not then extremal) in all cases except when $T = A_3 \oplus 4A_1$.

Let us consider the last case $T = A_3 \oplus 4A_1$ and show (as in Sect. 3.4.5) that it does not correspond to K3. As in Sect.3.4.4, one can calculate Dynkin diagram $\Gamma = \Gamma(P(\mathcal{M}_+^{(2,4)}))$. It has exactly one transparent double vertex a and eight simple transparent vertices c_v, $v \in V(K)$, where $V(K)$ is the set of vertices of a 3-dimensional cube K with distinguished two opposite 2-dimensional faces α, $\alpha' \in \gamma(K)$ where $\gamma(K)$ is the set of all 2-dimensional faces of K. Black vertices of Γ are f_γ, $\gamma \in \gamma(K)$, and one more black vertex f_0. Simple transparent vertices of Γ which are connected by a simple edge with a are either $b_{\overline{\gamma}}$, $\overline{\gamma} \in \overline{\gamma(K)}$, where $\overline{\gamma(K)}$ is the set of pairs of opposite 2-dimensional faces of K, or b_t, $t \in \overline{V(K)}$. Here $\overline{V(K)}$ consists of two elements corresponding to a choice of one vertex from each pair of opposite vertices of K in such a way that neither of three of them are contained in a 2-dimensional face $\gamma \in \gamma(K)$ (they define a regular tetrahedron with edges which are diagonals of 2-dimensional faces of K).

Let us describe edges of Γ different from above. Thick edges connect c_v corresponding to opposite vertices $v \in V(K)$, vertices b_{t_1} and b_{t_2}

where $\{t_1, t_2\} = \overline{V(K)}$, vertices b_t and c_v where $v \in t$. Simple edges connect f_0 with f_α and $f_{\alpha'}$. Double simple edges connect c_v with f_γ, if $v \in \gamma$, and $b_{\overline{\gamma}}$ with f_γ, if $\gamma \in \overline{\gamma} - \{\alpha, \alpha'\}$, and the vertex $b_{\overline{\alpha}}$ with f_0.

Assume that Γ corresponds to a K3 pair (X, θ). On its DPN surface, let us contract exceptional curves corresponding to c_v, $v \in t$; $b_{\overline{\gamma}}$, $\overline{\gamma} \in \overline{\gamma(K)}$; f_0 and $b_{t'}$, $t' \ne t$ (here $t \in \overline{V(K)}$ is fixed). Then curves corresponding to f_v, $v \in V(K)$, and the vertex a define Fano's configuration of lines in $\mathbb{P}^2$ which can exist only in characteristic 2.

Cases 20f,g. In these cases, $\widetilde{T} = D_4 \oplus 4A_1$ or $\widetilde{T} = 8A_1$. As for analogous cases 10, everything is reduced to prove that $T = 7A_1$ does not correspond to a K3 pair (X, θ).

In this case, $\Gamma = \Gamma(P(\mathcal{M}_+^{(2,4)}))$ is as follows. Let $I = \{1, 2, 3, 4\}$ and $J = \{1, 2\}$. The Γ has: exactly one double transparent vertex a; black vertices f_{ij}, $i \in I$, $j \in J$, and $(i, j) \ne (4, 2)$; simple transparent vertices b_i, $i = 1, 2, 3$, and $b_{4(+)}$, $b_{4(-)}$ which are connected by a simple edge with a; simple transparent vertices $c_{j_1 j_2 j_3 j_4}$ where $j_1, j_2, j_3 \in J$, $j_4 \in \{1, -2, +2\}$ and $j_1 + j_2 + j_3 + j_4 \equiv 0 \mod 2$ which are disjoint to a.

Edges of Γ which are different from above, are as follows.

Double edges connect b_i with f_{ij}, if $i = 1, 2, 3$, and $b_{4(+)}$, $b_{4(-)}$ with f_{41}, and $c_{j_1 j_2 j_3 j_4}$ with f_{1j_1}, f_{2j_2}, f_{3j_3}, and $c_{j_1 j_2 j_3 1}$ with f_{41}.

Thick edges connect $b_{4(\pm)}$ with $c_{j_1 j_2 j_3 (\mp 2)}$, and $c_{j_1 j_2 j_3 j_4}$ with $c_{j_1' j_2' j_3' j_4'}$, if $j_1 \ne j_1'$, $j_2 \ne j_2'$, $j_3 \ne j_3'$, $|j_4| \ne |j_4'|$, and $c_{j_1 j_2 j_3 (+2)}$ with $c_{j_1' j_2' j_3' (-2)}$, if $(j_1, j_2, j_3) \ne (j_1', j_2', j_3')$.

Assume that Γ corresponds to a K3 pair (X, θ). On its DPN surface, let us contract exceptional curves corresponding to b_1, b_2, b_3, $b_{4(+)}$, f_{11}, f_{21}, f_{31}, f_{41}, $c_{222(+2)}$. The curve corresponding to a gives a conic in $\mathbb{P}^2$. Corresponding to f_{12}, f_{22}, f_{32} curves give lines touching to the conic and having a common point. This is possible in characteristic 2 only. $\square$

This finishes the proof of Theorem 3.6

3.5. **Classification of DPN surfaces of elliptic type.** Each non-symplectic involution of elliptic type (X, θ) of K3 gives rise to the "good" DPN pair (Y, C) where

$$Y = X/\{1, \theta\},\ C = \pi(X^\theta) \in |-2K_Y|, \tag{29}$$

$\pi : X \to Y$ the quotient morphism; and vice versa. ¿From Theorems 3.6, 3.7, we then get classification of "good" DPN pairs (Y, C) and DPN surfaces Y of elliptic type. See Sect. 2 and especially Sects 2.1 and 2.7. It is obtained by the reformulation of Theorems 3.6 and 3.7

and by redrawing of the diagrams. But, for readers convenience, we do it below.

Theorem 3.16 (Classification Theorem for "good" DPN surfaces of elliptic type in the extremal case)**.** *Over $\mathbb{C}$, a "good" DPN surface Y of elliptic type is called* **extremal,** *if the number of its exceptional curves with the square* (-2) *(i. e. the number of black vertices in the diagram* Γ *below) is maximal for the fixed main invariants* (r, a, δ) *(equivalently,* (k, g, δ)*).*

Moreover, the dual diagram $\Gamma = \Gamma(Y)$ of all exceptional curves on Y is one of diagrams given in Table 3, and vice versa any diagram of Table 3 corresponds to some of the Y.

