Equivalence theorems for complex affine hypersurfaces

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Introduction. In this paper we study equivalence of complex hypersurfaces in $\mathbb{C}^{n+1}$, where $\mathbb{C}^{n+1}$ is considered as a homogeneous space under the action of the group AGL(n+1,$\mathbb{C}$).

Depending on the choice of transversal vector fields one gets various approaches to complex affine differential geometry of complex hypersurfaces in $\mathbb{C}^{n+1}$. Assume we have a complex hypersurface f: M $\to$ $\mathbb{C}^{n+1}$. If ξ is an arbitrary (i.e. real of class C^∞) transversal vector field for f on some open set U $\subset$ M , then it induces on U (by formulas (1.1), (1.2) in Section 1) a complex torsion-free connection ∇, a symmetric $\mathbb{C}$-bilinear form h, called the second fundamental form, a (1,1)-tensor field S (in general neither complex nor anti-complex), called the shape operator, and a $\mathbb{C}$-valued $\mathbf{R}$-linear 1-form τ. These objects determine f and ξ modulo AGL(n+1,$\mathbb{C}$), see Theorem 2.1. One can consider holomorphic transversal vector fields, but then the induced connection is not Kähler unless it is flat. With the aim of getting affine geometry compatible with Kähler geometry K.Nomizu, U.Pinkall and F.Podesta introduced in [NPP] the notion of affine Kähler connection and affine Kähler hypersurface. Namely a complex torsion-free connection on a complex manifold M is called affine Kähler if its curvature tensor R satisfies the condition R(JX,JY) = R(X,Y) for any X,Y, where J is the complex structure on M. A complex hypersurface is called affine Kähler if it is endowed with an anti-holomorphic transversal vector field ξ. The connection induced by an anti-holomorphic transversal vector field is affine Kähler.

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Instead of anti-holomorphic transversal vector fields, however, we propose to consider transversal vector fields for which the corresponding shape operator is anti-complex. It turns out that metric transversal vector fields as well as anti-holomorphic ones have this property. A metric transversal vector field is never anti-holomorphic unless the given hypersurface is part of a hyperplane in $\mathbb{C}^{n+1}$.The anti-complexity of the shape operator is , in fact, a property of the complex transversal vector bundle spanned by ξ. The connection induced by such a transversal vector field is affine Kähler. Conversely, if $\operatorname{rk} h > 1$ at some point of M and the induced connection is affine-Kähler, then the corresponding shape operator is anti-complex at each point of M, see Lemma 3.1.

In Section 2 we prove two basic theorems (Theorems 2.1.,2.2.) for hypersurfaces endowed with arbitrary transversal vector fields. From the affine point of view the most important object induced on a hypersurface is the induced connection. In Section 3 we shall prove some results in which the equality only of the induced connections implies affine equivalence. In particular, a complex affine analogue ot the Killing - Beez theorem is given (Theorem 3.2). Except for this theorem Section 3 deals with hypersurfaces endowed with transversal vector fields whose shape operators are anti-complex. We prove, for instance, that if for two complex hypersurfaces equipped with such transversal vector fields the induced connections are equal and non-flat, then the hypersurfaces are affine equivalent (Theorems 3.5.). Using this result one easily gets the classical equivalence theorem for Kähler hypersurfaces in $\mathbb{C}^{n+1}$ as well as a theorem about homothetical equivalence of Kähler hypersurfaces (Theorems 3.7., 3.8.).

1. Preliminaries. Let M be a connected complex n-dimensional manifold and $f : M \to \mathbb{C}^{n+1}$ a holomorphic immersion. We shall denote by J the complex structure on M as well as the standard one in $\mathbb{C}^{n+1}$. The tangent space T_xM has a natural structure of a complex vector space where the multiplication by i is given by J. Throughout the paper we shall use the notation $iX = JX$ for X tangent to M or $X \in \mathbb{C}^{n+1}$. Let ξ be an arbitrary (i.e. of class C^∞ - in the real sense) vector field transversal to f on some open set $U \subset M$. We can write the formulas of Gauss and Weingarten:

(1.1) $D_X f_* Y = f_* \nabla_X Y + h(X,Y)\xi$,

(1.2) $D_X \xi = -f_* SX + \tau(X)\xi$

where D is the standard connection on $\mathbf{C}^{n+1}$ and X,Y are tangent to M. For any transversal vector field ξ the ∇ is a torsion-free complex (i.e. $\nabla J = 0$) connection, h is a symmetric $\mathbf{C}$-valued, $\mathbf{C}$-linear 2- form, S an $\mathbf{R}$-linear (1,1)-tensor field and τ is a $\mathbf{C}$-valued $\mathbf{R}$-linear 1-form. All the objects are of class C^∞. A transversal vector field is holomorphic iff $SJ = JS$ and $\tau J = i\tau$. It is anti-holomorphic if and only if $SJ = -JS$ and $\tau J = -i\tau$. If ξ' is another transversal vector field defined on U, then $\xi' = f_* Z + \varphi\xi$, where Z is tangent to M and φ is a nowhere vanishing complex valued function of class C^∞. Then

