# On the geometry of some strata of uni-singular curves

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**Abstract**

We study the global properties of some equi-singular families (strata) of uni-singular curves. The singularities of closures of the strata are resolved and the resolutions are presented as projective bundles. This enables to study their geometry. In particular we calculate the intersection rings of the closures of the strata, the Picard groups of the proper strata and check the affinity of the proper strata. The rational equivalence classes of some geometric cycles on the (resolved closures of the) strata are calculated.

## Contents



## 1 Introduction

### 1.1 Formulation and results

We work with (complex) algebraic curves in $\mathbb{P}^2$. Identify the complete linear system $|dL|$ (the parameter space of plane curves of degree $d$) with the projective space $\mathbb{P}^{N_d}_f$. Here $N_d = \binom{d+2}{2} - 1$, the subscript $f$ is due to the defining equation of the curves, $f(x) = 0$.

The parameter space is stratified according to the *embedded topological singularity type* of curves. The generic point of $\mathbb{P}^{N_d}_f$ corresponds to a smooth curve. The set of points corresponding to the singular curves is called *the discriminant* ($\Sigma$). It is a (projective) hypersurface in the parameter space.

An *equisingular stratum* is the (quasi-projective) variety of points corresponding to the curves with the given topological type of singularity. The generic point of the discriminant lies in the stratum of nodal curves ($\Sigma_{A_1}$). Other strata correspond to higher singularities (e.g. curves with r-nodes $\Sigma_{(A_1)^r}$ or $\Sigma_{A_k}$, $\Sigma_{D_k}$, $\Sigma_{E_k}$ etc..). For a comprehensive introduction cf. [GLSbook]. In this paper we study some of the strata and their natural compactifications. To be clear we sometimes call the strata themselves: *the proper strata*.

The degree of curves, $d$, is assumed to be sufficiently high (for a specified singularity type). Then the proper strata possess good geometric properties: are non-empty, quasi-projective, irreducible, reduced,

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smooth, rational algebraic varieties of expected dimension (for recent review cf. [GLS06]). A sufficient condition for this is:

if the curve has $r$ singularities of types $\{\mathbb{S}_i\}$, with orders of determinacy $o.d.(\mathbb{S}_i)$, then the degree must be not less than $\sum o.d.(\mathbb{S}_i) + r - 1$ [Dimca, proposition I.3.9].

In the previous work [Ker06] the (partial) resolutions of the compactified strata of uni-singular curves were constructed as subvarieties of some multi-projective spaces. This enabled us to calculate their cohomology classes $[\widetilde{\Sigma}_{\mathbb{S}}] \in H^*(\mathbb{P}^{N_d}_f, \mathbb{Z})$.

The goal of this paper is to study the geometry of the strata, generalizing the work by Diaz and Harris [DiazHarris86, DiazHarris88]. We restrict the consideration to curves with one singular point of *linear* type (the precise definition is in §2.3). The simplest examples of linear singularity types are $A_{k\leq 3}$, $D_{k\leq 6}$, $E_{k\leq 8}$, $X_9$, $J_{10}$, $Z_{k\leq 13}$ etc. (In this paper the low codimension singularity types are denoted according to the tables in [AGV, section II.15]). We discuss compactifications and partial resolutions of the linear strata.

The above cohomology classes are used to obtain the information about rational equivalence on the strata. In particular we calculate the intersection rings of the (resolved compactifications of the) strata. As an example we give explicit formulae in the following cases (the type is specified by the normal form): $x_1^{p+1} + x_2^{p+1}$ (ordinary multiple point), $x_1^p + x_2^{p+1}$ (generalized cusp, e.g. $A_2, E_6..$), $x_1^p + x_1 x_2^p + x_2^{p+2}$ (e.g. $A_3, E_7..$), $x_1^{p+1} + x_1^2 x_2^{p-1} + x_2^{p+2}$ (e.g. $A_2, D_5..$), $x_1^{p+1} + x_1^2 x_2^{p-1} + x_2^{p+3}$ for $p \geq 3$.

The knowledge of the intersection ring is important for a whole class of enumerative problems concerning the specified singularity type (e.g. consideration of curves with a restriction on the position of singular point, on the tangent cone etc.).

We study the boundary divisors of the compactification $\Sigma \subset \overline{\mathbf{\Sigma}}$. Their equivalence classes were mostly obtained in the previous work, now they are used to calculate the Picard group of the proper strata $\Sigma$ or their partial compactifications. In addition, they often allow us check whether the proper stratum is affine (cf. §1.1.4). In particular we present examples of *non-affine proper strata*.

### 1.1.1 Historic overview.

The enumerative questions and questions about irreducibility, dimension, smoothness of the strata are very old (with results starting from the 19'th century, cf. [GLSbook, Kaz4]). But the questions on the geometry and intersection theory of the strata seem to be almost untouched (to the best of author's knowledge).

• In [DiazHarris86, DiazHarris88] the geometry of the Severi variety $(\Sigma_{(A_1)^r}, \overline{\mathbf{\Sigma}}_{(A_1)^r})$ was studied and (the classes of) many of its divisors have been described. In particular they classified the top dimensional components of the boundary and the singularities of the variety at these components. It was proved that the proper stratum is affine and conjectures on the structure of $Pic(\Sigma_{(A_1)^r})$ and $Pic(\overline{\mathbf{\Sigma}}_{(A_1)^r})$ were formulated. For example: $Pic(\Sigma_{(A_1})^r)$ is torsion.

• In [Treger94] it was proved that the stratum $\overline{\overline{\mathbf{\Sigma}}}_{rA_1}(y_1..y_r) = \overline{\{C \text{ has nodes at } y_1..y_r\} \subset \mathbb{P}^{N_d}_f \times Sym^r(\mathbb{P}^2)}$ is uni-branched everywhere.

• In [MiretXambo94] the Picard groups of $\Sigma_{A_1}$ and $\overline{\mathbf{\Sigma}}_{A_1}$ were calculated (verifying in particular the above conjecture).

• In [MiretValls05] the result was generalized to the stratum of plane curves with ordinary multiple point $\overline{\mathbf{\Sigma}}_{x_1^{p+1}+x_2^{p+1}}$. They identified the boundary divisors of $\Sigma_{x_1^{p+1}+x_2^{p+1}} \subset \overline{\mathbf{\Sigma}}_{x_1^{p+1}+x_2^{p+1}}$ and calculated the intersection rings $A^*(\overline{\mathbf{\Sigma}}_{x_1^{p+1}+x_2^{p+1}})$ and $A^*(\overline{\mathbf{\Sigma}}_{x_1^{p+1}+x_2^{p+1},A_1})$. In particular, they verified the above conjecture for $Pic(\Sigma_{x_1^{p+1}+x_2^{p+1}})$.

• [Edidin94] has showed that $Pic(\Sigma_{(A_1)^r})$ is torsion for curves sufficiently many nodes (for a given degree of curve $d$).

### 1.1.2 On compactification and resolution.

An equi-singular stratum $\Sigma_\mathbb{S}$ is defined as a subvariety of the parameter space $\mathbb{P}^{N_d}_f$. Correspondingly, ity has a natural compactification: the topological closure (denoted by $\overline{\Sigma}_\mathbb{S}$). This compactification is highly singular (in codimension 1). Another unpleasant feature is that the boundary $\overline{\Sigma}_\mathbb{S}\setminus\Sigma_\mathbb{S}$ is in general not of pure dimension (in particular not a divisor). A simple example of this is: $\partial\overline{\Sigma}_{A_2}=\overline{\Sigma}_{A_3}\cup\overline{\Sigma}_{A_2A_1}\cup\overline{\Sigma}_{D_4}$.

A natural approach to resolve the singularities is to consider the *universal curve* (i.e. the lifting to a bigger ambient space)

$$\overline{\widetilde{\Sigma}}_\mathbb{S}(x)=\overline{\{(x,f)|\text{The curve }\{f(x)=0\}\text{ has the singularity }\mathbb{S}_x\text{ at the point }x\}}\overset{i}{\hookrightarrow}\mathbb{P}^2_x\times\mathbb{P}^{N_d}_f \tag{1}$$

The projection $\overline{\widetilde{\Sigma}}_\mathbb{S}(x)\to\overline{\Sigma}_\mathbb{S}$ is surjective and generically 1:1. In the simplest case of ordinary multiple point this "lifted" stratum is already a smooth variety (cf. §1.2). In general, one must lift further and consider "generalized universal curves", taking into account other parameters of the singular germ: tangent lines ($l$), osculating conics etc. The next lifting is:

$$\begin{aligned}&\overline{\widetilde{\Sigma}}_\mathbb{S}(x,\{l_i\})=\overline{\left\{(x,\{l_i\},C)\middle|\begin{array}{l}C\text{ has the singularity of type }\mathbb{S}\\ \text{at }x\text{ with the tangent cone }T_C=(l_1^{p_1}l_k^{p_k})\end{array}\right\}}\subset Aux\times\mathbb{P}^{N_d}_f\\ &Aux=\{(x,\{l_i\})|\forall i:\ x\in l_i\}\subset\mathbb{P}^2_x\times\prod(\mathbb{P}^2_{l_i})^*\end{aligned} \tag{2}$$

Here $Aux$ is the auxiliary space. We denote the lifted strata by $\widetilde{\Sigma},\overline{\widetilde{\Sigma}}$ and usually assign the parameters of lifting, as above.

Not all the elements of the tangent cone are of the same importance. A line $l_i$ with $p_i=1$ corresponds to a smooth branch, not tangent to any other branches of the germ. We call such branches: *free*. Usually, when lifting we take into account only tangents to the non-free branches.

We consider mostly generalized Newton-non-degenerate types (the precise definition in §2.3). In this case the number of tangents with $p_i>1$ is at most 2 (theorem 2.9). So, there are three cases:

- All the branches are free. Then $\mathbb{S}$ is necessarily an ordinary multiple point. The lifting is minimal $\overline{\widetilde{\Sigma}}_\mathbb{S}(x)$.
- $p_1>1$, $p_{i>1}\le 1$. For example all the ADE's satisfy this. The lifting is $\overline{\widetilde{\Sigma}}_\mathbb{S}(x,l)$.
- $p_1,p_2>1$, $p_{i>2}\le 1$. An example of such a type is that, with the normal form $x_1^ax_2^b+x_1^{k_a}+x_2^{k_b}$, with $a,b\ge 2$, $k_a,k_b\ge a+b+1$. The lifting is $\overline{\widetilde{\Sigma}}_\mathbb{S}(x,l_1,l_2)$.

**Remark 1.1** Sometimes, the non-free branches tangent to $l_1,l_2$ have the same type and thus are "indistinguishable" (e.g. in the previous example for $k_a=k_b$ and $a=b$). Then the lifting $\widetilde{\Sigma}\to\Sigma$ is actually a $2:1$ covering. This does not bring any complication, one only need to divide by 2 in the necessary places. Usually we omit this case, for the sake of brevity only.

For linear singularities the above lifting is already a resolution with some good properties:

**Proposition 1.2** *Let $\mathbb{S}$ be a linear singularity type and $\overline{\widetilde{\Sigma}}$ the corresponding lifting. (I.e. $\overline{\widetilde{\Sigma}}_\mathbb{S}(x)$, $\overline{\widetilde{\Sigma}}_\mathbb{S}(x,l)$ or $\overline{\widetilde{\Sigma}}_\mathbb{S}(x,l_1,l_2)$.) Then*

- *The stratum $\overline{\widetilde{\Sigma}}_\mathbb{S}\subset Aux\times\mathbb{P}^{N_d}_f$ is a smooth rational variety. The projection $\overline{\widetilde{\Sigma}}_\mathbb{S}\to\overline{\Sigma}_\mathbb{S}$ is 1:1 when restricted to the proper stratum $\Sigma_\mathbb{S}$ (note the remark 1.1).*
- *The boundary of the lifted stratum $\overline{\widetilde{\Sigma}}\setminus\widetilde{\Sigma}$ is of pure co-dimension 1 (i.e. a true hypersurface). So, the resolution is divisorial.*

The first statement was proved in [Ker06] by explicit construction (cf. also §3). The second is proved in §4 by classification of the nearest adjacent types for linear singularities.