In the diagrams Γ, simple transparent vertices correspond to curves of the 1st kind (i. e. to non-singular rational irreducible curves with the square (-1)*), double transparent vertices correspond to non-singular rational irreducible curves with the square* (-4)*, black vertices correspond to non-singular rational irreducible curves with the square* (-2)*, a k-multiple edge (or an edge with the weight k when k is large) means the intersection index k for the corresponding curves. Any exceptional curve on Y is one of these curves.*

For a not necessarily extremal "good" DPN surface Y of elliptic type the dual diagram $\Gamma = \Gamma(Y)$ of all exceptional curves on Y also consists of simple transparent, double transparent and black vertices which have exactly the same meaning as in Theorem 3.16 above. All black vertices of Γ define the*Du Val part* $\operatorname{Duv}\Gamma$ of Γ. All double transparent vertices of Γ, and all simple transparent vertices of Γ which are connected by two edges with double transparent vertices of Γ (there are always two of these double transparent vertices) define the *logarithmic part* $\operatorname{Log}\Gamma$ of Γ. The rest (vertices different from vertices of $\operatorname{Duv}\Gamma$ and $\operatorname{Log}\Gamma$) define the *varying part* $\operatorname{Var}\Gamma$ of Γ. In Theorem below we identify vertices of $\Gamma(Y)$ with elements of Picard lattice $\operatorname{Pic}\widetilde{Y}$, then weights of edges are equal to the corresponding intersection pairing in this lattice which makes sense to the descriptions of the graphs $\operatorname{Var}\Gamma(Y)$ and $\Gamma(Y)$.

Theorem 3.17 (Classification Theorem for DPN surfaces in the non-extremal, i. e. general, case of elliptic type)**.** *Over $\mathbb{C}$, dual diagrams $\Gamma(Y)$ of all exceptional curves of not necessarily extremal "good" DPN surfaces Y of elliptic type are described by arbitrary (without any restrictions) subdiagrams $D \subset \operatorname{Duv}\Gamma$ of extremal DPN surfaces described in Theorem 3.16 above with the same main invariants* (r, a, δ) *(equivalently* (k, g, δ)*).*

Moreover, $\operatorname{Duv}\Gamma(Y) = D$, $\operatorname{Log}\Gamma(Y) = \operatorname{Log}\Gamma$, *and these subdiagrams are disjoint to each other;*

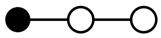
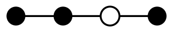
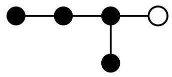
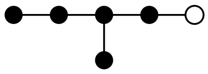
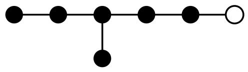
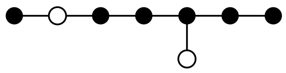
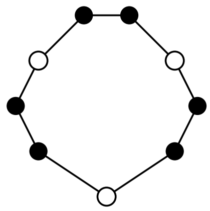
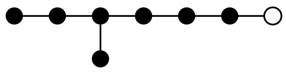
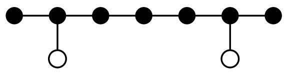
$$\operatorname{Var}\Gamma(Y) = \{f \in W(\operatorname{Var}\Gamma) \mid f \cdot D \geq 0\}$$

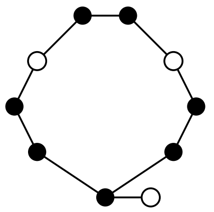
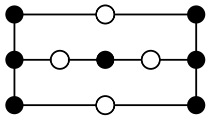
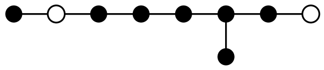
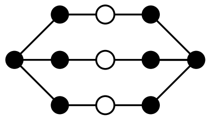
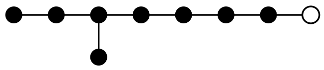
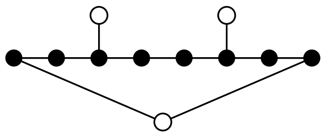
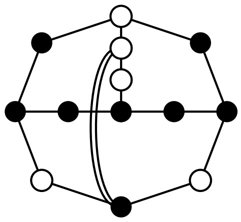
where W *is the subgroup of automorphisms of the Picard lattice of the extremal DPN surface (the Picard lattice is defined by the diagram* Γ*), generated by reflections in elements with square* -2 *corresponding to all vertices of* $\operatorname{Duv}\Gamma$.

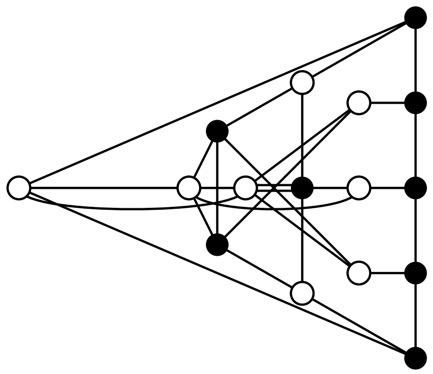
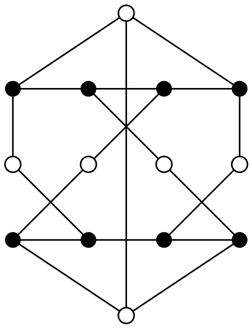
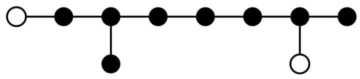
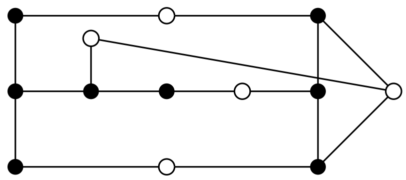
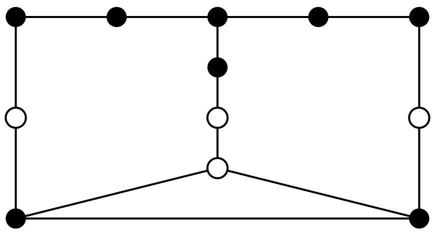
Two such subdiagrams $D \subset \operatorname{Duv}\Gamma$ *and* $D' \subset \operatorname{Duv}\Gamma'$ *(with the same main invariants) give DPN surfaces* Y *with isomorphic diagrams* $\Gamma(Y)$*, if and only if they have isomorphic root invariants* $([D]^+, \xi^+)$ *and* $([D']^+, {\xi'}^+)$ *(see Theorem 3.7).*

