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AND
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INTRODUCTION. Let L be a numerically effective and big line bundle on a smooth projective surface S. Questions about the spannedness and very ampleness of $K_S \otimes L$ arise naturally, e.g. [Bom], [So-V]. Recently, Reider [Rdr] introduced a technique which yields answers to these questions that are not obtainable by previous methods.

Motivated by Bombieri's classical work [Bom] we are interested in using Reider's method to answer the following question.

<u>Question.</u> Let S be a smooth projective surface on which K_S is ample (this is relaxed to nef and big in §3). What is the smallest integer $t > 0$ so that the map associated to $\Gamma(K_S^t)$ gives a "k-th order embedding".

The first problem is to decide what we mean by a k-th order embedding. We introduce the concept of k-spannedness. Let $\mathcal{L}$ be a line bundle on S (resp. on a nonsingular curve C). We say that $\mathcal{L}$ is k-spanned for $k \geq 0$ if for any distinct points $z_1, \ldots, z_r$ on S (resp. on C) and any positive integers $k_1,\ldots,k_r$ with $\sum_{i=1}^{r} k_i = k+1$, the map

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{\mathfrak{Z}})$$

is onto, where $(\mathfrak{Z},\mathcal{O}_{\mathfrak{Z}})$ is an (admissible) 0-cycle defined by the ideal sheaf $\mathcal{I}_{\mathfrak{Z}}$ where $\mathcal{I}_{\mathfrak{Z}}\mathcal{O}_{S,z}$ is isomorphic to $\mathcal{O}_{S,z}$ (respectively $\mathcal{O}_{C,z}$) for $z \notin (z_1,\ldots,z_r)$ and $\mathcal{I}_{\mathfrak{Z}}\mathcal{O}_{S,z_i}$ is generated by $(x_i, y_i^{k_i})$ at z_i, with (x_i,y_i) local coordinates at z_i on S, $i = 1,\ldots,r$ (respectively $\mathcal{I}_Z$ is generated by $y_i^{k_i}$, y_i local coordinate at z_i on C).

Note that 0-spannedness is equivalent to $\mathcal{L}$ be spanned and 1-spannedness is equivalent to $\mathcal{L}$ being very ample.

There are a number of other notions of k-th order embedding (see §4 for some discussion). Our choice of the above definition was guided by two criteria:

1) the definition should be the weakest definition that includes the obvious examples (e.g. if L is very ample then L^k should give a k-th order embedding) but for which strong results can be proven;

2) there should be a strong criterion for $K_S \otimes L$ to give a k-th order embedding, where L is nef and big.

In this article we show that there is a very satisfactory answer to 2). We use this to answer the question for which positive t the line bundle K_S^t (respectively K_S^{-t}) is k-spanned where K_S (respectively K_S^{-1}) is ample. In the article [Be-So] by the first and the last author a detailed investigation of k-spannedness is made.

In §0 we introduce background material that we need.

In §1 we prove that if

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{I}_{\mathcal{Z}} \longrightarrow 0$$

is an exact sequence with $\mathcal{E}$ locally free rank 2 vector bundle on S and $L \cdot L \geq 4k + 5$ where $(\mathcal{Z}, \mathcal{O}_S/\mathcal{I}_{\mathcal{Z}})$ is a 0-cycle of length $k + 1 \geq 1$, then there is an effective divisor D containing $\mathcal{Z}$ and such that

$$L \cdot D - k - 1 \leq D \cdot D < L \cdot D/2 < k + 1.$$

This follows from Bogomolov's instability theorem along the lines of Reider's results.

In §2 we show that if $K_S \otimes L$ is not k-spanned and $L \cdot L \geq 4k + 5$ these appropriate exact sequences as above exist. We give an explicit construction of the sequences and bundles. Van de Ven explained this construction to the third author for zero cycles $p_1+\dots+p_{k+1}$ where the p_i's are all distinct. One can also construct the bundles by a general result of Catanese [C].

In §3 we derive a number of applications. We refer the reader to theorems (3.2) and (3.8).

Finally in §4 we discuss the definition of k-th order embedding that we would have preferred to use. This other notion is aesthetically quite nice and it has a good geometric interpretation. It also agrees with k-spannedness for $k = 0,1,2$ (see (4.1)). Unfortunately it is technically hard to verify which L give k-th order embeddings with respect to this alternate definition.

We would like to call attention to [Sa] where Reider's method is also studied.

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§0. Notation and background material

We work over the complex numbers. Throughout the paper, S always denotes a smooth connected projective surface. We denote its structure sheaf by $\mathcal{O}_S$ and the canonical sheaf of the holomorphic 2-forms by K_S. For any coherent sheaf $\mathcal{F}$ on S we shall denote by $h^i(\mathcal{F})$ the complex dimension of $H^i(\mathcal{F})$, where $H^i(\mathcal{F})$ stands for $H^i(S,\mathcal{F})$.

Let $\mathcal{L}$ be a line bundle on S. $\mathcal{L}$ is said to be numerically effective, nef for short, if $\mathcal{L} \cdot C \geq 0$ for each irreducible curve C on S, and in this case $\mathcal{L}$ is said to be big if $c_1(\mathcal{L})^2 > 0$, where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$.

(0.1) We fix some more notation.
$\sim$ (resp. $\approx$) the numerical (resp. linear) equivalence of divisors; $\chi(\mathcal{L}) = \sum(-1)^i h^i(\mathcal{L})$, the Euler characteristic of a line bundle $\mathcal{L}$; $|\mathcal{L}|$, the complete linear system associated to $\mathcal{L}$ and $\Gamma(\mathcal{L})$, the space of its global sections.

As usual we don't distinguish between locally free sheaves and vector bundles, nor between line bundles and Cartier divisors. Hence we shall freely switch from the multiplicative to the additive notation and viceversa.

(0.2) Given an ideal sheaf $\mathcal{I}_{\mathfrak{Z}}$ definining a 0-dimensional scheme $\mathfrak{Z}$ on S, we set

$$\deg \mathfrak{Z} = \text{length } (\mathcal{O}_S/\mathcal{I}_{\mathfrak{Z}}).$$

(0.3) THEOREM (Bogomolov, [Bog], [Re]) Let $\mathcal{E}$ be a locally free rank 2 vector bundle on S, such that $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. Then there exist line bundles $\mathcal{L}$, $\mathcal{M}$, a 0-dimensional scheme $\mathfrak{X}$ on S and an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \otimes \mathcal{I}_{\mathfrak{X}} \longrightarrow 0.$$

Furthermore,

i) $(\mathcal{L} - \mathcal{M}) \cdot (\mathcal{L} - \mathcal{M}) > 0$;

ii) $(\mathcal{L} - \mathcal{M}) \cdot H > 0$ for any ample line bundle H on S.

(0.3.1) Note that $\mathcal{L} - \mathcal{M}$ in (0.3) is Q-effective. Indeed, $\chi(m(\mathcal{L} - \mathcal{M}))$ goes to the infinity as m^2 by i). Then $h^0(m(\mathcal{L} - \mathcal{M}))$ goes to the infinity as m^2 since $h^2(m(\mathcal{L} - \mathcal{M})) = 0$ for $m >> 0$ in view of ii).