(1.3) $h(X,Y) = \varphi h'(X,Y)$,

(1.4) $S'X = \varphi SX - \nabla_X Z + \tau'(X)Z$

(1.5) $\varphi\tau'(X) = \varphi\tau(X) + X\varphi + h(X,Z)$

where h', S', τ' are the objects induced by ξ' . By (1.3) it is clear that the rank of a complex form h_x is independent of the choice of ξ. The rank will be called the type number of f at x and denoted by tf_x. Around any point of M it is possible to find a holomorphic transversal vector field ξ. If ξ is a holomorphic transversal vector field on U and $X_1,\ldots,X_n$ is a holomorphic complex frame on U, then the matrix $[h(X_i,X_j)]_{1 \le i,j \le n}$ is holomorphic and so are its minors. Hence we have

Lemma 1.1. *For every $r \in \mathbf{N}$ the set*

$$M^r = \{ x \in M,\ tf_x > r\}$$

is empty or open and dense in M. In particular, the rank of h is constant on an open dense subset of M.

Assume that an arbitrary transversal vector field ξ is given on an open set $U \subset M$. At every point $x \in M$ there is a complex basis $e_1,\ldots,e_n$ of T_xM such that $h(e_k,e_j) = 0$ and $h(e_j,e_j) = 1$ or 0 for $j,k = 1,\ldots,n$, $j \neq k$.. We shall order $e_1,\ldots,e_n$ in such a way that if there are vectors e_j for which $h(e_j,e_j) = 1$, then they are at the beginning of the sequence. Such a basis will be said to be adapted to h.

As in real affine geometry we have the equations of Gauss , Ricci and Codazzi:

(1.6) $R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$ - Gauss

(1.7) $h(X,SY) - h(Y,SX) = 2d\tau(X,Y)$ - Ricci

(1.8) $\nabla h(X,Y,Z) - \nabla h(Y,X,Z) = h(X,Z)\tau(Y) - h(Y,Z)\tau(X)$ - Codazzi I

(1.9) $\nabla S(X,Y) - \nabla S(Y,X) = \tau(X)SY - \tau(Y)SX$ - Codazzi II

for $X,Y,Z \in T_xM$, $x \in U$.

In what follows we shall mean by a function a complex valued function of class C^∞. If a transversal vector field ξ for a hypersuface $f : M \to \mathbb{C}^{n+1}$ is given, then ∇, h, S, τ will automatically denote the objects defined by formulas (1.1), (1.2) for the given f and ξ.

2. Basic equivalence theorems. Let $f^1, f^2 : M \to \mathbb{C}^{n+1}$ be two complex hypersurfaces. They are said to be affine equivalent if and only if there is $B \in AGL(n+1,\mathbb{C})$ such that $f^2 = B f^1$. Since the mappings f^1, f^2 are holomorphic, their affine equivalence on some open subset of M implies their global equivalence. Assume now that we have one immersion $f: M \to \mathbb{C}^{n+1}$ and a transversal vector field ξ on a connected open subset U. Let P be the principal fibre bundle of all $\mathbb{C}$-linear frames over U. The projection of P onto U will be denoted by π. Since $\nabla J = 0$, ∇ is covariant derivation coming from a connection on P. We define the mapping $F: P \to AGL(n+1, \mathbb{C})$ by

(2.1) $F(l) = (d_xf \circ l, \xi, f(\pi(l)))$

where $(d_xf \circ l, \xi)$ is the linear part and $f(\pi(l))$ the translation part of $F(l)$. Let ω' be the Maurer-Cartan form on $AGL(n+1,\mathbb{C})$. One can check that the pull back $\omega := F^*\omega'$ depends only on ∇, h, S, τ. It can be described in the following way. We shall use the index range

$$1 \le i,j \le n+2,$$
$$1 \le \alpha,\beta \le n.$$

The i-th row of the matrix $\omega = (\omega^i{}_j)_{1 \le i,j \le n+2}$ will be denoted by ω^i and the j-th column by ω_j. It is straightforward to verify

(2.2) $(\omega^\alpha{}_\beta)_{\alpha,\beta}$ = the connection form of ∇ on P

$(\omega^{n+1}{}_\alpha)_l(Y) = h(\pi_* Y, e_\alpha)$, where $l = (e_1,\dots,e_n)$,

$\omega^{n+2} = 0$,

$\omega^{n+1}{}_{n+2} = 0$,

$(\omega^{\alpha}{}_{n+2})_{\alpha}$ = the canonical form on P,

$(\omega^{\alpha}{}_{n+1})_{l}(Y) = (-\ l^{-1}(S\pi_{*}Y))^{\alpha}$, where $(y)^{\alpha}$ denotes the α-th coordinate of $y \in \mathbb{C}^n$ relative to the canonical basis.