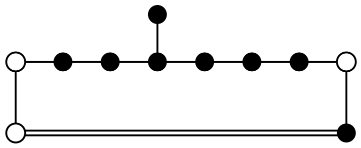
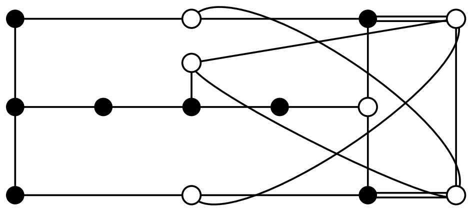
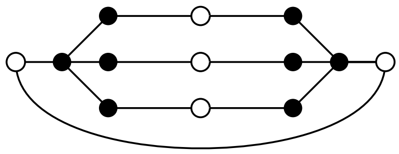
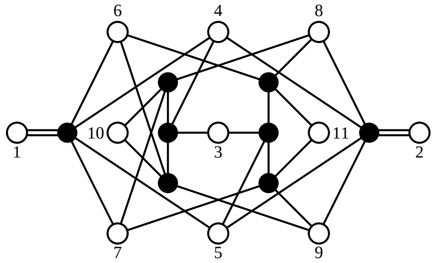
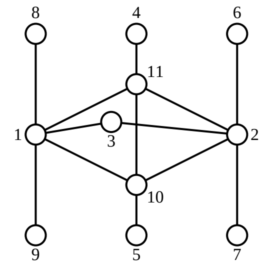
In Theorem 3.17, to calculate the root invariant $([D]^+, \xi^+)$ of a DPN surface, one has to go back from the graph Γ of Table 3 to the corresponding graph of Tables 1 or 2.

Table 3. *Dual diagrams* Γ *of all exceptional curves of extremal "good" DPN surfaces (over* $\mathbb{C}$*) of elliptic type*

N	r	a	δ	k	g	$\widetilde{r}$	Γ
1	1	1	1	0	10	1	$\Gamma = \emptyset$, $\mathbb{P}^2$
2	2	2	0	0	9	1	 $\mathbb{F}_0$ or $\mathbb{F}_2$
3	2	2	1	0	9	2	 $\mathbb{F}_1$
4	3	3	1	0	8	2	
5	4	4	1	0	7	1	
6	5	5	1	0	6	1	
7	6	6	1	0	5	1	
a							
b							
8	7	7	1	0	4	1	
a							
b							
c							
9	8	8	1	0	3	1	
a							
b							

N	r	a	δ	k	g	$\widetilde{r}$	Γ
9	8	8	1	0	3	1	
c							
d							
e							
f							
10	9	9	1	0	2	1	
a							
b							
c							

N	r	a	δ	k	g	$\widetilde{r}$	Γ
10	9	9	1	0	2	1	
d							
e							
f							
g							
h							

N	r	a	δ	k	g	$\widetilde{r}$	Γ
10	9	9	1	0	2	1	
i							
j							
k							
l							

N	r	a	δ	k	g	$\widetilde{r}$	Γ
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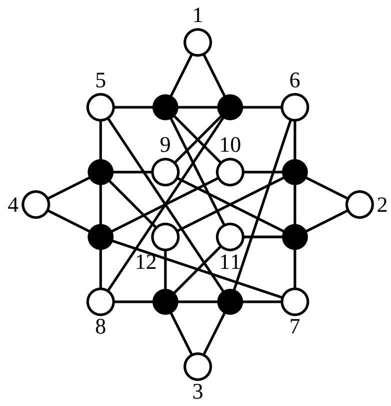
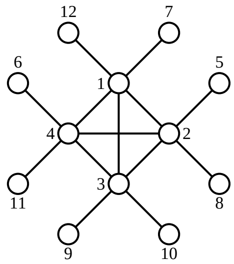
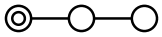
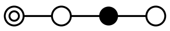
m							 
11	2	0	0	1	10	1	 $\mathbb{F}_4$
12	3	1	1	1	9	2	
13	4	2	1	1	8	2	
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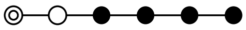
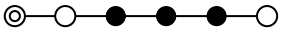
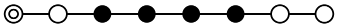
15	6	4	0	1	6	1	
16	6	4	1	1	6	2	
17	7	5	1	1	5	2	
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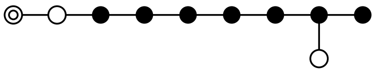
19	9	7	1	1	3	1	