**Remark 1.3**
- Note, that the resolution is "geometrical", i.e. every point of the resolved variety $\overline{\widetilde{\Sigma}}$ corresponds to a

specific plane curve. The resolution is not a good one - the exceptional divisors intersect non-transversally.
• For non-linear singularities the situation is much more complicated. The (naturally) lifted strata can be still singular and it is not clear whether they can be desingularized by lifting only. We consider example of $A_k$ (§3.2).

Once the resolution $\overline{\overline{\Sigma}}_{\mathbb{S}}$ have been constructed, one immediate profit is a characterization of some singularities of the original variety $\overline{\Sigma}_{\mathbb{S}}$. When is $\overline{\Sigma}_{\mathbb{S}}$ uni-branched? More precisely, given a point $pt \in \Sigma_{\mathbb{S}'} \subset \overline{\Sigma}_{\mathbb{S}}$, let $L \subset \mathbb{P}^{N_d}_f$ be the generic plane through $pt$, such that $dim(L) + dim(\Sigma_{\mathbb{S}'}) = N_d$. Then $(L \cap \Sigma_{\mathbb{S}}, pt)$ is a germ of variety of dimension $(dim(\Sigma_{\mathbb{S}} - dim(\Sigma_{\mathbb{S}'})$, possibly singular at $pt$. (it is a section of the mini-versal deformation.) So, the question is: whether $(L \cap \Sigma_{\mathbb{S}}, pt)$ is locally irreducible?

We use a simple criterion: *if the germ $(A, pt)$ is reducible and $\tilde{A} \xrightarrow{\pi} A$ is a resolution then the fiber $\pi^{-1}(pt)$ is also reducible.*

Applying this we get in some simple cases:

**Example 1.4**
• $\mathbb{S} = x_1^{p+1} + x_2^{p+1}$. Let $\overline{\Sigma}_{\mathbb{S}'} \subset \overline{\Sigma}_{\mathbb{S}}$ be the locus of curves with at least two singular points, each of multiplicity at least $p+1$. (Thus, $codim_{\overline{\Sigma}_{\mathbb{S}}} \overline{\Sigma}_{\mathbb{S}'} = \binom{p+2}{2} - 2$.) Then $\overline{\Sigma}_{\mathbb{S}} \setminus \overline{\Sigma}_{\mathbb{S}'}$ is uni-branched (or smooth) at each point.
• $\mathbb{S} = x_1^p + x_2^{p+1}$. Let $\overline{\Sigma}' \subset \overline{\Sigma}_{\mathbb{S}}$ be the locus of curves with either a point of multiplicity at least $p+1$, or at least two points of type $\mathbb{S}$. (Thus, $codim_{\overline{\Sigma}_{\mathbb{S}}} \overline{\Sigma}' = 2$, for the top-dimensional component.) Then $\overline{\Sigma}_{\mathbb{S}} \setminus \overline{\Sigma}'$ is uni-branched (or smooth) at each point.

The crucial importance of this resolution is that it is a projective bundle.

**Proposition 1.5**
• *The projection $\overline{\overline{\Sigma}}_{\mathbb{S}} \rightarrow Aux$ is a locally trivial fibration over the smooth base. (Here $Aux = \mathbb{P}^2_x$ or $Aux = \{(x, l) | l(x) = 0\}$ or $Aux = \{(x, l_1, l_2) | l_1(x) = 0 = l_2(x)\}$.) The fibres are projective spaces, linearly embedded into the space $\mathbb{P}^{N_d}_f$. Correspondingly, the lifting is the projectivization of a vector bundle: $\overline{\overline{\Sigma}}_{\mathbb{S}} = \mathbb{P}roj \hat{\hat{\Sigma}}_{\mathbb{S}}$.*
• *The Segre class of the vector bundle $\hat{\hat{\Sigma}}_{\mathbb{S}}$ is determined by the cohomology class of the embedding: $[\overline{\overline{\Sigma}}_{\mathbb{S}}] \in H^*(Aux \times \mathbb{P}^{N_d}_f, \mathbb{Z})$. In particular, let $[\overline{\overline{\Sigma}}](X, F)$ (or $[\overline{\overline{\Sigma}}](X, L, F)$ or $[\overline{\overline{\Sigma}}](X, L_1, L_2, F)$) be the class of the lifted stratum (considered as a homogeneous polynomial in $X, L_i, F$). Then the total Segre class is $[\overline{\overline{\Sigma}}](X, 1)$ (or $[\overline{\overline{\Sigma}}](X, L, 1)$ or $[\overline{\overline{\Sigma}}](X, L_1, L_2, 1)$).*

As in the previous proposition, the first statement was proved in [Ker06] by explicit construction. The second statement is a standard fact from intersection theory (cf. proposition 2.2).

**1.1.2.1 Partial compactifications** are often important. In particular it is natural to consider the compactification $\Sigma_{\mathbb{S}} \subsetneq \bar{\Sigma}_{\mathbb{S}} \subsetneq \overline{\Sigma}_{\mathbb{S}}$ with preserved topological type of the *chosen* singularity. So, the curve that belongs to $\bar{\Sigma}$ can be singular at other points (e.g. $\bar{\Sigma}_{A_k}$ contains $\bar{\Sigma}_{A_k, A_1}$ but is disjoint to $\Sigma_{A_{k+1}}$). We call it the *semi-compactification*. It is a standard fact that $\overline{\Sigma}_{\mathbb{S}}$ is smooth at $\Sigma_{\mathbb{S}, A_1}$ (for $\mathbb{S} \neq A_1$).

**1.1.2.2 Equi-generic compactification** is another interesting partial compactification $\Sigma_{\mathbb{S}} \subset \bar{\Sigma}_{\delta(\mathbb{S})} \subsetneq \overline{\Sigma}_{\mathbb{S}}$, i.e. the singularity type $\mathbb{S}$ can change, but the (geometric) genus of the curve is preserved. By the classical formula $g = \frac{(d-1)(d-2)}{2} - \delta$ equi-generic means $\delta(\mathbb{S}) = const$. We discuss some properties of this compactification in §4.3.

### 1.1.3 On rational equivalence and the intersection ring.

Having constructed a lifted stratum (a smooth projective variety) one can study various divisors/cycles in it. There are several types of divisors/cycles in $\overline{\overline{\Sigma}}$ (in the spirit of [DiazHarris88]):

**1.1.3.1 Boundary divisors/cycles** are of two types:
• those corresponding to the curves with additional singular point (i.e. the strata of semi-compactification).
• degenerations of the singularity type $\mathbb{S}$. In particular:
⋆ $\delta = const$ degenerations
⋆ degenerations (not) changing the degree of determinacy (e.g. $A_k \to D_{k+1}$ or $A_k \to A_{k+1}$)
⋆ degenerations (not) changing the tangent cone (e.g. $x_1^p + x_2^p \to x_1^p + x_2^{p+1}$ or $x_1^p + x_1 x_2^p \to x_1^p + x_2^{p+2}$).

**1.1.3.2 Intrinsic divisors.** Various divisors/cycles of the ambient space (a multi-projective space) are pulled-back to the lifted stratum. For example, for the minimal lifting (as in (1)), let $X, F$ be the classes dual to the corresponding hyperplanes in $\mathbb{P}^2_x \times \mathbb{P}^{N_d}_f$. Then the cycle $i^*(X^j F^k)$ corresponds to the family of plane curves (with singularity of a prescribed type) that pass through $k$ fixed (generic) points of $\mathbb{P}^2_x$ and whose singularity lies on a fixed generic $2-j$ plane.

**1.1.3.3 Extrinsic divisors/cycles** are defined by the properties of (embedded) curves. For example, the cycles of curves with hyperflexes or multi-tangents, singular points whose smooth branches have flexes or tangent lines are also tangent at other points etc..

We are interested in (rational equivalence) classes of these divisors/cycles. It appears that classes of all the cycles are expressible through the classes of the intrinsic divisors (the precise statement is below).

For boundary divisors the corresponding expression is either obtained directly (using the methods of [Ker06, Appendix A]) or one first should calculate the cohomology class of a (lifted) boundary divisor (e.g. $\overline{\widetilde{\mathbf{\Sigma}}}_{*,A_1} \subset Aux \times \mathbb{P}^{N_d}_f$) and then to represent it as a product $[\overline{\widetilde{\mathbf{\Sigma}}}_*][D]$ [Ker07].

To express the class of an extrinsic divisor/cycle through the classes of intrinsic one should make additional calculations (e.g. to impose the conditions of tangency and to intersect the lifted stratum with the corresponding hypersurfaces). Here we consider only the cycle of curves whose smooth branches have (hyper)flexes at the singular point.

As was noticed above, intrinsic divisors generate the intersection ring. More precisely (for linear singularity types):

**Proposition 1.6**
• *The intersection ring $A^*(\overline{\widetilde{\mathbf{\Sigma}}})$ is generated by the pullbacks $i^*(X), i^*(L), i^*(F)$ of the divisors in the ambient space. Namely:*

$$A^*(\overline{\widetilde{\mathbf{\Sigma}}}) = \frac{A^*(Aux)[i^*(F)]}{Ch(\widehat{\Sigma}_{\mathbb{S}})} \quad where \quad A^*(Aux) = \frac{\mathbb{Z}[i^*(X)]}{i^*(X)^3} \quad or \quad A^*(Aux) = \frac{\mathbb{Z}[i^*(X), i^*(L)]}{i^*(X)^3,\ \ i^*(L)^3,\ \ i^*(X)^2 - i^*(L)i^*(X) + i^*(L)^2}$$
$$or \quad A^*(Aux) = \frac{\mathbb{Z}[i^*(X), i^*(L_1), i^*(L_2)]}{i^*(X)^3,\ \ i^*(L_1)^3,\ \ i^*(L_2)^3,\ \ i^*(X)^2 - i^*(L_1)i^*(X) + i^*(L_1)^2,\ \ i^*(X)^2 - i^*(L_2)i^*(X) + i^*(L_2)^2} \tag{3}$$

*Here $Ch(\widehat{\Sigma}_{\mathbb{S}}(x,l)) = \sum_{i \geq 0} F^{r-i} c_i$, with $\{c_i\}$ the total Chern class of the vector bundle (the inverse of the Segre class from the proposition 1.5).*
• *the Picard group for a (lifted, closed) stratum of linear singularity is of rank 2: $Span_{\mathbb{Z}}[i^*(X), i^*(F)]$ (for the ordinary multiple point) or of rank 3: $Span_{\mathbb{Z}}[i^*(X), i^*(L), i^*(F)]$ or of rank 4: $Span_{\mathbb{Z}}[i^*(X), i^*(L_1), i^*(L_2), i^*(F)]$*

This follows from the propositions 2.1 and 2.2.

**1.1.3.4 Canonical class.** The total Chern class of the bundle $\widehat{\Sigma}_{\mathbb{S}}(x,l)$ determines uniquely those of the tangent bundle $T_{\overline{\widetilde{\mathbf{\Sigma}}}_{\mathbb{S}}(x,l)}$ and the normal bundle $\mathcal{N}_{\overline{\widetilde{\mathbf{\Sigma}}}_{\mathbb{S}}(x,l)}$, namely:

$$0 \to \mathcal{O} \to \pi^* \widehat{\Sigma}_{\mathbb{S}}(x,l) \otimes \mathcal{O}(1) \to T_{\overline{\widetilde{\mathbf{\Sigma}}}_{\mathbb{S}}(x,l)} \to T_{Aux} \to 0 \qquad 0 \to T_{\overline{\widetilde{\mathbf{\Sigma}}}_{\mathbb{S}}(x,l)} \to T_{\mathbb{P}^2_x \times (\mathbb{P}^2_l)^* \times \mathbb{P}^{N_d}_f} \to \mathcal{N}_{\overline{\widetilde{\mathbf{\Sigma}}}_{\mathbb{S}}(x,l)} 0 \tag{4}$$

In particular we get the canonical class

$$K_{\overline{\overline{\Sigma}}_{\mathbb{S}}} = K_{Aux} - c_1(\widehat{\widehat{\Sigma}}_{\mathbb{S}}) - rc_1\mathcal{O}_{\mathbb{P}_f^{N_d}}(1) \tag{5}$$

**1.1.3.5 Intersections on the proper stratum** From the intersection ring of the closed stratum one proceeds to that of the semi-compactification and further to the proper stratum. To find the corresponding Picard group, one should factor by the boundary divisors. In this case we obtain:

**Proposition 1.7** *For a given linear singularity type and the "generic" degree of curve $d$ the classes of the boundary divisors generate the intersection ring $A^*(\overline{\overline{\Sigma}}) \otimes \mathbb{Q}$.*

Here by generic we mean that $d$ does not satisfy a specific Diophantine equation (or a system of equations) which is fixed uniquely by the singularity type. In all the cases checked, this equation has no integer solutions.