(0.4) k-spannedness. Let $\mathcal{L}$ be a line bundle on S (resp. on a nonsingular curve C). We say that $\mathcal{L}$ is k-spanned for $k \geq 0$ if for any distinct points, $z_1, \ldots, z_r$ on S (resp. on C) and any positive integers $k_1, \ldots, k_r$ with $\sum_{i=1}^{r} k_i = k + 1$, the map

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{\mathfrak{Z}})$$

is onto, where $(\mathfrak{Z},\mathcal{O}_{\mathfrak{Z}})$ is a 0-dimensional subscheme defined by the ideal sheaf $\mathcal{J}_{\mathfrak{Z}}$ where $\mathcal{J}_{\mathfrak{Z}}\mathcal{O}_{S,z}$ is $\mathcal{O}_{S,z}$ (resp. $\mathcal{O}_{C,z}$) for $z \notin \{z_1,\dots,z_r\}$ and $\mathcal{J}_{\mathfrak{Z}}\mathcal{O}_{S,z_i}$ is generated by $(x_i,y_i^{k_i})$ at z_i, with (x_i,y_i) local coordinates at z_i on S, $i = 1,\dots,r$ (resp. $\mathcal{J}_{\mathfrak{Z}}$ is generated by $y_i^{k_i}$, y_i local coordinate at z_i on C). We call a 0-cycle $\mathfrak{Z}$ as above <u>admissible</u>.

(0.4.1) Note that 0-spanned is equivalent to $\mathcal{L}$ being spanned by $\Gamma(\mathcal{L})$ and 1-spanned is equivalent to very ample.

Note also that if $\mathcal{L}$ is k-spanned, then $\mathcal{L} \cdot C \geq k$ for every irreducible reduced curve C on S, with equality only if C is a smooth rational curve in $\mathbb{P}^k$.

□

For any further background material we refer to [Rdr], [Re] and [Be-So].

§1. Numerical conditions

Let $\mathcal{E}$ be a locally free rank 2 vector bundle on a surface S. Assume that $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$ and let $\mathcal{E}$ given by an exact sequence

$$(1.1) \qquad 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{J}_{\mathfrak{X}} \longrightarrow 0$$

where the ideal sheaf $\mathcal{J}_{\mathfrak{X}}$ defines some 0-dimensional analytic set $\mathfrak{X}$ and L is a nef and big line bundle. Let

$$(1.2) \qquad 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \otimes \mathcal{J}_{\mathfrak{X}} \longrightarrow 0$$

be the exact sequence given by Bogomolov's theorem (0.3). Note that sequences (1.1) and (1.2) lead to

$$(1.3) \qquad c_1(\mathcal{E}) \approx L \approx \mathcal{L} + \mathcal{M} \quad .$$

The following generalizes some numerical conditions obtained by Reider's method [Rdr].

(1.4) PROPOSITION. <u>Let</u> L <u>be a nef and big line bundle on a surface</u> S <u>and let</u> $\mathcal{E}$ <u>be a locally free rank</u> 2 <u>vector bundle on</u> S <u>with</u> $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. <u>With the notation as above, assume</u> $\mathcal{J}_{\mathfrak{X}}$ <u>is non-trivial</u>. <u>Then the sequence</u>

$$\mathcal{L} \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{I}_{\mathcal{Z}}$$

obtained from (1.1) *and* (1.2) *is non trivial,* $\mathcal{L} \approx L - D$ *where* D *is an effective divisor containing* $\mathcal{Z}$, $\mathcal{M} \approx \mathcal{O}_S(D)$ *and* $L - 2D$ *is* $\mathbb{Q}$-*effective*. *Furthermore*

(1.4.1) $\quad L \cdot D - \deg \mathcal{Z} \leq D \cdot D < L \cdot D/2 < \deg \mathcal{Z}$.

If $(L \cdot D)^2 = (L \cdot L)(D \cdot D)$, *then* $L \sim \lambda D$ *for some* $\lambda \in \mathbb{Q}$, $2 < \lambda \leq 1 + \deg \mathcal{Z}/D \cdot D$.

Proof. First, note that the composition $\mathcal{L} \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{I}_{\mathcal{Z}}$ cannot be the zero morphism. If it was, then $\mathcal{L} \subseteq \ker(\mathcal{E} \longrightarrow L \otimes \mathcal{I}_{\mathcal{Z}}) = \mathcal{O}_S$ that is $\mathcal{L} \approx \mathcal{O}_S(-\mathfrak{D})$ for some effective divisor $\mathfrak{D}$ and $L \approx \mathcal{M} - \mathfrak{D}$ by (1.3). Hence, for an arbitrary ample line bundle H, $(\mathcal{L} - \mathcal{M}) \cdot H = (-L - 2\mathfrak{D}) \cdot H < 0$ since L is nef and $\mathfrak{D}$ is effective, this contradicting (0.3), ii). Thus we have, if $\mathcal{M} \approx \mathcal{O}_S(D)$,

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \mathcal{O}_S(D) \otimes \mathcal{I}_{\mathcal{Z}} & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{E} & \longrightarrow & L \otimes \mathcal{I}_{\mathcal{Z}} & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & \mathcal{L} & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where $\mathcal{L} \approx L - D$ by (1.3). We claim that D is effective and it contains $\mathcal{Z}$. To see this, tensorize the sequences above by $\mathcal{L}^{-1}$. Then $h^0(\mathcal{E} \otimes \mathcal{L}^{-1}) \neq 0$, so after proving that $h^0(\mathcal{L}^{-1}) = 0$, we get an embedding

$$0 \longrightarrow H^0(E \otimes \mathcal{L}^{-1}) \longrightarrow H^0(\mathcal{O}_S(D) \otimes \mathcal{I}_{\mathcal{Z}})$$

and we are done. Now, for any ample line bundle H on S, we have $L \cdot H = (\mathcal{L} + D) \cdot H \geq 0$ since L is nef and further $(\mathcal{L} - D) \cdot H > 0$ by (0.3),ii). Therefore $\mathcal{L} \cdot H > 0$ which implies that $\mathcal{L}^{-1}$ is not effective.

Now, again by Bogomolov's theorem (0.3), we have

(1.4.2) $$(L - 2D) \cdot L > 0$$

while from the above diagram we see that

(1.4.3) $$c_2(E) = (L - D) \cdot D + \deg \mathfrak{X} = \deg \mathcal{Z}$$

and hence

(1.4.4) $$L \cdot D - \deg \mathcal{Z} \leq D \cdot D.$$

The Hodge index theorem and (1.4.2) yield

(1.4.5) $$2D \cdot D < L \cdot D$$

so we find $D \cdot D < \deg \mathfrak{X}$ by combining (1.4.4) and (1.4.5). Therefore $L \cdot D < 2 \deg \mathfrak{X}$ again by (1.4.4), which proves (1.4.1).

The fact that $L - 2D$ is $\mathbb{Q}$-effective follows from (1.4.2) as in (0.3.1).

Finally if $(L \cdot D)^2 = (L \cdot L)(D \cdot D)$ it has to be $L \cdot D \neq 0$ since otherwise $D \cdot D = 0$, contradicting (1.4.5); then $L \sim \lambda D$, for some $\lambda \in \mathbb{Q}$, by the Hodge index theorem. From (1.4.4) we get $\lambda \lesssim 1 + \deg \mathfrak{X}/D \cdot D$, while (1.4.2) yields $\lambda > 2$.

§2. Construction of vector bundles.

Let L be an arbitrary line bundle on a surface S. In this section we construct, in an explicit geometric way, locally free rank 2 vector bundles $\mathcal{E}$ on S given by exact sequences of type (1.1) of previous section, for admissible 0-cycles $\mathcal{Z}$ as in Definition (0.4). For simplicity we carry out a detailed computation in one special case which, however, completely describes all the techniques that are to be used. This approach, in the case of 0-cycles $\mathcal{Z} = p_1+\ldots+p_r$ with all p_i's distinct was shown to the third author by A. Van de Ven.

The existence of the vector bundles we construct also follows from necessary and sufficient conditions due to Catanese [C], for certain extensions of sheaves to be locally free.

