Removable Singularities for

the Yang-Mills-Higgs equations

in two dimensions

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1. Introduction

In this paper we prove a removable singularities theorem for the coupled Yang-Mills-Higgs equations over a two dimensional base manifold M. This problem is local so at no loss of generality we assume that $M=B_4^2-\{0\}$, where $B_4^2-\{0\}$ is the punctured 2-ball of radius 4 centered at the origin. We also assume that every connection has some gauge in which it is C^1 over the punctured ball.

Let M be a domain in R^2 and η be a vector bundle over M with compact structure group $G \subset O(n)$ and Lie algebra $\mathfrak{G}$. Let the metric on G be induced by the trace inner product on O(n) and let η have a metric compatible with the action of G. Let d be exterior differentiation, δ its adjoint, and let [,] denote the Lie bracket in $\mathfrak{G}$.

A connection determines a covariant derivative D which within a local trivialization defines a Lie algebra valued 1-form A by D = d+A. On p-forms we have locally $D\omega = d\omega + [A,\omega]$, $D^*\omega = \delta\omega + *[A,*\omega]$, where D^* is the adjoint of D. We denote the curvature 2-form by F and have $F = dA + \frac{1}{2}[A,A]$ in this local trivialization.

Gauge transformations are sections of Aut η which act on connections and curvature forms according to the transformations:

$$A^g = g^{-1}Ag + g^{-1}dg$$

$$F^g = g^{-1}Fg \ .$$

The pair (A,F) is gauge equivalent to $(\overline{A},\overline{F})$ iff there is a gauge transformation g such that $\overline{A} = A^g$ and $F = F^g$.

We now follow [Sb2] exactly and define the Higgs field φ using the determinant bundle. We denote by L the determinant bundle raised to the $\frac{1}{2}$-power. Sections of this bundle are constant in a fixed co-ordinate system but we have weight 1 under scale transformations.

The Higgs field φ is a section of $\eta \otimes L$. Therefore, in a fixed co-ordinate system φ may be regarded as a matrix-valued function. Under scale charges $y = rx$, $\varphi(y) = \frac{\varphi(x)}{r}$ (cf.: [P] [SB2]).

The Yang-Mills-Higgs equations are:

(YMH 1) $$D^{*}F = [D\varphi,\varphi]$$

(YMH 2) $$D^{*}D\varphi = \frac{\lambda}{2}(|\varphi|^{2} - m^{2})\varphi \ ;$$

where λ is a fixed real constant and where m is a section of L constant in a fixed co-ordinate system but having weight 1 under scale changes. Thus under the transformation $y = rx$ we have $m' = m/y$. The equations (YMH1,2) are thus invariant under the scale transformation $y = rx$.

Certain norms are invariant under scale transformations. For example $\|\varphi\|_{L^2}$ is invariant and if ψ is any p-form $\|\psi\|_{L^{2/p}}$ is invariant. We also have an important fact used in [U1].

<u>Fact [U1]</u>

Suppose $\psi \in L^{2/p}$ with $\|\psi\|_{L^{2/p}}$ invariant. Then, given a domain D in R^2 and $\gamma > 0$ there is a metric g_0 conformally equivalent to the Euclidean metric in which on bounded sets in R^2; $\int_D |\psi|^{2/p}\, dx < \gamma$.

This fact follows from conformal invariance and the continuity of the L^p-norms. See [UF] for details.

1.b. <u>Statement of the Main Theorem</u>

Now we state our Main Theorem:

Theorem M

Let $M = B_4^2 - \{0\}$ and let η be as above. Let A be a connection on η that satisfies condition H(2), defined in section 1.c. Let F be the curvature form of A and let F be C^∞ over M. Let (F,φ) satisfy (YMH1) and (YMH2) over M. Let $F \in L^1(B_4^2)$.

If $\lambda \geq 0$ let $\varphi \in H_2^1(B_4^2)$. If $\lambda < 0$ let $\varphi \in L^{2+\epsilon}(B_4^2)$ and

$$\overline{\lim_{t\to 0}} \int_{B_1/B_t} \frac{|\varphi|^2}{|x|^2 \log^2\left(\frac{1}{t}\right)} = 0.$$

Then, there exists a continuous gauge transformation such that (F,φ) is gauge equivalent to a C^∞-pair over B_4^2 and the bundle extends continuously to a bundle over B_4^2.

A theorem of this type was first proved by K. Uhlenbeck for the pure Yang-Mills equations over R^4 in [U1]. Later Parker [P] extended the result to the coupled Yang-Mills-Higgs equations over R^4. Papers of L.M. and R.J. Sibner [SB1], [SB2], [SB3] proved similar theorems for dimension 3 and for all higher dimensions. This paper fills the two-dimensional gap in the literature.

We would like to thank L.M. Sibner for suggesting this problem and C. Taubes for a useful abelian example suggesting that holonomy would be important.

1.c. Auxiliary Gauges

Condition H

We wish to introduce a condition on the connection A that insures that the bundle is trivial over the punctured disk M above. This condition is a "holonomy" condition. We call it condition H.

We use the conventions of [KN1] Vol. 1 pg. 71-72. We first define some useful paths.

<u>Definition</u>: Let $\ell_R : [0,1] \to S^1_R$ be given by $\ell_R : t \mapsto (R \cos 2\pi t, R \sin 2\pi t)$ with $S^1_R = \{x \in R^2 \,|\, |x|=R\}$. We say that ℓ_R is the standard loop for S^1_R. Let $L_\theta : [0,1] \to R$ be given by $L_\theta : t \mapsto (Rt,0)$. We call L_θ the standard ray.

For each R, let g(R) be the holonomy of A around the loop ℓ_R.

<u>Definition 1.1.</u>: The map $C_R : (0,4] \to G$ given by $R \to g(R)$ is a path denoted by C_R.

Now, we define condition H(K) and condition H.