**Corollary 1.8** For a given linear singularity type $\mathbb{S}$ and the "generic" degree of curve $d$ the proper stratum $\Sigma_{\mathbb{S}}$ does not contain complete curves and the group $Pic(\Sigma_{\mathbb{S}})$ is finite.

### 1.1.4 When is the proper stratum affine?

Having identified the (irreducible) boundary divisors $D_i$ and calculated their equivalence classes, one can check whether the proper stratum $\Sigma_{\mathbb{S}}$ is affine. Note that in view of non-pure dimensionality of the boundary (§1.1.2) the affinity is not at all obvious. We work with the lifted stratum, whose boundary is of codimension 1. Somewhat unexpectedly, it appears that the result depends on the degree of curves $d$ (i.e. its relation to parameters of the singularity). To check the affinity we use the classical criterion:

**Proposition 1.9** *Let $A$ be a quasi-projective variety, such that $A \subset \overline{A}$. Suppose the boundary is pure dimensional, of codimension 1 (i.e. a true hypersurface) and its decomposition into irreducible components is $\overline{A} \setminus A = \bigcup_i D_i$. If $\sum a_i D_i$ is ample on $\overline{A}$ for some $\{a_i \in \mathbb{N}\}_i$ then $A$ is affine.*

The following observation is useful. Suppose the component $D_1$ of the boundary is ample. Then by choosing big $a_1$ one can assure that $a_1 D_1 + \sum_{i\geq 2} D_i$ is ample too. Therefore, to prove that the stratum is affine it suffices to find just one ample component of the boundary. Alternatively, in this case $\overline{A} \setminus D_1$ is already affine. And all further deletions of divisors preserve affinity.

The converse to the proposition 1.9 is in general wrong, due to a counterexample by Zariski (cf. [Goodman]). We use a weaker statement:

**Proposition 1.10** *Under the previous assumptions, if $A$ is affine then for some $\{a_i \in \mathbb{N}\}_i$ the divisor $D = \sum a_i D_i$ is numerically effective (nef). That is, for every subscheme $Y \subset \overline{A}$ of dimension $r$ we have $D^r Y \geq 0$ [Debarre, chapter 1].*

To prove that a divisor $D \subset \overline{\overline{\Sigma}}$ is not nef, we can consider its maximal self-intersection: $D^{dim(D)+1}$. It corresponds to a zero dimensional cycle on $\overline{\overline{\Sigma}}$. The cycle is pushed forward to the ambient (multi-projective) space, where it has the degree. So, if the degree is negative the initial divisor is not nef.

Applying this idea to the boundary component of a stratum, corresponding to the singularity type with higher order of determinacy, we obtain the following criterion:

**Proposition 1.11** *Let $\mathbb{S}$ be a linear singularity type, with the multiplicity $p$ and the order of determinacy $q$. Then the proper stratum $\Sigma_{\mathbb{S}}$ is affine for $d \geq \frac{3pq}{p+q}$.*

This criterion misses the region $q \leq d < \frac{3pq}{p+q}$, for example for ordinary multiple point: $p \leq d < \frac{3}{2}p$. The bound can often be improved (by considering other components of the boundary). However, there is almost always a small region $d \gtrsim p+1$ where the stratum is *non-affine.*

So, the affinity of the Severi vartiety [DiazHarris88] is generalized to other strata provided the degree is high enough.

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## 1.2 Example: curves with an ordinary multiple point

### 1.2.1 Explicit construction and the class of the embedding

The simplest example is the stratum of curves with ordinary multiple point $f = x_1^{p+1} + x_2^{p+1}$. In this case the calculations are immediate. The lifted stratum is just the universal curve [Ker06]:

$$\overline{\widetilde{\Sigma}}(x) = \{(x,f)|\ f|_x^{(p)} = 0\} \subset \mathbb{P}^2_x \times \mathbb{P}^{N_d}_f \tag{6}$$

All the defining equations are transverse. Therefore the lifted closed stratum is T-smooth.

The projection $\overline{\widetilde{\Sigma}}(x) \to \overline{\Sigma}$ is 1:1 over the proper stratum. It is $r : 1$ over the points corresponding to curves with $r$ points of multiplicity at least $(p+1)$. The projection has one-dimensional fibers over the points corresponding to the non-reduced curves (with a component of multiplicity at least $p+1$).

Since $\overline{\widetilde{\Sigma}}(x)$ is a globally complete intersection, its cohomology class is just the product of the classes of defining hypersurfaces (for the details and notations cf. §2)

$$[\overline{\widetilde{\Sigma}}(x)] = \Big((d-p)X + F\Big)^{\binom{p+2}{2}} \in H^*(\mathbb{P}^{N_d}_f \times \mathbb{P}^2_x, \mathbb{Z}) \tag{7}$$

Here $F, X$ are the generators of the ring $H^*(\mathbb{P}^{N_d}_f \times \mathbb{P}^2_x)$.

The coefficients of this polynomial have direct enumerative meaning, being related to the intrinsic divisors:

• The number of curves in a generic linear system (of appropriate dimension) with a point of multiplicity $p+1$ (lying anywhere in the plane) is the coefficient of $X^2 F^{\binom{p+2}{2}-2}$.

• The number of curves in a generic linear system (of appropriate dimension) with a point of multiplicity $p+1$ lying on a fixed generic line, is the coefficient of $XF^{\binom{p+2}{2}-1}$. If the singular point lies on a curve the result should be multiplied by the degree of the curve.

• The number of curves in a generic linear system (of appropriate dimension) with a point of multiplicity $p+1$ fixed in the plane, is the coefficient of $F^{\binom{p+2}{2}}$. It is obviously 1.

Thinking of the parameter space $\mathbb{P}^{N_d}_f$ as the projectivization of a vector space $\mathbb{P}^{N_d}_f = Proj(\widehat{\mathbb{P}^{N_d}_f})$, the lifted stratum is the projectivization of a vector bundle $\widehat{\overline{\widetilde{\Sigma}}} \subset \mathbb{P}^2_x \times \widehat{\mathbb{P}^{N_d}_f}$. Correspondingly its intersection ring is completely fixed by propositions 2.1 and 2.2. Namely, $A^*(\overline{\widetilde{\Sigma}}(x)) = A^*(\mathbb{P}^2_x)[F] \big/ \Big(F^r + F^{r-1}c_1 + .. + c_r\Big)$. Here $X, F \in A^2(\overline{\widetilde{\Sigma}}(x))$ are the pullbacks of $X, F \in H^2(\mathbb{P}^{N_d}_f \times \mathbb{P}^2_x)$ and the total Chern class $1 + c_1 .. + c_r$ is fixed by the total Segre class $s = \Big(1 + (d-p)X\Big)^{\binom{p+2}{2}}$. This illustrates the proposition 1.6.

### 1.2.2 Resolution of the vector bundle.

In this case it is also easy to write explicit resolution for the vector bundle, in particular the intersection ring was obtained in [MiretValls05]. A curve belongs to the (closure of the) stratum iff $jet_p(f) = 0$. Thus (in notations of §2) $f \in S^p Q^* \otimes S^{d-p}(\widehat{\mathbb{P}^2})^*$. So, we have just the standard Kozsul resolution:

$$0 \to S^{d-p-1}(\widehat{\mathbb{P}^2})^* \otimes \wedge^2 Q^* \otimes S^{p-1} Q^* \xrightarrow{\alpha} S^{d-p}(\widehat{\mathbb{P}^2})^* \otimes S^p Q^* \xrightarrow{\beta} \widehat{\widetilde{\Sigma}}(x) \to 0 \tag{8}$$

with the maps $\alpha: \; f_1 \otimes (\xi_1 \wedge \xi_2) \otimes f_2 \to (f_1\xi_1) \otimes (f_2\xi_2) - (f_1\xi_2) \otimes (f_2\xi_1)$ and $\beta: \; f_1 \otimes f_2 \to (f_1 f_2)$.

From the resolution the total Chern class of the bundle $\tilde{\bar{\bar{\Sigma}}}(x)$ is directly calculated (cf. sec 2.1.3). Finally (cf. [MiretValls05, proposition 1.2])

$$A^*(\tilde{\bar{\bar{\mathbf{\Sigma}}}}(x)) = \frac{\mathbb{Z}[X,F]}{\Big(X^3, \;\; F^r - \binom{p+2}{2}(d-p)XF^{r-1} + \Big(\binom{p+2}{2} + \binom{\binom{p+2}{2}}{2}\Big)(d-p)^2 X^2 F^{r-2}\Big)}, \qquad r = N_d + 1 - \binom{p+2}{2} \tag{9}$$

The relations in the intersection ring $A^*(\tilde{\bar{\bar{\mathbf{\Sigma}}}}(x))$ are obtained most easily from the cohomology class (7):

$$X^2F^{r-1} = 1[pt], \quad XF^r = \binom{p+2}{2}(d-p)[pt], \quad F^{r+1} = \binom{\binom{p+2}{2}}{2}(d-p)^2[pt] \tag{10}$$

here $[pt]$ is the class of a point in $\tilde{\bar{\bar{\mathbf{\Sigma}}}}$ (smooth rational variety).

### 1.2.3 Some extrinsic divisors/cycles

can be described very explicitly and their classes can be written out immediately. For example, consider a cycle in $\tilde{\bar{\bar{\mathbf{\Sigma}}}}$, consisting of curves with the ordinary multiple point, such that the $i$'th (smooth) branch and its tangent line $l_i$ have tangency of order $k_i \geq 2$. (Here $l_i$'s are 1-forms defining the lines.)

Start from the branch decomposition: $f = (l_1 + ..)(l_2 + ..)..(l_{p+1} + ..) + \textit{higher order terms}$. The condition of prescribed order of tangency gives:

$$f = (l_1(1 + ..) + m_{k_1})(l_2(1 + ..) + m_{k_2})..(l_{p+1}(1 + ..) + m_{k_{p+1}}) + m_{k_1 + .. + k_{p+1} + 1} \tag{11}$$

(here $m_i$ is the corresponding local ideal).

So the corresponding (lifted) cycle is defined as (cf. [Ker06])

$$\tilde{\bar{\bar{\mathbf{\Sigma}}}}_{k_1..k_{p+1}}(x, l_i, A_{k_i}, B_{k_i - 2}) = \left\{ \begin{pmatrix} x, \{l_i\}, \{A_{k_i}\} \\ \{B_{k_i-2}\}, f \end{pmatrix} \Bigg| \begin{array}{l} f|_x^{\sum k_i - (p+1)} \sim SYM(A_{k_1}..A_{k_{p+1}}), \\ A_{k_i}(x) \sim SYM(l_i, B_{k_i-2}), \quad l_i(x) = 0 \end{array} \right\} \tag{12}$$

here $A_{k_i}, B_{k_i}$ are some auxiliary forms, all the notations are from §2. As always, the definition is a combination of standard proportionality conditions, which are mutually transverse. Correspondingly the cohomology class is written immediately. By projecting out the auxiliary variables $(l_i, A_{k_i}, B_{k_i})$ one obtains the class of the minimal lifting of the cycle: $[\tilde{\bar{\bar{\mathbf{\Sigma}}}}_{k_1..k_{p+1}}(x)]$. Finally, representing this class as the product $[\tilde{\bar{\bar{\mathbf{\Sigma}}}}(x)][C]$ we get the expression $[C]$ for the cycle in terms of pulled back generators $X, F$.

### 1.2.4 Boundary and (non-)affinity of the proper stratum.

**1.2.4.1 Semi-compactification $\bar{\Sigma} \subset \bar{\mathbf{\Sigma}}$.** The boundary divisor corresponds to the "degenerate" multiple point. As the original point is characterized by its (non-coinciding) tangents, the only minimal (topological) degeneration is: two tangents merge. Correspondingly, the boundary (the stratum adjacent in codimension 1) is $\bar{\mathbf{\Sigma}}_{x_1^{p+1} + x^2 x_2^{p-1} + x_2^{p+2}}$, which is a hypersurface.

