Finite Generation of Canonical Rings and Flip Conjecture (detailed version)

Hajime Tsuji

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Starsse 26
W-5300 Bonn3
Federal Republic of Germany

Tokyo Metropolitan University
Department of Mathematics
Minamiohsawa Hachiôji
Tokyo
Japan

MPI/91-18

Since

$$\log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}v)^n}{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\psi_c)^n} = \int_0^1 \dot{\Delta}_a(v - \psi_c)da,$$

where $\dot{\Delta}_a$ is the Laplacian with respect to the Kähler form

$$\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\{(1-a)\phi_c + av\}$$

, by the maximum principle, we obtain

$$v \geq \psi_c \quad \text{on } K \times [0, T).$$

On the other hand, trivially

$$\hat{\Delta}_t v \geq -n \quad \text{on } K \times [0, T)$$

holds, where $\hat{\Delta}_t$ is the Laplacian with respect to the Kähler form $\hat{\omega}_t$. Let h be the C^∞-function on $\bar{K} \times [0, T)$ such that

$$\begin{cases} \hat{\Delta}_t h & = & -n & \text{on } K \times [0, T) \\ h & = & 0 & \text{on } \partial K \times [0, T). \end{cases}$$

Then by the maximum principle, we have

$$v \leq h \quad \text{on } K \times [0, T).$$

Hence we have

$$\psi_c \leq v \leq h \quad \text{on } K \times [0, T). \tag{12}$$

Now to fix C^k-norms on $X_\nu - D$, we shall construct a complete Kähler-Einstein form on $X_\nu - D$.

We quote the following theorem.

Theorem 4.3 *([18]) Let $\bar{M}$ be a nonsingular projective manifold and let B be an effective divisor with only simple normal crossings. If $K_{\bar{M}} + B$ is ample, then there exists a unique (up to constant multiple) complete Kähler-Einstein form on $M = \bar{M} - B$ with negative Ricci curvaure.*

By the construction of D, D is a divisor with simple normal crossings and $K_{X_\nu} + D$ is ample. Hence by Theorem 4.3, there exists a complete Kähler-Einstein form ω_D on $X_\nu - D$ such that

$$\omega_D = -\mathrm{Ric}_{\omega_D}.$$

Then we have

$$\| dv \| \leq \max\{\| dh \|, \| d\psi_c \|\} \quad \text{on } \partial K \times [0, T),$$

where $\| \quad \|$ is the pointwise norm with respect to ω_D. To make this estimate independent of K, we need to use special properties of ω_D.

Finite Generation of Canonical Rings and Flip Conjecture

Hajime Tsuji

dedicated to Professor T. Nagano
on the occasion of his 60-th birthday

Abstract

We prove the finite generation of canonical rings of projective varieties of general type and the flip conjecture in all dimension. As a consequence we prove the minimal model conjecture up to dimension 4 which is previously known to be true up to dimension 3 by Mori ([19]).

Contents

1 Introduction **2**

2 Preliminaries **4**

2.1 Zariski decomposition 5

2.2 Structure of d-closed positive $(1,1)$-currents 7

3 Deformation of Kähler form I **13**

3.1 Hamilton's equation 13

3.2 Reduction to the parabolic Monge-Ampère equation 13

3.3 C^0-estimate 14

3.4 C^2-estimate 15

4 Deformation of Kähler form II 18
4.1 Kähler-Einstein currents 18
4.2 Kodaira's Lemma 19
4.3 Construction of a suitable ample divisor 21
4.4 The Dirichlet problem 22
4.5 C^0-estimate 23
4.6 C^1-estimate on ∂K 25
4.7 C^1-estimate on K 29
4.8 C^2-estimate on ∂K 30
4.9 C^2-estimate on K 36
4.10 Costruction of Kähler-Einstein currents 38
4.11 Finite orderness of the Kähler-Einstein current ω_E 43

5 L^2-vanishing theorem 49
5.1 L^2-estimate on a complete Kähler manifold 49
5.2 Relation between the asymptotic behavior of the base locus of the pluricanonical system and the Lelong number of the Kähler-Einstein current ω_E 51
5.3 Use of the argument of Benveniste-Kawamata-Shokurov 55

6 Zariski Decomposition of Canonical Divisor 62
6.1 An algorithm to construct a Zariski decomposition 62
6.2 Termination of the algorithm 64

1 Introduction

The *classification theory of algebraic varieties* is the attempt to study all algebraic varieties by decomposing them into 3 kinds of particles :

1. varieties with negative K_X,
2. varieties with numerically trivial K_X,
3. varieties with positive K_X

and their fibre spaces. In this sense the classification theory of algebraic varieties can be considered as a higher dimensional generalization of Riemann's uniformization theorem in the one dimensional case.

In 1976 S.-T. Yau solved the Calabi's conjecture([28]) (We should note that T. Aubin has also contributed to the solution independently in the case of the negative first Chern class ([2]). But his analysis seems to be less geometric than in [28]). This breakthrough gave me a confidence that there is a strong relation between the classification theory of algebraic varieties and Kähler-Einstein metrics. Roughly speaking [28] enables us to translate properties of the canonical bundle of any compact Kähler manifold to propeties of the Ricci curvature of a Kähler metric constructed by using a partial differential equation of Monge-Ampère type. In fact [28] gives a new proof for Riemann's uniformization theorem although [28] is difficult.

As for the particles of the 1-st kind, S. Mori invented his cone theorem ([20]) to single out these particles. The purpose of this article is to single out the particles of the 3-rd kind globally (the existence of canonical model) and locally (the flip conjecture). In comparison with Mori's theory, the method in the present paper is quite transcendental in nature. In my opinion it seems to be very difficult to obtain the results in this paper by a purely algebraic method because the canonical ring of an algebraic variety seems to be a quite transcendental object.

As for the 2nd particles, there are no essential ways to single out these particles at present. This problem is called *the abundance conjecture.* Our method does not work to single out the particles of 2nd kind.

The following conjecture is one of the central problem in the classification theory of algebraic varieties.