(2.1) PROPOSITION. *Let* L *be a line bundle on a surface* S *and let* $\mathcal{Z}$ *be an admissible* 0-*cycle such that* $\deg \mathcal{Z} = n$ *and* $\mathrm{Supp}(\mathcal{Z})$ *is a single point* z (*respectively* n *distinct points* $z_1,\ldots,z_n$) *on* S. *Assume that* $K_S + L$ *is* $(n-1)$-*spanned and the map*

$$\Gamma(K_S \otimes L) \longrightarrow \Gamma(K_S \otimes L \otimes \mathcal{O}_{\mathcal{Z}})$$

is not onto, i.e. $|K_S + L|$ does not separate n-infinitely near points at z (resp. n distinct points $z_1,\dots,z_n$). Then there exists a locally free rank 2-vector bundle $\mathcal{E}$ on S given by the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{I}_{\mathfrak{Z}} \longrightarrow 0.$$

Proof. Here we only carry out the case when $\mathrm{Supp}(\mathfrak{Z})$ is a single point z. The proof in the case $\mathrm{Supp}(\mathfrak{Z}) = \{z_1,\dots,z_n\}$ is a straightforwad modification of it.

Let (x,y) be local coordinates at z. Then $\mathcal{I}_{\mathfrak{Z}}\mathcal{O}_{S,p} = \mathcal{O}_{S,p}$ for $p \neq z$ and $\mathcal{I}_{\mathfrak{Z}}\mathcal{O}_{S,z} = (x,y^n)$. Let $\mathcal{I}'$ be the ideal of $\mathcal{O}_S$ defined by $\mathcal{I}'\mathcal{O}_{S,p} = \mathcal{O}_{S,p}$ if $p \neq z$ and $\mathcal{I}'\mathcal{O}_{S,z} = (x,y^{n-1})$. Hence from the assumptions made we get

$$(2.2) \qquad \Gamma(K_S \otimes L \otimes \mathcal{I}_{\mathfrak{Z}}) \cong \Gamma(K_S \otimes L \otimes \mathcal{I}')$$

To see this look at the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{C}_z & \longrightarrow & \mathbb{C}_z^n & \longrightarrow & \mathbb{C}_z^{n-1} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & K_S \otimes L & \xrightarrow{\sim} & K_S \otimes L & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & & & K_S \otimes L \otimes \mathcal{I}_{\mathfrak{Z}} & \longrightarrow & K_S \otimes L \otimes \mathcal{I}' & \longrightarrow & \mathbb{C}_z \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & . &
\end{array}$$

If (2.2) is not true, then

$$n \geq \dim(\Gamma(K_S \otimes L)/\Gamma(K_S \otimes L \otimes \mathcal{I}_{\mathfrak{Z}})) > \dim(\Gamma(K_S \otimes L)/\Gamma(K_S \otimes L \otimes \mathcal{I}')) = n-1$$

and hence $\Gamma(K_S \otimes L) \longrightarrow \mathbb{C}^n_z$ is onto, a contradiciton.

From the bottom line of the above diagram and (2.2) we see that there is an injection

$$0 \longrightarrow \mathbb{C} \longrightarrow H^1(K_S \otimes L \otimes \mathcal{I}_{\mathfrak{Z}}). \tag{2.2.1}$$

Note that in the case when $\mathrm{Supp}(\mathfrak{Z}) = \{z_1, \ldots, z_n\}$, isomorphism (2.2) is equivalent to say that $\mathfrak{Z}$ satisfies the <u>Cauley-Bacharach property relatively to</u> $|K_S \otimes L|$: that is, for any divisor $A \in |K_S \otimes L|$ passing through $\mathrm{Supp}(\mathfrak{Z}) - z_j$, $j \in \{1, \ldots, n\}$, then A contains $\mathrm{Supp}\,(\mathfrak{Z})$ entirely (compare with [G-H]).

To go on, we need to fix some more notation. Let $\pi_i : S_i \longrightarrow S_{i-1}$ be the blowing up of S_{i-1} at a point p_{i-1}, $E_i = \pi_i^{-1}(p_{i-1})$ the exceptional divisors, $i = 1, \ldots, n$, $\overline{E}_i$ the proper transform of E_i under $\pi_{i+1} \circ \ldots \circ \pi_n$, $i = 1, \ldots, n-1$, and $\overline{E}_n = E_n$. Further one sees that

$$\begin{cases} \overline{E}_i \cdot \overline{E}_j = 1 \text{ if } |i-j| = 1; \ \overline{E}_i \cdot \overline{E}_j = 0 \text{ if } |i-j| > 1; \\ \\ \overline{E}_i \cdot \overline{E}_i = -2 \ \text{ if } \ i \neq n; \ \ \overline{E}_n \cdot \overline{E}_n = -1. \end{cases} \tag{2.3}$$

Let $\bar{S} = S_n$, $S_0 = S$ and denote by $\pi : \bar{S} \longrightarrow S$ the composition of the π_i's. Now, by a suitable choice of the points p_i's , a straightforward check shows that

$$(2.4)\qquad \begin{cases} \pi^*\mathcal{I}' \approx \pi^*\mathcal{I}_{\mathfrak{X}} + \bar{E}_n \approx -\bar{E}_1 - 2\bar{E}_2 - \ldots - (n-1)\bar{E}_n; \\ \pi^*\mathcal{I}_{\mathfrak{X}} \approx -\bar{E}_1 - 2\bar{E}_2 - \ldots - n\bar{E}_n; \\ K_{\bar{S}} \approx \pi^* K_S + \bar{E}_1 + 2\bar{E}_2 + \ldots + n\bar{E}_n . \end{cases}$$

Let $\bar{L} = \pi^* L$. Therefore by Leray's spectral sequence and the Serre duality,

$$H^1(K_S \otimes L \otimes \mathcal{I}_{\mathfrak{X}}) \cong H^1(K_{\bar{S}} + \bar{L} - 2\sum_i i\bar{E}_i) \cong H^1(2\sum_i i\bar{E}_i - \bar{L})$$

and hence $H^1(2 \sum_i iE_i - \bar{L}) \neq (0)$ by (2.2.1). Then there exists a locally free rank 2 vector bundle $\mathcal{G}$ on $\bar{S}$ and a non trivial extension

$$\#) \qquad 0 \longrightarrow \mathcal{O}_{\bar{S}}(\sum_i i\bar{E}_i) \longrightarrow \mathcal{O}_{\bar{S}}(\mathcal{G}) \longrightarrow \mathcal{O}_{\bar{S}}(\bar{L} - \sum_i i\bar{E}_i) \longrightarrow 0 .$$

Putting $\Delta = \sum_i i\bar{E}_i$ one has

$$\Delta \cdot \bar{E}_n = -1; \quad \Delta \cdot \bar{E}_i = 0, \ i = 1,\ldots,n-1$$

by (2.3) and hence Δ cuts on Δ_{red} a divisor δ , of degree -1, which lies on the component $\bar{E}_n$. Look at the restriction

$$\#)_{\overline{E}_n} \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(\delta) \longrightarrow \mathcal{G}_{\overline{E}_n} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-\delta) \longrightarrow 0 .$$

(2.5) CLAIM. The sequence $\#)_{\overline{E}_n}$ is a non trivial extension with $\mathcal{G}_{\overline{E}_n} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $(\pi_n)_* \mathcal{G}$ is a locally free rank 2 vector bundle on S_{n-1}.

