CONFORMAL GEOMETRY OF THE IRRATIONAL ROTATION ALGEBRA

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Abstract: In this paper we shall show how the non-unimodularity of the volume element of a non-commutative torus, endowed with its canonical conformal structure, affects the formula for the value at the origin, $\zeta(0)$, of the zeta function on this torus. Using in a critical manner the adaption of the pseudo-differential calculus to this simplest example of a non-commutative Riemannian manifold, we give a general formula for $\zeta(0)$. Instead of vanishing, this computed value involves modified logarithms of the modular operator. In order to concentrate on the computational aspect, we bypass in the present version of this paper the important task of casting our discussion of conformal structure within the framework of the theory of positive cyclic cohomology.

§1 Preliminaries

Recall that for a classical Riemann surface Σ with metric g, to the Laplacian $\square_g = d^*d$, where d is the de-Rham differential operator acting on the Riemann surface, one associates the zeta function

$$\zeta(s) = \sum \lambda_j^{-s}\ ,\ \mathrm{Re}(s) > 1,$$

where the summation is over the non-zero eigenvalues λ_j of $\square_g$. The meromorphic continuation of $\zeta(s)$ to $s=0$, where it has no pole, gives the important information

$$\zeta(0) = (1/24\pi)\int_\Sigma S\ -\ \mathrm{Card}\{j \mid \lambda_j = 0\} = (i/12)\, c_1(\Sigma), \qquad (i = \sqrt{(-1)}),$$

where S is the scalar curvature and

$$c_1(\Sigma) = (1/2\pi i)\int_\Sigma \partial\bar\partial \log(g)$$

the first Chern number (= Euler-Poincaré characteristic); this vanishes when Σ is the classical 2-torus $\mathbb{R}^2/\mathbb{Z}^2$, for example, and is an invariant within the conformal class of the metric, that is under the transformation $g \to e^f g$ for f a smooth real valued function on Σ. Moreover, recall that one may define the determinant of the Laplacian $\square_g$ by

$$\log\det \square_g = -\zeta'(0).$$

Now fix a real irrational number θ. The dynamical system given by an irrational rotation of the circle S^1 is embodied in the C^*-algebra A_θ. That is, one has both an action of the group $\mathbb{Z}$ and of the algebra $C(S^1)$ of continuous functions on S^1 given by

$$[1]\, f(s) = f(s-\theta) \quad \text{(irrational rotation)}$$

$$[e^{2\pi i s}]\, f(s) = e^{2\pi i s} f(s) \quad \text{(regular representation)}$$

for all f in $C(S^1)$. These operations do not commute; instead, one has

$$[e^{2\pi i s}][1] = e^{2\pi i \theta}[1][e^{2\pi i s}].$$

We may represent these actions on a Hilbert space by passing to $\mathcal{H}=L^2(\mathbb{R})$, the completion of the algebra of compactly supported functions on $\mathbb{R}$ with respect to the inner product

$$\langle f,g\rangle = \int_{\mathbb{R}} f(s)\overline{g(s)}\, ds,$$

which contains the subspace $S(\mathbb{R})$ of functions of rapid decay. One defines the following operators on elements $\xi=\xi(s)$ of $S(\mathbb{R})$:

$$(U\xi)(s) = \xi(s-\theta)$$

$$(V\xi)(s) = e^{2\pi i s}\xi(s)$$

These operators satisfy

$$VU = e^{2\pi i\theta}UV,\ U^* = U^{-1},\ V^* = V^{-1}.$$

The norm closure of this algebra is the C^*-algebra A_θ. Notice that if we replace θ by zero, U by $e^{2\pi i x}$ and V by $e^{2\pi i y}$ in the above definitions we recover the algebra of continuous functions on $\mathbb{R}^2/\mathbb{Z}^2$. If one wishes to express the elements of A_θ as certain series

$$\sum a(n,m)U^n V^m,\ a(n,m)\in\mathbb{C},$$

where the summation is over the elements (n,m) of $\mathbb{Z}^2$, one finds by direct computation using the commutation relations that the C^*- algebra norm $\|\ \|$ of the above series is given by the number

$$\sup\{\sum|b(p,q)|^2 \mid b(p,q)=\sum a(n,m)e^{2\pi i n r\theta}\xi(r,s), \sum|\xi(r,s)|^2=1\}.$$