Conjecture 1.1 *(Minimal Model Conjecture) Let X be a normal projective variety. Assume that X is not uniruled. Then there exists a minimal projective variety X_{min} (cf. Definition 2.5) which is birational to X.*

This conjecture is trivial in the case of an algebraic curve and is known to be true classically in the case of $\dim X = 2$. Recently S. Mori solved the conjecture in the case of $\dim X = 3$([19]). His method depends on the close study of 3-dimensional terminal singularities and it seems to be difficult to generalize his method to the case of higher dimensional varieties. I hope that the present paper will give a perspective of the conjecture in all dimension because our method is independent of the dimension of the variety. In fact, we prove *the Flip Conjecture*(existence of flip) in all dimension in this paper. Hence to prove the minimal model conjecture , we only need to prove the termination of flips. In particular since the termination of flips is known in the case of $\dim X \leq 4$ ([16]), we have a solution of *Minimal Model Conjecture* in the case of $\dim X \leq 4$.

In this paper all varieties and morphisms are defined over $\mathbf{C}$.

The the following theorems are main results in this paper.

Theorem 1.1 *Let X be a smooth projective variety of general type. Then the canonical ring*

$$R(X, K_X) = \oplus_{\nu \geq 0} H^0(X, \mathcal{O}_X(\nu K_X))$$

is finitely generated. Hence the canonical model

$$X_{can} = Proj\, R(X, K_X)$$

exists.

In the case of $\dim X = 2$, this theorem was proved by D. Mumford([30, appendix]) and recently S. Mori proved Theorm 1.1 in the case of $\dim X = 3$ in terms of the existence of minimal models of 3-folds ([19]).

The following conjecture is essential in the construction of minimal models in the case of dimension greater than 2.

Conjecture 1.2 *(Flip Conjecture) Let X be a projective variety with only terminal singularities. Let $\varphi : X \longrightarrow X'$ be a birational contraction of an extremal ray (cf.[15, 20]). Then*

$$R(X/X', K_X) = \oplus_{\nu \geq 0} \varphi_* \mathcal{O}_X([\nu K_X])$$

is finitely generated as an $\mathcal{O}_{X'}$-algebra.

As a corollary of Theorem 1.1, we have:

Theorem 1.2 *Flip conjecture holds in all dimensions.*

This theorem implies the existence of a minimal model in the case of $\dim X \leq 4$.

Theorem 1.3 *Let X be a normal projective variety of dimension ≤ 4. If X is not uniruled, then there exists a minimal algebraic variety X_{min} which is birational to X.*

The proof of Theorem 1.1 is closely related to the cone theorem of Mori and Kawamata([20, 14]) although it is purely analytic in nature. Mori proved his cone theorem by his method *bend and break curves*. Instead of curves we *bend and break Kähler forms* by Hamilton's heat flow.

Our method depends on the analysis of complex Monge-Ampère equations in [28, 4].

In the course of writing up this paper, I received many valuable suggestions and remarks from Professors S. Bando, T. Fujita, Y. Kawamata, R. Kobayashi, B. Shiffman, Y.-T. Siu and others. I would like to express my hearty thanks to them. In particular during my visit to U.S. from Noevember to December in 1990, I enjoyed very helpful and valuable discussions with Professors Y.-T. Siu and B. Shiffman.

This work has been completed during my stay at Max-Planck-Institut für Mathematik. The last but not least, I would like to express my hearty thanks to the institute for the hospitality.

2 Preliminaries

In this section, we fix the basic notations and introduce basic notions. We shall prove some results about the structure of d-closed positive $(1,1)$-currents for the later use.

2.1 Zariski decomposition

Let X be a normal projective variety of dimension n. We denote by $Z_{n-1}(X)$ (resp. $Div(X)$), the group of Weil (resp. Cartier) divisor on X. The canonical divisor K_X is defined by

$$K_X = i_*\Omega^n_{X_{reg}},$$

where X_{reg} denote the regular part of X and $i : X_{reg} \longrightarrow X$ is the canonical injection. K_X is an element of $Z_{n-1}(X)$. An $\mathbf{R}$−divisor D is an element of $Z_{n-1}(X) \otimes \mathbf{R}$, i.e. $D = \sum d_j D_j$ (finite sum). where $d_j \in \mathbf{R}$ and the D_j are mutually distinct prime divisor on X.

If $D \in Div(X) \otimes \mathbf{R}$, we say that D is $\mathbf{R}$-Cartier. We define round up $\lceil D \rceil$, the integral part $[D]$, the fractional part $\{D\}$ and the round off $\langle D \rangle$ by

$$\lceil D \rceil = \sum \lceil d_j \rceil D_j, [D] = \sum [d_j] D_j,$$
$$\{D\} = \sum \{d_j\} D_j, \langle D \rangle = \sum \langle d_j \rangle D_j,$$

where $\lceil r \rceil, [r]$ and $\langle r \rangle$ for $r \in \mathbf{R}$ are integers such that

$$r - 1 < [r] \leq r \leq \lceil r \rceil < r + 1$$

$$r - \frac{1}{2} \leq \langle r \rangle < r + \frac{1}{2}$$

and

$$\{r\} = r - [r].$$

Definition 2.1 *$D \in Div(X) \otimes \mathbf{R}$ is said to be nef if $D \cdot C \geq 0$ holds for every effective curve on X.*

Definition 2.2 *Let X be a normal projective variety. We say that X has only canonical (resp. terminal) singularities, if K_X is $\mathbf{Q}$-Cartier, i.e. $K_X \in Div(X) \otimes \mathbf{Q}$ and there is a resolution of singularity $\mu : Y \longrightarrow X$ such that the exceptional locus F of μ is a divisor with normal crossings and*

$$K_Y = \mu^*(K_X) + \sum a_j F_j,$$

where $a_j \geq 0$ (resp $a_j > 0$).

The following definition is more general.

Definition 2.3 *A pair (X, Δ) for $\Delta \in Z_{n-1}(X) \otimes \mathbf{Q}$ is said to be logcanonical (resp. logterminal) if the following conditions are satisfied.*

1. $[\Delta] = 0$ *and* $K_X + \Delta \in Div(X) \otimes \mathbf{Q}$.

2. *There is a resolution of singularity $\mu : Y \longrightarrow X$ such that the union F of the exceptional locus of μ and the inverse image of the support of Δ is a divisor with normal crossings and*

$$K_Y = \mu^*(K_X + \Delta) + \sum a_j F_j, a_j \geq -1 (resp. > -1).$$

Definition 2.4 *A normal projective variety X is said to be $\mathbf{Q}$-factorial, if every Weil divisor is $\mathbf{Q}$-Cartier.*

In this paper, we use the notion of minimal varieties in the following sense.