N	r	a	δ	k	g	$\widetilde{r}$	Γ
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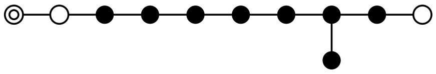
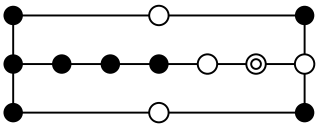
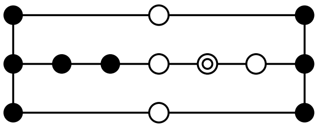
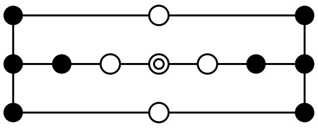
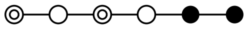
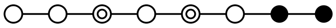
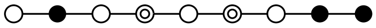
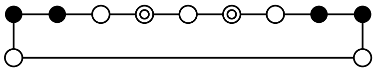
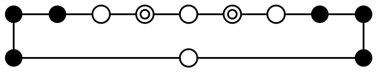
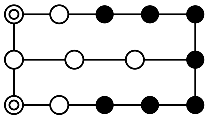
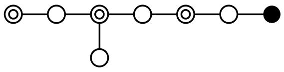
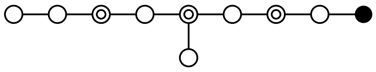
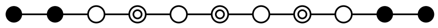
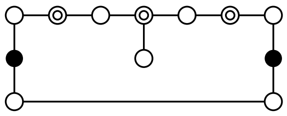
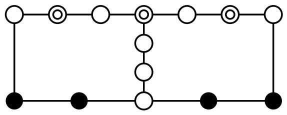
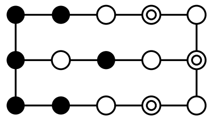
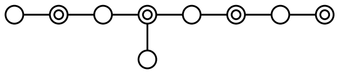
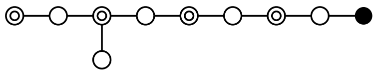
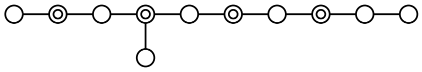
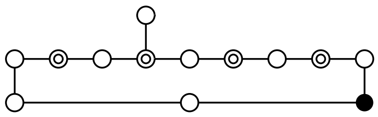
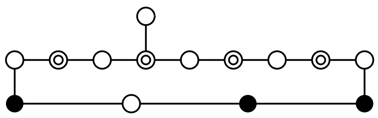
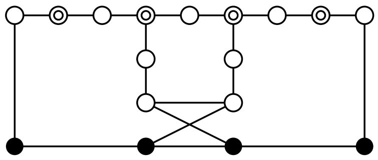
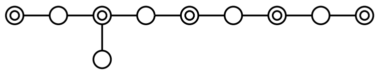
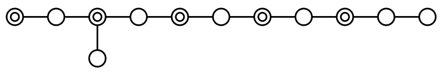
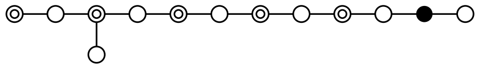
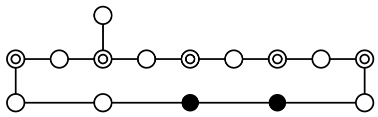
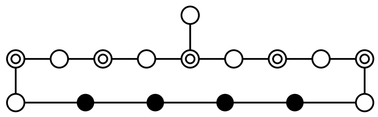
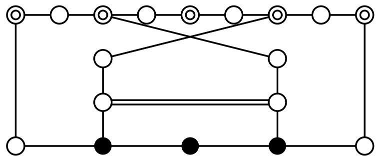
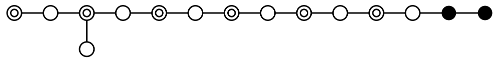
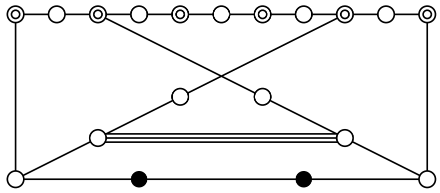
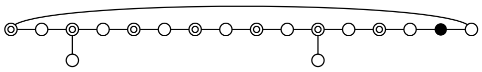
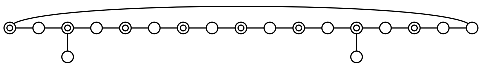
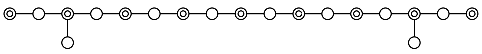
a							
b							
c							
d							
21	6	2	0	2	7	1	
22	7	3	1	2	6	2	
23	8	4	1	2	5	2	
24	9	5	1	2	4	2	
25	10	6	0	2	3	1	
26	10	6	1	2	3	1	
27	11	7	1	2	2	1	
28	8	2	1	3	6	2	
29	9	3	1	3	5	3	
30	10	4	0	3	4	1	

N	r	a	δ	k	g	$\widetilde{r}$	Γ
31	10	4	1	3	4	3	
32	11	5	1	3	3	2	
33	12	6	1	3	2	1	
34	9	1	1	4	6	2	
35	10	2	0	4	5	2	
36	10	2	1	4	5	3	
37	11	3	1	4	4	3	
38	12	4	1	4	3	2	
39	13	5	1	4	2	2	
40	10	0	0	5	6	1	
41	11	1	1	5	5	2	

N	r	a	δ	k	g	$\widetilde{r}$	Γ
42	12	2	1	5	4	2	
43	13	3	1	5	3	2	
44	14	4	0	5	2	1	
45	14	4	1	5	2	2	
46	14	2	0	6	3	1	
47	15	3	1	6	2	2	
48	16	2	1	7	2	2	
49	17	1	1	8	2	2	
50	18	0	0	9	2	1	

3.6. **Application: On classification of plane sextics with simple singularities.** Let Y be a "good" DPN surface of elliptic type which were classified in Theorems 3.16 and 3.17. Let $\Gamma(Y)$ be the dual diagram of all exceptional curves on Y. By definition of "good" DPN surfaces, there exists a non-singular curve

$$C = C_g + E_{a_1} + \cdots + E_{a_k} \in |-2K_Y| \tag{30}$$

where $E_{a_1}, \ldots, E_{a_k}$ are exceptional curves with square (-4) corresponding to all double transparent vertices $a_1, \ldots, a_k$ of $\Gamma(Y)$ and $g > 1$ the genus of the irreducible non-singular curve C_g. Here (k, g, δ) (equivalent to (r, a, δ)) are the main invariants of Y.

We denote by E_v the exceptional curve on Y corresponding to a vertex $v \in V(\Gamma(Y))$. If v is black, then $C \cdot E_v = C_g \cdot E_v = 0$. If v is simple transparent, then $C \cdot E_v = 2$.

If v is simple transparent and v is not connected by any edge with double transparent vertices of $\Gamma(Y)$ (i. e. $E_v \cdot E_{a_i} = 0$, $i = 1, \dots, k$) then $C_g \cdot E_v = 2$. This intersection index can be obtained in two ways:

$$C_g \text{ intersects } E_v \text{ transversally in two points;} \tag{31}$$

$$C_g \text{ simply touches } E_v \text{ in one point.} \tag{32}$$

(For example, in Case 47 of Table 3 we have two such vertices v.)

Up to this ambiguity, we know (from the diagram $\Gamma(Y)$) how components of C intersect exceptional curves. Which of possibilities (31) or (32) does take place is defined by the generalized root invariant which we don't consider in this work.

Let $t_1, \dots, t_{r-1} \in V(\Gamma(Y))$ be a sequence of vertices such that the contraction of exceptional curves $E_{t_1}, \dots, E_{t_{r-1}}$ gives a morphism $\sigma : Y \to \mathbb{P}^2$ which is a sequence of contractions of curves of the 1st kind. By Sect. 2.1, the image $D = \sigma(C) \subset \mathbb{P}^2$ is then a sextic (it belongs to $|-2K_{\mathbb{P}^2}|$) with simple singularities. What components and what singularities the curve D does have is defined by the subgraph $\Gamma(t_1, \dots, t_{r-1})$ generated by vertices $t_1, \dots, t_{r-1}$ in $\Gamma(Y)$. We formalize that below.