$(\omega^{n+1}{}_{n+1})(Y) = \tau(\pi_{*}Y)$.

We have the following theorem, see for instance [G]:

Let F^1, F^2 be two smooth mappings of a connected manifold N into a Lie group G. Then $F^2 = A F^1$ for some $A \in G$ if and only if $F_1{}^{*}\omega' = F_2{}^{*}\omega'$, where ω' is the Maurer-Cartan form on G.

Using this fact and formulas (2.2) we obtain

Theorem 2.1. *Let $f^1, f^2 : M \to \mathbb{C}^{n+1}$ be complex hypersurfaces and ξ^1, ξ^2 vector fields transversal to f^1, f^2 respectively on some open set $U \subset M$. Assume that*

$$\nabla^1 = \nabla^2,\quad h^1 = h^2,\ S^1 = S^2,\ \tau^1 = \tau^2,$$

where ∇^i, h^i, S^i, τ^i are the objects defined by formulas (1.1), (1.2) for i=1,2. Then there is $B \in AGL(n+1,\mathbb{C})$ such that $f^2 = Bf^1$ on M and $\xi^2 = B\xi^1$ on U.

Similarly to the real case (see [O]) we can prove the following

Theorem 2.2. *Let $f^1, f^2 : M \to \mathbb{C}^{n+1}$ be complex hypersurfaces and $tf^1 > 1$ at some point of M. If there exist vector fields ξ^1, ξ^2 transversal to f^1 and f^2 on some open set $U \subset M$ such that*

$$\nabla^1 = \nabla^2,\ h^1 = \psi h^2,$$

for some nowhere-vanishing function ψ, then there is $B \in AGL(n+1,\mathbb{C})$ such that $f^2 = Bf^1$ on M and $\xi^2 = \psi B\xi^1$ on U.

Proof. At first we assume that $\psi \equiv 1$. We set $\nabla = \nabla^1 = \nabla^2$ and $h = h^1 = h^2$. Let $x \in U \cap M^1$ and $e_1, \dots, e_n$ be a basis of T_xM adapted to h. Then $h(e_1,e_1) = h(e_2,e_2) = 1$ and by the Codazzi equation we get

$$\tau^1(e_k) = \nabla h(e_1,e_k,e_1) - \nabla h(e_k,e_1,e_1) = \tau^2(e_k)$$

for $k > 1$ and

$$\tau^1(e_1) = \nabla h(e_2,e_1,e_2) - \nabla h(e_1,e_2,e_2) = \tau^2(e_1).$$

Since h is $\mathbb{C}$-bilinear, we have also

$$\tau^1(Je_k) = \nabla h(e_1,Je_k,e_1) - \nabla h(Je_k,e_1,e_1) = \tau^2(Je_k)$$

for $k > 1$ and

$$\tau^1(Je_1) = \nabla h(e_2,Je_1,e_2) - \nabla h(Je_1,e_2,e_2) = \tau^2(Je_1).$$

Therefore $\tau^1 = \tau^2$ at each point of $U \cap M^1$ and so on U. By using the Gauss equation in a similar way we get $S^1 = S^2$ on U.The assertion now follows from Theorem 2.1. In the case where ψ is not identically 1, we can replace ξ^1 by $\psi\xi^1$ and use formulas (1.3) - (1.5). The proof is complete.

By using Theorems 2.1. and 2.2. one can prove various equivalence theorems depending on properties of transversal vector fields . For instance, we have

Proposition 2.3. *Let $f^1, f^2 : M \to \mathbb{C}^{n+1}$ be complex hypersurfaces and ξ^1, ξ^2 vector fields transversal to f^1, f^2 respectively , defined on an open subset U of M. Assume that the induced connections ∇^1, ∇^2 are equal. Then each of the following conditions 1) - 4) (holding at each point of U) implies that f^1, f^2 are affine equivalent.*

1) a) $S^1 = S^2$

b) $\dim_{\mathbb{C}}(\mathrm{span}_{\mathbb{C}} \mathrm{im} S^1) > 1$

2) a) $h^1 = h^2$,

b) $S^k J = -JS^k$ for $k = 1,2$,

c) $\tau^1_1 = \tau^2_1$ or $\tau^1_2 = \tau^2_2$, where $\tau^k = \tau^k_1 + i\tau^k_2$ and τ^k_j are real valued forms for $k,j = 1,2$.

3) a) ξ^1, ξ^2 are anti-holomorphic,

b) $h^1 = h^2$.