Definition 2.5 *Let X be a normal projective variety. X is said to be minimal, if the following condition is satisfied.*

1. *X has only terminal singularities.*

2. *K_X is nef.*

3. *X is $\mathbf{Q}$-factorial.*

Definition 2.6 *$D \in Div(X) \otimes \mathbf{Q}$ is said to be big, if $\kappa(X, D) = \dim X$*

Now we shall define Zariski decomposition.

Definition 2.7 *And expression $D = P + N$, $(D, P, N \in Div(X) \otimes \mathbf{R})$ is called a Zariski decomposition of D if the following conditions are satisfied.*

1. *D is big.*

2. *P is nef.*

3. *N is effective.*

4. *The natural homomorphisms*

$$H^0(X, \mathcal{O}_X([mP])) \longrightarrow H^0(X, \mathcal{O}_X([mD]))$$

are bijective for all positive integers m.

Conjecture 2.1 *Let X be a smooth projective variety of general type. Then there exists a modification*

$$f : \tilde{X} \longrightarrow X$$

such that f^*K_X has a Zariski decomposition.

By [14] to prove Theorem 1.1 it is sufficient to solve Conjecture 2.1. In this paper we shall prove Theorem 1.1 by solving Conjecture 2.1.

2.2 Structure of d-closed positive $(1,1)$-currents

To solve Conjecture 2.1, we shall use the theory of currents which is considered to be a generalization of the notion of subvarieties.

Let M be a complex manifold of dimension n.

Definition 2.8 *The current $\mathcal{D}^{p,q}(M)$ of type (p,q) are the continuous linear functional on the compactly supported C^∞ forms of type $(n-p, n-q)$, $A_c^{n-p,n-q}(M)$ with the C^∞-topology.*

$$\partial : \mathcal{D}^{p,q}(M) \longrightarrow \mathcal{D}^{p+1,q}(M),\ \bar\partial : \mathcal{D}^{p,q}(M) \longrightarrow \mathcal{D}^{p,q+1}(M)$$

are defined by

$$\partial T(\varphi) = (-1)^{p+q+1}T(\partial\varphi),\ \bar\partial T(\varphi) = (-1)^{p+q+1}T(\bar\partial\varphi)$$

for $T \in \mathcal{D}^{p,q}(M)$ and we set $d = \partial + \bar\partial$. A (p,p)-current T is real in case $T = \bar T$ in the sense that $\overline{T(\varphi)} = T(\bar\varphi)$v for all $\varphi \in A_c^{n-p,n-p}(M)$ and a real current T is positive in case

$$(\sqrt{-1})^{p(p-1)/2}T(\eta \wedge \bar\eta) \geq 0, \eta \in A_c^{n-p,0}(M).$$

Let V be a subvariety of codimension p in M. Then

$$V(\varphi) = \int_V \varphi,\quad \varphi \in A_c^{n-p,n-p}(M)$$

is a d-closed positive (p,p)-current. Hence we can consider subvarieties as d-closed positive currents. On the other hand, every $C^\infty(p,p)$-form ψ on M defines a (p,p)-current T_ψ by

$$T_\psi(\varphi) = \int_M \psi \wedge \varphi,\quad \varphi \in A_c^{n-p,n-p}(M).$$

The current of this type is called a smooth current. As we explain below, a general d-closed positive current is basically somewhere between the smooth currents and those supported by analytic varieties.

Now we shall introduce an important invariant for d-closed positive (p,p)-currents. Let T be a d-closed positive (p,p)-current on M. For each point $x \in M$ we define a number

$$\Theta(T,x)$$

defined as follows. Let (U,z) be a local coordinate around x ($z(x) = O$). We set

$$B[r] = \{y \in U \mid \| z(y) \| < 1\},$$

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^n dz_i \wedge d\bar z_i,$$

$\chi[r]$: the characteristic function of $B[r]$.

We define $\Theta(T,x)$ by

$$\Theta(T,x) = \lim_{r\downarrow 0} \frac{1}{\pi^{n-p}r^{2n-2p}} T(\chi[r]\omega^{n-p})$$

and call it the Lelong number of T at x. The Lelong number exists and finite for all d-closed positive (p,p)-current (cf. [9, pp.390-391]) and it is independent of the choice of the coordinate ([23]).

We shall summarize the basic properties of the Lelong number.

It is easy to see that the Lelong number is upper semicontinuous as a function on the manifold. And clearly if T is a smooth d-closed positive (p,p)-current, $\Theta(T,x) = 0$ for every $x \in M$. On the other hand we have:

Theorem 2.1 *([9, p. 391]) Let $V \subset M$ be a subvariety of codimension p in M. Then we have*

$$\Theta(V,x) = mult_x V.$$

The following theorem describes the positive Lelong number locus of a d-closed positive (p,p)-current.

Theorem 2.2 *([23]). Let T be a d-closed positive (p,p)-current on M. Then for every positive number ε*

$$S_\varepsilon(T) = \{x \in M \mid \Theta(T,x) \geq \varepsilon\}$$

is a subvariety of codimension $\geq p$.

The following example shows that $S_\varepsilon(T)$ may have a large codimension in general.

Example 2.1 *Let T be a d-closed positive $(1,1)$-current on $\mathbf{C}^n$ defined by*

$$T = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\sum_{i=1}^{n} \mid z_i \mid^2).$$

Then

$$\Theta(T,x) = \begin{cases} 1 & if\ x = O \\ 0 & otherwise \end{cases}$$

To define the pullback of a d-closed positive $(1,1)$-current we need the following Poincaré lemma.

Lemma 2.1 *($\partial\bar{\partial}$ Poincaré lemma). Let T be a d-closed positive $(1,1)$-current defined on the unit ball $B = B(1)$ in $\mathbf{C}^n$ with center O Then there exists a plurisubharmonic function F on B such that*

$$T = \sqrt{-1}\partial\bar{\partial}F$$

and the difference of two such F is a pluriharmonic function on B.