Let

$$\widetilde{D} = C_g + \sum_{v_i \in \{a_1, \dots, a_k\} - \{t_1, \dots, t_{r-1}\}} E_{v_i}$$

be the curve of components of C which are not contracted by σ. Then $\sigma : \widetilde{D} \to D$ is the normalization of D. In pictures, we denote $\widetilde{D}$ (or D) by the symbol $\otimes$ and evidently denote the intersection of this curve and its local branches at the corresponding singular point with the components E_{t_j} which are contracted to this point. For connected components of $\Gamma(t_1, \dots, t_{r-1})$ we then have possibilities presented in Table 4 below depending on types of the corresponding singular points of D.

By Table 4, the ambiguity (31) or (32) takes place only for singularities of the types A_{2k-1} or A_{2k}. Thus, we have to introduce the notation $\mathfrak{A}_{2k-1}$ for the singularity of the type A_{2k-1} or A_{2k} of the component $\sigma(C_g)$ of D of the geometric genus $g > 1$.

In the "good" column of Table 4, we denote by $\mathcal{A}_n$, $\mathcal{D}_n$, $\mathcal{E}_n$ connected components of graphs $\Gamma(t_1, \dots, t_{r-1})$ corresponding to singularities A_n, D_n and E_n of the curve D respectively. Obviously, finding of all possible contractions $\sigma : Y \to \mathbb{P}^2$ reduces to enumeration of all subgraphs $\Gamma \subset \Gamma(Y)$ with the connected components $\mathcal{A}_n$, $\mathcal{D}_n$, $\mathcal{E}_n$ and with the common number $r-1$ of vertices. A choice of such a subgraph $\Gamma \subset \Gamma(Y)$ defines the sextic D with the corresponding irreducible components and simple singularities, and the related configuration of rational curves

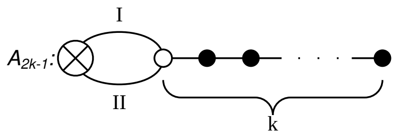
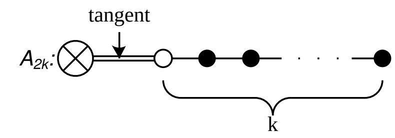
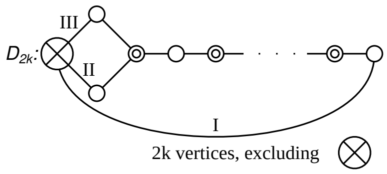
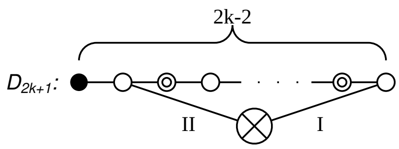
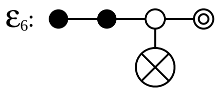
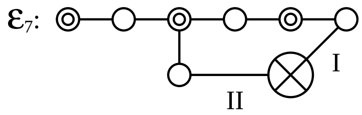
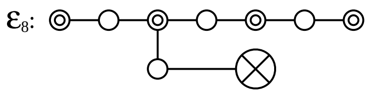
$$\sigma(E_v),\ v \in V(\Gamma(Y)) - (\{a_1, \dots, a_k\} \cup \{t_1, \dots, t_{r-1}\})\,, \tag{33}$$

which one can call as *the exceptional curves of a sextic D with simple singularities.*

Thus, the classification in Theorems 3.16 and 3.17 of DPN surfaces of elliptic type implies a quite delicate classification of sextics D having an irreducible component of the geometric genus $g \geq 2$. For this classification, we correspond to a sextic $D \subset \mathbb{P}^2$ a subgraph $\Gamma \subset \Gamma(Y)$ up to isomorphisms of graphs $\Gamma(Y)$ which send the subgraphs Γ to one another. The analogous classification can be repeated for to classify curves with simple singularities in $|-2K_{\mathbb{F}_n}|$, $n = 0, \dots, 4$. One should only replace $r-1$ by $r-2$. We also note that a choice of different subgraphs $\Gamma \subset \Gamma(Y)$ for the same curve C defines birational transformations of the corresponding rational surfaces ($\mathbb{P}^2$ or $\mathbb{F}_n$) which transform the curves D to one another. Thus, the graph $\Gamma(Y)$ itself classifies the corresponding curves D up to some their birational equivalence.

A complete enumeration of all cases has no principal difficulties, and it is only related to a long enumeration using Theorem 3.17 of all possible diagrams $\Gamma(Y)$ and their subdiagrams Γ. Unfortunately, it seems, number of cases is enormous. But the complete enumeration can be important in some problems of real algebraic geometry and singularity theory. For example, it could be important for classification of irreducible quartics in $\mathbb{P}^3$ with double rational singularities by the method of projection from a singular point. To remove the ambiguity (31) or (32), one has to perform similar (to ours) classification of generalized root invariants.

Table 4. *Correspondence between connected components of Γ and singularities of $D = \sigma(\widetilde{D})$.*

Type of Singular point of D	Equations of the Singularity and its Branches	Connected Components of Γ, and Curve $\widetilde{D}$ (denoted by $\otimes$)
$A_{2k-1} = \mathfrak{A}_{2k-1}$	$y^2 - x^{2k} = 0$ I: $y - x^k = 0$ II: $y + x^k = 0$	
$A_{2k} = \mathfrak{A}_{2k-1}$	$y^2 - x^{2k+1} = 0$	
D_{2k}	$xy^2 - x^{2k-1} = 0$ I: $x = 0$ II: $y - x^{k-1} = 0$ III: $y + x^{k-1} = 0$	
D_{2k+1}	$xy^2 - x^{2k} = 0$ I: $x = 0$ II: $y^2 - x^{2k-1} = 0$	
E_6	$y^3 - yx^4 = 0$	
E_7	$y^3 - yx^3 = 0$ I: $y = 0$ II: $y^2 - x^3 = 0$	
E_8	$y^3 - x^5 = 0$	

4. Classification of log del Pezzo surfaces of index ≤ 2 and applications

4.1. **Classificaion of log del Pezzo surfaces of index ≤ 2.** From results of Sects 1 — 3 we obtain

Theorem 4.1. *(over $\mathbb{C}$) For any log del Pezzo surface Z of index ≤ 2 there exists and a unique resolution of singularities $\sigma : Y \to Z$ (it is called "good") such that Y is a "good" DPN surface of elliptic type and σ contracts exactly all exceptional curves of the Du Val and the logarithmic part of $\Gamma(Y)$. Vice versa, if Y is a "good" DPN surface of elliptic type, then there exists and a unique morphism $\sigma : Y \to Z$ of contraction of all exceptional curves corresponding to the Du Val and the logarithmic part of $\Gamma(Y)$ which gives resolution of singularities of log del Pezzo surface Z of index ≤ 2.*

Thus, classifications of log del Pezzo surfaces of index ≤ 2 and "good" DPN surfaces of elliptic type are equivalent, and they are given by Theorems 3.16 and 3.17.