Clearly the C^* or continuity condition is hard to control in terms of the coefficients. On the other hand, the smoothness condition is easy to control in this way. To see this, we introduce another dynamical system given by the action of $T^2 = \{z\in C \mid |z|=1\}^2$ on A_θ by the 1-parameter groups of automorphisms $\{\alpha_s\}$, $\{\beta_t\}$ determined by

$$\alpha_s(U)= \exp(2\pi i s)U,\ \alpha_s(V)=V,\ (s\in\mathbb{R}),$$

$$\beta_t(U)=U,\ \beta_t(V)= \exp(2\pi i t)V,\quad (t\in\mathbb{R}).$$

We define the sub-algebra A_θ^∞ of smooth elements of A_θ to be those x in A_θ such that the mapping

$$\mathbb{R}^2 \to A_\theta$$
$$(s,t) \mapsto \alpha_s \beta_t(x)$$

between Banach spaces is smooth. Expressed as a condition on the coefficients, this imposes that they be of rapid decay, namely that $\{|n|^k |m|^q |a(n,m)|\}$ be bounded for any positive k, q. The derivations associated to the above groups of automorphisms are given by their action on an element a of A_θ as follows

$$\delta_1(a)=\lim_{\varepsilon\to 0}(\alpha_\varepsilon(a) - a)/\varepsilon,$$
$$\delta_2(a)=\lim_{\varepsilon\to 0}(\beta_\varepsilon(a) - a)/\varepsilon,$$

so that one has the defining relations,

$$\delta_1(U) = 2\pi i U,\ \delta_1(V) = 0,$$
$$\delta_2(U) = 0,\ \delta_2(V) = 2\pi i V.$$

The derivations δ_1, δ_2 are analogues of the differential operators $\partial/\partial x$, $\partial/\partial y$ on the smooth functions on $\mathbb{R}^2/\mathbb{Z}^2$. One also has inner derivations arising from commutators which are trivial in the commutative case. These are the derivations associated to the 1-parameter families $\{\sigma_t\}$, $t\in\mathbb{R}$, of inner automorphisms

$$\sigma_t(x) = e^{-ift}\, x\, e^{ift}$$

for $f = f^*$ a non-constant self-adjoint element of A_θ^∞. The derivation corresponding to this last group is given by $i \log \Delta$ where

$$\Delta(x) = e^{-f}\, x\, e^{f}$$

and

$$(\log \Delta)(x) = [x,f],\ x\in A_\theta^\infty.$$

As θ is supposed irrational, there is a unique trace τ_0 on A_θ determined by the orthogonality properties

$$\tau_0(U^n V^m)=0 \text{ if } (n,m)\neq(0,0), \text{ and } \tau_0(1)=1.$$

We can construct a Hilbert space $\mathcal{H}_0$ from A_θ by completing with respect to the inner product

$$<a,b>=\tau_0(b^* a),\ a,b\in A_\theta,$$

and using the derivations δ_1, δ_2, introduce a complex structure by defining

$$\partial= \delta_1 + i\delta_2,\ \partial^*= -\delta_1 + i\delta_2$$

where (extending ∂, ∂^* to unbounded operators on $\mathcal{H}_0$) ∂^* is the adjoint of ∂ with respect to the inner product defined by τ_0. As an appropriate analogue of the space of

(1,0)-forms on the classical 2-torus, we propose that one takes the unitary bi-module over A_θ^∞ given by the closure of the space of finite sums $\sum a\,\partial b$, $a,b\in A_\theta^\infty$, with respect to the inner product (or metric in this context) given as above by

$$<a\partial b, a'\partial b'>=\tau_0((a')^* a(\partial b)(\partial b')^*),\ \ a,a',b,b'\in A_\theta^\infty.$$