Proof of the Claim. Look at the exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{S}}(2\Lambda - \overline{L} - \overline{E}_n) \longrightarrow \mathcal{O}_{\overline{S}}(2\Lambda - \overline{L}) \longrightarrow \mathcal{O}_{\overline{E}_n}(2\Lambda - \overline{L}) \longrightarrow 0 .$$

From Leray's spectral sequence, Serre's duality and (2.2) we get an isomorphism

$$H^2(2\Lambda - \overline{L} - \overline{E}_n) \cong H^2(2\Lambda - \overline{L}).$$

Then the restriction map

$$\rho_n : H^1(2\Lambda - \overline{L}) \longrightarrow H^1(\overline{E}_n, 2\Lambda - \overline{L}) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(-2))$$

is onto. Now, $H^1(\overline{E}_n, 2\Lambda - \overline{L})$ is a 1-dimensional vector space which parametrizes the non-trivial extensions of type $\#)_{\overline{E}_n}$. Therefore since ρ_n is surjective, we can choose $\mathcal{G}$ in #) in such a way that $\mathcal{G}_{\overline{E}_n} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Indeed by tensoring with $\mathcal{O}_{\mathbb{P}^1}(-1)$ the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathcal{T}_{\mathbb{P}^1} \longrightarrow 0$$

we find a non-trivial extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

To prove the second part of the Claim, let $U \subset S_{n-1}$ be an affine neighborhood of $p_{n-1} = \pi_n(\bar{E}_n)$. From the exact sequence

$$0 \longrightarrow \mathcal{J}_{\bar{E}_n} \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_{\bar{E}_n} \longrightarrow 0$$

we find by the above a map

$$\varphi : \Gamma(\pi_n^{-1}(U), \mathcal{G}) \longrightarrow \mathbb{C} \oplus \mathbb{C}$$

If φ is surjective, we are done. Indeed, $s_0 = \varphi^{-1}(0,1)$, $s_1 = \varphi^{-1}(1,0)$ are independent on $\Gamma(\pi_n^{-1}(U), \mathcal{O}_{\bar{S}}) \cong \Gamma(U, \mathcal{O}_{S_{n-1}})$ in a neighborhood of $\bar{E}_n$, so that, since $\mathcal{G}$ is locally free of rank 2, one has

$$\Gamma(U, (\pi_n)_* \mathcal{G}) \cong s_0 \Gamma(U, \mathcal{O}_{S_{n-1}}) \oplus s_1 \Gamma(U, \mathcal{O}_{S_{n-1}})$$

which says that $(\pi_n)_* \mathcal{G}$ is free at p_{n-1}. To see that φ is surjective, write $\mathcal{J}_n = \mathcal{J}_{\bar{E}_n}$ and set $\mathcal{F}_m = \mathcal{J}_n \mathcal{G} \otimes \mathcal{O}_{\bar{S}} / \mathcal{J}_n^m$, $m \geq 1$. Look at the exact sequence

$$0 \longrightarrow \mathcal{J}_n\mathcal{G} \otimes \mathcal{J}_n^m/\mathcal{J}_n^{m+1} \longrightarrow \mathcal{F}_{m+1} \longrightarrow \mathcal{F}_m \longrightarrow 0$$

which becomes by the above

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(m+1) \oplus \mathcal{O}_{\mathbb{P}^1}(m+1) \longrightarrow \mathcal{F}_{m+1} \longrightarrow \mathcal{F}_m \longrightarrow 0 .$$

Then $H^1(\mathcal{F}_{m+1}) \cong H^1(\mathcal{F}_m)$, $m \geq 1$. Since $H^1(\mathcal{F}_1) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = (0)$, we get $H^1(\mathcal{F}_m) = 0$, $m \geq 1$. Thus Grothendieck's theorem on formal functions says that $R^i(\pi_n)_*(\mathcal{J}_n\mathcal{G}) = 0$, $i \geq 1$, and hence Leray's spectral sequence gives us $H^1(\pi^{-1}(U), \mathcal{J}_n\mathcal{G}) \cong H^1(U, (\pi_n)_*\mathcal{J}_n\mathcal{G}) = (0)$, so φ is surjective.

□

Note that

(2.6) $$\mathcal{G}_{\Delta_{red}} \cong \mathcal{O}_{\Delta_{red}} \oplus \mathcal{O}_{\Delta_{red}}$$

Indeed, since $\Delta \cdot \bar{E}_i = 0, i \neq n$, we get from #) an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{G}_{\bar{E}_i} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

Therefore $\mathcal{G}_{\bar{E}_i} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ since $H^1(\mathcal{O}_{\mathbb{P}^1}) = (0)$. Then (2.6) follows by using the fact that Δ_{red} is a tree of smooth rational curves, i.e. $\mathcal{G}_{\bar{E}_i}$ is trivial for all $i = 1,\ldots,n$ and since there are no cycles on the graph associated to Δ_{red}, the trivializations patch to a global trivialization on Δ_{red}. Note also that

$$(\pi_n)_*\mathcal{G}\Big|_{\sum_{i=1}^{n}(\pi_n)_*\overline{E}_i} \cong \mathcal{O} \oplus \mathcal{O}.$$

This is immediate from Claim (2.5) and (2.6) above. Now, a straightforward check shows that $E_{n-1}^2 = -1$ on S_{n-1} and further $(\pi_n)_*\Delta \cdot (\pi_n)_*\overline{E}_i = 0$, $i = 1,\ldots,n-2$, $(\pi_n)_*\Delta\cdot E_{n-1} = -1$. Again, by using this and the arguments as in the proof of Claim (2.5) it thus follows that $(\pi_{n-1} \circ \pi_n)_*\mathcal{G}$ is free at $\pi_{n-1}(E_{n-1}) = p_{n-2}$ on S_{n-2}. By going on in this way, a step by step proof shows that $\mathcal{E} = \pi_*\mathcal{G}$ is a locally free rank 2 vector bundle on S. Then by pushing forward sequence #) we find an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{J}_{\mathfrak{Z}} \longrightarrow 0$$

where $\deg \mathfrak{Z} = n$. This proves Proposition (2.1).

§3. Applications

From the results of sections 1,2 we deduce the following

(3.1) THEOREM. *Let* L *be a nef and big line bundle on a surface* S *and let* $L \cdot L \geq 4k + 5$. *Then either* $K_S + L$ *is* k-*spanned or there exists an effective divisor* D *such that* $L - 2D$ *is* $\mathbb{Q}$-*effective,* D *contains some admissible* 0-*cycle of degree* $\leq k + 1$ *where the* k-*spannedness fails and*

$$L \cdot D - k - 1 \leq D \cdot D < L \cdot D/2 < k + 1.$$

Proof. If $K_S + L$ is not k-spanned there exists a 0-dimensional cycle $\mathfrak{Z}$ on S on the same type as in (0.4) and of degree $\deg \mathfrak{Z} \leq k + 1$ such that

$$\Gamma(K_S \otimes L) \longrightarrow \Gamma(K_S \otimes L \otimes \mathcal{O}_{\mathfrak{Z}})$$

is not onto. Then by (2.1) there exist a locally free rank 2 vector bundle $\mathcal{E}$ on S and an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{J}_{\mathfrak{Z}} \longrightarrow 0$$

where $\mathcal{O}_{\mathfrak{Z}} = \mathcal{O}_S/\mathcal{J}_{\mathfrak{Z}}$. Since $c_1^2(\mathcal{E}) = L \cdot L$ and $c_2(\mathcal{E}) = \deg \mathfrak{Z}$, the assumption $L \cdot L \geq 4k + 5$ leads to $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, so Proposition (1.4) applies to give the result.

□

The previous results allows us to give some answer to the following

Question. Let L be a nef and big line bundle on a surface S. What is the smallest integer $t > 0$ such that the map associated to $\Gamma(K_S \otimes L^t)$ gives a "k-th order embedding" where by k-th order embedding we mean $K_S \otimes L^t$ to be k-spanned.

□

Since a lot of results are known in the "classical" cases $k = 0,1$, we shall assume from now on that $k \geq 2$.