The boundary divisor is irreducible (since it represents an equisingular stratum). Its class is $2pF + p(2d - 3(p+1))X$ (cf. [Ker06, Appendix A.1.2]). So, it is very ample for $2d \geq 3(p+1)$ (as a positive combination of very ample divisors). Thus, from the discussion in 1.1.4 it follows that both $\tilde{\bar{\Sigma}}$ and the proper stratum $\Sigma$ are *affine* for $d \geq \frac{3(p+1)}{2}$.

To check whether for $p + 1 \leq d < \frac{3(p+1)}{2}$ the divisor is nef, we consider its maximal (non-empty) self-intersection: $\Big(2pF + p(2d - 3(p+1))X\Big)^{r+1} \in A^*(\tilde{\bar{\bar{\mathbf{\Sigma}}}})$. The resulting cycle of the intersection is zero dimensional. Push it forward to the ambient space and calculate the resulting degree.

Direct check shows that for $d=p+1$ it is negative for all $p$, for $d=p+2$ it is negative if $p\geq 8$ etc.. So, there exists a region near $(p+1)$ for which the semi-compactification is not affine.
The Picard group of the semi-compactification is $\frac{Span_{\mathbb{Z}}(X,F)}{2pF+p\Big(2d-3(p+1)\Big)X}$.

**1.2.4.2 Proper stratum.** The boundary divisor of $\widetilde{\Sigma}\subset\bar{\widetilde{\Sigma}}$ is the stratum $\bar{\Sigma}_{x_1^{p+1}+x_2^{p+1},A_1}$, its points correspond to the curves with an additional node.

Its divisor class is obtained in [Ker07], it is

$$\alpha X+\beta F,\quad \alpha=-pd(4+3p)+3p(2+3p+p^2),\qquad \beta=3(d-1)^2-4p-3p^2 \tag{13}$$

Correspondingly the Picard group of the proper stratum is $\frac{Span_{\mathbb{Z}}(X,F)}{2pF+p(2d-3(p+1))X,\ \alpha X+\beta F}$ recovering the known results of [MiretXambo94] and [MiretValls05, section 5] (the later up to a misprint).

Again, this divisor is not nef for $d\geq p+2$. Therefore for $p\geq 8$, there exists a region of values of $d$ (lying inside $(p+1,\frac{3(p+1)}{2})$) for which the proper stratum is *not affine*.

**1.2.4.3 The components of equi-generic compactification** are directly obtained by applying the method of §4.3. In particular they are in 1:1 correspondence with the possible degenerations of the homogeneous form $x_1^{p+1}+..+x_2^{p+1}$. The codimension one component is the one described above. The component of the minimal dimension corresponds to the generalized cusp: $x_1^{p+1}+x_2^{p+2}$.

# 2 Some definitions, notations

## 2.1 The relevant notions and results

### 2.1.1 The ambient space and the cohomology.

In this paper we deal with many rational equivalence classes of various varieties, embedded into various (products of) projective spaces. To simplify the formulae we adopt the following notation. If we denote a point in the space $\mathbb{P}^2_x$ by the letter $x$, then the homogeneous coordinates are $(x_0,x_1,x_2)$. The generator of the cohomology (or intersection) ring of this $\mathbb{P}^2_x$ is denoted by the upper-case letter $X$, so that $H^*(\mathbb{P}^2_x)=\mathbb{Z}[X]/(X^3)$. Alternatively $X$ is the first Chern class of the dual tautological bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

By the same letter we also denote the hyperplane class in homology of $\mathbb{P}^n_x$. Since it is always clear, where we speak about coordinates and where about (co)homology classes, no confusion arises. To demonstrate this, consider the hypersurface

$$V=\{(x,y,f)|\ f(x,y)=0\}\subset\mathbb{P}^n_x\times\mathbb{P}^n_y\times\mathbb{P}^{N_d}_f \tag{14}$$

Here $f$ is a bi-homogeneous polynomial of bi-degree $d_x,d_y$ in homogeneous coordinates $(x_0,\dots,x_n)$, $(y_0,\dots,y_n)$, the coefficients of $f$ are the homogeneous coordinates in the parameter space $\mathbb{P}^{N_d}_f$. The cohomology class of this hypersurface is

$$[V]=d_xX+d_yY+F\in H^2(\mathbb{P}^n_x\times\mathbb{P}^n_y\times\mathbb{P}^{N_d}_f) \tag{15}$$

A (projective) line through the point $x\in\mathbb{P}^2_x$ is defined by a 1-form $l$ (so that $l\in(\mathbb{P}^2_l)^*$, $l(x)=0$). Correspondingly the generator of $H^*((\mathbb{P}^2_l)^*)$ is denoted by $L$.

For projective space homology and rational equivalence coincide. For other varieties we consider rational equivalence of cycles. For a subvariety of multi-projective space $\Sigma\overset{i}{\hookrightarrow}\mathbb{P}^{n_1}\times..\times\mathbb{P}^{n_r}$ we are interested in its intersection ring $A^*(\Sigma)$. A divisor $D$ in the ambient space gives the pullback $[i^*(D)]\in A^*(\Sigma)$. To avoid messy notation we often omit the pullback sings (in this case we specify the relevant intersection ring).

### 2.1.2 Symmetric forms.

We often work with symmetric $p-$forms $\Omega^p \in S^p(\widehat{\mathbb{P}^2})^*$ (here $(\widehat{\mathbb{P}^2})^*$ is a 3-dimensional vector space of linear forms). Thinking of the form as of a symmetric tensor with $p$ indices $(\Omega^{(p)}_{i_1,\ldots,i_p})$, we write $\Omega^{(p)}(\underbrace{x,\ldots,x}_{k})$ as a shorthand for the tensor, multiplied $k$ times by the point $x \in \widehat{\mathbb{P}^2}$

$$\Omega^{(p)}(\underbrace{x,\ldots,x}_{k}) := \sum_{0\leq i_1,\ldots,i_k\leq 2} \Omega^{(p)}_{i_1,\ldots,i_p} x_{i_1}\ldots x_{i_k} \tag{16}$$

So, for example, the expression $\Omega^{(p)}(x)$ is a $(p-1)-$form. Unless stated otherwise, we assume the symmetric form $\Omega^{(p)}$ to be generic (in particular non-degenerate, i.e. the corresponding hypersurface $\{\Omega^{(p)}(\underbrace{x,\ldots,x}_{p}) = 0\} \subset \mathbb{P}^n_x$ is smooth).

Symmetric forms typically occur as tensors of derivatives of order $p$, e.g. $f^{(p)}$. Sometimes, to emphasize the point at which the derivatives are calculated we assign it. So, e.g. $f|^{(p)}_x(\underbrace{y,\ldots,y}_{k})$ means: the tensor of derivatives of order $p$, calculated at the point $x$, and contracted $k$ times with $y$.

### 2.1.3 Relevant bundles.

We constantly work with projectivization of vector bundles (vector spaces). To avoid messy notation we adopt the following convention: if a projective bundle/space is denoted by $Z$ then the corresponding vector bundle/space is $\hat{Z}$. For example $\mathbb{P}^n$ =Proj$(\widehat{\mathbb{P}^n})$. An equisingular stratum $\Sigma$ is the projectivization of the corresponding bundle $\widehat{\Sigma} \subset \widehat{\mathbb{P}^{N_d}_f}$.

The following bundles constantly occur. The tautological and quotient bundles are related by $0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \widehat{\mathbb{P}^n}|_{\mathbb{P}^n} \to Q \to 0$. Here $\mathcal{O}_{\mathbb{P}^n}(-1) = \{(x,v) \in \mathbb{P}^n_x \times \widehat{\mathbb{P}^n} |\ v \in x\}$. The dual sequence $0 \to Q^* \to \widehat{\mathbb{P}^n}^*|_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$ defines the bundle $Q^*$ whose fiber over $x$ consists of one-forms vanishing at $x$. It is of rank $n$, its Chern class is $c(Q^*) = \frac{1}{1+X}$.

The symmetric power $S^pQ^*$ has as its fiber polynomials (of degree $p$) vanishing at a given point up to the order $p-1$. Its total Chern class is computed by the general rule [Fulton98, section 3.2]: $c(S^pQ^*) = 1 - \binom{p+1}{2}X + \binom{\binom{p+1}{2}+1}{2}X^2$

For further reference we mention also the tensor product $E \otimes \mathcal{L}$ of rank $r$ bundle with a line bundle. Its total Chern class is

$$c(E\otimes\mathcal{L}) = \sum_{j=0}^{r} c_1^j(\mathcal{L}) \sum_{i=0}^{r-j} \binom{r-i}{j} c_i(E) \tag{17}$$

To calculate the intersection ring of the projectivization of a bundle we use the following classical theorem:

**Theorem 2.1** *Let a vector bundle $E_B$ of rank $r$ and its projectivization $\mathbb{P}E$ be as in the diagram, $\xi = c_1(\mathcal{O}(1)_{\mathbb{P}E})$ be the first Chern class of the dual tautological bundle. Then $A^*(\mathbb{P}E) = A^*(B)[\xi]/(\xi^r + \pi^*(c_1(E))\xi^{r-1} + ..\pi^*(c_r(E)))$*

$$\begin{tikzcd} E \arrow[d] \arrow[dr, "proj"] & \pi^*(E)\supset\mathcal{O}(-1) \arrow[d] \arrow[d] \\ B & \mathbb{P}E \arrow[l, "\pi"'] \end{tikzcd}$$

In our case, $\widetilde{\overline{\mathbf{\Sigma}}} \overset{i}{\hookrightarrow} \mathbb{P}^{N_d}_f \times Aux$ the first Chern class is: $\xi = c_1(\mathcal{O}_{\widehat{\Sigma}}(1)) = c_1(i^*\mathcal{O}_{\widehat{\mathbb{P}^{N_d}_f}}(1)) = i^*c_1(\mathcal{O}_{\mathbb{P}^{N_d}_f}(1)) = i^*(F)$.

To apply this theorem we need the total Chern class of the bundle $E \to B$ (i.e. $\widetilde{\overline{\mathbf{\Sigma}}} \to Aux$). In this paper we constantly meet the situation where the total space of the bundle is embedded (cf. the diagram), so that the fibres of $E$ are linear subspaces of $\widehat{\mathbb{P}^{N_d}_f}$. And the cohomology class $[Proj(E)] \in H^*(B \times \mathbb{P}^{N_d}_f)$ is known [Ker06].

$$\begin{tikzcd} E \subset B \times \widehat{\mathbb{P}^{N_d}_f} \arrow[d] \arrow[d] \\ B \end{tikzcd}$$

This fixes the total Chern class completely:

**Proposition 2.2** *Let the class* $[Proj(E)]$ *be given by a (homogeneous) polynomial* $P(..., F)$ *in the generators of the intersection ring of* $B \times \mathbb{P}_f^{N_d}$. *Then the polynomial is monic in* $F$ *and the total Segre class of the bundle* $E \to B$ *is* $P(.., 1)$.

**proof:**
• Let $[pt] \in H^*(B)$ be the class of a point. Then $[pt][Proj(E)] \in H^*(B \times \mathbb{P}_f^{N_d})$ is the class dual to the fiber over a point. So it is non-zero, being the class of a hyperplane in $\mathbb{P}_f^{N_d}$. Thus the polynomial is monic in $F$.
• From the previous proposition we get the identity: $F^r + \pi^*(c_1(E))F^{r-1} + ..\pi^*(c_r(E)) = 0 \in A^*(Proj(E))$. Here $F$ is the pullback of the hyperplane class in $\mathbb{P}_f^n$, while $\pi^*(c_i(E))$ are some classes on $B$. By pushing-forward the identity to the ambient space $B \times \mathbb{P}_f^{N_d}$ we get $\Big(F^r + \pi^*(c_1(E))F^{r-1} + ..\pi^*(c_r(E))\Big)[Proj(E)] = 0 \in A^{(N_d+1)}(B \times \mathbb{P}_f^{N_d})$.

As the class $[Proj(E)]$ is monic in $F$, the Chern classes are restored uniquely from this identity. In fact the identity can be lifted to an identity in the bigger polynomial ring :

$$\Big(F^r + \pi^*(c_1(E))F^{r-1} + ..\pi^*(c_r(E))\Big)[Proj(E)] = F^{N_d+1} \in A^{(N_d+1)}(B)[F] \tag{18}$$

Thus the statement for the total Segre class follows. ■

## 2.2 On the spaces of approximating k-jets

Usually the defining conditions of singular germs are formulated in terms of k-jets of functions (where $k = o.d. - 1$, here $o.d.$ being the order of determinacy of the singularity). Therefore we consider the parameter spaces of the projectivized k-jets.