Then, in order to introduce the conformal class of a metric we consider the family of states $\varphi=\varphi_f$, $f=f^*\in A_\theta^\infty$, defined on A_θ by

$$\varphi(a)=\tau_0(ae^f),\ \ a\in A_\theta .$$

Note that, whereas for τ_0 we have the trace relation

$$\tau_0(b^*a)=\tau_0(ab^*),\ \ a,b\in A_\theta,$$

for φ we have

$$\varphi(ab^*)=\varphi(b^*e^f a e^{-f})=\varphi(b^*\sigma\sqrt{(-1)}(a)),\ \ a\in A_\theta.$$

We define the inner product (,) on A_θ by

$$(a,b)=\varphi(b^*a),\ \ a,b\in A_\theta.$$

Viewed as a metric within the same conformal class as < , > determining a closure $\mathcal{H}_f$ of the space of finite sums $\sum a\partial b$, $a,b\in A_\theta^\infty$, we see that

$$<(\partial b)k,(\partial c)k>=(\partial b,\partial c),\qquad b,c\in A_\theta^\infty,$$

where $k=e^{f/2}$, so that the appropriate correction to the operator ∂ is $k\partial$, where here right multiplication by k is understood. With these remarks in mind, for an invertible non-constant self-adjoint element k of A_θ^∞, we introduce the correction

$$D=(k\,\partial)(k\,\partial)^*=k\square k,$$

to the Laplacian

$$\square=\partial\partial^*=-(\delta_1^2+\delta_2^2)\ .$$

It is the dependence on k in the computations of the behaviour of the zeta function of D at the origin that will feature in what follows.

§2 Statement of the theorem

With the notation of §1, we study the Laplacian zeta function defined for Re(s)>1 by the Mellin transform

$$\zeta(s)=(1/\Gamma(s))\int_0^\infty \mathrm{Trace}^+(e^{-tD})t^{s-1}dt=\mathrm{Trace}(D^{-s}),$$

where $D=TT^*$, $T=k\partial$, and

$$\mathrm{Trace}^+(e^{-tD})=\mathrm{Trace}(e^{-tD})-\mathrm{Dim Ker}(T),$$

where by Trace(.) we understand the ordinary trace of the operator. The definition of $\zeta(s)$ can be extended by meromorphic continuation to all values of s, barring $s=1$ where the function has a simple pole. We now state the main result.

Theorem: Let θ be an irrational number and k a self-adjoint non-zero element of A_θ^∞. Then the value at the origin of the zeta function $\zeta(s)$ of the operator $D= k\Box k$ is given by

$$\zeta(0)=\tau_0(h(\theta,k))$$

where

$$h(\theta,k) = (\pi/3)k^{-1}\delta_i\delta_i(k) - (2\pi/3)k^{-1}\delta_i(k)\delta_i(k)k^{-1} + (2\pi)\mathfrak{D}_1\,(k^{-1}\delta_i(k))\delta_i(k)k^{-1}$$
$$-(4\pi)\mathfrak{D}_2(1+\Delta^{1/2})(k^{-1}\delta_i(k))\delta_i(k)k^{-1} + (2\pi)\mathfrak{D}_3(1+2\Delta^{1/2}+\Delta)(k^{-1}\delta_i(k))\delta_i(k)k^{-1}.$$

Here, summation of repeated indices over $j=1,2$ is understood and $\mathfrak{D}_m$, m a positive integer, stands for the modified logarithm

$$\mathfrak{D}_m = (-1)^m(\Delta-1)^{-(m+1)}\{\log\Delta - \textstyle\sum_{j=1}^{m}((-1)^{(j+1)}/j)(\Delta-1)^j\}. \quad \square\square\square$$

As pointed out in §1, in the commutative case the corresponding value for the zeta function at the origin is zero.