<u>Definition 1.2: (condition H(K))</u>: If as $R\downarrow 0$ the elements g(R) considered as points on the carrier of the path C_R approach the identity element in the C^K-topology we say the connection satisfies <u>condition</u> H(K).

<u>Theorem 1.1</u>: The following is equivalent to condition H(1) : There exists a trivialization over a small ball $B_{R_0} - \{0\}$, $\exists R_0$, $0 < R_0 \le 4$ centered at the origin, in which the connection defines a local co-variant derivative $D = d+A$, $A = A_r(r,\theta)dr + A_\theta(r,\theta)d\theta$ with $A_r(r,\theta)$, $A_\theta(r,\theta) \in \Gamma(\mathfrak{G} \otimes T^*(B_{R_0} - \{0\}))$ and with $\lim_{r\to 0} A_\theta(r,\theta) = 0$, with the limit taken in the sup-norm topology on $\mathfrak{G}$.

<u>Proof (1→2)</u> Choose an orthonormal framing $\{v_i(r,\theta)\}$ of η over the ray $\{(r,0|0 \le r \le \epsilon\}$. Extend this to a framing $\{v_i(r,\theta)\}$ by parallel translation around the circles ℓ_R. Then, $\nabla_\theta v_i = 0$, $v_i(r,\theta) = v_i(r,0)\bullet g(r,\theta)$ for some $g(r,\theta) \in G$. In particular, $v_i(r,2\pi) = v_i(r,0)\bullet g(r)$ for some $g(r) = g(r,2\pi) \in G$. The hypothesis implies that for small ϵ, the element g(r) is close to the identity so that

$g(r) = \exp(h(r))$ for some $h(r) \in \mathfrak{G}$. Let $\varphi : [0,2\pi] \to [0,1]$ be a smooth function which vanishes near 0 and is 1 near 2π. Then $w_i(r,\theta) = v_i(r,\theta)\bullet\exp(-\varphi(\theta)h(r))$ is a smooth orthonormal framing of η over $B_2-\{0\}$. In this framing the connection form is: $(A_\theta)^i_j =$ $< \nabla_\theta w_i, w_j > = < [\nabla_\theta(v_i\bullet\exp(-\varphi(\theta)h(r))], w_j > = -\varphi'(\theta)h(r)\delta_{ij}$. Hence $|A_\theta|$ $\leq c|h(r)| \downarrow 0$ as $r \downarrow 0$.

$(2\to1)$. This follows from standard O.D.E. estimates on integrating the parallel transport equation for each horizontal lift of ℓ_R. Q.E.D.

Remark 1.1.: Thus condition H(1) implies that the bundle η is trivial over $B^2_{R_0} - \{0\}$.

1.b. The Auxiliary Gauge

We will give in Section 3 a gauge-independent proof that under the conditions of Theorem M, the curvature F is actually in $L_p(B_R)$ for $1 \leq p < \infty$ if R is small enough, in any smooth gauge over $B_R - \{0\}$. This estimate, coupled with the existence of an "auxillary" gauge in which the connection form A is L_p - norm close to zero (flat connection), will enable us to use a new gauge-fixing argument [U3] of Uhlenbeck to build a Coulomb gauge over $B_R - \{0\}$, bypassing the original broken Hodge gauge argument of [U1]. Thus this paper is *much simplified* compared to the author's Max-Planck preprint [S] which preceded it.

In this section we construct the "auxillary" gauge and show that the L_p - norm of the induced connection form is small.

Lemma 1.1.: Under the conditions of Theorem 1.1, let the connection satisfy condition H(2). Then, there exists a local trivialization in which the connection induces the local co-variant derivative $D = d+A$, $A := A_r(r,\theta)dr + A_\theta(r,\theta)d\theta$ and we have:

$\lim_{r\to 0} A_r(r,0) = 0$, $\lim_{r\to 0} A_\theta(r,\theta) = 0$, $\lim_{r\to 0} \frac{d}{dr}(A_\theta(r,\theta)) = 0$.

Proof: We start with the orthonormal framing v_i over the standard ray L_θ used in the beginning of the proof of Theorem 1.1. We use this framing to give a local trivialization for the bundle restricted to have the standard ray as a base space. The connection restricts and we denote the restricted connection by ∇_r. This connection defines $\{\bar{A}_r(r,0)\}^i_j := < \nabla_r v_i(r,0),\ v_j(r,0) >$. Now we define $\hat{s}(r) \in G$ as the solution to: $\frac{d\hat{s}(r)}{dr} = -\bar{A}_r(r,0)\hat{s}(r)$, $\hat{s}(R_0) = I$, $\ni_{R_0}$, $0 < R_0 < 1$. Now define $\bar{v}_i(r,0) := v_i(r,0) \bullet \bar{s}(r)$.

Note that

$$\{\tilde{A}_r(r,0)\}^i_j := <\nabla_r\bar{v}_i(r,0),\bar{v}_j(r,0)> = \hat{s}^{-1}(r)\bar{A}_r(r,0)\hat{s}(r)+\hat{s}^{-1}(r)\ \frac{d\hat{s}(r)}{dr}$$

and thus; $\lim_{r\to 0} \{\tilde{A}(r,0)\}^i_j = 0 = \lim_{r\to 0} < \nabla_r\bar{v}_i(r,0),\ \bar{v}_j(r,0) >$.

Now carry out the proof of Theorem 1.1 with $\{v_i\}$ replaced by $\{\bar{v}_j\}$. Note that in the gauge constructed for which $\lim_{r\to 0} A_\theta(r,\theta) = 0$ we have

$$\lim_{r\to 0} \{A_r(r,0)\}^i_j = \lim_{r\to 0} < \nabla_r w_i(r,0),w_j(r,0)>= \lim_{r\to 0} < \nabla_r\bar{v}_i(r,0),\bar{v}_j(r,0)>=0$$

Note also that; $\lim_{r\to 0} \{\frac{d}{dr}(A_\theta(r,\theta))\}^i_j = \lim_{r\to 0}(h'(r)\varphi'(\theta)\delta_{ij}) = 0$ by condition H(2); since this follows from the formula for $(A_\theta(r,\varphi))^i_j$. (line 8 of the Proof (1-2)) in the proof of Theorem 1.1. Q.E.D.