(3.2) THEOREM. Let L be a nef and big line bundle on a surface S and let $t^2 L \cdot L \geq 4k + 5$. Then $K_S \otimes L^t$ is k-spanned, outside of a finite numbers of curves, for $t = k, k+1, k+2, k+3$ unless there exists an effective divisor D with the numerical properties listed in the table below

t	L·D	D·D	L	L·L	k	see examples
k	1	-1				(3.2.1 d)
		0				(3.2.1 c)
		1	~D	1		(3.2.1 a)
	2	1	~λD, λ∈ℤ	≤ 4	2	(3.2.1 b)
		2		2	2 or 3	
		3		1	2,3 or 4	
k+1	1	1	~D			(3.2.1 a)
		0				(3.2.1 c)
k+2	1	1	~D	1		(3.2.1 a)

Furthermore, under the extra assumptions $k \geq 3$ and $L \cdot L \geq 5$, $K_S \otimes L^{k-1}$ is k-spanned unless there exists an effective divisor D such that either $L \cdot D = 1$, $D \cdot D = -2,-1,0$ or $L \cdot D = 2$, $D \cdot D = 0$, $k = 3$ (see example (3.2.1 e)).

Proof. In view of Theorem (3.1), if $K_S \otimes L^t$ is not k-spanned, there exists an effective divisor D, containing an admissible 0-cycle of degree $\leq k + 1$ where the k-spannedness fails and such that

$$*) \qquad tL \cdot D - k - 1 \leq D \cdot D < tL \cdot D/2 < k + 1 .$$

Note that, since we are looking for the k-spannedness outside of a finite number of curves, we can assume that $L \cdot D > 0$. Indeed, standard arguments show that the set $\mathcal{U}$ of the irreducible, reduced curves C of S such that $L \cdot C = 0$ is finite. Then clearly $L \cdot D > 0$ if D passes through a point $x \in S \backslash \mathcal{U}$. Now a case by case analysis for each considered value of t, simply using *) and the Hodge index theorem gives the results.

(3.2.1) Let us give some examples showing that the above result is sharp.

(3.2.1 a) Let $S = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(1)$. Since, given a line ℓ on S, $(K_S + (k+i)L) \cdot \ell = k + i - 3 < k$ for $i = 0,1,2$ we see that $K_S + (k+i)L$ is not k-spanned for $i = 0,1,2$. In this case $D = \ell, L \sim D$ and $L^2 = 1$.

(3.2.1 b) Let $S = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(2)$. Note that $(K_S + 2L)\cdot\ell < 2$ and thus $K_S + 2L$ is not k-spanned for $k \geq 2$. Here $D = \ell$, $L^2 = 4$, $L \cdot D = 2$, $D^2 = 1$, $L \sim 2D$.

(3.2.1 c) Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}_S(1,1)$. Note that $(K_S + (k+i)L) \cdot H = k + i - 2 < k$ for $i = 0,1$, where H is a curve of type $(1,0)$ or $(0,1)$. Thus $K_S + (k + i)L$ is not k-spanned for $i = 0,1$. Here $D = H$, $D \cdot D = 0$, $L^2 = 2$, $L \cdot D = 1$.

(3.2.1 d) Let S be a cubic surface in $\mathbb{P}^3$. Let $L = \mathcal{O}_{\mathbb{P}^3}(1)|_S$. Note that $(K_S + kL) \cdot D = k - 1$ for any line D on S. Thus $K_S + kL$ is not k-spanned. Here $D \cdot D = -1$, $L \cdot D = 1$.

(3.2.1 e) Let (S,L) be a conic bundle, i.e. S is a birationally ruled surface and L is a nef and big line bundle on S such that $L \cdot f = 2$, f a fibre of the ruling, and L is relatively ample with respect to the ruling. Then $(K_S + 2L)\cdot f = 2$ and hence $K_S + 2L$ is not 3-spanned. Here $D = f$, $D \cdot D = 0$, $L \cdot D = 2$.

□

We derive now from (3.1) and (3.2) a number of consequences.

(3.3) COROLLARY. Let S <u>be a surface with ample canonical bundle</u>. <u>Assume</u> $t^2K_S \cdot K_S \geq 4k + 5$. <u>Then</u> K_S^t <u>is</u> k-<u>spanned for</u>

$t = k + 4$ and is k-spanned for $t = k + 3, k + 2, k + 1$ unless there exists an effective divisor D such that either $t = k+3, k+2$, $K_S \cdot D = D \cdot D = 1$, $K_S \sim D$, or $t = k+1$, $K_S \cdot D = 1$ and either $D \cdot D = -1, 1$, or $k=2,3$ and $K_S \cdot D = D \cdot D = 2$, $K_S \sim D$.

Furthermore, if $k \geq 3$ and $K_S \cdot K_S \geq 5$, then K_S^k is k-spanned unless there exists an effective divisor D such that either $K_S \cdot D = 1$, $D \cdot D = -1$ or $k = 3$ and $K_S \cdot D = 2$, $D \cdot D = 0$.

Proof. It follows from Theorem (3.2), by noting that $K_S \cdot D$ and $D \cdot D$ have the same parity by the genus formula.

(3.4) COROLLARY. Let S be as in Corollary (3.3). Further assume that the cotangent bundle $\mathcal{T}_S^*$ is ample. Then the two cases in (3.3) with $K_S \cdot D = 1$, $D \cdot D = -1$ can't occur.

Proof. If $K_S \cdot D = 1$, $D \cdot D = -1$, then there exists an irreducible reduced component C of D such that $K_S \cdot C = 1$, $C^2 \leq 0$. Therefore $g(C) = 0$ or 1, where $g(C)$ denotes the arithmetic genus of C. Let $\nu : \tilde{C} \longrightarrow C$ be the normalization of C. Then the cotangent bundle $\mathcal{T}_{\tilde{C}}^*$ is ample since $\nu^* \mathcal{T}_S^* \subset \mathcal{T}_{\tilde{C}}^*$ and $\nu^* \mathcal{T}_S^*$ is ample being ν a finite morphism. Therefore $g(\tilde{C}) \geq 2$, a contradiction.

(3.5) COROLLARY. Let S be a surface with $K_S \sim 0$ and let L be a nef and big line bundle on S with $t^2 L \cdot L \geq 4k + 5$.

Then L^t is k-spanned for $t = k+2$ and is k-spanned for $t = k+1, k$ unless there exists an effective divisor D such that either, $t = k+1, k$, $L \cdot D = 1$, $D \cdot D = 0$, or $t = k$ with $k = 2, 3$, $L \cdot D = 2$, $D \cdot D = 2$, $L \sim D$.

Further, if $k \geq 3$ and $L \cdot L \geq 5$, L^{k-1} is k-spanned unless there exists an effective divisor D such that either $L \cdot D = 1$, $D \cdot D = 0, 2$, or $L \cdot D = 2$, $D \cdot D = 0$, $k = 3$.

Proof. It follows from Theorem (3.2) by noting that $D \cdot D$ is even by the genus formula.

(3.6) COROLLARY. Let S be a Del Pezzo surface (i.e. $-K_S$ ample). Then K_S^{-t} is k-spanned if and only if

$t \geq k+2$ if $K_S \cdot K_S = 1$;
$t \geq k/3$ if $S = \mathbb{P}^2$;
$t \geq k/2$ if $S = \mathbb{P}^1 \times \mathbb{P}^1$;
$t \geq k$ if $K_S \cdot K_S \geq 3$ or $K_S \cdot K_S = 2$ and $k \neq 1$.

Further if $K_S \cdot K_S = 2$, K_S^{-t} is very ample iff $t \geq 2$.