Proof. Let Z be a log del Pezzo surface of index ≤ 2. In Sect. 1, a "canonical" (i. e. uniquely defined) resolution of singularities $\sigma : Y \to Z$ had been suggested such that Y is a "good" DPN surface of elliptic type. First, a minimal resolution of singularities $\sigma_1 : Y' \to Z$ is taken, and second, the blow-up of all intersection points of components of curves in preimages of non Du Val singularities K_n of Z is taken. Let us show that σ contracts exactly exceptional curves of $\operatorname{Duv}\Gamma(Y)$ and $\operatorname{Log}\Gamma(Y)$.

Let E be an exceptiona curve of Y corresponding to a vertex of $\operatorname{Duv}\Gamma(Y)$ of $\operatorname{Log}\Gamma(Y)$. Let $\widetilde{C} \in |-2K_Z|$ be a non-singular curve of Z which does not contain singular points of Z (it does exist by Theorem 1.5), and $C_g = \sigma^{-1}(\widetilde{C_g})$. Then (see Sects 1.5 and 2) $C_g+E_1+\cdots+E_k \in |-2K_Y|$ where E_i are all exceptional curves on Y with the square (-4) and C_g a non-singular irreducible curve of genus $g \ge 2$. By Sect. 2, then $E \cdot C_g = 0$. If σ does not contract E, then for the curve $\sigma(E)$ on Z we have $\sigma(E)\cdot \widetilde{C_g} = \sigma(E)\cdot(-2K_Z) = 0$. Then $-K_Z$ is not ample. We get a contradiction. Vice versa, by construction, σ contracts only curves from $\operatorname{Duv}\Gamma(Y)$ and $\operatorname{Log}\Gamma(Y)$. This shows that σ is a "good" resolution.

Now, let Y be a "good" DPN surface of elliptic type and $\sigma : Y \to Z$ a contraction of all exceptional curves corresponding to vertices $\operatorname{Duv}\Gamma(Y)$ and $\operatorname{Log}\Gamma(Y)$ (it does exist analytically because $\operatorname{Duv}\Gamma(Y)\cup \operatorname{Log}\Gamma(Y)$ is negative, and we show that it does exist algebraically by the direct construction below). To prove Theorem, we should prove that Z is a log del Pezzo surface of index ≤ 2, and σ the "good" resolution of singularities of Z.

Second statement becomes obvious if one decomposes σ as the composition of contractions of all exceptional curves of 1st kind from $\operatorname{Log}\Gamma(Y)$ (they don't intersect each other) and further contraction of the remaining exceptional curves.

To prove the first statement, one can use the double covering $\pi : X \to Y$ with the involution θ (see Sect. 2), the relation between exceptional curves of Y and (X, θ) (see Sect. 2), and that the contraction of $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$ configurations of (-2) curves on X does exit and gives the corresponding quotient singularities $\mathbb{C}/G_i$ where $G_i \subset SL(2, \mathbb{C})$ are finite subgroups. Using these considerations (i. e. first we consider the corresponding contraction, and second the quotient by involution), and Brieskorn's results [Bri68], we obtain that all non Du Val singularities of Z are $\mathbb{C}/\widetilde{G_i}$ where $\widetilde{G_i} \subset GL(2, \mathbb{C})$ and $\widetilde{G_i} \cap SL(2, \mathbb{C}) = G_i$ have index 2 in $\widetilde{G_i}$, and $\widetilde{G_i}/G_i = \{1, \theta\}$. It follows that Z is a complete algebraic surface with log-terminal singularities of index ≤ 2.

Let us show that $-K_Z$ is ample. By Kleiman's criterion, it is enough to show that $(-K_Z)^2 > 0$ and $(-K_Z) \cdot D > 0$ for any curve D on Z. We have (see Sect. 1.5)

$$4(-K_Z)^2 = (-2K_Z)^2 = (\sigma^*(-2K_Z))^2 = (C_g)^2 > 0.$$

since Y is a DPN surface of elliptic type. Moreover,

$$-2K_Z \cdot D = -2\sigma^* K_Z \cdot \sigma^* D = C_g \cdot \sigma^* D \geq 0$$

because C_g is irreducible with $(C_g)^2 > 0$ and $\sigma^* D$ is effective. Moreover, we get here zero, only if the effective divisor $\sigma^* D$ consists of exceptional curves F on Y with $C_g \cdot F = 0$. But such curves F correspond to vertices of the logarithmic or the Du Val part of $\Gamma(Y)$. They are contracted by σ into points of Z which is impossible for the divisor $\sigma^* D$. $\square$

Using Theorem 4.1, we can transfer to log del Pezzo surfaces Z of index ≤ 2 *the main invariants* (r, a, δ), *equivalently* (k, g, δ), *the root invariant, the root subsystem, the exceptional curves* which are defined for the surface Y of the "good" resolution $\sigma : Y \to Z$. In particular, the *Picard number of* Z is

$$\widetilde{r} = \operatorname{rk} \operatorname{Pic} Z = r - \#V(\operatorname{Duv}\Gamma(Y)) - \#V(\operatorname{Log}\Gamma(Y)). \tag{34}$$

In Theorem 3.16 we have shown the Picard number $\widetilde{r}$ in the extremal (for Y) case. Obviously, surfaces Z with the extremal Y are distinguished by the minimal Picard number $\widetilde{r}$ for the fixed main invariants (r, a, δ) (equivalently, (k, g, δ)). Since the $\operatorname{Log}\Gamma(Y)$ is prescribed by the main invariants and is then fixed, this is equivalent to have the maximal rank (i. e. the number of -2 curves for the minimal resolution of singularities) for Du Val singularities of Z.