Proof. We set $\nabla = \nabla^1 = \nabla^2$. The curvature tensor of ∇ will be denoted by R. If $S^1 = S^2$, then we shall denote both by S. Similarly, we set $h = h^1 = h^2$ if $h^1 = h^2$. In the sequal we shall omit the case where $h \equiv 0$. In this case both hypersurfaces are totally geodesic and it is easy to see that they are $AGL(n+1,\mathbb{C})$ - equivalent if the induced connections are equal. According to the cases *1) - 3)* we have

1) The Gauss equation yields

$$h^1(Y,Z)SX - h^1(X,Z)SY = h^2(Y,Z)SX - h^2(X,Z)SY \tag{2.3}$$

for any X, Y, Z. Take arbitrary $X, Z \in T_xM$, $x \in U$. There is $Y \in T_xM$ such that SY does not belong to the complex vector line $\mathbb{C} \cdot SX$. By inserting these X, Y, Z into (2.3) we obtain $h^1(X,Z) = h^2(X,Z)$. In a similar way one can use the second Codazzi equation to get the equality $\tau^1 = \tau^2$.

2) Assume that $\mathrm{rk}\, h > 0$ on U. Let $x \in U$ and let $e_1, \dots, e_n$ be a basis of T_xM

adapted to h. By the Gauss equation and the assumption b) we obtain

$$2S^1(Je_1) = R(Je_1, e_1)e_1 = 2S^2(Je_1)$$

We have also

$$S^1 e_j = R(e_j, e_1)e_1 = S^2 e_j$$

for $j > 1$. Since S^1 and S^2 are anti-complex, we get $S^1 = S^2$. Similarly the Codazzi equation yields

(2.4) $\quad \tau^1(e_j) = \nabla h(e_1, e_j, e_1) - \nabla h(e_j, e_1, e_1) = \tau^2(e_j)$

and

(2.5) $\quad \tau^1(Je_j) = \nabla h(e_1, Je_j, e_1) - \nabla h(Je_j, e_1, e_1) = \tau^2(Je_j)$

for $j > 1$. Therefore $\tau^1 = \tau^2$ on the complex space spanned by $e_2, \dots, e_n$. Using the Codazzi equation for $\nabla h(e_1, Je_1, e_1) - \nabla h(Je_1, e_1, e_1)$ we get

$$\tau^1{}_1(Je_1) + \tau^1{}_2(e_1) = \tau^2{}_1(Je_1) + \tau^2{}_2(e_1)$$

and

$$\tau^1{}_2(Je_1) - \tau^1{}_1(e_1) = \tau^2{}_2(Je_1) - \tau^2{}_1(e_1).$$

Assumption c) now implies: $\tau^1(e_1) = \tau^2(e_1)$ and $\tau^1(Je_1) = \tau^2(Je_1)$. Therefore $\tau^1 = \tau^2$.

3) As in 2) we assume that $\mathrm{rk}\,h > 0$ on U. Also as in 2) we have $S^1 = S^2$ on U and $\tau^1{}_x = \tau^2{}_x$ on the complex space spanned by $e_2, \dots, e_n$, where $e_1, \dots, e_n$ is a basis of T_xM adapted to h and x is any point of U. By the Codazzi equation we get

$$2\tau^1(Je_1) = \nabla h(e_1, Je_1, e_1) - \nabla h(Je_1, e_1, e_1) = 2\tau^2(Je_1).$$

Since τ^1 and τ^2 are anti-complex, we also have $\tau^1(e_1) = \tau^2(e_1)$. The proof is completed.

Remark. A transversal vector field ξ defines on its domain a complex volume element ϑ_C and a real volume element ϑ by

(2.6) $\vartheta_C(X_1, \dots, X_n) = \det_{\mathbf{C}}(f_* X_1, \dots, f_* X_n, \xi)$

(2.7) $\vartheta(X_1, \dots, X_{2n}) = \det_{\mathbf{R}}(f_* X_1, \dots, f_* X_{2n}, \xi, J\xi)$

It is clear that if the assumptions of one of the above theorems are satisfied and there is a point $x \in M$ such that $\vartheta_C{}^1 = \vartheta_C{}^2$ (resp. $\vartheta^1 = \vartheta^2$) at x, then f^1, f^2 are $ASL(n+1, \mathbf{C})$ (resp. $AGL(n+1, \mathbf{C}) \cap ASL(2n+2, \mathbf{R})$) - equivalent. Therefore, from Theorem 2.2. one can easily get complex versions of the classical Radon theorem (see [B] p.158) about equivalence of non-degenerate hypersurfaces relative to the special affine group.