§3 Pseudo-differential calculus

With the notation of the preceding sections, we introduce in the present one the notion of a pseudo-differential operator given the triple $(A_\theta^\infty,\delta_1,\delta_2)$. See also [B]. First of all, for a non-negative integer h, we define the vector space of differential operators of order at most h to be those polynomial expressions in δ_1,δ_2 of the form

$$P(\delta_1,\delta_2)=\textstyle\sum_{|j|\leq h}a_j\,\delta_1^{j_1}\,\delta_2^{j_2}\ ,\ a_j\in A_\theta^\infty,\ j=(j_1,j_2)\in\mathbb{Z}^2_{\geq 0},\ |j|=j_1+j_2.$$

To extend this definition, let $\mathbb{R}_2$ be the group dual to $\mathbb{R}^2$ and introduce the class of operator valued distributions given by those complex linear functions $P: C^\infty(\mathbb{R}_2)\rightarrow A_\theta^\infty$ which are continuous with respect to the semi-norms p_{i_1,i_2} determined by

$$p_{i_1,i_2}(P(\varphi)) = \|\delta_1^{i_1}\,\delta_2^{i_2}(P(\varphi))\|,\ i_1,i_2\in\mathbb{Z}_{\geq 0},\ \varphi\in C^\infty(\mathbb{R}_2).$$

We use the notation $y_1= e^{2\pi i\xi_1}, y_2= e^{2\pi i\xi_2}, \xi=(\xi_1,\xi_2)\in\mathbb{R}^2$, for the canonical coordinates of $\mathbb{R}_2$, and $\partial_1=\partial/\partial\xi_1$, $\partial_2=\partial/\partial\xi_2$ for the corresponding derivations. An example of an operator valued distribution is provided by the δ-function $\delta=\delta(\xi)$ which has the formal Fourier series

$$\delta(\xi)=\sum_{n,m\in\mathbb{Z}} e^{2\pi i n\xi_1+2\pi i m\xi_2}$$

representing and determined by its value on Fourier expansions of elements of $C^\infty(\mathbb{R}_2)$. We may now introduce the algebra of pseudo-differential operators via the algebra of operator valued symbols.

Definition: An element $\rho=\rho(\xi)=\rho(\xi_1,\xi_2)$ of $C^\infty(\mathbb{R}_2, A_\theta^\infty)$ is a symbol of order the integer h if and only if for all non-negative integers i_1,i_2,j_1,j_2

$$p_{i_1,i_2}(\partial_1^{j_1}\partial_2^{j_2}\rho(\xi))\leqslant c(1+|\xi|)^{h-|j|},$$

where c is a constant depending only on ρ, and if there exists an element $k=k(\xi_1,\xi_2)$ of $C^\infty(\mathbb{R}_2-\{1,1\}, A_\theta^\infty)$ such that

$$\lim_{\lambda\to\infty}\lambda^{-h}\rho(\lambda\xi_1,\lambda\xi_2)=k(\xi_1,\xi_2). \quad ■■■$$

We denote the space of symbols of order h by S_h, the union $S=\bigcup_{h\in\mathbb{Z}}S_h$ forming an algebra. Symbols of non-integral order are not required for this paper. For integers n,m set $\delta_{n,m}=\delta_{n,m}(\xi)=\delta(\xi_1-2\pi n,\xi_2-2\pi m)$. An example of a symbol of order h a positive integer is provided by the polynomial $\rho(\xi)=\sum_{|j|\leqslant h}a_j(i)^{|j|}\xi_1^{j_1}\xi_2^{j_2}$, $a_j\in A_\theta^\infty$, and one has $\delta_{n,m}(\rho)=\sum_{|j|\leqslant h}a_j(2\pi i)^{|j|}n^{j_1}m^{j_2}$ so that $\delta_{n,m}(\rho)U^nV^m=\sum_{|j|\leqslant h}a_j\delta_1^{j_1}\delta_2^{j_2}(U^nV^m)$. For an element $a=\sum_{n,m}a(n,m)U^nV^m$ of A_θ^∞ one therefore has $\sum_{n,m}\delta_{n,m}(\rho)a(n,m)U^nV^m=\sum_{|j|\leqslant h}a_j\delta_1^{j_1}\delta_2^{j_2}(a)$, associating to the symbol ρ the differential operator $P_\rho=P(\delta_1,\delta_2)=\sum_{|j|\leqslant h}a_j\,\delta_1^{j_1}\delta_2^{j_2}$ on A_θ^∞. Indeed, for every integer h, a symbol ρ of that order determines an operator on A_θ^∞ via the map ψ: $\rho\mapsto P_\rho$ given by the formula

$$P_\rho(a)=\sum_{n,m\in\mathbb{Z}}\delta_{n,m}(\rho)a(n,m)U^nV^m,\quad a=\sum_{n,m}a(n,m)U^nV^m.$$

For example, the image under ψ of the symbol $(1+|\xi|^2)^{-k}, k\geqslant 1$, of order $-2k$ acts on A_θ^∞.