The above lemma states every d-closed positive $(1,1)$-current has locally a plurisubhamonic potential. The set where a potential takes $-\infty$ is independent of the choice of the potential. We call the set the pluripolar set of the d-closed positive $(1,1)$-current. It is well known the pluripolar set is alway of measure 0.

Let $f : X \longrightarrow Y$ be a morphism between two connected complex manifolds. Let S be a d-closed positive $(1,1)$-current on Y. Suppose that $f(X)$ is not contained in the pluripolar set of S. Then we can define the pullback f^*S by using $\partial\bar{\partial}$ Poincaré lemma as follows.

Let $\mathcal{U} = \{U_\alpha\}$ be a sufficiently fine open covering of Y such that for every U_α there exists a plurisubharmonic function F_α on U_α such that

$$S \mid U_\alpha = \sqrt{-1}\partial\bar{\partial}F_\alpha.$$

Then we define the pullback f^*S by

$$f^*S \mid f^{-1}(U_\alpha) = \sqrt{-1}\partial\bar{\partial}(f^*F_\alpha).$$

This definition is well defined and independnet of the choice of $\mathcal{U}$ and $\{F_\alpha\}$.

Theorem 2.2 describes the positive Lelong number locus of a d-closedpositive (p,p)-current. On the other hand the following lemma describes the basic property of the zero Lelong number locus of a d-closed positive $(1,1)$-current.

Lemma 2.2 *Let X be a projective manifold of dimension n and let L be a line bundle on X. Suppose that there exists a d-closed positive $(1,1)$-current T on X such that $c_1(L) = [T]$, where $[T]$ is the de Rham cohomology class of T. Let x be a point on X such that $\Theta(T,x) = 0$ and let C be an arbitrary irreducible reduced curve through x. Then the intersection number $c_1(L)\cdot C$ is nonnegative.*

Proof. Let ω be a smooth Kähler form on X and let r_x be thedistance function from x with respect to ω. Then there exists a neighbourhood U of x such that $\log r_x$ is strictly plurisubharmonic on U. Let ρ be a nonnegative C^∞-function on U with compact support which is identically 1 on some neighbourhood of x. We set

$$\psi = (2n+2)\rho\log r_x.$$

Let (H,h) be a hermitian line bundle on X such that

$$\sqrt{-1}\partial\bar{\partial}\psi - \sqrt{-1}\partial\bar{\partial}\log h > c\omega$$

holds for some $c > 0$. By the $\partial\bar{\partial}$-Poincaré lemma for d-closed positive $(1,1)$ currents, there exists a singular hermitian metric a on L such that

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log a = T.$$

Let σ,τ be local holomorphic generators around x of H and L respectively. By taking U small enough, we may assume that σ and τ are defined on U. Let m be an arbitrary fixed positive integer and we set

$$F = \bar{\partial}(\rho\sigma \otimes \tau^{\otimes m}).$$

Since $\Theta(T, x) = 0$ by Lemma 2.1 , we have

$$F \in L^2_{(0,1)}(X, H \otimes L^{\otimes m}),$$

where $L^2_{(p,q)}(X, H\otimes L^{\otimes m})$ denote the Hilbert space of the $H\otimes L^{\otimes m}$ valued L^2 forms of type (p, q) with respect to the singular hermitian metric $e^{-\psi}h\otimes a^{\otimes m}$. By Hörmander's L^2 estimates for $\bar{\partial}$-operator, there exists $\varphi \in L^2_{(0,0)}(X, H \otimes L^{\otimes m})$ such that

$$\bar{\partial}\varphi = F.$$

Then

$$\mu = \rho\sigma \otimes \tau^{\otimes m} - \varphi$$

is a global holomorphic section of $H \otimes L^{\otimes m}$. Let a_0 be a smooth hermitian metric on L. Since $T = -\sqrt{-1}\partial\bar{\partial}\log a$ is positive, a/a_0 is bounded from below by a positive constant on X. This implies that

$$\mu(x) = (\sigma \otimes \tau^{\otimes m})(x)$$

holds by the definition of ψ. Hence

$$c_1(H \otimes L^{\otimes m}) \cdot C > 0$$

holds. Since m is an arbitrary positive integer, we have

$$c_1(L) \cdot C \geq 0.$$

This completes the proof of the lemma. Q.E.D.

Remark 2.1 *The original form of Lemma 2.2 was Cororally 2.1 below. To polish up this lemma into the present form the discussion with Professor B. Shiffman was very helpful. I would like to express my thank here.*

Cororally 2.1 *Let X,L,T be as in Lemma 2.2. Suppose that $\Theta(T, x) = 0$ holds for all $x \in X$. Then L is nef.*

Remark 2.2 *Professor S. Bando kindly informed me that one may use the smoothing by heat deformation of currents to prove this lemma. In particular his proof implies that every d-closed positive* $(1,1)$*-current with vanishing Lelong number on a compact Kähler manifold represents a cohomology class on the closure of the Kähler cone of the manifold.*

Remark 2.3 *After I finished up writing this paper, Professor S. Bando kindly sent me the preprints of J.-P. Demailly ([6, 7]). These works are closely related to Lemma 2.2, Cororally 2.1 and also Section 5 below.*

The Lelong number of a d-closed positive $(1,1)$-current indicates the growth of a local potential function of the current around a point.

Lemma 2.3 *([23, p.85, Lemma 5.3]) Let* φ *be an arbitrary plurisubharmonic function on an open subset* U *of* $\mathbf{C}^n$ *and let* x *be a point in* U*. Let*

$$T = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi.$$

Then the followings are true.

1. *If* $\Theta(T,x) < 1$*, then* $e^{-\varphi}$ *is locally integrable at* x*.*
2. *If* $\Theta(T,x) \geq n$*, then* $e^{-\varphi}$ *is not locally integrable at* x*.*

The above lemma is not enough for our purpose because the two cases are not complement each other. But the following lemma is well known.