Definition 1.3.: We call the gauge defined by Lemma 1.1 the auxiliary gauge.

Lemma 1.2.: Let the conditions of Theorem 1.1 hold. Let the connection satisfy condition H(2). Let the curvature be in $L^P(B_{R_0})$ $1 \leq P < \infty$. Then in the auxiliary gauge we have: $\int_0^R |A_r(r,\theta)|^P rdr < \infty$, $0 < R < R_0$.

Proof: In the auxiliary gauge we have:

$$\frac{\partial A_r}{\partial \theta} - \frac{\partial A_\theta}{\partial r} + \frac{1}{2}[A_r, A_\theta] = F_{r,\theta} \text{ and } \int_0^{2\pi}\int_0^{R_0} \frac{|F_{r,\theta}|^P}{r^P} \cdot rdrd\theta = \|F\|_{L^P(B_{R_0})}.$$

Fix S, $0 < S \leq R_0$ and integrate:

$$A_r(S,\theta) = A_r(S,0) + \int_0^\theta \frac{\partial A_\theta}{\partial r}(S,t)dt$$
$$- \frac{1}{2}\int_0^\theta [A_r(S,t), A_\theta(S,t)]dt - \int_0^\theta F_{r,\theta}(S,t)dt$$

$0 \leq \theta < 2\pi$

Thus:

$$|A_r(S,\theta)| \leq |A_r(S,0)| + \int_0^\theta \left|\frac{\partial A_\theta(S,t)}{\partial r}\right| dt$$
$$+ \int_0^\theta |F_{r,\theta}(S,t)| dt + 2\int_0^\theta |A_r(S,t)||A_\theta(S,t)| \, dt \quad \text{for all } S;$$

$0 < S < R_0$.

Thus, by Lemma 1.1 and since $F \in L_P$, $1 \leq P < \infty$. We obtain (by elementary computations):

$$\int_{R/2}^R |A_r(S,\theta)|^P SdS \leq 0(R) + k\int_0^\theta \left[\int_{R/2}^R |A_r(S,t)|^P SdS\right] \cdot |A_\theta(S,t)|^P dt,$$

$0 < R < R_0$, with K independent of R.

Now, apply Gronwall's inequality, pg. 189 [AMR], to get:

$$(*) \quad \int_{R/2}^{R} |A_r(S,\theta)|^P S dS \le 0(R) \ \exp\left[K \cdot \int_0^\theta |A_\theta(S,t)|^P dt\right]$$

$$\le \tilde{K} \cdot 0(R), \ 0 < R < R_0,$$

with K and $\tilde{K}$ independent of R, since $\lim\limits_{S\to 0} |A_\theta(S,t)| = 0$.

Now applying (*) with R replaced by $R/2^m$, $m = 1,2,\ldots$ and summing we obtain:

$$\int_0^R |A_r(S,\theta)|^P S dS \le 0(R) \qquad \text{Q.E.D.}$$

Lemma 1.3: Under the hypothesis of Lemma 1.2. in the auxilliary gauge we have $\|A\|_{L^P(B_R-\{0\})} < 0(R)$, $1 \le P < \infty$.

Proof: Apply Lemma 1.1 to estimate $\|A_\theta\|_{L^P(B_R-\{0\})}$ and Lemma 1.2 to estimate $\|A_r\|_{L^P(B_R-\{0\})}$. Q.E.D.

2. Some Improvements on Morrey's Theorem

In this section we state some improved versions of Morrey's theorem in 2-dimensions that will be used later.

First we state Morrey's theorem in 2-dimensions.

Theorem 2.1. (Morrey's theorem in 2-dimensions) [MO]. Let $u \in H_2^1(\Omega)$ with $u \ge 0$ and suppose that: Ω is a locally Lipshitz domain in R^2, and $\int_\Omega \nabla u \nabla \xi + f\cdot u \, dx \le 0$ for all non-negative $\xi \in C_0^\infty(\Omega)$. Let f satisfy the Morrey Condition:

$\int_{B_R \subset \Omega} |f|^{1+\epsilon} dx \le c\, R^\beta$ for all $B_R \subset \Omega$ and some $\epsilon, \beta > 0$ then

$\sup_{B(x_0,\rho)} |\, u(x)\, |^2 \le \frac{c}{a^2} \int_{B(x_0,\rho+a)} |\, u(y)\, |^2 dy$ for all

$B(x_0,\rho) \subset B(x_0,\rho+a) \subset \Omega$.

Proof: Identical to the proof of Theorem 5.3.1 of [MO], pg. 137, except that we need our somewhat stronger Morrey condition because the inequality $\int g|w|^2 \le c_n[\int |\, \nabla w\, |^2\, dx + \int |g\, |^{n/2} dx]$ fails in 2-dimensions due to critical Sobolov exponents.

We would now like to note that if $u \in C^\infty(\Omega)$ we can state an improvement of Morrey's estimate involving $\frac{K}{a^2} \int_{B(x_0,\rho+a)} |u(y)| dy$. This improvement follows from an iteration argument of E. Bombieri. See [BO], pg. 66.

Theorem 2.2. (Bombieri). Let Ω be compact. Let $u \in C^\infty$ in Ω and let $u \ge 0$. Let u satisfy:

$\sup_{B_\rho} (u(x))^2 \le \frac{c}{(R-\rho)^2} \int_{B_R} u^2 dx$ for all concentric B_R, $B_\rho \subset \Omega$, $0 < \rho < R$. Then $\sup_{B_\rho} u(x) \le \frac{c}{(R-\rho)^2} \int_{B_R} u\, dx$ where B_R and B_ρ are as above.