Proof. If $S = \mathbb{P}^2$, $K_S \otimes \mathcal{O}_S(k+3) \approx \mathcal{O}_S(k)$ is k-spanned by (3.2), so that $K_S^{-t} \approx \mathcal{O}_S(3t)$ is k-spanned if and only if $3t \geq k$.

If $S = \mathbb{P}^1 \times \mathbb{P}^1$, again by (3.2), $K_S \otimes \mathcal{O}_S(k+2) \approx \mathcal{O}_S(k)$ is k-spanned except possibly if $\mathcal{O}_S(1) \cdot D = D \cdot D = 1$ for some effective divisor D . If D is a curve of type (a,b) this leads to $2ab = 1$, a contradiction. Therefore $\mathcal{O}_S(k)$ is k-spanned and hence $K_S^{-t} \approx \mathcal{O}_S(2t)$ is k-spanned too if and only if $2t \geq k$.

Now, let $K_S \cdot K_S \geq 3$ or $K_S \cdot K_S = 2$, $k \neq 1$ and $t \geq k$. Note that condition $(t+1)^2 K_S \cdot K_S \geq 4k + 5$ is satisfied, then, if K_S^{-t} is not k-spanned, there exists by (3.1) an effective divisor D such that

$$(t+1)K_S^{-1} \cdot D - k - 1 \leq D \cdot D < (t+1)K_S^{-1} \cdot D/2 < k+1$$

Therefore we have $D \cdot D \geq 0$, $K_S^{-1} \cdot D = 1$ and hence $D \cdot D \geq 1$ since $K_S^{-1} \cdot D$, $D \cdot D$ have the same parity by the genus formula. But $(D \cdot D)(K_S \cdot K_S) \leq (K_S \cdot D)^2$ by the Hodge index theorem, so we get $D \cdot D = 1$ and $D \sim K_S^{-1}$, a contradiction. Thus K_S^{-t} is k-spanned if $t \geq k$. To see the converse note that there exists a line ℓ on S with $\ell^2 = K_S \cdot \ell = -1$, since S is not minimal. Then if K_S^{-t} is k-spanned, $t = K_S^{-1} \cdot \ell \geq k$.

Let $K_S \cdot K_S = 1$ and $t \geq k+2$. Then the condition $(t+1)^2 K_S \cdot K_S \geq 4k+5$ is satisfied, so the same argument as above shows that K_S^{-t} is k-spanned whenever $t \geq k+2$. As to the converse, note that the general element of $|K_S^{-1}|$ is a smooth elliptic curve E . Then, if K_S^{-t} is k-spanned, one has by [Be-So], (1.4.1), $t = K_S^{-t} \cdot E \geq k+2$.

Finally, in the special case $K_S \cdot K_S = 2$, again the same argument as above shows that K_S^{-t} is very ample if $t \geq 2$; on the other hand it is easy to see that K_S^{-1} is not very ample.

Note also that for $k \leq 1 = K_S \cdot K_S$, one sees that $K_S^{-(k+1)}$ is not k-spanned.

□

It is worth to point out the following general fact.

(3.7) REMARK. Certain natural conditions imply there are no rational curves on a surface S (e.g. $\mathcal{T}_S^*$ nef), as well as certain natural conditions imply there are no rational or elliptic curves (e.g., $\mathcal{T}_S^*$ ample or S hyperbolic in the sense of Kobayashi). Let L be a line bundle on S. Then the following hold.

(3.7.1) If (S,L) contains no lines and L is ample and spanned with $L \cdot L \geq 5$, then $K_S \otimes L^t$ is k-spanned if $t \geq (k+2)/2$.

(3.7.2) Assume there are no rational curves on S and let L be very ample with $L \cdot L > 9$. Then $K_S \otimes L^t$ is k-spanned for $t \geq (k+1)/3$.

(3.7.3) Assume there are no rational or elliptic curves on S and let L be very ample with $L \cdot L > 16$. Then $K_S \otimes L^t$ is k-spanned if $t \geq (k+2)/4$.

Let us show (3.7.1). If $K_S \otimes L^t$ is not k-spanned for $t \geq (k+2)/2$, we have from (3.1)

$$L \cdot D(k+2)/2 - k - 1 \leq D \cdot D < L \cdot D(k+2)/4 < k+1$$

for some effective divisor D. Then, since $L \cdot D \geq 2$, the only possible cases are $L \cdot D = 2$ or 3. If $L \cdot D = 2$, then $D \cdot D \geq 1$ and the Hodge index theorem leads to the contradiction $5 \leq (D \cdot D)(L \cdot L) \leq (L \cdot D)^2 = 4$. If $L \cdot D = 3$, then $D \cdot D \geq 2+k/2$ which gives the contradiction $(2+k/2)5 \leq (D \cdot D)(L \cdot L) \leq 9$. Similarly for (3.7.2), (3.7.3). Note that the assumptions made lead to $L \cdot D > 2$, $L \cdot D > 3$, respectively.

□

We conclude this section by looking at the k-spannedness of $K_S \otimes L$ for low values of k.

(3.8) THEOREM. Let L be a nef and big line bundle on a surface S and let $L \cdot L \geq 4k + 5$. Then $K_S + L$ is k-spanned for $k = 2,3$, outside of a finite number of curves, unless there exists an effective divisor D with the numerical properties listed in the table below

k	L·D	D·D
2	1	-2,-1 or 0
	2	-1 or 0
	3	0
	4	1
3	1	-3,-2,-1 or 0
	2	-2,-1 or 0
	3	-1 or 0
	4	0
	5	1
	6	2

<u>Proof.</u> If $K_S \otimes L$ is not k-spanned outside of a finite number of curves, then by (3.1),

$$L\cdot D - k-1 \leq D\cdot D < L\cdot D/2 < k+1$$

for some effective divisor D and we can assume $L\cdot D > 0$.

Let $k = 2$, so $L\cdot L \geq 13$. Note that by the Hodge index theorem it has to be $L\cdot D \leq 4$ since otherwise $D\cdot D \geq 2$ and $(L\cdot L)(D\cdot D) \leq 25$, which leads to a contradiction. In the same way the case $L\cdot D = 3$, $D\cdot D = 1$ is ruled out.

Let $k = 3$, so $L\cdot L \geq 17$. Again, by the Hodge index theorem we see that $L\cdot D \leq 6$ as well as the cases $L\cdot D = 2$,

$D \cdot D = 1$; $L \cdot D = 3$, $D \cdot D = 1$; $L \cdot D = 4$, $D \cdot D = 1$ and $L \cdot D = 5$, $D \cdot D = 2$ lead to contradictions.

(3.9) COROLLARY. _Let_ S _be a surface with ample canonical bundle and with_ $K_S \cdot K_S \geq 13$. _Then_ K_S^2 _is_ 2-_spanned unless there exists an effective divisor_ D _such that either_ $K_S \cdot D = 1$, $D \cdot D = -1$, _or_ $K_S \cdot D = 2$, $D \cdot D = 0$.

Proof. It follows from (3.8) by noting that $K_S \cdot D$ and $D \cdot D$ have the same parity by the genus formula.

(3.10) COROLLARY. _Let_ S _be a surface with_ $K_S \sim 0$ _and let_ L _be an ample line bundle on_ S _with_ $L \cdot L \geq 13$. _Then_ L _is_ 2-_spanned unless there exists an effective divisor_ D _such that either_ $L \cdot D = 1$, $D \cdot D = -2, 0$, _or_ $L \cdot D = 2, 3$, $D \cdot D = 0$.

Proof. It follows from (3.8), by noting that $D \cdot D$ is even by the genus formula.