For a variety $Z = Spec(A)$ the classical k-jets scheme is defined as $\mathcal{L}_k(Z) = Spec(A[\epsilon]/\epsilon^{k+1})$. Correspondingly, its elements are jets of smooth or uni-branch curves. As the closure of every equisingular stratum contains points corresponding to multi-branch singularities, we have to consider more general spaces of jets: jets of functions. So, to a germ $(f, 0)$ we assign its k-jet $b_k = jet_k f(x_1, x_2)$.

Here we consider $b_k$ both as a polynomial (in local coordinates) and as a symmetric $k-$form (in homogeneous coordinates). Correspondingly the definition can be written as $f|_x^{(k)} \sim b_k$. Such a jet defines a plane curve $\{b_k(x..x) = 0\} \subset \mathbb{P}^2$, abusing notations we denote it by the same letter $b_k$.

Thus, naively we define the parameter space of k-jets as the incidence variety (the basic point, the jet): $\{(x, b_k)| \ \ b_k(x..x) = 0\} \subset \mathbb{P}_x^2 \times \mathbb{P}_{b_k}$.

**Example 2.3** In the simplest case ($k = 1$), the curve $b_1$ is just the tangent line and the space of projectivized 1-jets is $\mathbb{P}(T^*\mathbb{P}^2)(-1) = \{(x, l)|l(x) = 0\} \subset \mathbb{P}_x^2 \times (\mathbb{P}_l^2)^*$. Projectivization turns the space of one-forms vanishing at $x$ into the lines through the point. For higher $k$ we have a conic osculating the curve $f = 0$, the cubic etc.

For a generic point of the variety of k-jets, by the natural reduction $b_k \to b_{k'} = jet_{k'}(b_k)$ we get all the lower jets (approximations to the curve). However for jets with singularities at $x$ this is not the case.

To carry all the information we have to blowup the parameter space along the loci of jets with an ordinary multiple point (i.e. $jet_{k'}(b_k) = 0$ or $b_k(\underbrace{x..x}_{k-k'}) = 0$). Thus we define

**Definition 2.4** *The parameter space of projectivized jets is*

$$\mathbb{P}Jet_k := \{(x, l, b_2..b_k)|l(x) = 0, \quad b_2(x) \sim l, .., b_k(x) \sim b_{k-1}\} \subset \mathbb{P}_x^2 \times (\mathbb{P}_l^2)^* \times \mathbb{P}_{b_2} \dots \mathbb{P}_{b_k} \tag{19}$$

Here the conditions are that the point $x$ lies on the line $l$, the line $l$ is tangent to the conic $b_2$ at $x$ etc..

**Proposition 2.5**
• *The parameter space is smooth and the map $\mathbb{P}Jet_k \to \mathbb{P}Jet_{k-1}$ is a (projective) bundle of rank $k+1$.*
• *The projection $(x, l, b_2..b_k) \to (x, l, b_2..b_{i-1}, b_{i+1}..b_k)$ is the blowup over the locus $b_{i+1}(x) = 0$.*

**proof:**
• First, note that each fiber is evidently a projective space. Thus one only has to show a trivialization over some open sets and to check that the transition maps are linear on fibers.

The automorphisms of the plane ($PGL(3)$) act linearly on the parameter space and all the orbits are isomorphic. Therefore it is sufficient to prove local triviality for the restriction of the bundle to a cycle in the parameter space: fixed point $x = (1, 0, 0)$. For a fixed $x$, consider $b_{k-1}$ as a point in a big projective space. In fact, due to the condition $b_{k-1}(x..x) = 0$ the point lies in some hyperplane. Cover this hyperplane by standard affine sets $U_{i_1..i_{k-1}} = \{b_{i_1..i_{k-1}} = 1\}$. Thinking of each such set as being $\mathbb{C}^n$, there is a unique translation from any point to the origin: $Tr_{b_{k-1}}$. This translation induces a *linear* transformation on the fiber $Tr^*_{b_{k-1}}(b_k)$, which is the needed local trivialization.

Note that the trivialization is achieved by linear transformations, therefore the transition functions are also linear.
• The projection $(x, l, b_2, ..b_k) \to (x, l, b_2..b_{i-1}, b_{i+1}..b_k)$ is generically 1:1 (as $b_i \sim b_{i+1}(x)$). It is not 1:1 over the locus with $b_{i+1}(x) = 0$. Here the only restriction is $b_i(x) \sim b_{i-1}$, this gives the dimension of the fiber: $\binom{i+2}{2} - \binom{i+1}{2}$. ■

Note that the variety $\mathbb{P}Jet_k$ is defined as a subvariety of multi-projective space by a combination of standard set of conditions. Therefore its cohomology class in $H^*(\mathbb{P}^2_x \times \dots (\mathbb{P}_{b_k})^*, \mathbb{Z})$ is written immediately:

$$[\mathbb{P}Jet_k] = [l(x) = 0][b_2(x) \sim l] \dots [b_k(x) \sim b_{k-1}] \tag{20}$$

where the classes in square brackets are just the classes of diagonals (cf. §A.1).

Alternatively, from the explicit definition above we can calculate the Chow rings of the projectivized jet spaces. For this, we represent $Jet_k$ as a vector bundle over $\mathbb{P}Jet_{k-1}$ (of rank $r = k+1$). The above definition immediately leads to the free resolution:

$$0 \to \mathcal{O}(-1)_{\mathbb{P}Jet_{k-1}} \otimes Q^* \stackrel{\alpha}{\to} S^k Q^* \oplus \left(\mathcal{O}(-1)_{\mathbb{P}Jet_{k-1}} \otimes \widehat{\mathbb{P}2}^*_x\right) \stackrel{\beta}{\to} Jet_k \to 0 \tag{21}$$

Here the maps are: $\alpha: \ \xi \otimes q {\to} \xi q \oplus (-\xi \otimes q)$ and $\beta: \ q \oplus \xi \otimes f {\to} q + \xi f$.

Now, the total Chern class is

$$c(Jet_k) = \frac{c(S^k Q^*) c\left(\mathcal{O}(-1)_{\mathbb{P}Jet_{k-1}} \otimes \widehat{\mathbb{P}2}^*_x\right)}{c(\mathcal{O}(-1)_{\mathbb{P}Jet_{k-1}} \otimes Q^*)} \tag{22}$$

and thus is expressed in terms of the Chern classes of $Q^*$ and $Jet_{k-1}$. Once the total Chern class is known, the Chow ring is determined by the theorem 2.1.

**Example 2.6** For $k = 1$ we have (cf. the previous example): $c(Jet_1) = 1 - X + X^2$ and the intersection ring is $A^*(\mathbb{P}Jet_1) = A^*(\mathbb{P}^2_x) / (L^2 - LX + X^2)$. Here $X, L$ are the pullbacks of the corresponding first Chern classes from the ambient space $\mathbb{P}^2_x \times (\mathbb{P}^2_l)^*$.

## 2.3 On the singularity types

**Definition 2.7** *Let $(C_x, x) \subset (\mathbb{C}^2_x, x)$ and $(C_y, y) \subset (\mathbb{C}^2_y, y)$ be two germs of isolated curve singularities. They are topologically equivalent if there exist a homeomorphism $(\mathbb{C}^2_x, x) \mapsto (\mathbb{C}^2_y, y)$ mapping $(C_x, x)$ to $(C_y, y)$. The corresponding equivalence class is called topological singularity type. The variety of points (in the parameter space $\mathbb{P}^{N_d}_f$), corresponding to curves with singularity of the same (topological) type $\mathbb{S}$ is called the* **equisingular stratum** $\Sigma_{\mathbb{S}}$

The topological type can be specified by a (simple polynomial) representative of the type: the **normal** form. For example for several simplest types (all the notations are from [AGV], we ignore the moduli of analytic types):

$$\begin{array}{l} A_k : x_2^2 + x_1^{k+1}, \quad D_k : x_2^2 x_1 + x_1^{k-1}, \quad E_{6k} : x_2^3 + x_1^{3k+1}, \quad E_{6k+1} : x_2^3 + x_2 x_1^{2k+1}, \quad E_{6k+2} : x_2^3 + x_1^{3k+2} \\ J_{k\geq 1, i\geq 0} : x_2^3 + x_2^2 x_1^k + x_1^{3k+i}, \quad Z_{6k-1} : x_2^3 x_1 + x_1^{3k-1}, \quad Z_{6k} : x_2^3 x_1 + x_2 x_1^{2k}, \quad Z_{6k+1} : x_2^3 x_1 + x_1^{3k} \\ X_{k\geq 1, i\geq 0} : x_2^4 + x_2^3 x_1^k + x_2^2 x_1^{2k} + x_1^{4k+i}, \quad W_{12k} : x_2^4 + x_1^{4k+1}, \quad W_{12k+1} : x_2^4 + x_2 x_1^{3k+1} \end{array} \tag{23}$$

Using the normal form $f = \sum a_{\mathbf{I}} \mathbf{x}^{\mathbf{I}}$ one can draw the *Newton diagram* of the singularity. Namely, one marks the points $\mathbf{I}$ corresponding to non-vanishing monomials in $f$, and takes the convex hull of the sets $\mathbf{I} + \mathbb{R}^2_+$. The envelope of the convex hull (the chain of segment-faces) is the Newton diagram.

**Definition 2.8**
• *The singular germ is called Newton-non-degenerate with respect to its diagram if the truncation of its polynomial to every face of the diagram is non-degenerate (i.e. the truncated polynomial has no singular points in the torus $(C^*)^2$).*
• *The germ is called generalized Newton-non-degenerate if it can be brought to a Newton-non-degenerate form by a locally analytic transformation.*
• *The singular type is called Newton-non-degenerate if it has a (generalized) Newton-non-degenerate representative.*

For Newton-non-degenerate types the normal form is always chosen to be Newton-non-degenerate . So, the Newton-non-degenerate type $\mathbb{S}$ can be specified by giving the Newton diagram of its normal form $\mathbb{D}_{\mathbb{S}}$.

Newton-non-degeneracy implies strong restrictions on the tangent cone:

**Proposition 2.9** *Let $T_C = \{(l_1, p_1)...(l_k, p_k)\}$ be the tangent cone of the germ $C = \cup C_j$ (here all the tangents $l_i$ are different, $p_i$ are the multiplicities, so that $\sum_i p_i = mult(C)$). If the germ is generalized Newton-non-degenerate then $p_i > 1$ for at most two tangents $l_i$.*

So, for a generalized Newton-non-degenerate germ there are at most two distinguished tangents. We always orient the coordinate axes along these tangents.

As we consider the topological types, one could expect that to bring a germ to the Newton diagram of the normal form, one needs local homeomorphisms. However for curves the locally analytic transformation always suffice. In this paper we restrict consideration further to the types for which only linear transformations suffice.

**Definition 2.10** *A (generalized Newton-non-degenerate ) singular germ is called linear if it can be brought to the Newton diagram of its type by projective transformations only (or linear transformations in the local coordinate system centered at the singular point). A linear stratum is the equisingular stratum, whose open dense part consists of linear germs. The topological type is called linear if the corresponding stratum is linear.*

The linear types happen to be abundant due to the following observation

**Proposition 2.11** *[Ker06, section 3.1] The Newton-non-degenerate topological type is linear iff every segment of the Newton diagram has the bounded slope: $\frac{1}{2} \leq tg(\alpha) \leq 2$.*

**Example 2.12** The simplest class of examples of linear singularities is defined by the series: $f = x^p + y^q, \quad p \leq q \leq 2p$. In general, for a given series only for a few types of singularities the strata can be linear. In the low modality cases the linear types are:
• Simple singularities (no moduli): $A_{1\leq k\leq 3}, \quad D_{4\leq k\leq 6}, \quad E_{6\leq k\leq 8}$
• Unimodal singularities: $X_9 (= X_{1,0}), \quad J_{10} (= J_{2,0}), \quad Z_{11\leq k\leq 13}, \quad W_{12\leq k\leq 13}$
• Bimodal: $Z_{1,0}, \quad W_{1,0}, \quad W_{1,1}, \quad W_{17}, \quad W_{18}$

Most singularity types are nonlinear. For example if a curve has an $A_4$ point, the best we can do by projective transformations is to bring it to the Newton diagram of $A_3$ $a_{0,2}x_2^2 + a_{2,1}x_2x_1^2 + a_{4,0}x_1^4$.