Proof: Use the iteration at the top of pg. 66 of [BO]. Q.E.D.

3. A Regularity Theorem for the Higgs Field

In this section we assume that the Higgs field is a C^∞ solution of the field equation:

$$\text{(YMH2)}\quad D^*D\phi = \frac{\lambda}{2}\,(|\phi|^2 - m^2)\phi$$

in the punctured unit ball $B^2 - \{0\}$. As in [Sb2] the assumptions on ϕ near the origin depend on the sign of λ.

Because of the criticality of the Sobolev exponent $\frac{2n}{n-2}$ for L_2 functions in 2-dimensions, we require several technical changes from the argument in [SB2]. This is where we use the estimates of section 2.

The main result of this section is:

Theorem 3.1. Let ϕ be a C^∞ solution of (YMH2) in $B^2 - \{0\}$ in R^2. We assume:

(a) $\varphi \in H_2^1(B^2)$ if $\lambda > 0$

(b) $\varphi \in H_2^1(B^2)$ if $\lambda = 0$

(c) $\varphi \in L^{2+\epsilon}(B^2)$ for some $\epsilon > 0$ and

$$\overline{\lim_{t\to 0}} \int_{B_1/B_t} \frac{|\varphi|^2}{|x|^2 \log^2(\frac{1}{t})} = 0, \text{ if } \lambda < 0.$$

Then $\varphi \in L^\infty(B^2 - \{0\})$.

Remark 3.1: That condition (c) is natural follows by considering the case when the structure group is commutative (i.e., the real numbers) and looking at the scalar inequality

$$\Delta u + u^3 \le 0.$$

then, $u = -\ln r + r$ is an unbounded function satisfying the above inequality and $-\ln r + r$ is in all L^p $1 \le p < \infty$. Also note that, if r is small enough, the function $-\sqrt{-\ln r + r} = u$ also satisfies the above inequality and is in all L^p $1 \le p < \infty$.

Also note that our condition (c) is weaker than $\varphi = o(|\log|x||^{1/2})$ and that $\varphi \in O(|\log|x||^{1/2})$ is weaker than (c).

Similarly, we see that conditions (b) and (a) are natural by considering $\Delta u = 0$ in $B^2 - \{0\}$. Then $u = \ln\left(\frac{|x|}{2}\right)$ is an unbounded solution of $\Delta u + u^3 \leq 0$ with $u \notin H_2^1(B^2)$.

To prove 7.1 we make strong use of the fact that $u = |\varphi|$ is a weak solution in $B^2 - \{0\}$ of: $(\Delta|\varphi|) \geq \frac{\lambda}{2}(|\varphi|^2 - m^2)|\varphi|$, where Δ is the ordinary Laplacian on functions. This follows from Weitzenblock - like identities and details may be found in [Sb2] (formula 2 and Lemma 1.2).

At no loss of generality we assume $u \geq 1$.

For example, in case (b) the function $|\phi|$ is subharmonic. We dispose of case (b).

Proof:(case (b)). First we show that u is a weak solution of $\int_{B^2} \nabla u \cdot \nabla \eta dx \leq 0$ for all $\eta \in C_0^\infty(B^2)$, $\eta \geq 0$. Let $\epsilon > 0$. Let ψ_3 be in $C_0^\infty(B^2)$ with $\psi_\epsilon = \psi_\epsilon(|x|)$ $\psi_3 = 1$ on B_ϵ, $\psi_\epsilon = 0$ on $B_{2\epsilon}$, ψ_ϵ monotone decreasing in $|x|$, $|\nabla\psi_\epsilon| \leq \frac{K}{\epsilon}$. We multiply $\Delta u \geq 0$ by $\eta\psi_\epsilon$ to obtain: $\int_{B^2}(\nabla u \cdot \nabla\eta)\psi_\epsilon dx \leq \sqrt{\int_{\text{supp }\psi_\epsilon}|\nabla u|^2 dx}\ \sqrt{\int_{\text{supp }\psi_\epsilon}|\nabla\psi_\epsilon|^2 dx}$.

Let $\epsilon \downarrow 0$ and note that the right hand side tends to zero. By Lebesques dominated convergence theorem we have:

$$\int_{B^2}(\nabla u \cdot \nabla\eta)dx \leq 0.$$

Now, we apply the argument of [Giaq] p. 119 and the "reverse" Sobolev estimate [Giaq] p. 122, or equivalently apply Theorem 2.1, pg. 136 of [Giaq] to get $\nabla u \in L_{2+\epsilon}({B^2}_{1/2})$. Since $u \in H_2^1(B^2) \subset R^2$ Sobolev's embedding theorem gives $u \in L_{2+\epsilon}(B^2_{1/2})$. Now extend u to $\hat{u}$

where $\hat{u}$ is defined in B_1^2 with compact support and with

$\|\hat{u}\|_{H^1_{2+\epsilon}}(B_1^2) \le K\|u\|_{H^1_{2+\epsilon}}(B_{1/2})$.

Thus, by the Sobolev embedding theorem and the inequality of [GT] pg. 155 we have:

$$\sup_{B^2_{1/2}}|u| \le \sup_{B^2_1}|\hat{u}| \le C\ \|\hat{u}\|_{H^1_{2+\epsilon}}(B_1) \le K\|u\|_{H^1_{2+\epsilon}}(B_{1/2}).$$

Since u is bounded in $B_2^2 - B_{1/2}^2$ we see that u is bounded in B_2^2. Q.E.D.

We now dispose of Case (a).