Definition: The space ψ of pseudo-differential operators is given by the image of the algebra S under the map ψ. ■■■

Definition: The equivalence $\rho\sim\rho'$ between two symbols ρ,ρ' in S_k, $k\in\mathbb{Z}$, holds if and only if $\rho-\rho'$ is a symbol of order h for all integers h. ■■■

Definition: The class of pseudo-differential operators is the space ψ modulo addition by an element of $\psi(Z)$ where Z is the sub-algebra of S with elements equivalent to the zero symbol. ◻◻◻

It is possible to invert the map ψ to obtain for each element P of ψ a unique symbol $\sigma(P)$ up to equivalence. Recall from §1 that the trace τ_0 on A_θ^∞ enables one to define the adjoint of operators acting on A_θ^∞ via their extension to $\mathcal{H}_0$. By direct analogy with [G], Chapter 1, Theorem, p16, one may deduce the following result.

Proposition. For an element P of ψ with symbol $\sigma(P)=\rho=\rho(\xi)$, the symbol of the adjoint P^* satisfies

$$\sigma(P^*)\sim\sum_{(l_1,l_2)\in(\mathbb{Z}_{\geq 0})^2}(1/(l_1)!(l_2)!)[\,\partial_1^{l_1}\partial_2^{l_2}\delta_1^{l_1}\delta_2^{l_2}(\rho(\xi))^*].$$

If Q is an element of ψ with symbol $\sigma(Q)=\rho'=\rho'(\xi)$, then the product PQ is also in ψ and has symbol

$$\sigma(PQ)\sim\sum_{(l_1,l_2)\in(\mathbb{Z}_{\geq 0})^2}(1/(l_1)!l_2)!)[\,\partial_1^{l_1}\partial_2^{l_2}(\rho(\xi))\delta_1^{l_1}\delta_2^{l_2}(\rho'(\xi))].$$ ◻◻◻

Notice that in the above Proposition as throughout the present paper, given symbols $\{\rho_j\}_{j=0}^{\infty}$ the relation $\rho\sim\sum_{j=0}^{\infty}\rho_j$ signifies that there exists a positive integer H such that for all $h>H$, the difference $\rho-\sum_{j=0}^{h}\rho_j$ is in S_k, for all integers k. The elliptic pseudo-differential operators are those whose symbols fulfil the critereon which follows.

Definition. Let h be an integer and ρ a symbol of order h. Then $\rho=\rho(\xi)$ is elliptic if it is invertible within the algebra $C^\infty(\mathbb{R}_2,A_\theta^\infty)$ and if its inverse satisfies

$$\|\rho(\xi)^{-1}\|\leqslant c(1+|\xi|)^{-h}$$

for a constant c depending only on ρ and for $|\xi|=(\xi_1^2+\xi_2^2)^{1/2}$ sufficiently large. ◻◻◻

An example of an elliptic operator is provided by the Laplacian $\square=-(\delta_1^2+\delta_2^2)$ on A_θ^∞ introduced in §1 which has the corresponding invertible symbol $\sigma(\square)=|\xi|^2$.

§4 Understanding the computations

The arguments of this section are kept brief, being direct analogues of standard ones. Bearing in mind the definition of the zeta function given in §2, we observe that by Cauchy's formula we have

$$e^{-tD}=(1/2\pi i)\int_C e^{-t\lambda}(\lambda I-D)^{-1}d\lambda$$

where λ is a complex number but not real non-negative, and C encircles the non-negative real axis in the anti-clockwise direction without touching it. One then obtains a workable estimate of $(\lambda I-D)^{-1}$ by passing to the algebra of symbols. Using the definition of a symbol, one can replace the trace in the formula for the zeta function by an integration in the symbol space (argument along the diagonal), namely,

$$\zeta(s)=(1/\Gamma(s))\int_0^\infty\int\tau_0(\sigma(e^{-tD})(\xi))t^{s-1}d\xi dt.$$