3. Equivalence of hypersurfaces with the same induced connection. If R is the curvature tensor of a complex connection ∇ on a complex manifold M, then we set

(3.1) $\quad \operatorname{im} R_x = \operatorname{span}_{\mathbf{R}} \{R(X,Y)Z,\ X,Y,Z \in T_xM\}.$

(3.2) $\quad \ker R_x = \bigcap_{X,Y \in T_x M} \ker R(X,Y)$

Since $\nabla J = 0$, the mapping $V \to R(X,Y)V$ is $\mathbf{C}$-linear and hence $\operatorname{im} R_x$ and $\ker R_x$ are complex subspaces of T_xM. Let $f\colon M \to \mathbf{C}^{n+1}$ be a complex hypersurface and ξ a transversal vector field for f on some open set $U \subset M$. If h and S are the second fundamental form and the shape operator corresponding to ξ, then we set

(3.3) $\quad \ker h_x = \{X \in T_xM;\ h(X,Y) = 0 \text{ for every } Y \in T_xM\}$

(3.4) $\quad \operatorname{im}_{\mathbf{C}} S_x = \operatorname{span}_{\mathbf{C}} \operatorname{im} S_x$

Clearly, $\ker h_x$ is a complex subspace of T_xM. We shall need

Lemma 3.1. *Let $f\colon M \to \mathbf{C}^{n+1}$ be a complex hypersurface and ξ a transversal vector field for f on U. At every point x of U we have.*

(3.5) $\quad \operatorname{im} R_x \subset \operatorname{im}_{\mathbf{C}} S_x$

(3.6) $\quad \ker h_x \subset \ker R_x.$

If $S_xJ = -JS_x$ and $S_x \neq 0$, then the equality holds in (3.6).

If S is anti-complex at each point of U, then ∇ is affine Kähler. Conversely, if the induced connection is affine Kähler and $tf > 1$ at some point of M, then the shape operator is anti-complex at each point of U.

Proof. Inclusions in (3.5) and (3.6) trivially follow from the Gauss equation. Assume that S is anti-complex. Let $Z \in \ker R$, i.e. for every X,Y we have

$$h(Y,Z)SX - h(X,Z)SY = 0$$

$$h(JY,Z)SX - h(X,Z)SJY = 0.$$

By multiplying the first equality by i and using the complexity of h and the anti-complexity of S we get

$$ih(Y,Z)SX - ih(X,Z)SY = ih(Y,Z)SX + ih(X,Z)SY = 0.$$

Since there is X such that $SX \neq 0$, we have $h(Y,Z) = 0$ for every Y.

If S is anti-complex, then, by the Gauss equation, the induced connection is affine Kähler. Assume that the induced connection is affine Kähler and $tf^1 > 1$. Let $x \in U \cap M^1$ and $e_1, \ldots, e_n$ be a basis of T_xM adapted to h. If

$j > 1$, then

$$Se_j = R(e_j,e_1)e_1 = R(Je_j,Je_1)e_1 = JSJe_j.$$

We have also

$$Se = R(e_1,e_2)e_2 = R(Je_1,Je_2)e_2 = JSJe_1,$$

which finishes the proof.

The following theorem is a complex analogue of the affine Beez-Killing theorem proved in [O].

Theorem 3.2. *Let f^1, $f^2 : M \to \mathbf{C}^{n+1}$ be complex hypersurfaces equipped with transversal vector fields ξ^1, ξ^2 on an open set $U \subset M$. Assume that $\nabla^1 = \nabla^2$. If*

1) $tf^1 > 1$ at some point of M,

2) $\dim_{\mathbf{C}} im R^1 > 2$ at some point of U,

then f^1 and f^2 are $AGL(n+1,\mathbf{C})$ - equivalent.

Proof. By assumption 2) and Lemma 3.1. we know that $rk_{\mathbf{C}} S^k > 2$ at some point of U for $k = 1,2$. We can assume that $tf^1 > 1$ and $rk_{\mathbf{C}} S^k > 2$ at each point of U. Take $x \in U$. We shall prove that any h^1-orthogonal basis of T_xM is also h^2-orthogonal. Let $e_1,\dots,e_n$ be an h^1-orthogonal basis of T_xM. Since $rk_{\mathbf{C}} S^2 > 2$, at least three of the vectors $S^2e_1,\dots,S^2e_n$, $S^2Je_1,\dots,S^2Je_n$ are $\mathbf{C}$-linearly independent. With the aim of proving that $e_1,\dots,e_n$ is h^2-orthogonal it is sufficient to consider two cases:

a) Among $S^2e_1,\dots,S^2e_n$ there exist three $\mathbf{C}$-linearly independent vectors.

b) S^2e_1, S^2e_2, S^2Je_1 are $\mathbf{C}$-linearly independent.