<u>Proof</u>: (Case (a)). In Case (a) we have that $u = |\phi|$ solves $\Delta u \ge \frac{\lambda}{2}(u^2 - m^2)u$ with $\lambda > 0$. Thus: $\Delta u \ge \frac{\lambda}{2}(u^2 - m^2)u$. Now consider the two sets.

$$A = \{x \in B^2 - \{0\} \text{ such that } u \le m\},$$
$$B = \{x \in B^2 - \{0\} \text{ such that } u > m\}.$$

These sets are pairwise disjoint. Now, because $u \in C^\infty$ on $B^2 - \{0\}$, the set B is open.

Cover B by a countable collection of small balls, each contained in B. Then on any such small ball in B we have $\Delta u \ge 0$ and by the estimate above used in the proof of case (b) we obtain:

$$\sup_B u \le K \| u \|_{H^1_2}(B^2 - \{0\}).$$

Now on A, u is bounded above by m. Hence u is bounded on $B^2 - \{0\}$. Q.E.D.

We now prove case (c). This requires some work because the proof of Proposition 2.3 of [Sb2] fails in 2-dimensions. The main problem is that when $n = 2$ inequality (1.14), page 7 of [Sb2], fails since $\frac{2n}{n-2} = \infty$ and $c_n = \infty$ when $n = 2$. Nevertheless we establish the same estimate as in the conclusion of Proposition 2.3 of [Sb2] using a modified technique.

First we prove the following proposition.

Proposition 3.1. (cf. Prop. 2.3 of [Sb2]). If condition (c) is satisfied, either we have:

$$\int_{B^2} \eta^2 |\nabla u|^2 dx \leq K \int_{B^2} |\nabla \eta|^2 u^2 dx$$

for all test functions η in $C_0^\infty(B^2)$ or u is bounded.

Proof: We use a sequence η_K of test functions that vanish for $|x| \leq \epsilon_K$, tend to 1 as ϵ_K tends to zero and such that $\int |\nabla \eta_K|^2 dx \to 0$ as $K \to \infty$. These are defined cf. [G] pg. 547 bottom, by:

$$\bar{\eta}_K = \bar{\eta}^{\epsilon_K}(|x|) = \begin{cases} 0 & \text{for } |x| \leq \epsilon_K \\ 1 & \text{for } |x| \geq 1 \\ \dfrac{1}{\log\left(\frac{1}{\epsilon_K}\right)} \cdot \log\left[\dfrac{|x|}{\epsilon_K}\right] & \text{for } \epsilon_K < |x| < 1 \end{cases}$$

Remark 3.2: Note that our growth condition in case (c) is chosen exactly to insure that $\int_{B_2} |u|^2 |\nabla \bar{\eta}_K| \to 0$ as $K \to \infty$.

Now let η be C_0^∞ and let $\bar{\eta}$ be a C^∞ function vanishing in a neighborhood of the origin. Use the test function $\tau = (\eta\bar{\eta})^2(u)$ as ξ in: $\int \nabla u \cdot \nabla \xi dx \leq \int hu\xi dx$ for all non-negative $\xi \in C_0^\infty(B^2 - \{0\})$. where $h = -\frac{\lambda}{2}(|\varphi|^2 - m^2)$ and $u = |\varphi|$. We get: $J_1(\bar{\eta}) =$

$K \int (\eta\bar{\eta})^2|\nabla u|^2dx \leq \int|2\eta\bar{\eta}\nabla u|\ \nabla(\eta\bar{\eta})u|\ dx + \int(\eta\bar{\eta})h\ u^2\ dx = I_1 + I_2.$

Now, $I_1 \leq \mu\int(\eta\bar{\eta})^2|\nabla u|^2dx + C(\mu)\int|\nabla(\eta\bar{\eta})|^2|u|^2dx$ and the first term on the right may be absorbed into the left hand side. Also, $\int |\ \nabla(\eta\bar{\eta})\ |^2u^2dx \leq K[\int|\nabla\eta|^2u^2dx + \int |\nabla\bar{\eta}|\ ^2u^2dx]$. Note that $\int|\nabla\bar{\eta}|^2dx \to 0$ if we set $\bar{\eta} = \eta_K$ and let $K \to \infty$. Do this. Thus in the limit as $K \to \infty$, $I_1 \leq \int |\nabla\eta|^2|u|^2dx$. Now, $I_2 = \int (\eta\bar{\eta})^2hu^2dx$ $\leq \int_{\text{supp}\eta \cap \text{supp}\bar{\eta}} (\eta\bar{\eta})^2hu^2dx\_$. Since $\lambda \leq 0$ we have $h = \frac{-\lambda}{2}(|\phi|^2 - m^2)$ $\leq \frac{-\lambda}{2}(|\phi|^2)$. $I_2 \leq K \int_{\text{supp}\ \eta \cap \text{supp}\ \bar{\eta}} (\eta\bar{\eta})^2|\phi|^2|\underline{u}|^2dx = J_2$.

We now estimate J_2:

<u>Remark:</u> The estimate of I_2 in the proof of proposition 2.3, pg. 11 of [Sb2], is based on the inequality: $\int gw^2dx \leq C_n\|g\|_{n/2}\int|\nabla w|^2dx$ which is proved using Sobolev's inequality. This inequality estimates I_2 from above by a sum of terms, the first of which is proportional to $\|\phi\|_{L^2}$. Then use is made of conformal scaling to make $\|\phi\|_{L^2}$ small.

In two dimensions however, the Sobolev estimate has a critical exponent and constant c_n corresponding to this exponent is <u>infinite.</u> Thus we need a new argument. This new estimate is contained in the proof of the following sublemma.

<u>Sublemma 3.1.</u> Let $B^2 -\{0\} \supset \Omega \supset \text{supp}\ \eta \cap \text{supp}\ \bar{\eta}$. Then: $J_2 \leq C[\int_\Omega|\phi|^2dx]\cdot[\int_\Omega(\eta\bar{\eta}u)^2dx + \int_\Omega|\nabla(\eta\bar{\eta}u)^2dx]$.