(3.10) REMARK. The analogues of (3.3), (3.9) and (3.5), (3.10) for K_S and L respectively nef and big are easily shown. Note that in the nef and big case we get results on k-spannedness outside of -2 curves with $K_S \cdot D = 0$ or $L \cdot D = 0$ respectively. We leave this to the reader.

§4. Further remarks.

It is worth to point out that in the special case $k = 2$ the notion of k-spannedness is related in a natural way to properties of the Douady Space $S^{[r]}$ of 0-dimensional subspaces $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of a smooth compact complex surface S with length $(\mathcal{O}_{\mathcal{Z}}) = r$, for $r \leq 3$. Recall that, for any $r, S^{[r]}$ is smooth by a Fogarty's result [F]. Further the natural map from the Hilbert scheme to the Chow scheme gives a birational morphism $\alpha_r : S^{[r]} \longrightarrow S^{(r)}$ where $S^{(r)}$ is the r-th symmetric product of S. Note that α_r is an isomorphism outside of $\alpha_r^{-1}(\Delta)$, where Δ is the diagonal of $S^{(r)}$. We also refer to [I1], [I2] and [Bea].

(4.1) LEMMA. Let $\mathcal{L}$ be a 2-spanned line bundle on S and let $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ be a 0-dimensional subscheme defined by the ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ with length $(\mathcal{O}_{\mathcal{Z}}) = 3$. Then the map $\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}})$ is onto.

Proof. Let $\mathfrak{m}_x$ be the maximal ideal in the analytic local ring $\mathcal{O}_{S,x}$ of a point $x \in S$.

We claim that either $\mathcal{I}_{\mathcal{Z}}/\mathfrak{m}_x^2 \cap \mathcal{I}_{\mathcal{Z}} \neq (0)$ or $\mathcal{I}_{\mathcal{Z}} = \mathfrak{m}_x^2$, where clearly $\mathcal{I}_{\mathcal{Z}}$ stands here for $\mathcal{I}_{\mathcal{Z}}\mathcal{O}_{S,x}$. Indeed, if $\mathcal{I}_{\mathcal{Z}}/\mathfrak{m}_x^2 \cap \mathcal{I}_{\mathcal{Z}} = (0)$, then $\mathcal{I}_{\mathcal{Z}} \subseteq \mathfrak{m}_x^2$; this leads to a surjective morphism $\mathcal{O}_{S,x}/\mathcal{I}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{S,x}/\mathfrak{m}_x^2$ which is in fact an isomorphism since length $(\mathcal{O}_{\mathcal{Z}}) = 3$. Hence $\mathcal{I}_{\mathcal{Z}} = \mathfrak{m}_x^2$.

Now one easily sees that the map

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{S,x}/\mathfrak{m}_x^2)$$

is onto since $\mathcal{L}$ is 2-spanned. To conclude it thus suffices to show that the map

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{S,x}/\mathcal{J}_{\mathfrak{X}})$$

is onto whenever $\mathcal{J}_{\mathfrak{X}}/\mathfrak{m}_x^2 \cap \mathcal{J}_{\mathfrak{X}} \neq (0)$.

To see this, note that there exists a linear element w in $\mathcal{J}_{\mathfrak{X}}$, $w \in \mathfrak{m}_x - \mathfrak{m}_x^2$, which can be considered as a local parameter at x. Let ν be the smallest integer satisfying the following property: there exists some $y \in \mathcal{J}_{\mathfrak{X}}$ with $y \in \mathfrak{m}_x^{\nu} - \mathfrak{m}_x^{\nu+1}$ and $y \notin w\mathcal{O}_{S,x}$.

Note that $\nu \neq 2$. Otherwise we can write, for some $u \in \mathfrak{m}_x$, $y = u^2 \bmod(\mathfrak{m}_x^3)$. Then clearly $y = u'^2$ where $u' = y^{1/2}$ is a local parameter at x. Therefore $(w,u'^2) \subseteq \mathcal{J}_{\mathfrak{X}}$ and hence we get a surjective morphism $\mathcal{O}_{S,x}/(w,u'^2) \longrightarrow \mathcal{O}_{S,x}/\mathcal{J}_{\mathfrak{X}}$. This leads to a contradiction since length $(\mathcal{O}_{S,x}/(w,u'^2)) = 2$.

If $\nu = 3$, we can write $y = u^3 \bmod (\mathfrak{m}_x^4)$ for some $u \in \mathfrak{m}_x$. Then $(w,u''^3) \subseteq \mathcal{J}_{\mathfrak{X}}$ where $u'' = y^{1/3}$ is a local parameter at x. Therefore $\mathfrak{X}$ is an admissible cycle with $\mathcal{J}_{\mathfrak{X}} = (w,u''^3)$ and hence the map $\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{S,x}/\mathcal{J}_{\mathfrak{X}})$ is onto, $\mathcal{L}$ being 2-spanned.

Thus we are done after proving that case $\nu \geq 4$ does not occur. Indeed, if $\nu \geq 4$, then $\mathcal{I}_{\mathfrak{X}} \subseteq (w, m_x^{\nu})$ and there exists a surjective morphism $\mathcal{O}_{S,x}/\mathcal{I}_{\mathfrak{X}} \longrightarrow \mathcal{O}_{S,x}/(w, m_x^{\nu})$; this leads again to a contradiction since length $(\mathcal{O}_{S,x}/(w, m_x^{\nu})) = \nu$.

□

It would be nice if the following was true.

(4.2) CONJECTURE. Assume L is a nef line bundle on a smooth surface S and k a non-negative integer. Let $c = \max\{ab, a+b = k+1\}$. Assume $L \cdot L \geq 4c + 1$. If there are no effective divisors D with

$$L \cdot D - c \leq D \cdot D < L \cdot D/2 < c$$

then for all length $k+1$ zero-cycles $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ the map

$$\Gamma(K_S \otimes L) \longrightarrow \Gamma(K_S \otimes L \otimes \mathcal{O}_{\mathfrak{X}})$$

is onto.

(4.3) REMARK. It should be noted that the statement of Lemma (4.1) above is <u>not true</u> if lenght $(\mathcal{O}_{\mathfrak{X}}) > 3$. E.g., let $\mathfrak{X}$ be the 0-cycle with $\mathrm{Supp}(\mathfrak{X})$ a single point z and let (x,y) be local coordinates at z. Take $\mathcal{I}_{\mathfrak{X}} = (x^2, y^2)$. Then length $(\mathcal{O}_{\mathfrak{X}})=4$ and the element xy can be obtained neither as a quotient modulo (x^4, y) nor (x, y^4).

(4.3.1) Note also that another <u>alternate definition</u> of k-spannedness would be the following: for any distinct points $p_1,\ldots,p_r$ on S and any positive integers $k_1,\ldots,k_r$ with $\sum_{i=1}^{r} k_i = k+1$ then the map

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_S/\prod_i m_i^{k_i})$$

is onto, where the m_i's are the maximal ideals sheaves for the p_i's.

If $k = 0,1$ one sees that definition (0.4) of k-spannedness implies the alternate definition above. This is no longer true when $k \geq 2$. E.g., let again $\mathcal{Z}$ be a 0-cycle with $\mathrm{Supp}(\mathcal{Z})$ a single point z and let (x,y) be local coordinates at z. Then the element $xy \in \mathcal{O}_S/m_z^3$ can be obtained neither as a quotient modulo (x^3,y) nor (x,y^3).