The function $\Gamma(s)$ has a simple pole at $s=0$ with residue 1 so that,

$$\zeta(0)=\mathrm{Res}_{s=0}\int_0^\infty\int\tau_0(\sigma(e^{-tD})(\xi))t^{s-1}d\xi dt.$$

Just as in the arguments employed in the derivation of the asymptotic formula (see for example [G]),

$$\int\tau_0(\sigma(e^{-tD})(\xi))d\xi\sim t^{-1}\sum_{n=0}^{\infty} B_{2n}(D)t^n,\ t\to 0+,$$

one may appeal to the Cauchy formula quoted above. In particular, if B_λ denotes (a chosen approximation) to the inverse operator of $(\lambda I-D)$, its symbol has an expansion of the form

$$\sigma(B_\lambda)=\sigma(B_\lambda)(\xi)=b_0(\xi)+b_1(\xi)+b_2(\xi)+\dots$$

where j ranges over the non-negative integers and $b_j(\xi)=b_j(\xi,\lambda)$ is a symbol of order $-2-j$. As we shall explain at more length in §5, these symbols may be calculated inductively using the symbol algebra formulae beginning with $b_0(\xi)=(\lambda-k^2|\xi|^2)^{-1}$ which is the principal (highest homogeneous degree in ξ) symbol of $(\lambda I-D)^{-1}$. It turns out that $\zeta(0)$ equals the coefficient of λ^{-1} in $\int\tau_0(b_2(\xi))d\xi$. By a homogeneity argument one has in fact

$$\zeta(0)=\int\tau_0(b_2(\xi))d\xi \quad \_(1).$$

§5 Computational proof of the Theorem

Following on from the arguments of §4, by homogeneity there is no loss of generality in placing $\lambda=-1$ throughout the computation of $\zeta(0)$ and multiplying the final answer by -1. The problem is then to derive in the symbol algebra a recursive solution of the form $\sigma=b_0(\xi)+b_1(\xi)+b_2(\xi)+\dots$ to the equation

$$\sigma.(\sigma(D+I))=1+O(|\xi|^{-3}).$$

The accuracy to order -3 in ξ on the right hand side is in practice sufficient as we are only interested in evaluating σ up to $b_2(\xi)$. Throughout this section the convention of summation over repeated indices in the range $i,j=1,2$ is observed.

Lemma 1: The operator D has symbol $\sigma(D)=a_2(\xi)+a_1(\xi)+a_0(\xi)$ where, with summation over repeated indices in the range $i=1,2$, one has

$$a_2=a_2(\xi)=k^2\xi_i\xi_i$$

$$a_1=a_1(\xi)=2\xi_i(k\delta_i(k))$$

$$a_0=a_0(\xi)=k\delta_i\delta_i(k).$$

These expressions are derived by applying the product formula within the algebra of symbols given in Proposition §3 to $\sigma_1(\xi)=\xi_i\xi_i$ and $\sigma_2(\xi)=k$ and then multiplying on the left by k. ◻◻◻

To begin the inductive calculation of the inverse of the symbol of $D+I$, set

$$b_0=b_0(\xi)=(k^2|\xi|^2+1)^{-1} \qquad \_(2)$$

and compute to order -3 in ξ the product $b_0.((a_2+1)+a_1+a_0)$. By singling out terms of the appropriate degree -1 in ξ and using the Proposition of §3, one obtains

$$b_1=-(b_0a_1b_0+\partial_i(b_0)\delta_i(a_2)b_0) \qquad \_(3).$$

In a similar fashion, collecting terms of degree -2 in ξ and using (3) one obtains

$$b_2=-(b_0a_0b_0+b_1a_1b_0+\partial_i(b_0)\delta_i(a_1)b_0+\partial_i(b_1)\delta_i(a_2)b_0+(1/2)\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)b_0) \quad \_(4).$$