Consider case a). Take $j,k \in \{1,\dots,n\}$, $j \neq k$. There is l such that $l \neq j$ and $S^2e_l \notin \mathbf{C}\cdot(S^2e_k)$. By the Gauss equation we have

$$\begin{aligned} &h^2(e_l,e_j)S^2e_k - h^2(e_k,e_j)S^2e_l \\ &\qquad = R(e_k,e_l)e_j = h^1(e_l,e_j)S^1e_k - h^1(e_k,e_j)S^1e_l = 0 \end{aligned}$$

Hence $h^2(e_k,e_j) = 0$. Assume b). The Gauss equation yields

$$\begin{aligned} &h^2(Je_1,e_2)S^2e_1 - h^2(e_1,e_2)S^2Je_1 \\ &\qquad = R(e_1,Je_1)e_2 = h^1(Je_1,e_2)S^1e_1 - h^1(e_1,e_2)S^1Je_1 = 0. \end{aligned}$$

Hence $h(e_1,e_2) = 0$. If $k > 2$, then by the Gauss equation we get

$$\begin{aligned} &h^2(e_2,e_k)S^2e_1 - h^2(e_1,e_k)S^2e_2 \\ &\qquad = R(e_1,e_2)e_k = h^1(e_2,e_k)S^1e_1 - h^1(e_1,e_k)S^1e_2 = 0. \end{aligned}$$

Therefore $h^2(e_2,e_k) = h^2(e_1,e_k) = 0$ for $k > 2$. Using these equalities and the Gauss equation:

$$h^2(e_k,e_l)S^2e_l - h^2(e_l,e_l)S^2e_k$$
$$= R(e_l,e_k)e_l = h^1(e_k,e_l)S^1e_l - h^1(e_l,e_l)S^1e_k = 0$$

we obtain $h^2(e_k,e_l) = 0$ for $k \neq l$, $k,l > 1$. We have proved that $e_1,\dots e_n$ is h^2 - orthogonal.

Let $j \in \{1,\dots,n\}$. Using again the Gauss equation we obtain

$$h^2(e_j,e_j)S^2e_k = R(e_k,e_j)e_j = h^1(e_j,e_j)S^1e_k$$

for every $k \neq j$. If $h^1(e_j,e_j) = 0$, then we can take $k \neq j$ such that $S^2e_k \neq 0$ and we get $h^2(e_j,e_j) = 0$. If $h^1(e_j,e_j) = 0$, then we choose $k \neq j$ so that $S^1 e_k \neq 0$. Then we have $h^2(e_j,e_j) \neq 0$.

Assume now that $e_1,\dots,e_n$ is adapted to h^1 and $\operatorname{rk} h^1 = r$. Then the basis is h^2 - orthogonal, $h^2(e_j,e_j) \neq 0$ for $j \leq r$ and $h^2(e_j,e_j) = 0$ for $j > r$. If $j \neq k$ and $h^1(e_j,e_j) = h^1(e_k,e_k)$, then $h^2(e_j,e_j) = h^2(e_k,e_k)$. Namely, if in the sequence $e_1,\dots e_n$ the vectors e_j and e_k are replaced by the vectors $e_j + e_k$ and $e_k - e_j$ respectively, then the new basis is also h^1 - orthogonal and hence h^2 - orthogonal. Thus $h^2(e_j + e_k, e_k - e_j) = 0$ and consequently $h^2(e_j,e_j) = h^2(e_k,e_k)$. Summing up, we have proved that $h^2 = \alpha h^1$ for some nowhere vanishing function α on U (obviously of class C^∞). To finish the proof it is sufficient to apply Theorem 2.2.

From now on we shall consider hypersurfaces on which the induced connection is affine Kähler. We shall start with the following

Lemma 3.3. *Let ∇ be an affine Kähler connection on a complex manifold M. If $R(JX,X)X = 0$ for every $X \in T_xM$, then $R_x = 0$.*

Proof. Since for every $X,Y \in T_xM$

$$0 = R(JX + JY, X+Y)(X+Y)$$
$$= R(JX,X)Y + R(JY,Y)X + 2R(JY,X)X + 2R(JX,Y)Y.$$

we have

$$R(JX,X)Y + 2R(JY,X)X = -R(JY,Y)X - 2R(JX,Y)Y \tag{3.7}$$

for every X,Y. After replacing X by $-X$ in (3.7), the left-hand term remains unchanged and the right-hand one changes the sign. Thus for every X,Y we have

(3.8) $R(JX,X)Y + 2R(JY,X)X = 0$

Using the first Bianchi identity and the fact that ∇ is affine Kähler we get from (3.8)

(3.9) $R(Y,X)JX + 3R(JY,X)X = 0.$

When we replace X by JX in (3.9) and use (3.7) and (3.9) we obtain $R(JX, Y)X = 9R(JX,Y)X$ and consequently

(3.10) $R(X,Y)X = 0$

for any X,Y. Hence $R(X,Y+Z)(Y+Z) = 0$ for any X,Y,Z. Using (3.10) and the Bianchi identity we obtain $R(X,Y)Z = 0$.