Lemma 2.4 *([23, p.95, Lemma 7.5]). Let* T *be a d-closed positive* $(1,1)$*-current on* $B(r) = \{z \in \mathbf{C}^n \mid \| z \| < r\}$ *for some* $r > 0$*. Let us consider* $\mathbf{P}^{n-1}$ *as a parameter space which parametrizes complex line through the origin* O*. Then for every* $L \in \mathbf{P}^{n-1}$ *such that the restriction* $T \mid B(r) \cap L$ *is well defined,*

$$\Theta(T,O) \leq \Theta(T \mid L \cap B(r), O)$$

holds. And for almost all such L *(in the sense of Lebesgue measure on* $\mathbf{P}^{n-1}$*),*

$$\Theta(T,O) = \Theta(T \mid L \cap B(r), O)$$

holds.

The following lemma is fundamental for consideration of the behavior of the Lelong number under a modification of the variety. Although the author does not know any references, it seems to be standard.

Lemma 2.5 *Let X be a complex manifold and let T be a d-closed positive $(1,1)$-current on X. Let $f : \tilde{X} \longrightarrow X$ be the blowing up with a smooth center Y. Then for every $p \in \tilde{X}$, we have*

$$\Theta(f^*T, p) \geq \Theta(T, f(p)).$$

Moreover for a general point $p \in f^{-1}(Y)$,

$$\Theta(f^*T, p) = \Theta(T, f(p))$$

holds, where "general" means that p is outside of a set of measure 0 with respect to the Lebesgue measure on $f^{-1}(Y)$ associated with an arbitrary hermitian metric on $f^{-1}(Y)$.

Proof. If $f(p)$ is not in Y, then it is clear that $\Theta(f^*T, p) = \Theta(T, f(p))$ holds. Let $(U, x_1, \ldots, x_n)$ be a local coordinate around p. Then for almost every line L through p (with respect to this coordinate), $f^*T \mid L$ is well defined and

$$\Theta(f^*T \mid L, p) = \Theta(f^*T, p)$$

holds. If we take such L generally, we may assume that $f(L)$ is smooth at $f(p)$. Then with respect to a suitable local coordinate $(y_1, \ldots, y_n)$ around $f(x)$, $f(L)$ is a line and by Lemma 2.4, we have

$$\Theta(T, f(x)) \leq \Theta(T \mid f(L), f(x)) = \Theta(f^*T \mid L, x) = \Theta(f^*T, x).$$

This completes the proof of the lemma. Q.E.D.

The following example shows that the Lelong number of a d-closed positive $(1,1)$-current is not invariant under a modification in general.

Example 2.2 *Let T be a d-closed positive $(1,1)$-current on $\mathbf{C}^2$ defined by*

$$T = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\mid z_1 \mid^2 + \mid z_2 \mid^{2m}),$$

where m is a positive integer. Let $f : \tilde{\mathbf{C}}^2 \longrightarrow \mathbf{C}^2$ be a blowing up with center O. Let E be the exceptional divisor of f and let p_0 be the intersection of E and the strict transform of z_2-axis. Then

$$\Theta(T, p) = \begin{cases} m & \text{if } p = p_0 \\ 1 & \text{if } p \in E - \{p_0\} \\ 0 & \text{otherwise} \end{cases}$$

The following lemma will be used in Section 6.

Lemma 2.6 *([23, p.87, Lemma 6.2]) Suppose T is a d-closed positive $(1,1)$-current on a complex manifold M and V is a divisor in M. Let c be a nonnegative number. Suppose $\Theta(T, x) \geq c$ for every $x \in V$. Then $T - cV$ is a d-closed positive $(1,1)$-current on M.*

3 Deformation of Kähler form I

In this section we shall consider Hamilton's equation on a smooth projective variety of general type and determine the maximal existence time for the smooth solution. This section is more or less independent from the other sections. Hence a reader who is familiar with Monge-Ampère equation may skip this section. But the basic method of the estimate of parabolic Monge-Ampère equations is introduced in this section.

3.1 Hamilton's equation

Let X be a smooth projective variety of general type and let $n = \dim X$. Let ω_0 be a C^∞-Kähler form on X. We consider the initial value problem:

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}_\omega - \omega \quad \text{on } X \times [0, T) \tag{1}$$

$$\omega = \omega_0 \quad \text{on } X \times \{0\}, \tag{2}$$

where

$$\mathrm{Ric}_\omega = -\sqrt{-1}\partial\bar{\partial}\log \omega^n$$

and T is the maximal existence time for C^∞-solution.

Since

$$\frac{\partial}{\partial t}(d\omega) = -d\omega \quad \text{on } X \times [0, T)$$

$$d\omega_0 = 0 \quad \text{on } X \times \{0\},$$

we have that $d\omega = 0$ on $X \times [0, T)$, i.e., the equation preserves the Kähler condition.

3.2 Reduction to the parabolic Monge-Ampère equation

Let ω denote the de Rham cohomology class of ω in $H^2_{DR}(X, \mathbf{R})$. Since $-(2\pi)^{-1}\mathrm{Ric}_\omega$ is a first Chern form of K_X, we have

$$[\omega] = (1 - \exp(-t))2\pi c_1(K_X) + \exp(-t)[\omega_0]. \tag{3}$$

Let Ω be a C^∞-volume form on X and let

$$\omega_\infty = -\mathrm{Ric}\,\Omega = \sqrt{-1}\partial\bar{\partial}\log\Omega.$$

We set

$$\omega_t = (1 - \exp(-t))\omega_\infty + \exp(-t)\omega_0. \tag{4}$$

Since $[\omega] = [\omega_t]$ on $X \times \{t\}$ for every $t \in [0, T)$, there exists a C^∞-function u on $X \times [0, T)$ such that

$$\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}u. \tag{5}$$

By (1), we have

$$\frac{\partial}{\partial t}(\omega_t + \sqrt{-1}\partial\bar{\partial}u) = \sqrt{-1}\partial\bar{\partial}\log(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n - (\omega_t + \sqrt{-1}\partial\bar{\partial}u).$$

Hence

$$\begin{aligned} &\exp(-t)(\omega_\infty - \omega_0) + \sqrt{-1}\partial\bar{\partial}(\frac{\partial u}{\partial t}) \\ = &\sqrt{-1}\partial\bar{\partial}\log(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n - \omega_\infty + \exp(-t)(\omega_\infty - \omega_0). \end{aligned}$$

Then (1) is equivalent to the initial value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u \quad \text{on } X \times [0,T) \\ u &= 0 \quad \text{on } X \times \{0\}. \end{aligned} \tag{6}$$

Let

$$A(X) = \{[\eta] \mid \eta : \text{Kähler form on } X\} \subset H^2_{DR}(X, \mathbf{R})$$

be the Kähler cone of X. Since $[\omega]$ moves on the segument connecting $[\omega_0]$ and $[\omega_\infty] = 2\pi c_1(K_X)$, we cannot expect T to be ∞, unless $2\pi c_1(K_X)$ is on the closure of $A(X)$ in $H^2_{DR}(X, \mathbf{R})$. We shall determine T. It is standard to see that $T > 0$ ([11]).