This quasi-homogeneous form is degenerated ($a_{2,1}^2 = 4a_{0,2}a_{4,0}$) and by quadratic (nonlinear!) change of coordinates the normal form of $A_4$ is achieved.

# 3 Geometry of some compactified strata

A natural approach to the resolution of a compactified stratum is to lift the stratum to a bigger ambient space. As a bonus the lifted strata often appear to be fibrations over some simple subvarieties of the jet space $\mathbb{P}Jet$. This enables us to study their geometry. For a linear singularity type $\mathbb{S}$ with order of determinacy $o.d. = k+1$ and the distinguished tangents $l_i$ (in the sense of the proposition 2.9) the lifting is defined as:

$$\overline{\widetilde{\Sigma}}_{\mathbb{S}}(x,\{l_i\},b_2..b_k) := \overline{\left\{(x,\{l_i\},b_2..b_k,f)\middle| \begin{array}{c} f_k \text{ has the singularity type } \mathbb{S} \\ \text{with the } k\text{'th jet } b_k \end{array}\right\}} \subset \mathbb{P}Jet_k \times \mathbb{P}_f^{N_d} \tag{24}$$

The simplest examples are considered in the introduction. The lifting of non-linear singularities is more complicated and is discussed later.

One general remark is worth to mention. The lifting is defined by first lifting the proper stratum and then taking the topological closure. The explicit defining equations of the points in the closure can be difficult to write. However (even for non-linear singularities) the so defined variety always surjects the initial one:

**Proposition 3.1** *Let $\widetilde{\Sigma}_1, \widetilde{\Sigma}_2$ be two consecutive liftings of the proper stratum, i.e. the projection $\widetilde{\Sigma}_2 \to \widetilde{\Sigma}_1$ eliminates one variable. Then the projection of closures $\overline{\widetilde{\Sigma}}_2 \to \overline{\widetilde{\Sigma}}_1$ is well defined and surjective.*

The proof follows easily from the consideration in classical topology (since both varieties are compact).

## 3.1 Linear strata

For linear singularity we use the action of the group $PGL(3)$ on the stratum to fix the singular point and a tangent line. As the linear singularity is completely fixed by its Newton diagram, the fiber over the pair (the point, the line) is a linear subspace of $\mathbb{P}_f^{N_d}$ and the whole lifted stratum is a projective bundle over the incidence variety.

So, for linear singularities the lifted stratum $\overline{\widetilde{\Sigma}}(x,l)$ (or $\overline{\widetilde{\Sigma}}(x,l_1,l_2)$) is already smooth. To simplify the strata we lift them further: blowup the parameter space along the strata of points of higher multiplicities. So we are led to the liftings $\widetilde{\Sigma}(x,l,b_2..b_{o.d.-1})$.

**Proposition 3.2** *[Ker06, section 3] For a linear singularity with order of determinacy: o.d., the (closure of the) lifted stratum is a subvariety $\overline{\widetilde{\Sigma}}(x,l,b_2..b_{o.d.-1}) \subset \mathbb{P}Jet_{o.d.-1} \times \mathbb{P}_f^{N_d}$. The stratum is the projectivization of a vector bundle over a subvariety of $\mathbb{P}Jet_{o.d.-1}$. The base space and the total space of the fibration are smooth. The projection to the base factors through the chain of fibrations: $\overline{\widetilde{\Sigma}}(x,l,b_2..b_{o.d.-1}) \to \overline{\widetilde{\Sigma}}(x,l,b_2..b_{o.d.-2}) \to .. \to \overline{\widetilde{\Sigma}}(x,l) \to \mathbb{P}Jet_{o.d.-1}$*

The simplest example: ordinary multiple points is considered in 1.2. As in that example, the definition of a lifted stratum can be translated into a free resolution of the corresponding vector bundle. Some more examples are in Appendix B.

The representation of the (closure) of the linear strata as projectivized vector bundles enables us to calculate the intersection rings of the strata (as formulated in theorem 1.6).

The theorem 2.1 reduces the problem to the intersection ring of the auxiliary space and the total Chern class of the bundle. The first is calculated in proposition 2.2, the second is fixed by the proposition 2.2, which in our case reads:

*Let the cohomology class of a lifted stratum* $\bar{\bar{\mathbf{\Sigma}}} \subset Aux \times \mathbb{P}^D_f$ *be given by a homogeneous polynomial* $P(X, L, B_2..B_k, F)$ *in the generators of the cohomology ring* $H^*(Aux \times \mathbb{P}^D_f)$*. Then the total Segre class of the vector bundle* $\hat{\hat{\Sigma}} \to Aux$ *is given by* $s = P(X, L, B_2..B_k, 1)$*.*

The general procedure is described in introduction. We consider some more examples in section B.

### 3.2 An example of non-linear strata $A_k:\quad x_1^2 + x_2^{k+1}$

To formulate the defining conditions, start from the normal form of the type: $f = x_1^2 + x_2^{k+1}$. As a consequence one has $jet_k(f) = x_1^2$. To obtain the defining equation of the stratum, note that the normal form is achieved here by locally analytic coordinate transformations $x_i \to x_i + \sum a_{\mathbf{I}} \mathbf{x}_{\mathbf{I}}^{\mathbf{I}}$. (This is because the singularity is semi-quasi-homogeneous and has no moduli.) In this transformation only the terms of the degree $\le k-1$ are relevant. Therefore the defining condition can be written as $jet_k(f) = jet_k(b_{k-1}^2)$, where $b_{k-1}$ is a polynomial of degree $k-1$ (alternatively a germ of smooth curve at the origin). To write this condition in a form covariant under $PGL(3)$ we pass to (symmetric) tensors of derivatives. Namely, let $b_{k-1}$ denote also a symmetric tensor of order $(k-1)$ in variables $x_0..x_2$ (homogeneous coordinates). Consider the stratum

$$\bar{\bar{\mathbf{\Sigma}}}(x, b_{k-1}) = \overline{\left\{ (x, b_{k-1}, f) \Big| b_{k-1} \text{ is smooth}, \quad b_{k-1}(\underbrace{x..x}_{k-1}) = 0, \quad f|_x^{(k)} \sim (b_{k-1} b_{k-1})(\underbrace{x...x}_{k-2}) \right\}} \subset \mathbb{P}^2_x \times \mathbb{P}_{b_{k-1}} \times \mathbb{P}^{N_d}_f \tag{25}$$

To understand the stratum at the locus of singular $b_{k-1}$, the flat limit should be taken. Naively imposing the condition $jet_1(b_{q-p}) = 0$ results in the singularity of multiplicity $Min(2p, q-p+1)$. This is certainly wrong since many types of lower multiplicities are adjacent and the lifted stratum surjects the original stratum (cf. proposition 3.1).

To simplify the variety we lift it further to the space of jets. So, we define the lifted stratum:

$$\bar{\bar{\mathbf{\Sigma}}}(x, l, b_2..b_{k-1} C_k) = \overline{\left\{ \begin{array}{l} (x, l, b_2..b_{k-1}, C_k, f) \\ b_i \text{ are smooth} \end{array} \middle| \begin{array}{l} f^{(k)}|_x \sim C_k, \quad C_k \sim (b_{k-1} b_{k-1})(\underbrace{x...x}_{k-2}), \\ b_{k-1}(x) \sim b_{k-2}, ..b_2(x) \sim l, \quad l(x) = 0 \end{array} \right\}} \subset \mathbb{P}Jet_{k-1} \times \mathbb{P}_{C_k} \times \mathbb{P}^{N_d}_f \tag{26}$$

Now, the projection $\bar{\bar{\mathbf{\Sigma}}}(x, l, b_2..b_{k-1} C_k) \to (x, l, b_2..b_{k-1} C_k)$ is a bundle (with fibers: linear subspaces of $\mathbb{P}^{N_d}_f$). So, the whole difficulty is transformed to the base space of the bundle:

$$Aux := \overline{\left\{ \begin{array}{l} (x, l, b_2..b_{k-1}, C_k) \\ b_i \text{ are smooth} \end{array} \middle| \begin{array}{l} C_k \sim (b_{k-1} b_{k-1})(\underbrace{x...x}_{k-2}), \\ b_{k-1}(x) \sim b_{k-2}, ..b_2(x) \sim l, \quad l(x) = 0 \end{array} \right\}} \subset \mathbb{P}Jet_{k-1} \times \mathbb{P}_{C_k} \tag{27}$$

**Proposition 3.3** *The projection* $Aux \to \mathbb{P}Jet_{k-1}$ *is an isomorphism. Correspondingly, base space* $Aux$ *is a smooth irreducible variety and the lifted stratum* $\bar{\bar{\mathbf{\Sigma}}}(x, l, b_2..b_{k-1} C_k)$ *is a locally trivial (smooth) projective bundle.*

**proof:** Outside the scheme $(b_{k-1} b_{k-1})(\underbrace{x...x}_{k-2}) = 0$ the projection is an isomorphism. Note, that $Aux$ is defined as the topological closure over the blown up variety $(\mathbb{P}Jet_{k-1} \to (x, b_{k-1}))$. So, the projection is possibly not an isomorphism only over the *strict transform* of this scheme. The strict transform of the equations $C_k \sim (b_{k-1} b_{k-1})(\underbrace{x...x}_{k-2})$ (from the space of triples $\{x, b_{k-1}, C_k\}$ to the space of tuples $\{x, l, b_2, ..b_{k-1}, C_k\}$) is obtained by opening the brackets and substituting: $b_{i+1}(x) \sim b_i$. Therefore we get: $C_k \in Span(b_{k-1} l, b_{k-2} b_2, .., )$.

To understand the scheme $(b_{k-1} b_{k-1})(\underbrace{x...x}_{k-2}) = 0$ contract the left hand side with $x$ to get

$$0 = (b_{k-1} b_{k-1})(\underbrace{x...x}_{2k-6}) \in Span(b_3 l, b_2 b_2) \tag{28}$$

Therefore the scheme lies inside the locus $(b_3b_1) \sim (b_2b_2)$. This equation can be resolved using an auxiliary 1-form $\tilde{l}$ as: $b_2 \sim l\tilde{l}$ and $b_3 \sim l\tilde{l}^2$. Continue this, contracting $(k-4-i)_{i\geq 1}$ times with $x$. We get: $b_i \sim l\tilde{l}^{i-1}$. Finally, substitute this into the initial expression $0 = (b_{k-1}b_{k-1})(\underbrace{x...x}_{k-2}) = (l^2\tilde{l}^{2k-4})(\underbrace{x...x}_{k-2})$, to get: $\tilde{l}(x) = 0$.

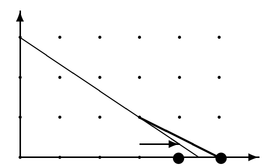
Therefore we have obtained:

- the "problematic" locus is $b_i \sim l\tilde{l}^{i-1}$, $\tilde{l}(x) = 0$.
- the fibers over it satisfy: $C_k \in Span(b_{k-1}l, b_{k-2}b_2, .., ) = l^2\tilde{l}^{k-2}$, i.e. for a fixed point $(x, l, b_2...b_{k-1}$ there is only one point in the fiber. ■

So, the lifted stratum is a projective bundle over a smooth variety. Note that the bundle is the restriction to *Aux* of the simple bundle: $\{(x, f, C_k)|\ f|_x^{(k)} \sim C_k\} \to \mathbb{P}Jet_{k-1}$. Therefore, the total Chern class of the bundle $\hat{\hat{\Sigma}}$ is the pullback of that of the simple bundle.

# 4 Boundary components of the full compactification

We prove that the boundary components are hypersurfaces (i.e. of pure codimension 1), as is stated in proposition 1.2. Note that the codimension one components of a lifted stratum can correspond to topological types of higher codimension. So, this problem differs from that of the classification of codimension 1 types.