Proposition 3.4. *Let $f^1, f^2 : M \to \mathbb{C}^{n+1}$ be complex hypersurfaces endowed with transversal vector fields ξ^1, ξ^2 on U. If the curvature tensors of the induced connections are equal and non-zero at a point $x \in U$, and the corresponding shape operators S^1, S^2 are anti-complex at x, then there is a non-zero complex number α such that $h^1 = \alpha h^2$ and $S^2 = \alpha S^1$.*

Proof. By Lemma 3.3. we know that there is a vector $X \in T_xM$ such that $R(JX,X)X \neq 0$. Since S^1 is anti-complex the Gauss equation gives

(3.11) $$2h^1(X,X)S^1JX = R(JX,X)X = 2h^2(X,X)S^2JX$$

Hence $h^1(X,X) \neq 0$, $h^2(X,X) \neq 0$, $S^1(X) \neq 0$, $S^2(X) \neq 0$. There is a basis $e_1,\dots,e_n$ of T_xM adapted to h^1 such that e_1 is proportional to X. By (3.11) we have $S^2e_1 = \alpha S^1e_1$ and $h^1(e_1,e_1) = \alpha h^2(e_1,e_1)$ for some non-zero complex number α. If $k > 1$, then the Gauss equation yields

(3.12) $$h^1(e_1,e_1)S^1e_k = h^2(e_1,e_1)S^2e_k - h^2(e_1,e_k)S^2e_1$$

Similarly we obtain

$$h^1(e_1,e_1)S^1Je_k = h^2(e_1,e_1)S^2Je_k - h^2(e_1,Je_k)S^2e_1$$

and consequently

(3.13) $$-ih^1(e_1,e_1)S^1e_k = -ih^2(e_1,e_1)S^2e_k - ih^2(e_1,e_k)S^2e_1$$

Multiplying (3.12) by $-i$ and comparing with (3.13) gives

(3.14) $h^2(e_1,e_k) = 0$ for every $k > 1$.

Formula (3.12) can now be rewritten as

(3.15) $h^1(e_1,e_1)S^1e_k = h^2(e_1,e_1)S^2e_k$.

Since $h^1(e_1,e_1) = \alpha h^2(e_1,e_1)$, we have $S^2e_k = \alpha S^1e_k$. Therefore $S^2 = \alpha S^1$.

If 1,k,j are mutually distinct, then the Gauss equation gives

$$0 = R(e_1,e_j)e_k = h^2(e_j,e_k)S^2e_1 - h^2(e_1,e_k)S^2e_j$$

Therefore, by (3.14), we have $h^2(e_j,e_k) = 0$ for any $k,j > 1$, $k \neq j$. Thus $e_1,\dots,e_n$ is h^2-orthogonal. Using once again the Gauss equation we get

$$h^1(e_k,e_k)S^1e_1 = R(e_1,e_k)e_k = h^2(e_k,e_k)S^2e_1$$

for $k > 1$. Since $S^2 = \alpha S^1$, we get $h^1(e_k,e_k) = \alpha\, h^2(e_k,e_k)$ and consequently $h^1 = \alpha h^2$. The proof is completed.

Theorem 3.5. *Let $f^1, f^2\colon M \to \mathbb{C}^{n+1}$ be complex hypersurfaces endowed with transversal vector fields ξ^1, ξ^2 whose corresponding shape operators are anti-complex at each point of some open subset $U \subset M$. If ξ^1, ξ^2 induce the same connection ∇ on U which is non-flat, i.e. the curvature tensor of ∇ is non-zero at some point of U, then $f^2 = B f^1$ for some $B \in AGL(n+1,\mathbb{C})$.*

Proof. We can assume that R is not zero at every point of U. By Proposition 3.4. we know that there is a nowhere vanishing function α on U such that $h^1 = \alpha h^2$ and $S^2 = \alpha S^1$. Replace now ξ^1 by $\alpha\xi^1$. The shape operator corresponding to $\alpha\xi^1$ is anti-complex and, of course, $\alpha\xi^1$ induces the connection ∇. Moreover, the shape operators and second fundamental forms corresponding to $\alpha\xi^1$ and ξ^2 are respectively equal. Hence we can assume that $\alpha \equiv 1$, i.e. $h^1 = h^2 = h$ and $S^1 = S^2 = S$. As in the proof of Proposition 3.4. we can find a basis $e_1,\dots,e_n$ of T_xM adapted to h such that $h(e_1,e_1) = 1$ and $Se_1 \neq 0$. By the first Codazzi equation we get

(3.16) $\tau^1(Je_1) - i\tau^1(e_1) = \nabla h(e_1,Je_1,e_1) - \nabla h(Je_1,e_1,e_1) = \tau^2(Je_1) - i\tau^2(e_1)$

The second Codazzi equation yields

$(i\tau^1(e_1) + \tau^1(Je_1))Se_1 = \nabla S(Je_1,e_1) - \nabla S(e_1,Je_1) = (i\tau^2(e_1) + \tau^2(Je_1))Se_1$

Since $Se_1 \neq 0$, we obtain

(3.17) $i\tau^1(e_1) - i\tau^2(e_1) = \tau^2(Je_1) - \tau^1(Je_1)$.