Theorem 3.1 *If $\omega_0 - \omega_\infty$ is a Kähler form, then T is equal to*

$$T_0 = \sup\{t > 0 \mid [\omega_t] \in A(X)\}.$$

The proof of Theorem 3.1 is almost parallel to that of [25, p.126, Theorem 3].

3.3 C^0-estimate

Lemma 3.1 *If $\omega_0 - \omega_\infty$ is a Kähler form, then there exists a constant C_0 such that*

$$\frac{\partial u}{\partial t} \leq C_0 \exp(-t).$$

Proof.

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \Delta_\omega\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - \exp(-t)tr_\omega(\omega_0 - \omega_\infty)$$

holds by defferentiating (5) by t. By the maximum principle, we have

$$\frac{\partial u}{\partial t} \leq (\max \log\frac{\omega_0^n}{\Omega})\exp(-t).$$

Q.E.D.

To estimate u from below, we modify (6) as

$$\begin{aligned} \frac{\partial u}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega_t^n} + f_t - u \quad \text{on } X \times [0, T_1) \\ u &= 0 \quad \text{on } X \times \{0\}, \end{aligned} \tag{7}$$

where

$$f_t = \log \frac{\omega_t^n}{\Omega} \tag{8}$$

and

$$T_1 = \min\{\sup\{t > 0 \mid \omega_t > 0\}, T\}. \tag{9}$$

If $t \in [0, T_1)$, we have

$$\log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega_t^n} = \int_0^1 \frac{d}{ds} \log \frac{(\omega_t + \sqrt{-1}s\partial\bar{\partial}u)^n}{\omega_t^n} ds = \int_0^1 \Delta_s u ds,$$

where Δ_s is the Laplacian with respect to the Kähler form $\omega_t + \sqrt{-1}s\partial\bar{\partial}u$. Then by the minimum principle, (7) and Lemma 2.2, we have

Lemma 3.2

$$u \geq -C_0 \exp(-t) + \min_X f_t \quad on \ X \times \{t\}, t \in [0, T_1).$$

We note that this estimate is depending on t and C_0 is independent of the choice of Ω.

3.4 C^2-estimate

For the next we shall obtain a C^2-estimate of u.

Lemma 3.3 *([28, p.351, (2.22)]) Let M be a compact Kähler manifold and let ω, $\tilde{\omega}$ be Kähler forms on M. Assume that there exists a C^∞-function φ such that*

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi.$$

We set

$$f = \log \frac{\tilde{\omega}^n}{\omega^n},$$

Then for every positive constant such that

$$C + \inf_{i \neq j} R_{i\bar{i}j\bar{j}} > 1,$$

$$\exp(C\varphi)\tilde{\Delta}(\exp(-C\varphi)(n+\Delta\varphi)) \geq$$
$$(\Delta f - n^2 \inf_{i\neq j} R_{i\bar{i}j\bar{j}}) - Cn(n+\Delta\varphi)$$
$$+(C+\inf_{i\neq j} R_{i\bar{i}j\bar{j}})(n+\Delta\varphi)^{\frac{n}{n-1}}\exp(-\frac{f}{n-1})$$

holds, where
$R_{i\bar{i}j\bar{j}}$ *: the bisectional curvature of* ω,
Δ *: the Laplacian with respect to* ω.

Applying this lemma to ω_t and $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}u$, we have:

Lemma 3.4 *For every* $C > 0$ *depending only on* $t \in [0, T_1)$ *such that*

$$C + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}(t) > 1 \quad \text{on } X \times \{0\},$$

$$\exp(Cu)(\Delta_\omega - \frac{\partial}{\partial t})(\exp(-Cu)tr_{\omega_t}\omega) \geq$$
$$-(\Delta_t \log\frac{\omega_t^n}{\Omega} + n^2\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(t) + n)$$
$$-C(n - \frac{1}{C} - \frac{\partial u}{\partial t})tr_{\omega_t}\omega - \exp(-t)tr_{\omega_t}((\omega_0 - \omega_\infty)\cdot\sqrt{-1}\partial\bar{\partial}u)$$
$$+(C+\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(t))\exp(\frac{1}{n-1}(-\frac{\partial u}{\partial t} - u + \log\frac{\omega_t^n}{\Omega}))(tr_{\omega_t}\omega)^{\frac{n}{n-1}}$$

holds, on $X \times \{t\}$) ($t \in [0, T_1)$), *where*
Δ_t *: Laplacian with respect to* ω_t,
$R_{i\bar{i}j\bar{j}}(t)$ *: the bisectional curvature of* ω_t
and $tr_{\omega_t}((\omega_0 - \omega_\infty)\cdot\sqrt{-1}\partial\bar{\partial}u)$ *denotes the trace (with respect to* ω_t*) of the product of the endomorphisms* $A, B \in End(TX)$ *defined by*

$$\begin{aligned}\omega_t(A(Z_1)\wedge\bar{Z}_2) &= (\omega_0 - \omega_\infty)(Z_1\wedge\bar{Z}_2)\\ \omega_t(B(Z_1)\wedge\bar{Z}_2) &= (\sqrt{-1}\partial\bar{\partial}u)(Z_1\wedge\bar{Z}_2),\end{aligned}$$

where the pair (Z_1, Z_2) *runs in* $TX \times_X TX$.