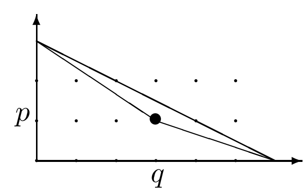


For definiteness, we fix the following convention: *every Newton diagram intersects the coordinate axes in the integral points.* This can always be achieved (by adding higher order terms) without changing the singularity type. In course of degeneration we move from a given type $(\mathbb{S}_1, \Sigma_{\mathbb{S}_1}, \mathbb{D}_{\mathbb{S}_1})$ to an adjacent $(\mathbb{S}_2, \Sigma_{\mathbb{S}_2}, \mathbb{D}_{\mathbb{S}_2})$. The adjacency $\Sigma_{\mathbb{S}_2} \subsetneq \overline{\boldsymbol{\Sigma}}_{\mathbb{S}_1}$ is called primitive if it cannot be further factorized as $\Sigma_{\mathbb{S}_2} \subsetneq \overline{\boldsymbol{\Sigma}}_{\mathbb{S}'} \subsetneq \overline{\boldsymbol{\Sigma}}_{\mathbb{S}_1}$

We consider typical examples of primitive adjacency. We also need the classes of the corresponding divisors in the intersection ring. As the topological type is given by the normal form and its Newton diagram, the degenerations of types are given in terms of the diagram or the normal form.

## 4.1 Examples of degenerations

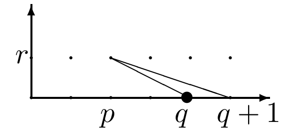
**Example 4.1**




Degeneration by removing a **vertex** (the intersection of two faces). The condition is: a monomial $x_1^p x_2^q$, $p \neq 0 \neq q$ should be absent (i.e. its coefficient must vanish). The class of this condition is given in §A.2 and the corresponding degeneration is:

$$[\overline{\overline{\boldsymbol{\Sigma}}}_1(x,l)]\Big(F + (d-p-2q)X + (q-p)L\Big) = [\overline{\overline{\boldsymbol{\Sigma}}}_2(x,l)] \qquad (29)$$

This gives the class of the divisor in $\mathbb{P}_x^2 \times (\mathbb{P}_l^2)^* \times \mathbb{P}_f^{N_d}$. The class of its pull-back to the stratum $\overline{\overline{\boldsymbol{\Sigma}}}_1(x,l)$ is obtained by pulling back the hyperplanes $X, L, F$, so it is: $i^*(F) + (d-p-2q)i^*(X) + (q-p)i^*(L)$.



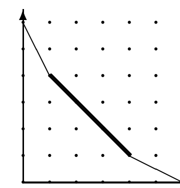

Another type of degeneration is to remove an **endpoint** (i.e. an intersection of the Newton diagram with a coordinate axis). Let $x_1^r x_2^p + .. + x_2^q$ be the quasi-homogeneous form corresponding to the face of the Newton diagram, intersecting the $x_2$ axis.

**Example 4.2** Note that if $r = 1$, then this form consists of two monomials only. And the endpoint $x_2^q$ can be removed by locally analytic transformation $x_1 \to x_1 + x_2^{q-p}$ (this does not change other parts of the Newton diagram).

Another case is the endpoint with $r > 1$ and $r$ divides $(q-p)$. In this case the endpoint can be removed by locally analytic shift: $x_1 \to x_1 + x_2^{\frac{q-p}{r}}$ (preserving the singularity type). So, erasing this point does not change the topological singularity type. When $r$ divides $(q-p)$ we call the endpoint **inessential**.

**Example 4.3** Consider the case of **essential endpoint** with $r > 1$ and $r$ does not divide $(q-p)$. Now, erasing the endpoint $x_2^q$ changes the topological type, we get an (primitively) adjacent type. The divisor class is given by the same formula as above: $i^*(F) + (d-2q)i^*(X) + qi^*(L)$.

**Example 4.4**
The last case is the degeneracy of a quasi-homogeneous form $x_1^* x_2^* (x_1^{ps} + .. + x_2^{qs})$, $(p,q) = 1$, corresponding to a face of the Newton diagram.In this case the resulting type is typically Newton-degenerate. The simplest degeneracy type is coincidence of the two roots, this realizes the primitive adjacency. The cohomology class of such degeneration is calculated in §A.2:



$$2(s-1)\Big(i^*(F) + i^*(X)(d - s(q+\frac{p}{2})) + i^*(L)\frac{s}{2}(q-p)\Big) \tag{30}$$

## 4.2 General description

In all the cases above the primitive adjacency is in codimension one i.e. $\Sigma' \subset \overline{\boldsymbol{\Sigma}}$ is a hypersurface. To prove the proposition 1.2 we prove that those are all the possible cases.

**Proposition 4.5** *Let $\Sigma$ be a linear stratum and the adjacency $\Sigma' \subset \overline{\boldsymbol{\Sigma}}$ be primitive. Then one of the following cases happens:*
• *The type of $\Sigma'$ is Newton-non-degenerate. Then $ND(\Sigma')$ is obtained from $ND(\Sigma)$ by either erasing a vertex (the intersection of two faces) or erasing an essential endpoint.*
• *The type of $\Sigma'$ is Newton-degenerate. Then $ND(\Sigma') = ND(\Sigma)$, while among the quasi-homogeneous forms corresponding to the faces of $ND(\Sigma)$, only one is degenerate. The degenerate form has one double root, all the other roots are simple.*

**proof:** Note that removing only inner points of a face does not change the diagram, and the type remains Newton-non-degenerate . So this does not change the topological type.

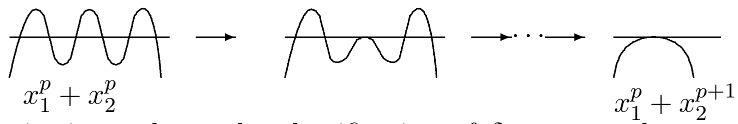
If at least one vertex is erased, then the topological type is changed. Thus any further degeneration will result in a non-primitive adjacency $\Sigma' \subset \overline{\boldsymbol{\Sigma}}$. The same is true for an essential endpoint.

Suppose no vertex and no essential endpoint are erased.
• If the type of $\Sigma'$ is Newton-non-degenerate then the only possibility is that an inessential endpoint of $\Sigma$ is removed. Since this does not change the type an additional point should be removed. It is immediately clear that this point must be the neighbor of the inessential endpoint. Then the topological type is changed and any further degeneration will result in non-primitive adjacency. This is the scenario of example 4.2.
• If the type of $\Sigma'$ is Newton-degenerate then at least one of the quasi-homogeneous forms corresponding to the faces has a multiple root. The double root already changes the topological type, thus any further degeneracy will result in a non-primitive adjacency. So here the example 4.4 is realized. ■

## 4.3 Equi-generic compactification

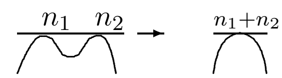
We discuss the compactification $\Sigma_{\mathbb{S}} \subset \bar{\Sigma}^g_{\mathbb{S}}$ consisting of curves of a fixed genus (cf. §1.1.2.2). As this is the $\delta = const$ stratum, the properties are readily obtained [GLSbook, section II.2.5]. For a given curve $C$ consider its normalization $\tilde{C} \to C$, obtained by repeated blowing up of the plane (the resolution is assumed to be minimal and good). Let $\{E_i\}$ be the corresponding tree of exceptional divisors. A $\delta$ constant degeneration lifts to a degeneration of the normalization $\tilde{C}$ such that:
• $\tilde{C}$ remains smooth
• all the intersection multiplicities $< \tilde{C}E_i >$ are preserved

**Example 4.6** For the ordinary multiple point the possible degenerations of the normalization $\tilde{C}$ relatively to the (unique) exceptional divisor $E$ are just the collisions of the intersection points $\tilde{C}E_i$.

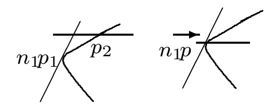



This criterion reduces the classification of $\delta = const$ degenerations to a simple chasing over the tree of exceptional divisors $\{E_i\}$. Consider first the two "primitive" degenerations.

**Example 4.7**




• $\tilde{C} \cap E_i \supset n_1p_1 + n_2p_2$ Here the normalization $\tilde{C}$ is tangent to $E_i$ at points $p_i$ with degree of tangency $p_i$. Merge the points $p_1 {\to} p_2$, with the result: $\tilde{C} \cap E_i \supset (n_1 + n_2)p$




• $\tilde{C} \cap E_1 \supset n_1p_1$, $\tilde{C} \cap E_2 \supset p_2$, $E_2 \cap E_1 \neq \emptyset$. Here only one multiplicity can be bigger than 1, otherwise the $\delta = const$ degeneration is impossible. Merge the point $p_i {\to} p = E_1 \cap E_2$, with the result: $\tilde{C} \cap E_i \supset (n_1 + n_2)p$

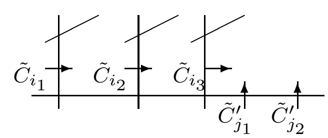
These two examples appear to be the building blocks for the general case.

**Proposition 4.8** *Every $\delta = const$ degeneration can be split into a chain of primitive degenerations, each corresponding to merging of just two intersection point of $\tilde{C} \cap (\bigcup E_i)$. Every primitive degeneration is of one of the types from example 4.7.*

**proof:** As the whole degeneration is just merging the points (preserving smoothness of $\tilde{C}$) it can be done pairwise. As the intersection $\tilde{C} \cap E_i$ should be preserved, both intersection points should remain on their components in course of degeneration. Thus either $p_1, p_2 \in E$ or $p_i \in E_i$ and $p_ip_j {\to} p = E_1 \cap E_2$. As the curve $\tilde{C}$ should remain smooth, in the first case the collision of points results in a bigger tangency. In the second case this argument causes that the curve is transversal to at least one exceptional divisor. Then the conclusion is immediate. ■

So, given a type $\mathbb{S}$ one applies (repeatedly) all the possible $\delta = const$ degenerations, to form the oriented graph: the $\delta-$*orbit*. The formula $\delta = \frac{\mu + r - 1}{2}$ immediately gives

**Corollary 4.9** Each primitive $\delta = const$ degeneration causes: $r {\to} r - 1$, $\mu {\to} \mu + 1$. In particular a unibranched type is necessarily an end of the $\delta$-orbit. An ordinary multiple point is necessarily the beginning.




Note that the $\delta$-orbit is not necessarily a tree and can have several ends. For example, for the configuration on the picture any of the branches $\tilde{C}'_j$ can be merged to glue with any of $\tilde{C}_i$ and for two different choices the configurations cannot be degenerated to a further common end.

# A Some cycles and their cohomology classes

## A.1 The diagonal

We often use the formula for the cohomology class of the diagonal $\Delta = \{x = y\} \subset \mathbb{P}^N_x \times \mathbb{P}^N_y$:

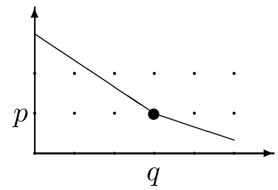
$$[\Delta] = \sum_{i=0}^{N} X^{N-i} Y^i \in H^{2N}(\mathbb{P}^N_x \times \mathbb{P}^N_y) \tag{31}$$

For example, a condition of proportionality of two symmetric forms $f^{(p)} \sim g^{(p)}$ is just the coincidence of the corresponding points in projective space.

### A.2 Some degenerations

For a given (closed) stratum we need the classes of divisors corresponding to curves with some properties (typically they have higher singularity types). We represent these as $[\Sigma_{\mathbb{S}_1}](degeneration) = [\Sigma_{\mathbb{S}_2}]$.

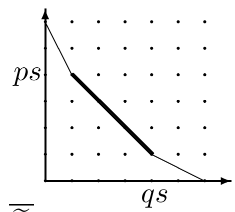
• Removing a monomial. The condition is: a monomial $x_1^p x_2^q$ should be absent in the normal form (i.e. its coefficient must vanish). The class of this divisor was calculated in [Ker06, section A.1.2] and the corresponding degeneration is:




$$[\overline{\overline{\mathbf{\Sigma}}}_{\mathbb{S}_1}(x,l)]\Big(F + (d-p-2q)X + (q-p)L\Big) = [\overline{\overline{\mathbf{\Sigma}}}_{\mathbb{S}_1}(x,l)] \tag{32}$$

This gives the class of the divisor in $\mathbb{P}^2_x \times (\mathbb{P}^2_l)^* \times \mathbb{P}^{N_d}_f$. The class of its pull-back to the stratum $\overline{\overline{\mathbf{\Sigma}}}_1(x,l)$ is obtained by pulling back the hyperplanes $X, L, F$, so it is: $i^*(F) + (d-p-2q)i^*(X) + (q-p)i^*(L)$.