<u>Remark:</u> The idea of the proof is that $V = |\phi|^2$ is a weak

sub-solution (in fact a C^∞ solution) of an elliptic equation on supp $\eta \cap$ supp $\bar{\eta} = \Omega_0$. Thus by a Morrey-like estimate (Bombieri's lemma) we can estimate $\sup_{B(R)\subset \Omega_0} |\phi| \le \frac{C}{R} [\int_{B(2R)\subset \Omega_0} \phi^2 dx]^{1/2}$. The sublemma then follows from a covering theorem. We do it now.

Let $V = \phi^2$, let all balls $B(r)$ be contained in Ω. Choose the balls B_R so that $B_R \subset B_{2R} \subset B_{3R} \subset \Omega_0$. Then Ω_0 is covered by a finite number of such balls. Since u is C^∞ in Ω_0 we can at no loss of generality assume that $u \ge 1$ on Ω_0. (If no such Ω_0 exists then u is bounded.) Recall that $u = |\phi|$ is a subsolution of $\Delta u \ge \frac{\lambda}{2} (|u|^2 - m^2)|u| \ge \frac{\lambda}{2} (|u|^2)|u|$ in Ω_0 since $\lambda < 0$. Thus $\Delta u - \frac{\lambda}{2} |u|^3 \ge 0$ in Ω_0. Now since $u \ge 1$, $u \in C^\infty$ on Ω_0, we have: $\Delta(|u|^2) = 2u\Delta u + 2|\nabla u|^2 \ge \Delta u$. Thus $V = |u|^2$ is a C^∞ subsolution in Ω_0 of $\Delta V + (\frac{-\lambda}{2} |u|)V \ge 0$. Note that $(\frac{-\lambda}{2} |u|)$ is in $L_{1+\varepsilon}, \exists_\varepsilon, \varepsilon>0$ (by our growth assumption $\phi \in L_{2+\varepsilon}(B^2)$). Now we apply Theorem 6.1 (Morrey's Theorem in 2-dimensions) and Theorem 6.2 (Bombieri's lemma) to get

$$\sup_{B(R)\subset \Omega_0} V \le \frac{C}{R^2} \int_{B(2R)\subset \Omega_0} |V|^1 \, dx, \quad \forall_{B(R),B(2R)} \text{ concentric in } \Omega_0.$$

Thus

$$\sup_{B(R)\subset \Omega_0} \phi \le \frac{C}{R} \left[\int_{B(2R)\subset \Omega_0} \varphi^2 \, dx \right]^{1/2}.$$

We now use the above inequality and Holder's inequality to achieve our estimate of J_2. Using Holder's inequality with

$p = 1 + \frac{\epsilon}{2}$, $q = \frac{2+\epsilon}{\epsilon}$ we get:

$$J_2 \le |\sup_{B(R)\cap\Omega_0} \phi|^{\frac{2\epsilon}{2+\epsilon}} \left[\int_{B(R)\cap\Omega_0} (\eta\bar{\eta}u)^{2\cdot(\frac{2+\epsilon}{\epsilon})}\right]^{(\frac{\epsilon}{2+\epsilon})} \cdot \left[\int_{B(R)\cap\Omega_0} \varphi^2\right]^{\frac{2+\epsilon}{\epsilon}} = J_3.$$

Now extend $\eta\bar{\eta}u$ to B_{4R} with the extension $E(\eta\bar{\eta}u)$ equal to zero on B_{4R}/B_{3R} and $\|E\eta\bar{\eta}u\|_{H_2^1(B_{4R})} \le \hat{K}\|\eta\bar{\eta}u\|_{H_2^1(B_{2R})}$. We can do this by Theorem 3.4.3 pg. 74[MO]. Thus:

$$J_3 \le \left[\int_{B_R} \sup \phi\right]^{\frac{2\epsilon}{2+\epsilon}} \left[\int_{B_R} \phi^2\right]^{\frac{2}{2+\epsilon}} \left[\int_{B_{4R}} E(\eta\bar{\eta}u)^{2(\frac{2+\epsilon}{\epsilon})}\right]^{2(\frac{\epsilon}{2+\epsilon})\frac{1}{2}\cdot 2}.$$

Now use Sobolev's inequality in the form:

$$\left[\int_{B_{4R}} u^t dx\right]^{1/t} \le CR^{2/t} \left[\int_{B_{4R}} |\nabla u|^2 dx\right]^{1/2} \quad \text{where} \quad t \ge 2$$

for $u \in H_2^1(B_{4R})$ with $u = 0$ on B_{4R}/B_{3R}. We let $u = E(\eta\bar{\eta}u)$ and $t = (2(2+\epsilon))/\epsilon$ to get:

$$\left[\int_{B_{4R}} |E(\eta\bar{\eta}u)|^{\frac{2(2+\epsilon)}{\epsilon}}\right]^{\frac{\epsilon}{2+\epsilon}} \le c\, R^{\frac{2\epsilon}{2+\epsilon}} \left[\int_{B_{4R}} |\nabla E(\eta\bar{\eta}u)|^2 dx\right]$$

and thus

$$\int_{B_R \subset \Omega_0} \phi^2 \eta^2 \bar{\eta}^2 u^2 dx \le (\sup_{B_R} \phi)^{\frac{2\epsilon}{2+\epsilon}} \cdot \left[\int_{B_{4R}} |\nabla E(\eta\bar{\eta}u)|^2 dx\right] \cdot$$

$$\left[\int_{B_R} \phi^2 dx\right]^{\frac{2}{2+\epsilon}} \left[CR^{\frac{2\epsilon}{2+\epsilon}}\right].$$