Formula (3.16) can be rewritten as

(3.18) $i\tau^2(e_1) - i\tau^1(e_1) = \tau^2(Je_1) - \tau^1(Je_1)$

Comparing formulas (3.17), (3.18) we get

(3.19) $\tau^1(e_1) = \tau^2(e_1)$ and $\tau^1(Je_1) = \tau^2(Je_1)$.

As in the proof of Proposition 2.3. for the case *2)* we also obtain $\tau^1 = \tau^2$ on the complex space spanned by $e_2,\dots,e_n$, see formulas (2.4), (2.5). Hence $\tau^1 = \tau^2$. The assertion now follows from Theorem 2.1.

Corollary 3.6. *Let $f^1, f^2 : M \to \mathbf{C}^{n+1}$ be complex hypersurfaces such that $\operatorname{tf}^1 > 1$ at a point of M. If there are transversal vector fields ξ^1, ξ^2 for f^1, f^2 on some open set $U \subset M$, inducing the same connection which is non-flat and affine Kähler, then f^1, f^2 are affine equivalent.*

Proof. Since $\operatorname{tf}^1 > 1$, we know by Lemma 3.1. that S^1 is anti-complex at each point of U. Take a point $x \in M^1 \cap U$. Then $S^1{}_x \neq 0$. By Lemma 3.1. we have $\ker h^1{}_x = \ker R_x$. Since $\operatorname{rk} h^1{}_x > 1$, we get $\dim \ker R_x \leq n-2$. We have also the inclusion $\ker h^2{}_x \subset \ker R_x$. Therefore $\dim \ker^2{}_x \leq n-2$ and consequently $\operatorname{rk} h^2{}_x > 1$. Hence, by Lemma 3.1, S^2 is anti-complex. We can now apply Theorem 3.5.

Consider now the case of Kähler hypersurfaces. In what follows $\mathbf{C}^{n+1}$ will be equipped with the standard Kähler structure. By a Kähler hypersurfaces we shall mean a hypersurface endowed with the induced Kähler structure. First we shall prove.

Theorem 3.7. *Let $f^1, f^2 : M \to \mathbf{C}^{n+1}$ be Kähler hypersurfaces. If f^1 is non-degenerate at a point of M and the Kähler connections induced by f^1 and f^2 are equal on some open subset U of M, then $f^1 = cBf^2$ for some $B \in U(n+1)$ and $c \in \mathbf{R}$.*

Proof. Let g^1 and g^2 be the metric tensor fields induced by f^1 and f^2, respectively. Assume that h^k, S^k are the Riemannian second fundamental forms and second fundamental tensors for f^k, $k = 1, 2$, defined on U. If $h^k = h^k{}_1 + ih^k{}_2$ is the decomposition of h^k into the real and imaginary part, then $h^k{}_1(X,Y) = g^k(S^kX,Y)$. The shape operators S^1, S^2 are anti-complex. Since f^1 is non-degenerate on a dense open subset of M, we can assume that the connection ∇ is not flat and S^1 is non-singular at every point of U. As in the proof of Theorem 3.5. we get a function α such that $h^1 = \alpha h^2$ and $S^2 = \alpha S^1$. Then we have

$$g^1(S^1X,Y) = |\alpha|^2 g^2(S^1X,Y)$$

Hence $g^1 = |\alpha|^2 g^2$. Since g^1, g^2 have the same Levi-Civita connection, the function $|\alpha|^2$ is constant. Set $c = |\alpha|$. If we multiply the immersion f^2 by c, then the new pair of hypersurfaces f^1, $f^{2'} = cf^2$ induce the same metric tensor field g^1 on U. On the other hand, by Theorem 3.5. we know that f^1 and f^2 are $AGL(n+1,\mathbf{C})$-equivalent and so are f^1 and $f^{2'}$. Let $f^1 =$

Bf^2. Since f^2 is not totally geodesic (because ∇ is non-flat),there are x, $y \in U$ such that $V_x = f^{2'}_*(T_xM)$ and $V_y = f^{2'}_*(T_yM)$ are not parallel. The transformation B restricted to V_x as well as to V_y preserves the scalar product in $\mathbb{C}^{n+1}$. Thus $B \in U(n+1)$.

We can also easily get the following classical result.

Corollary 3.8. *Let $f^1, f^2 : M \to \mathbb{C}^{n+1}$ be Kähler hypersurfaces. If the induced metric tensor fields are equal on some open set $U \subset M$, then f^1 and f^2 are $U(n+1)$-equivalent.*

Proof. In the case where the induced connection is non-flat the assertion follows from Theorem 3.5. and the arguments given at the end of the proof of Theorem 3.7. If ∇ is flat, then f^1 and f^2 are totally geodesic and hence $U(n+1,\mathbb{C})$-equivalent.

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