• Another type of degeneration is when a quasi-homogenous form corresponding to a face of the Newton diagram becomes degenerate. The simplest type of degeneration is the double root. Let $x_1^{p_0} x_2^{q_0}(a_0 x_1^{ps} + .. + a_s x_2^{qs})$ be the quasi-homogeneous form, with $(p,q) = 1$. The class of the corresponding degenerating divisor is computed by recursion.




**Proposition A.1** $[\overline{\overline{\mathbf{\Sigma}}}_{\mathbb{S}_1}(x,l)] = 2(s-1)\Big(i^*(F) + i^*(X)(d - s(q+\frac{p}{2})) + i^*(L)\frac{s}{2}(q-p)\Big)[\overline{\overline{\mathbf{\Sigma}}}_{\mathbb{S}_2}(x,l)]$

**proof:** Let $D$ be the needed divisor. Apply the degeneration procedure (as in [Ker06]) to arrive at a stratum with the quasi-homogeneous form being maximally degenerated (i.e. $x_1^{ps}$). For this demand that its coefficients vanish (one-by-one).

Let $D_{(j,s-j)}$ denote the divisor in $\Sigma_{\mathbb{S}_2}$ along which the coefficient of $x_1^{p_0+pj} x_2^{q_0+p(s-j)}$ vanishes. The first degeneration is: $\Sigma\mathbb{S}_2 \cap D_{(0,s)} = 2\Big(\Sigma_{\mathbb{S}_1} \cap D_{(0,s)} \cap D_{(1,s-1)}\Big) \cup \Big(\Sigma_{\mathbb{S}_2, D_{(0,s)}}\Big)$. Note that here $\Sigma_{\mathbb{S}_1} \cap D_{(0,s)} \cap D_{(1,s-1)}$ is a Newton-non-degenerate stratum, so its class is known.

Applying the degeneration $(1, s-1)$ we get: $\Big((\Sigma_{\mathbb{S}_2} \cap D_{(0,s)}) - 2\Sigma_{\mathbb{S}_1} \cap D_{(0,s)} \cap D_{(1,s-1)}\Big) \cap D_{(1,s-1)} = 2\Big(\Sigma_{\mathbb{S}_1} \cap D_{(0,s)} \cap D_{(1,s-1)} \cap D_{(2,s-2)}\Big) \cup \Big(\Sigma_{\mathbb{S}_2, D_{(0,s)}, D_{(1,s-1)}}\Big)$ Continuing in this way we reach the stratum with maximally degenerate quasi-homogeneous form: $x_1^{ps}$. This stratum can be represented as $\Sigma_{\mathbb{S}_1} \bigcap_{i=1}^{s-1} D_{(i,s-i)}$. This provides the needed equation for the cohomology class.

$$[\Sigma_{\mathbb{S}_2}][\bigcap_{j=0}^{s-2} D_{(j,s-j)}] = 2\sum_{j=0}^{s-2}[\Sigma_{\mathbb{S}_1, D_{(0,s)} \dots D_{(j,s-j)}}][D_{(j,s-j)}]..D_{(s-1,1)} + [\Sigma_{\mathbb{S}_2, D_{(0,s)} \dots D_{(s-1,1)}}] \tag{33}$$

The class of $D_{(j,s-j)}$ is given by the equation (32). As all the degenerations are invertible we divide by their classes and get:

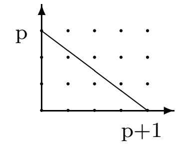
$$\begin{aligned} &[\Sigma_{\mathbb{S}_2}] = 2[\Sigma_{\mathbb{S}_1}]\textstyle\sum_{j=1}^{s-1}[D_{(j,s-j)}] = 2\sum_{i=1}^{s-1}\Big(F + (d - pi - 2q(s-i))X + (q(s-i) - pi)L\Big) = \\ &= 2(s-1)\Big(F + X(d - s(q+\tfrac{p}{2})) + L\tfrac{s}{2}(q-p)\Big) \end{aligned} \tag{34}$$

■

# B Examples of strata analysis

The simplest example: ordinary multiple point was treated in §1.2. Here we consider more examples, basically just repeating the analysis.

### B.1 The stratum of generalized cusps: $x_1^p + x_2^{p+1}$.




(For $p = 2$ it is $A_2$, for $p = 3$ it is $E_6$.) The lifted variety in this case is (cf. [Ker06]):

$$\widetilde{\overline{\Sigma}}(x,l) = \{(x,l,f)|\ f|_x^{(p)} \sim \underbrace{l \otimes .. \otimes l}_{p},\quad l(x)=0\} \subset \mathbb{P}^2_x \times (\mathbb{P}^2_l)^* \times \mathbb{P}^{N_d}_f \tag{35}$$

It is directly seen to be the projectivization of the corresponding vector bundle over the auxiliary space $(\mathbb{P}Jet_1)$. Its class $[\widetilde{\overline{\Sigma}}] \in H^*(\mathbb{P}Jet_1 \times \mathbb{P}^{N_d}_f)$ is $(l+x)\sum Q^{\binom{p+2}{2}-1-i}(pL)^i$, with $Q = (d-p)X + F$. This fixes the total Segre class of the bundle. Thus the Chow ring is:

$$A^*(\widetilde{\overline{\Sigma}}(x,l)) = \frac{\mathbb{Z}[i^*(X), i^*(L), i^*(F)]}{i^*(X)^3, i^*(L)^3, i^*(X)^2 - i^*(L)i^*(X) + i^*(L)^2, F^{r-3}(F+(d-p)X-pL)\begin{pmatrix} F^2 - \binom{p+2}{2}(d-p)FX + \\ \left(\binom{\binom{p+2}{2}}{2} - \binom{p+2}{2}^2\right)(d-p)^2X^2 \end{pmatrix}} \tag{36}$$

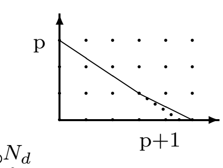
here $r = N_d + 1 - \binom{p+2}{2}$. The boundary of $\widetilde{\Sigma} \subset \widetilde{\overline{\Sigma}}$ consists of 3 irreducible divisors:
- Ordinary multiple point of higher multiplicity $x_1^{p+1} + x_2^{p+1}$. Its class if $F + (d-p)X - pL$
- The type $x_1^p + x_1x_2^p + x_2^{p+2}$. Its class is $F + (d-p-1)X + (p+1)(L-X)$.
- The curve acquires an additional node: $\Sigma_{x_1^p + x_2^{p+1}, A_1}$. The divisor class is $3(d-p-1)\Big((d+p-1)F - p^2X\Big)$.

From here we immediately get:

**Proposition B.1** *The Picard group of the proper stratum is* $\frac{Span_{\mathbb{Z}}(X,F)}{\Big(3(d-p-1)((d+p-1)F-p^2X)\Big),\Big((2p+1)F+(d(2p+1)-3p(p+1))X\Big)}$

We see that the stratum is affine for $d \geq \frac{3p(p+1)}{2p+1}$

### B.2 The stratum of points of type: $x_1^p + x_1x_2^p + x_2^{p+2}$




(For $p = 2$ it is $A_3$, for $p = 3$ it is $E_7$.) The lifted variety in this case is:

$$\widetilde{\overline{\Sigma}}(x,l,b_p) = \{(x,l,b_p,f)|\ f|_x^{(p+1)} \sim SYM(l,b_p),\ \ b_p(x) \sim (\underbrace{l..l}_{p-1}),\ \ l(x)=0\} \subset \mathbb{P}^2_x \times (\mathbb{P}^2_l)^* \times \mathbb{P}_{b_p} \times \mathbb{P}^{N_d}_f \tag{37}$$

(Note that here one need not consider the whole parameter space of jets $\mathbb{P}Jet_p$.) From this representation the cohomology class of $\widetilde{\overline{\Sigma}}(x,l,b_p)$ is obtained immediately, and from it one gets (by projection) the class $[\widetilde{\overline{\Sigma}}(x,l)]$.

Again, the stratum $\widetilde{\overline{\Sigma}}(x,l,b_p)$ is the projectivization of the corresponding vector bundle over the auxiliary space $(\mathbb{P}Jet_1)$. Thus the Chow ring is:

$$A^*(\widetilde{\overline{\Sigma}}(x,l)) = \frac{\mathbb{Z}[i^*(X), i^*(L), i^*(F)]}{i^*(X)^3, i^*(L)^3, i^*(X)^2 - i^*(L)i^*(X) + i^*(L)^2, \frac{1}{[\widetilde{\overline{\Sigma}}(x,l)]}} \tag{38}$$

here in the denominator by $\frac{1}{[\widetilde{\overline{\Sigma}}(x,l)]}$ we mean the polynomial expansion obtained after substitution $F \to 1$.

The boundary of $\widetilde{\Sigma} \subset \widetilde{\overline{\Sigma}}$ consists of 4 irreducible divisors:
- The type obtained by degenerating $..x_1x_2^p \to 0$. The normal form: $x_1^p + x_1x_2^p + x_2^{p+2}$. Its class is $\Big(F + (d-2p-1)X + (p-1)L\Big)$.
- The type $x_1^{p+1} + x_1x_2^p + x_2^{p+2}$, obtained by increasing multiplicity $...x_1^p \to 0$. Its class is $\Big(F + (d-p)X - pL\Big)$:

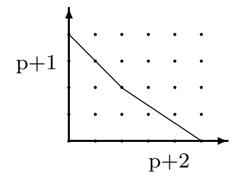
• The curve acquires an additional node: $\Sigma_{x_1^p+x_1x_2^p+x_2^{p+2},A_1}$. The divisor class is $\Big(F(3d^2-6d-1-3p^2)+(4+4p+3p^2-(d-p)(4+3p^2))X-4(p-1)L\Big)$.

From here we immediately get:

**Proposition B.2** *The Picard group of the proper stratum is*

$$\frac{Span_{\mathbb{Z}}(X,F)}{\Big(3(d-p-1)(d+p-1)F-p(4-3p+3dp)X\Big),\Big((1-2p)F+((d-p)(1-2p)+p(p+1))X\Big)}$$

## B.3 The stratum of points of type: $x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+2}$




(For $p=1$ it is $A_3$, for $p=2$ it is $D_5$.) The lifted variety in this case is:

$$\overline{\overline{\Sigma}}(x,l,b_p)=\{(x,l,f)|\ f|_x^{(p+1)}\sim SYM(l,l,b_p),\ \ b_p(x)=0,\ \ l(x)=0\}\subset\mathbb{P}^2_x\times(\mathbb{P}^2_l)^*\times\mathbb{P}_{b_p}\times\mathbb{P}_f^{N_d} \quad (39)$$

(Here $b_p$ is just an auxiliary p-form.) From this representation the cohomology class of $\overline{\overline{\Sigma}}(x,l,b_p)$ is obtained immediately, and from it one gets (by projection) the class $[\overline{\overline{\Sigma}}(x,l)]$.

Again, the stratum $\overline{\overline{\Sigma}}(x,l,b_p)$ is the projectivization of the corresponding vector bundle over the auxiliary space ($\mathbb{P}Jet_1$), its Chow ring is written as before.

The boundary of $\widetilde{\Sigma}\subset\overline{\overline{\Sigma}}$ consists of 4 irreducible divisors:

• The type obtained by degenerating $..x_1^2x_2^{p-1}\to 0$. The normal form: $x_1^{p+1}+x_1^3x_2^{p-2}+x_2^{p+2}$. Its class is $\Big(F+(d-2p)X+(p-3)L\Big)$.

• The type obtained by degenerating $..x_2^{p+2}\to 0$. The normal form: $x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+3}$. Its class is $\Big(F+(d-2(p+2))X+(p+2)L\Big)$.

• The type obtained by degenerating the homogeneous part $x_1^{p+1}+..+x_1^2x_2^{p-1}$. The resulting singularity is Newton degenerate. Its class is $(p-2)\Big(2F+3(d-(p-1))X\Big)$.

• The curve acquires an additional node: $\Sigma_{x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+2},A_1}$. The divisor class is $\Big(F(3d^2-6d-1-4p-3p^2)+(8+8p+5p^2-(d-p)(2+4p+3p^2))X-2(p-2)L\Big)$.

From here we immediately get:

**Proposition B.3** *The Picard group of the proper stratum is obtained by factorization of $Span_{\mathbb{Z}}(X,L,F)$ by the above classes.*

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