Recall that $\sup_{B_R} \phi \le (K/R)[\int_{B_{2R}} \phi^2 dx]^{1/2}$. Thus combining all our estimates we get:

$$\int_{B_R \subset \Omega_0} \phi^2\eta^2\bar{\eta}u^2 \le \left[\frac{C}{R}\left(\int_{B_{4R}}\phi^2 dx\right)^{1/2}\right]^{\frac{2\epsilon}{2+\epsilon}} \bullet \left[C\ R^{\frac{2\epsilon}{2+\epsilon}}\right] \bullet$$

$$\left[\int_{B_R}\phi^2 dx\right]^{\frac{2}{2+\epsilon}} \bullet \left[\int_{B_{4R}}|\nabla(E\eta\bar{\eta}u)|^2 dx\right].$$

$$\le \tilde{K}\left[\int_{B_{2R}}\phi^2 dx\right]\left[\int_{B_{4R}}(E(\eta\bar{\eta}u))^2 dx + \int_{B_{4R}}|\nabla(E\eta\bar{\eta}u)|^2 dx\right]$$

$$\le K\left[\int_{B_R}\phi^2 dx\right)\left[\int_{B_{2R}}(\eta\bar{\eta}u)^2 dx + \int_{B_{2R}}|\nabla(\eta\bar{\eta}u)|^2 dx\right].$$

Now using Besocovitch's covering lemma and changing constants appropriately we have $\int_\Omega\phi^2(\eta\bar{\eta})^2 \le C[\int_\Omega\phi^2][\int_\Omega(\eta\bar{\eta}u)^2 dx + \int_\Omega|\nabla(\eta\bar{\eta}u)|^2 dx.$ This completes the proof of the sublemma.

Q.E.D. Sublemma

Now we return to the main proof and use the sublemma. We have, using the sublemma and recalling that conformal invariance implies that we may choose $[\int\phi^2 dx]^{1/2} < \gamma$ (where γ may chosen small) that (II):

$$J_1(\bar{\eta}_k) = K\int_\Omega(\eta\bar{\eta}_k)^2|\nabla u|^2 dx \le \int_\Omega|\nabla\eta|^2u^2 dx + g(k) + c(\gamma)[(\int_\Omega|\nabla(\eta\bar{\eta}_k u)|^2 dx)$$

$$+ C(\int_\Omega|(\eta\bar{\eta}_k u)|^2 dx)] \text{ with } c(\gamma)\downarrow 0 \text{ if } \gamma\to 0 \text{ and } \lim_{k\to\infty} g(k) = 0.$$

Note that:

(III) $\int_\Omega|\nabla(\eta\bar{\eta}_k u)|^2 dx \le 2\int_\Omega\eta^2\bar{\eta}_k^2|\nabla u|^2 dx + 2\int_\Omega|\nabla(\eta\bar{\eta}_k)|^2u^2 dx$ so from (II) and (III) we obtain

(IV) $$K\int_\Omega(\eta\bar{\eta}_k)^2|\nabla u|^2 dx \le \int_\Omega|\nabla\eta|^2u^2 dx + g(k) + 2C(\gamma)[\int_\Omega\eta^2\bar{\eta}_k^2|\nabla u|^2 dx$$

$$+ 2\int_\Omega|\nabla\eta\bar{\eta}_k|^2u^2 dx] + C(\gamma)[\tilde{K}\int_\Omega(\eta\bar{\eta}_k u)^2 dx].$$

Now choosing γ small enough we absorb the term $2C(\gamma)\int_\Omega|\eta\bar{\eta}_k|^2|\nabla u|^2 dx$ in the left hand side of (IV) and we get:

$$\text{(V)}\qquad K\int_\Omega |\eta\bar\eta_k|^2|\nabla u|^2dx \le g(k) + \int|\nabla\eta|^2u^2dx$$

$$+ C(\gamma)[2\int_\Omega|\nabla\eta\bar\eta_k|^2u^2 + K\int_\Omega(\eta\bar\eta_k u)^2dx.$$

Now using growth condition c, we have

$$\int_\Omega|\nabla(\eta\bar\eta_k)|^2u^2dx \le 2\int|\nabla\eta|^2u^2dx + 2\int\eta^2(|\nabla\bar\eta_k|)^2u^2dx \le 2\int_\Omega|\nabla\eta|^2u^2dx$$

$$+ \ 2\int|u^2|\nabla\bar\eta_k|^2 \cdot \sup_\Omega \eta \le 2\int|\nabla\eta|^2u^2dx + h(k)$$

where $h(k) \downarrow 0$ as $k \to \infty$. Using this in (V) we obtain

$$\text{(VI)}\ K\int_\Omega|\eta\bar\eta_k|^2|\nabla u|^2dx \le h(k) + g(k) + \tilde K\int_\Omega|\nabla\eta|^2u^2dx + KC(\gamma)\int_\Omega(\eta\bar\eta_k u)^2dx$$

with $C(\gamma) \downarrow 0$ if $\gamma \to 0$. But

$$\int_\Omega(\eta\bar\eta_k u)^2dx = \int_{B^2}(\eta\bar\eta_k)^2u^2dx \le 2\int_{B^2}|\nabla(\eta\bar\eta_k)|^2u^2dx + 2\int_{B^2}\eta^2\bar\eta_k^2|\nabla u|^2dx.$$

Thus

$$\text{(VII)}\qquad K\int_{B^2}|\eta\bar\eta_k|^2|\nabla u|^2dx \le h(k)+ g(k) + \tilde K\int_{B^2}|\nabla\eta|^2|u|^2dx$$

$$+ 2K(\gamma)\int_{B^2}|\nabla(\eta\bar\eta_k)|^2u^2dx + 2K(\gamma)\int_{B^2}\eta^2\bar\eta_k^2|\nabla u|^2dx$$

(again with $K(\gamma) \downarrow 0$ as $\gamma \downarrow 0$.)