□

Let $S^{[r]}$ be denote the component of the Hilbert scheme of S parametrizing the 0-dimensional subschemes $(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$ of S with length $(\mathcal{O}_{\mathcal{Z}}) = r$. Let $\mathcal{M}$ be a line bundle on S. Given a zero-cycle $(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$ with length $(\mathcal{O}_{\mathcal{Z}}) = r \geq 1$ and

$$*) \qquad \Gamma(\mathcal{M}) \longrightarrow \Gamma(\mathcal{M} \otimes \mathcal{O}_{\mathcal{Z}})$$

onto, we define the associated rational map

$$\varphi_{r-1}(\mathcal{M}) : S^{[r]} \longrightarrow \mathrm{Grass}(\Gamma(\mathcal{M}),r)$$

at $(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$, where Grass $(\Gamma(\mathcal{M}),r)$ denotes the Grassmannian of all r-dimensional quotients of $\Gamma(\mathcal{M})$, by sending $(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$ into the quotient *).

If S is a surface in $\mathbb{P}^N$ which generates $\mathbb{P}^N$ we classically have a generically defined map $g_r : S^{(r)} \dashrightarrow \mathbb{P}^N$ for $r \leq N$ which sends $p_1+\ldots+p_r$ for distinct points p_i's to the $\mathbb{P}^{r-1}$ their images generate. Where it make sense, $\varphi_{r-1}(\mathcal{O}(1)) = g_r \circ \alpha_r$.

Note that if $\Gamma(\mathcal{M}) \longrightarrow \Gamma(\mathcal{M} \otimes \mathcal{O}_{\mathcal{Z}})$ is onto for all length r zero-cycles, then $\varphi_{r-1}(\mathcal{M})$ is a globally defined morphism. Further, if $r = 1$, this is the usual morphism to $\mathbb{P}(\Gamma(\mathcal{M}))$ associated to $\Gamma(\mathcal{M})$.

Note also that if a line bundle $\mathcal{L}$ is 0-spanned, $\varphi_0(\mathcal{L})$ is a morphism and if $\mathcal{L}$ is 1-spanned, i.e. $\mathcal{L}$ is very ample, $\varphi_i(\mathcal{L})$, for $i = 0,1$, are morphisms and $\varphi_0(\mathcal{L})$ is an embedding. The following shows that the k-th order embedding in the above sense (i.e. $\varphi_{k-1}(\cdot)$ is an embedding) is almost implied by the k-spannedness for $k = 2$.

(4.4) PROPOSITION. <u>With the notation as above, if</u> $\mathcal{L}$ <u>is 2-spanned, then the maps</u> $\varphi_i(\mathcal{L})$, <u>for</u> $i \leq 2$, <u>are everywhere defined morphisms and</u> $\varphi_1(\mathcal{L})$ <u>is one to one</u>.

<u>Proof.</u> The first part of the statement is clear. To prove that $\varphi_1(\mathcal{L})$ is one to one we have to show that if $(\mathcal{Z}_1, \mathcal{O}_{\mathcal{Z}_1})$, $(\mathcal{Z}_2, \mathcal{O}_{\mathcal{Z}_2})$ are length 2 0-cycles, then

$$\ker(\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}_1})) \neq \ker(\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}_2})).$$

If we had equality, then if s vanishes on $\mathcal{Z}_1$ it must vanish on $\mathcal{Z}_2$. If $\mathrm{red}(\mathcal{Z}_1) \neq \mathrm{red}(\mathcal{Z}_2)$ this is absurd since it would imply that

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}_3}) \tag{4.4.1}$$

can't be onto with $\mathcal{Z}_3 = \mathcal{Z}_1 \cup \{p\}$, $p \in \mathrm{red}(\mathcal{Z}_2)$, $p \notin \mathrm{red}(\mathcal{Z}_1)$. Indeed any $s \in \Gamma(\mathcal{L})$ which is zero on $\mathcal{Z}_1$ would be zero on $\mathcal{Z}_2$ and hence at p. Since length $(\mathcal{O}_{\mathcal{Z}_3}) = 3$, this contradicts the fact that (4.4.1) must be onto by the above Lemma. If $\mathrm{red}(\mathcal{Z}_1) = \mathrm{red}(\mathcal{Z}_2)$ and $\mathcal{Z}_1 \neq \mathcal{Z}_2$ we must have $\mathrm{red}(\mathcal{Z}_1)$ equal to a single point p. It is easy to check that if $s \in \Gamma(\mathcal{L})$ is zero on $\mathcal{Z}_1, \mathcal{Z}_2$, then it must belong to $\Gamma(\mathcal{L} \otimes \mathfrak{m}_p^2)$ where $\mathfrak{m}_p^2$ is the maximal ideal of $\mathcal{O}_{S,p}$. Thus if any $s \in \Gamma(\mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}_1})$ is automatically zero on $\mathcal{Z}_2$, it would follow that

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_{S,p}/\mathfrak{m}_p^2)$$

can't be onto. This contradicts the 2-spannedness (in fact 1-spannedness).

(4.5) REMARK. Let $\mathcal{A}^n(S)$ be the set of admissible zero cycles on S of length n. It is worth noting that given $s \in S^{(n)}$, $\alpha_r^{-1}(s) \cap \mathcal{A}^n(S)$ is a Zariski open subset of $\alpha_r^{-1}(s)$ whose closure is the unique irreducible component of $\alpha_r^{-1}(s)$ of maximal dimension ([I1], Cor. 1 and Thm. 2). From this and standard dimension counts, see e.g. [I1], we see that $\dim \overline{(S^{[n]} - \mathcal{A}^n(S))} = 2n - 4$. Thus if $\mathcal{L}$ is k-spanned, the associated rational map $S^{[k+1]} \longrightarrow \mathrm{Grass}(\Gamma(\mathcal{L}),k+1)$ is a morphism outside off a subset Z of codimension 4. Note also that $\mathrm{cod}(\alpha_r(Z)) \geq 6$ in light of (4.1) and the usual dimension counts.

(4.6) REMARK. It is natural to hope that by using Reider's method with a choice of different ideals instead of (x^n,y) the numerical condition in (3.1) would imply a strong form of k-spannedness for $K_S \otimes L$, L nef and big line bundle on S. Indeed by using the ideals of type (x^a,y^b) a slight bit more information can be obtained. In general though we have found this extra information very obscure. The one new exception comes from using the ideal (x^2,y^2). We can conclude in this case that if $L \cdot L \geq 17$ and there are no effective divisors D with

$$L \cdot D - 4 \leq D \cdot D < L \cdot D/2 < 4,$$

then $K_S \otimes L$ is 2-spanned with respect to the alternate definition (4.3.1).

The proof runs along these lines. By the above numerical conditions $K_S \otimes L$ is 2-spanned with respect to definition (0.4). Then for a given point $p \in S$, the morphism

$$\Gamma(K_S \otimes L) \longrightarrow \Gamma(K_S \otimes L \otimes \mathcal{O}_{S,p}/\mathfrak{m}_p^3)$$

is onto or the sections that vanish to the second order at x only generate a 2-dimensional subspace V of $K_S \otimes L \otimes \mathfrak{m}_p^2/\mathfrak{m}_p^3$. We can choose two linear functions A,B in local coordinates at p such that A^2,B^2 tensor a non identically zero section of $K_S \otimes L$ generate V. We now construct a vector bundle $\mathcal{E}$ fitting in an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{I}_{\mathfrak{X}} \longrightarrow 0$$

where $\mathcal{I}_{\mathfrak{X}}$ is the ideal sheaf generated by A^2,B^2 at x and 1 elsewhere. This construction follows by a modification of the construction we give in § 2 of this paper and also immediately from [C]. Note that length $(\mathcal{O}_{\mathfrak{X}}) = 4$. Thus by the numerical conditons there exists a certain section s of $K_S \otimes L$ which at x is of the form $A^2f(A,B) + B^2g(A,B) + AB$. This shows that $V = K_S \otimes L \otimes \mathfrak{m}_p^2/\mathfrak{m}_p^3$.

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