It is extremely useful during the computation to exploit the fact that in the target formula for $\zeta(0)$ given in (1), §5, one invokes the trace, so that members of the factors of the individual summands may be permuted cyclically without loss of generality for the answer. Moreover, using integration by parts with respect to ξ the expression $\partial_i(b_1)\delta_i(a_2)b_0$ may be replaced by $-b_1(\partial_i\delta_i(a_2)b_0+\delta_i(a_2)\partial_i(b_0))$. One may therefore without loss of generality replace the right hand side of (4) by the sum of the following list of nine symbols,

(i) $-b_0^2a_0$ (ii) $b_0^2a_1b_0a_1$ (iii) $b_0\partial_i(b_0)\delta_i(a_2)b_0a_1$ (iv) $-b_0\partial_i(b_0)\delta_i(a_1)$ (v) $-b_0^2a_1b_0\partial_i\delta_i(a_2)$ (vi) $-\partial_i(b_0)b_0a_1b_0\delta_i(a_2)$ (vii) $-b_0\partial_j(b_0)\delta_j(a_2)b_0\partial_i\delta_i(a_2)$ (viii) $-\partial_i(b_0)\partial_j(b_0)\delta_j(a_2)b_0\delta_i(a_2)$ (ix) $-(1/2)b_0\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)$.

By formula (1), one then has to sum the integrals of each of these nine terms over the whole ξ-plane. The full details of the computation are lengthy and to include them in the present article would make the reading of the document too cumbersome. It is worth commenting however that the integrands in the expression $\int b_2(\xi,-1)\,d^2\xi$ which depend upon ξ are rational functions whose denominators invoke only powers of $(k^2|\xi|^2+1)$ and whose numerators invoke only powers of $k|\xi|$. In the commutative case $(k=1)$, the computation invariably reduces to terms involving integrals of the form

$$I_m=\int_0^\infty v^m/(v+1)^{m+2}dv=1/(m+1),$$

where m is a positive integer (in practice one encounters only the range $m=0,1,2,3$). In the non-commutative case, when in particular k and $\delta_i(k)$, $i=1,2$, do not commute, the computation reduces to terms involving either an integral I_m or an integral of the form,

$$\int_0^\infty(k^{2m}u^m/(k^2u+1)^{m+1})(L)(1/(k^2u+1))d(k^2u)=(\mathcal{J}_m.\Delta)(L),\quad i=1,2,\ \text{some } L\in A_\theta^\infty,$$

where $$\mathcal{J}_m=\int_0^\infty(x^m/(x+1)^{m+1})(1/(x\Delta+1))\,dx\ ,$$

is in fact the modified logarithm function $\mathfrak{D}_m$ defined in §2 in the statement of the Theorem and Δ is the operator introduced in §1.

These remarks invoke the following lemma.

Lemma 2: For every element L of A_θ^∞ and every non-negative integer m one has,

$$\int_0^\infty(k^{2m}u^m/(k^2u+1)^{m+1})(L)(1/(k^2u+1))d(k^2u)=(\mathfrak{D}_m.\Delta)(L).$$

Proof: On affecting the change of variables $s'=\log(u)+f$ one obtains

$$\int_0^\infty(k^{2m}u^m/(k^2u+1)^{m+1})(L)(1/(k^2u+1))d(k^2u)$$
$$=\int_{-\infty}^\infty(e^{ms'}/(e^{s'}+1)^{m+1}(L)(1/(e^{s'}+1)d(e^{s'})$$
$$=\int_{-\infty}^\infty(e^{(m+1/2)s'}/(e^{s'}+1)^{m+1})(\Delta^{1/2}(L)\,(e^{s'/2}/(e^{s'}+1))ds'$$
$$=-\int_{-\infty}^\infty(e^{(m+1/2)s'}/(e^{s'}+1)^{m+1})(\Delta^{1/2}(L))\int_{-\infty}^\infty(e^{its'}/(e^{\pi t}+e^{-\pi t})\,dt\,ds'$$
$$=-\int_{-\infty}^\infty(e^{(m+1/2)s'}/(e^{s'}+1)^{m+1})\int_{-\infty}^\infty(e^{its'}/(e^{\pi t}+e^{-\pi t})\sigma_t(\Delta^{1/2}(L))dt\,ds'$$
$$=\int_{-\infty}^\infty(e^{(m+1/2)s'}/(e^{s'}+1)^{m+1}(e^{(s'+\log(\Delta))/2}/(e^{(s'+\log(\Delta)}+1))ds'(\Delta^{1/2}(L))$$
$$=\int_0^\infty(x^m/(x+1)^{m+1}(1/(x\Delta+1))dx(\Delta(L))=(\mathcal{J}_m.\Delta)(L).$$