In Mori Theory, see [Mor82] and [Rei83], log del Pezzo surfaces Z with $\operatorname{rk}\operatorname{Pic} Z = 1$ are especially important. They give relatively minimal models (different from $\mathbb{P}^1 \times \mathbb{P}^1$) in the class of rational surfaces with log-terminal singularities: any rational surface with log-terminal singularities has a contraction morphism on such a model. ¿From Theorems 4.1 and 3.6, we obtain classification of such models with log-terminal singularities of index ≤ 2. By Theorem 3.16, they correspond to exactly extremal DPN surfaces of elliptic type with

$$\widetilde{r} = r - \#V(\operatorname{Duv}\Gamma(Y)) - \#V(\operatorname{Log}\Gamma(Y)) = 1,$$

and Theorem 3.16 gives their classification.

Theorem 4.2. *(over $\mathbb{C}$) There exist, up to isomorphism, exactly 46 relatively minimal and different from $\mathbb{P}^1 \times \mathbb{P}^1$, i. e. with* $\operatorname{Pic} Z = \mathbb{Z}$, *log del Pezzo surfaces Z of index ≤ 2. The DPN surfaces Y of their "good" resolution of singularities are extremal and correspond to exactly the following cases of Theorem 3.16 where we show in brackets the type of their singularities.*

(35)

$$\begin{gathered} 1(Y=\mathbb{P}^2),\ 2(A_1),\ 5(A_1A_2),\ 6(A_4),\ 7a(D_5),\ 7b(2A_1A_3),\ 8a(E_6),\\ 8b(A_1A_5),\ 8c(3A_2),\ 9a(E_7),\ 9b(A_7),\ 9c(A_2A_5),\ 9d(A_12A_3),\\ 9e(A_1D_6),\ 9f(3A_1D_4),\ 10a(E_8),\ 10b(A_8),\ 10c(A_1A_7),\\ 10d(A_1A_2A_5),\ 10e(2A_4),\ 10f(D_8),\ 10g(A_3D_5),\ 10h(A_2E_6),\\ 10i(A_1E_7),\ 10j(2A_1D_6),\ 10k(2D_4),\ 10l(2A_12A_3),\ 10m(4A_2);\\ 11(K_1),\ 15(K_1A_4),\ 18(K_1A_1A_5),\ 19(K_1A_7),\ 20a(K_1D_8),\\ 20b(K_12A_1D_6),\ 20c(K_1A_3D_5),\ 20d(K_12D_4);\ 21(K_2A_2),\\ 25(2K_1A_7),\ 26(K_22A_3),\ 27(K_2A_7),\ 30(K_32A_2);\ 33(K_3A_1A_5);\\ 40(K_5),\ 44(K_5A_4);\ 46(K_6A_2);\ 50(K_9). \end{gathered}$$

In particular, the isomorphism class of Z is defined by types of its singularities. The number of non-Du Val singularities is at most one except when the singularities are $2K_1A_7$.

Proof. We need only to show that the corresponding to all these cases surfaces are unique up to isomorphisms. As in the proof of Proposition 3.12, we can choose an appropriate sequence of vertices of $\Gamma(Y)$ such that the contraction of the corresponding sequence of the curves of the 1st kind gives a morphism of Y onto $\mathbb{P}^2$ or $\mathbb{F}_n$, $n = 0, 2, 3, 4$.

Considering images of other exceptional curves of Y, one can see (by direct inspection in each case) that the contracting exceptional curves give a rigid configuration of points on the corresponding surface. □

Remark 4.3. In the Gorenstein case (i. e. for index ≤ 1, when there are no singularities K_n) Theorem 4.2 had been proved by completely different method in [BBD84]. This case gives the first 28 cases 1 — 10m of Theorem 4.2, the remaining 18 of its cases 11 — 50 are non-Gorenstein.

If we consider log del Pezzo surfaces Z of index ≤ 2 and without Du Val singularities, we get an opposite case to the previous one. Thus, any singularity of Z must have index 2. This case includes and is surprisingly similar to the classical case of non-singular del Pezzo surfaces when there are no singularities at all. Applying Theorem 4.1 we get the following

Theorem 4.4. *(over $\mathbb{C}$) Up to deformation, there exist exactly 50 types of log del Pezzo surfaces Z with singularities of index exactly* 2 *(if a singularity does exist). The DPN surfaces Y of their "good" resolution of singularities have empty Du Val part* $\operatorname{Duv}\Gamma(Y)$*, zero root invariants, and are defined by their main invariants* (r, a, δ) *(equivalently* (k, g, δ)*), up to deformation. The diagram* $\Gamma(Y)$ *can be obtained from the diagram* Γ *of cases 1 — 50 of Table 3 (with the same main invariants) as follows:* $\Gamma(Y)$ *consists of* $\operatorname{Log}\Gamma(Y) = \operatorname{Log}\Gamma$ *and*

$$\operatorname{Var}\Gamma(Y) = W(\operatorname{Var}\Gamma) \tag{36}$$

where W is generated by reflections in all vertices of $\operatorname{Duv}\Gamma$ *(i. e. one should take $D = \emptyset$ in Theorem 3.17). In cases 7, 8, 9, 10, 20 one can consider only diagrams Γ of cases 7a, 8a, 9a, 10a and 20a (diagrams 7b, 8b,c, 9b—f, 10b—m, 20b—d give the same).*

Proof. This case corresponds to Y with empty $D \subset \operatorname{Duv}\Gamma$ of Theorem 3.17. Then the root invariant is 0. Thus, all cases 7, 8, 9, 10 or 20 give the isomorphic root invariants and the same diagrams, and we can consider only the corresponding cases 7a, 8a, 9a, 10a and 20a to calculate the diagrams.

Let us show that moduli spaces of DPN surfaces Y with the same main invariants (r, a, δ) and zero root invariant (i. e. $D = 0$) are connected.