Proof. Let
$$f = \log\frac{\omega^n}{\omega_t^n} = \frac{\partial u}{\partial t} + u - \log\frac{\omega_t^n}{\Omega}.$$
Then by Lemma 3.3, we have

$$\exp(Cu)\Delta_\omega(\exp(-Cu)tr_{\omega_t}\omega)$$
$$\geq (\Delta_t f - n^2\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(t)) - Cn(n+\Delta_t u)$$
$$+(C+\inf_{i\neq j} R_{i\bar{i}j\bar{j}})(tr_{\omega_t}\omega)^{\frac{n}{n-1}}\exp(-\frac{f}{n-1}).$$

Since

$$\Delta_t f = \Delta_t(\frac{\partial u}{\partial t} + u - \log\frac{\omega_t^n}{\Omega})$$

$$= \Delta_t\frac{\partial u}{\partial t} + tr_{\omega_t}\omega - n - \Delta_t\log\frac{\omega_t^n}{\Omega}$$

and

$$\exp(Cu)\frac{\partial}{\partial t}(\exp(-Cu)tr_{\omega_t}\omega)$$

$$= -C\frac{\partial u}{\partial t}tr_{\omega_t}\omega + tr_{\omega_t}\frac{\partial \omega}{\partial t} - tr_{\omega_t}\frac{\partial \omega_t}{\partial t}\cdot\omega$$

$$= -C\frac{\partial u}{\partial t}tr_{\omega_t}\omega + \Delta_t\frac{\partial u}{\partial t} - \exp(-t)tr_{\omega_t}(\omega_0 - \omega_\infty) + \exp(-t)tr_{\omega_t}(\omega_0 - \omega_\infty)\cdot\omega,$$

we obtain the lemma. Q.E.D.

Let ε be an arbitrary small positive number. We set

$$T_1(\varepsilon) = \min\{\sup\{t > 0 \mid \omega_t > 0\} - \varepsilon, T\}$$

and let C be a positive number such that

$$C + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}(t) > 1$$

for all $t \in [0, T_1(\varepsilon)]$. Then since the function $x\exp(-x)$ is bounded on $[0, \infty)$, by the maximum principle and Lemma 3.4, we have that if $\exp(-Cu)tr_{\omega_t}\omega$ take its maximum at $(x_0, t_0) \in X \times [0, T_0(\varepsilon)]$, we have

$$tr_{\omega_t}\omega(x_0, t_0) < C_\varepsilon$$

for some $C_\varepsilon > 0$ depending only on ε. Then by the C^0-estimate of u , Lemma 3.1 and Lemma 3.2, by the maximum principle for parbolic equations we have that there exists a positive constant $C'_{1,\varepsilon}$ such that

$$tr_{\omega_t}\omega < C'_{1,\varepsilon}.$$

Hence we obtain:

Lemma 3.5 *There exists a positive constant $C_{1,\varepsilon}$ depending only on $T_1(\varepsilon)$ such that*

$$\| u \|_{C^2(X)} \leq C_{2,\varepsilon}$$

for every $t \in [0, T_1(\varepsilon))$, where $\| \ \|_{C^r(X)}$ is the C^2-norm with repsect to ω_0.

Now by [26], for every $r \geq 2$ there exists a positive constant $C_{r,\varepsilon}$ depending only on $T_1(\varepsilon)$ such that

$$\| u \|_{C^r(X)} \leq C_{r,\varepsilon}.$$

Letting ε tend to 0, we have that

$$T \geq T_1$$

holds. Since $[\omega_{T_0}]$ is on the closure of the Kähler cone $A(X)$, by changing Ω properly, we can make $T_0 - T_1 > 0$ arbitarary small. Hence we conclude that $T = T_0$. This completes the proof of Theorem 3.1. Q.E.D.

4 Deformation of Kähler form II

In this section we shall construct a Kähler-Einstein form on a Zariski open subset of X by using a initial value problem similar to (1) in the last section. In this section we use the same notation as in the last section.

4.1 Kähler-Einstein currents

To state our theorem we need the following definitions.

Definition 4.1 *Let D be a $\mathbf{R}$-Cartier divisor on a projective variety Y. Then the stable base locus of D is defined by*

$$SBs(D) = \cap_{\nu>0} Supp Bs \mid [\nu D] \mid .$$

Definition 4.2 *Let D be a Cartier divisor on a projective variety Y and let $\Phi_{|D|}; Y - \cdots \rightarrow \mathbf{P}^{N(\nu)}$ be the rational map associated with $\mid \nu D \mid$. Let $\mu_\nu : Y_\nu \longrightarrow Y$ be a resolution of the base locus of $\mid \nu D \mid$ and let $\tilde{\Phi}_{|\nu D|}; \tilde{X} \longrightarrow \mathbf{P}^{N(\nu)}$ be the associated morphism. We set*

$$\cdot \quad E(\nu D) = \overline{\mu_\nu(\tilde{E}(\nu D) \cap (Y - Supp\ Bs \mid \nu D \mid)} \ \text{(Zariski closure)}$$

and call it the exceptional locus of $\mid \nu D \mid$. It is easy to see that $E(\nu D)$ is independent of the choice of the resolution of the base locus μ_ν. We set

$$SE(D) = \cap_{\nu>0} E(\nu D)$$

and call it the stable exceptional locus of D.

We set

$$S = \mathrm{SBs}(K_X) \cup SE(K_X).$$

The main result in this section is the following theorem.

Theorem 4.1 *There exists a d-closed positive* $(1,1)$*-current* ω_E *on* X *such that*

1. ω_E *is smooth on a nonempty Zariski open subset* U *of* X.

2. $-Ric_{\omega_E} = \omega_E$ *holds on* U.

3. $[\omega_E] = 2\pi c_1(K_X)$.

The following remark will be important in differential geometry.

Remark 4.1 *As an immediate consequence of Theorem 1.1, we can take* U *to be* $X - S$ *by using the result in [24]. But to prove Theorem 1.1, we do not need to take* U *to be* $X - S$.

4.2 Kodaira's Lemma

The following lemma of Kodaira is well known and fundamental in the proof of Theorem 4.1.

Lemma 4.1 *(Kodaira's lemma) Let* D *be a big divisor (cf. Definition 2.6) on a smooth projective variety* M. *Then there exists an effective* $\mathbf{Q}$*-divisor* E *such that* $D - E$ *is an ample* $\mathbf{Q}$*-divisor.*

Proof. Let H be a very ample divisor on M. Then

$$0 \to H^0(M, \mathcal{O}_M(mD - H)) \to H^0(M, \mathcal{O}_M(mD)) \to H^0(H, \mathcal{O}_H(mD \mid H))$$

is exact. Since D is big, for a sufficiently large m, $\mid mD - H \mid$ is nonempty. This completes the proof of the lemma. Q.E.D.