Now for $\Delta=1$ the integral $\mathcal{J}_m$ equals I_m for every positive integer m. On the other hand, by inspection one sees that $\mathcal{J}_m$ is of the form

$$\mathcal{J}_m=(c_m/(\Delta-1)^{m+1})\{\log(\Delta)-P(\Delta)\},$$

where P is a polynomial of degree at most m. In the neighbourhood of $\Delta=1$ one has

$$\log(\Delta)=\sum_{j=1}^{\infty}((-1)^{j+1}/j)(\Delta-1)^j,$$

and from the its value at $\Delta=1$, where $\mathcal{J}_m$ is non-singular one sees from this last expression that $\mathcal{J}_m$ is the modified logarithm $\mathfrak{D}_m$ introduced in §2 where,

$$\mathfrak{D}_m=((-1)^m/(\Delta-1)^{m+1})\ \{\log(\Delta)-\sum_{j=1}^{m}((-1)^{j+1}/j)(\Delta-1)^j\}.$$

This completes the proof of the lemma. ◘◘◘

Applying these considerations, we find that the respective contributions of the entire list of summands quoted above (with $\lambda=-1$) is

(i) $-\pi k^{-1}\delta_i\delta_i(k)$

(ii) $2\pi(\mathfrak{D}_1\Delta^{1/2})(k^{-1}\delta_i(k))\delta_i(k)k^{-1}$

(iii) $-2\pi(\mathfrak{D}_2(\Delta^{1/2}+\Delta))(k^{-1}\delta_i(k))\delta_i(k)k^{-1}$

(iv) $\pi(k^{-1}\delta_i(k)\delta_i(k)k^{-1}+k^{-1}\delta_i\delta_i(k))$

(v) $-2\pi(\mathfrak{D}_1(1+\Delta^{1/2}))(k^{-1}\delta_i(k))\delta_i(k)k^{-1}$

(vi) $2\pi(\mathfrak{D}_2(1+\Delta^{1/2}))(k^{-1}\delta_i(k))\delta_i(k)k^{-1}$

(vii) $2\pi(\mathfrak{D}_2\Delta^{1/2}(2+\Delta^{-1/2}+\Delta^{1/2}))(k^{-1}\delta_i(k))\delta_i(k)k^{-1}$

(viii) $-2\pi(\mathfrak{D}_3\,\Delta^{1/2}(2+\Delta^{-1/2}+\Delta^{1/2}))(k^{-1}\delta_i(k))\delta_i(k)k^{-1}$

(ix) $-\pi/3(k^{-1}\delta_i(k)\delta_i(k)k^{-1}+k^{-1}\delta_i\delta_i(k))$

Summing the above one obtains:

$(-\pi/3)k^{-1}\delta_i\delta_i(k)+(2\pi/3)k^{-1}\delta_i(k)\delta_i(k)k^{-1}-(2\pi)\mathfrak{D}_1\,k^{-1}\delta_i(k)\delta_i(k)k^{-1}$
$+(4\pi)\mathfrak{D}_2(1+\Delta^{1/2})(k^{-1}\delta_i(k))\delta_i(k)k^{-1}-(2\pi)\mathfrak{D}_3(1+2\Delta^{1/2}+\Delta)(k^{-1}\delta_i(k))\delta_i(k)k^{-1}$, which on multiplication by -1 gives the result of the Theorem. Placing $\Delta=1$ in the above expression yields $\pi/3(k^{-1}\delta_i(k)\delta_i(k)k^{-1}-k^{-1}\delta_i\delta_i(k))$ which under the trace τ_0 gives $(-\pi/6)\,\delta_i\delta_i(\log k)$ and $\tau_0(\delta_i(a))$ appropriately vanishes for all a in A_θ^∞.

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