It is enough to show connectedness of the corresponding "good" DPN pairs (Y, C) where $C \in |-2K_Y|$ is non-singular. Taking double covering $\pi : X \to Y$ ramified in C, it is enough to prove connectedness of moduli of K3 surfaces with non-symplectic involutions (X, θ) and

$(S_X)_+ = S$ where S has invariants (r, a, δ). General such pairs have zero root invariant, as we want, because general (X, θ) have $S_X = S$. Using Global Torelli Theorem [PS-Sh71] and surjectivity of Torelli map [Kul77] for K3, the connectedness of moduli reduces to the uniqueness, up to automorphisms, of the primitive embedding $S \subset L_{K3}$ where L_{K3} is the cohomology lattice of K3. This uniqueness had been proved in [Nik80.2]. $\square$

We remark that equivalent to Theorem 4.4 results had been first obtained in [Nik83].

We remark that cases 1 — 10 of Theorem 4.4 give classical non-singular del Pezzo surfaces. Therefore, Theorem 4.4 and all results of this work show that log del Pezzo surfaces of index ≤ 2 are very similar to classical non-singular del Pezzo surfaces.

4.2. **Application: Minimal projective compactifications of affine surfaces in $\mathbb{P}^2$ by relatively minimal log del Pezzo surfaces of index ≤ 2.** Let us consider one of the 45 relatively minimal surfaces of Theorem 4.2 which are different from $\mathbb{P}^2$ (i. e. except the case 1). Let $\sigma : Y \to Z$ be its minimal resolution of singularities, and $v_1, \ldots, v_{r-1}$ a sequence of vertices of $\Gamma(Y)$ such that the corresponding exceptional curves on Y give a contraction of the sequence of curves of the 1st kind $\tau : Y \to \mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be the union of images by τ of all exceptional curves E_v, $v \in (V(\mathrm{Duv}(\Gamma(Y))) \cup V(\mathrm{Log}(\Gamma(Y))))$. Then, the embedding $f = (\sigma\tau^{-1}) : V \to Z$ gives a compactification of the affine surface $V = \mathbb{P}^2 - C$ of $\mathbb{P}^2$. The morphism f is minimal in the sense that f cannot be extended through components of C (see [BBD84] for details). The description of all such affine surfaces V and compactifications is then reduced to the description of subdiagrams of $\Gamma(Y)$ (defined by $v_1, \ldots, v_{r-1})$) which were described by their connected components in Sect. 3.6.

This is similar to [BBD84] in the Gorenstein case.

4.3. **Some questions.** Here we discuss several questions of general interest.

4.3.1. *Dimension of the moduli space.* What is the dimension of the moduli space $\mathcal{M}_Z$ of log del Pezzo surfaces Z of index ≤ 2 and with the fixed invariants

$$(k,\ g,\ \delta,\ \ \textit{root invariant})\,, \tag{37}$$

equivalently,

$$(r,\ a,\ \delta,\ \ \textit{the dual diagram of exceptional curves}\ \Gamma(Y))? \tag{38}$$

We have the obvious formula (where we identify $\mathcal{M}_Z$ with the moduli $\mathcal{M}_Y$ of the corresponding "good" DPN surfaces Y of the "good" resolution of singularities of Z):

$$\dim \mathcal{M}_Y + \dim |-2K_Y| - \dim \operatorname{Aut} Y = \dim \mathcal{M}_{(Y,C)} \tag{39}$$

where $\mathcal{M}_{(Y,C)}$ is the moduli space of "good" DPN pairs (Y,C) where $C \in |-2K_Y|$ is non-singular. Here we used the fact that the group $\operatorname{Aut}(Y,C)$ (equivalently $\operatorname{Aut}(X,\theta)$ for the double covering) is finite. We remind that $\operatorname{Aut} X$ is 0-dimensional and discrete for a K3 surface X. The local Torelli Theorem and the local surjectivity of Torelli map for K3 surfaces then give

$$\dim \mathcal{M}_{(Y,C)} = 20 - r - \#V(\operatorname{Duv}(\Gamma)) = 9 + g - k - \#V(\operatorname{Duv}(\Gamma)). \tag{40}$$

Moreover,

$$\dim |-2K_Y| = \dim |C_g| = 3g - 3 \tag{41}$$

(see Sects 1.4 and 1.5). It follows that

$$\begin{aligned}\dim \mathcal{M}_Z = \dim \mathcal{M}_Y =& 12 - 2g - k - \#V(\operatorname{Duv}(\Gamma(Y)) + \dim \operatorname{Aut} Y = \\ & \frac{r+3a}{2} - 10 - \#V(\operatorname{Duv}(\Gamma(Y)) + \dim \operatorname{Aut} Y\end{aligned} \tag{42}$$

where we can assume that Y is general. The group $\operatorname{Aut}(Y)$ can be easily computed as follows: We can choose an appropriate sequence $v_1, \dots, v_k \in \Gamma(Y)$ such that the contraction of the corresponding curves of the 1st kind gives a morphism $\tau : Y \to V$ where V is one of relatively minimal rational surfaces $\mathbb{P}^2$ or $\mathbb{F}_n$, $n = 0$, 2, 3 or 4. Then the connected component of identity $(\operatorname{Aut} Y)_0$ gives a subgroup of $\operatorname{Aut} V$ which preserves the configuration $\mathcal{K}$ of images of the the exceptional curves $E_{v_1}, \dots, E_{v_k}$ and corresponding to $\operatorname{Duv} \Gamma$ and $\operatorname{Log} \Gamma$. It can be easily computed in concrete cases. Finally we obtain $\dim \mathcal{M}_Y = \dim \mathcal{M}_Z$ using (42). Clearly, $\dim \operatorname{Aut} Y$ must be zero in all interesting cases.

Is there a different from (42) formula for $\dim \mathcal{M}_Z$ where $\dim \operatorname{Aut} Y$ is computed?

4.3.2. *Description of moduli spaces.* It would be very interesting to describe the moduli spaces $\mathcal{M}_Z$ in the form it is known for the ordinary non-singular del Pezzo surfaces. In the notations above, this reduces to the explicit description of moduli spaces of all possible configurations $\mathcal{K}$.

4.3.3. *Finite characteristic.* It would be very interesting to generalize results of this work to finite characteristic. As we had mentioned in Remark 3.14, it seems, the main problem is to generalize Theorem 1.5. We think that our results are valid in characteristic ≥ 3. As we have seen, in characteristic 2 the number of cases increases.

4.3.4. There are many results (e. g. see [Man86], [MT86] and [CT88]) where arithmetics of classical non-singular del Pezzo surfaces is studied. What is the arithmetics of log del Pezzo surfaces of index ≤ 2 and equivalent DPN surfaces of elliptic type?

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