Now choose γ small enough and absorb the last right hand term on the left hand side.

$$\text{(VIII)}\qquad K\int_{B^2}|\eta\bar\eta_k|^2|\nabla u|^2dx \le h(k) + g(k) + \tilde K\int_{B^2}|\nabla\eta|^2u^2dx$$

$$+ 2K(\gamma)\int_{B^2}|\nabla(\eta\bar\eta_k)|^2u^2dx.$$

But, $\int_{B^2}|\nabla(\eta\bar\eta_k)|^2u^2dx = \int_\Omega|\nabla(\eta\bar\eta_k)^2u^2dx = A$ (we have already shown $A \le 2\int_\Omega|\nabla\eta|^2u^2dx + h(k)$, with $h(k) \downarrow 0$ as $k \to \infty$. Thus combining terms we obtain

(IX) $K\int_{B^2}|\eta\bar{\eta}_k|^2|\nabla u|^2dx \leq m(k) + \tilde{K}\int_{B^2}|\nabla\eta|^2u^2dx$

with $m(k) \downarrow 0$ if $k \to \infty$.

Now, let $k \to \infty$ and we get:

(X) $\int_{B^2}|\eta|^2|\nabla u|^2dx \leq K\int_{B^2}|\nabla\eta|^2u^2dx$ with K independent of u.

Q.E.D.

Now we prove Theorem 3.1.

Proof: Theorem 3.1 now follows from De-Georgi iteration, pg. 76 [LU] which uses the estimate of Proposition 3.1 as its basic inequality.

Q.E.D.

We now conclude this section with a final corollary.

Corollary 3.1. Under the hypothesis of Theorem 3.1, $D\phi$ is in $L^2(B^2)$.

Proof: This is the same as the proof of Corollary 2.4 of [Sb2].

Q.E.D.

4. A Growth Estimation for F

We show F is actually in L_P for all $1 \leq P < \infty$, in any smooth gauge over $B_R - \{0\}$, $0 < R < R_0$.

Theorem 4.1: Under the conditions of theorem M, F is in any L^P, $1 \leq P < \infty$, in any smooth gauge over $B_R - \{0\}$, $0 < R < R_0$.

Proof: Since $*F$ is a smooth function on the punctured ball it follows from inequality 6.7, pg. 269 of [JT] and from YMH1 that :

$\|d|*F|\|_{L^2} \leq 2\ \mathrm{MAX}|\varphi|\ \|D\varphi\|_{L^2} < \infty$ by our estimates on φ. Now, since $F \in L^1$ it follows that $*F$ is in $W^1_1(B_R-\{0\})$. Thus by Sobolev's theorem F is in $L^2(B_R-\{0\})$. Now, we have $F \in H^1_2\ (B_R-\{0\})$ and

Sobolev's embedding theorem gives that $F \in L^P(B_R-\{0\})$ for all P, $1 \leq P < \infty$. Q.E.D.

5. Proof of the Main Theorem

Corollary 5.1: (F, φ) is a weak solution of the field equations in the full ball B_R, $0 < R < R_0$.

Proof: Same as Corollary 5.3 of [Sb2].

Lemma 5.1: $\int_{B_R} |D\varphi| dr \leq kR, \quad 0 < R < R_0$

Proof: $D\varphi \in L^2(B_R)$; apply Holders inequality. Q.E.D.

Theorem 5.1: Under the conditions of theorem M, there exists a smooth gauge over $B_{R_0} - \{0\}$ in which the induced covariant derivative is $d + A$ and $A \in H^1_q(B_{R_0})$ with $q < 2$.

Proof: By lemma 1.3 we have an auxillary gauge in which the induced covariant derivative is $d + A_{aux}$ and $\|A_{aux}\|_{L^P(B_R - \{0\})} \leq 0(R)$; $0 < R \leq R_0$, $1 \leq p < \infty$. As in the proof of Corollary 4.3 of [U3] we solve (by now this is standard): $d^*(g^{-1}dg + g^{-1}A_{aux}g) = 0$ for g in the space $L^1_P(B^2-\{0\}, G) \subset C^0(B_{R_0} - \{0\}, G)$; if $P > 2$. Let $P > 4$. Note, in this gauge, the connection form, (again denoted by A), is in $L_P(B_{R_0}-\{0\}) = L_P(B_{R_0})$. Now, in this gauge, as in Prop. 5.4 [Sb2], we have $dA = F - \frac{1}{2}[A,A] => dA \in L_{P/2}(B_{R_0})$. Since $\delta A = 0$ in $B_{R_0} - \{0\}$ we have $\nabla A \in L^{P/2}(B_{R_0})$. Thus $A \in H^1_q(B_{R_0}) \subset C^0(B_{R_0})$; with $q > 2$. Q.E.D.

Remark: Note that for clarity and consistency we have followed the function space notation in [Sb2]. In more precise notation

$A \in L^P(B_{R_0} - \{0\})$ would be $A \in L^P(B_{R_0} - \{0\}, \mathfrak{G} \otimes \Lambda^{1^*} R^2)$,

$A \in H^1_q(B_{R_0})$ would be $A \in H^1_q(B_{R_0}, \mathfrak{G} \otimes \Lambda^{1^*} R^2)$, etc.

Theorem 5.2: Under the conditions of Theorem M, there exists a smooth gauge over B_{R_0} in which the induced curvature form F and the induced connection from A satisfy:

1) $\delta A = 0$
2) $A \in H^1_q(B_{R_0})$ with $q > 2$
3) $\|A\|_{H^1_2(B_{R_0})} \le C \|F\|_{L^2(B_{R_0})}$.

Proof: Using the gauge given by Theorem 5.1, apply Lemma 1.3 of [U1]. Note that $\|F\|_{q,B_{R_0}} < k(n)$ as required in Lemma 1.3, if R_0 is small enough, since $F \in L_P(B_{R_0})$ for all $1 \le P < \infty$ (apply Holder's inequality). Q.E.D.

At this point, the proof of Theorem M follows exactly the proof on the last two pages of [pg. 15-16] of [Sb2]. Q.E.D.

We are finished.

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