The following cororally is trivial by Lemma 4.1 and Kleiman's criterion for ampleness([17]).

Cororally 4.1 *([13, Lemma 3 and 4]) Let* D *be a nef and big divisor an a smooth projective variety* M. *Then there extists an effective* $\mathbf{Q}$*-divisor* E *such that for every sufficiently small positive rational number* ε, $D - \varepsilon E$ *is an ample* $\mathbf{Q}$*-divisor on* M.

Let ν be a sufficiently large positive integer such that

1. $\mid \nu K_X \mid$ gives a birational rational map from X into a projective space.

2. Supp Bs$\mid \nu K_X \mid = \mathrm{SBs}(K_X)$.

Let $f_\nu : X_\nu \to X$ be a resolution of the base locus of $\mid \nu K_X \mid$ and let

$$F_\nu = \sum_i b_i^\nu F_i^\nu$$

be the fixed part of $\mid f_\nu^*(\nu K_X) \mid$. We take f_ν so that F^ν is a divisor with normal crosssings. We set

$$\tilde{b}_i^\nu = b_i^\nu / \nu.$$

Let σ_i^ν be a global holomorphic section of $\mathcal{O}_{X_\nu}(F_i^\nu)$ with divisor F_i^ν. Then there exist hermitian metrics $\| \ \|$ on $\mathcal{O}_{X_\nu}(F_i^\nu)$'s such that

$$\omega_\infty^\nu = f_\nu^* \omega_\infty + \sum \sqrt{-1} \tilde{b}_i^\nu \partial\bar{\partial} \log \| \sigma_i^\nu \|^2$$

is positive on $f_\nu^{-1}(X - S)$, if ν is sufficiently large. We may assume

$$\log \| \sigma_i^\nu \| \leq 0$$

holds for every i. We set

By Lemma 4.1, there exists an effective $\mathbf{Q}$-divisor

$$R_\nu = \sum r_j^\nu R_j^\nu$$

on X_ν such that

$$f_\nu^*(K_X) - \sum \tilde{b}_i^\nu F_i^\nu - R^\nu$$

is an ample $\mathbf{Q}$-divisor on X_ν. We note that εR_ν has the same property as R_ν for $\varepsilon \in [0,1]$. Let τ_j^ν be a global section of $\mathcal{O}_{X_\nu}(R_j^\nu)$ with divisor R_j^ν. Then there exists hermitian metrics $\| \ \|$ on $\mathcal{O}_{X_\nu}(R_j^\nu)$ such that

$$\omega_\infty^\nu + \sum_j \sqrt{-1} r_j^\nu \partial\bar{\partial} \log \| \tau_j^\nu \|^2$$

is a smooth Kähler form on X_ν and $\| \tau_j^\nu \| \leq 1$ holds on X_ν for all j. We set

$$\delta_\nu = \sum_j \sqrt{-1} r_j^\nu \log \| \tau_j^\nu \|^2 .$$

Then for every $\varepsilon \in [0,1]$,

$$\omega_\infty^\nu + \varepsilon \sqrt{-1} \partial\bar{\partial} \delta^\nu$$

is a smooth Kähler form on X_ν.

We set

$$\xi_\nu = \sum_i \tilde{b}_i^\nu \log \| \sigma_i^\nu \|^2 .$$

4.3 Construction of a suitable ample divisor

To construct Kähler-Einstein current on X, we use the Dirichlet problem for parabolic Monge-Ampère equation. Hence we shall construct a strongly pseudoconvex convex exhaustion of a Zariski open subset of X_ν with certain properties. We fix sufficiently large ν hereafter. Let

$$\Phi : X_\nu \longrightarrow \mathbf{P}^N$$

be a embedding of X_ν into a projective space. Let

$$\pi_\alpha ; X_\nu \longrightarrow \mathbf{P}^n (\alpha = 1, \dots m)$$

be generic projections and we set

$$W_\alpha : \text{the ramification divisor of } \pi_\alpha, H_\alpha := \pi_\alpha^*(\{z_0 = 0\}),$$

where $[z_0 : \dots : z_n]$ be the homogeneous coordinate of $\mathbf{P}^n$. For simplicity we shall denote the support of a divisor by the same notation as the one, if without fear of confusion. If m is sufficiently large, we may assume the following conditions:

1. $\bigcap_{\alpha=1}^m (W_\alpha + H_\alpha) = \phi$,
2. $D := F_\nu + \sum_{\alpha=1}^m (W_\alpha + H_\alpha)$ is an ample divisor with normal crossings.
3. D contains $S \cup R_\nu$.
4. $K_{X_\nu} + D$ is ample.

Then $X_\nu - D$ is strongly pseudoconvex and the following lemma is necessary for our purpose.

Lemma 4.2 *There exists a positive strongly plurisubharmonic exhaustion function φ of $X_\nu - D$ such that $\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\varphi$ is a complete Kähler form on $X_\nu - D$.*

Proof. Let $D = \sum_k D_k$ be the irreducible decomposition of D and let λ_k be a global holomorphic section of $\mathcal{O}_{X_\nu}(D_k)$ with divisor D_k. Then there exist hermitian metrics $\| \ \|$'s on $\mathcal{O}_{X_\nu}(D_k)$'s such that

$$-\sum_k \sqrt{-1}\partial\bar{\partial} \log \| \lambda_k \|^2$$

is a smooth Kähler form on X_ν. We set for a positive number ι

$$\varphi = -\sum_k \log \| \lambda_k \| - \iota \log\log \frac{1}{\| \lambda_k \|}.$$

Then if we choose ι sufficiently small, then

$$\omega_{\varphi} = \sqrt{-1}\partial\bar{\partial}\varphi$$

is a complete Kähler form on $X_{\nu} - D$. Clearly by adding a sufficiently large positive number, we can make the exhuastion φ to be positive on $X_{\nu} - D$. Q.E.D.

Remark 4.2 *As one see in Lemma 4.8 below, ω_{φ} has a bounded Poincaré growth.*

4.4 The Dirichlet problem

We set for $c > 0$,

$$K_c = \{x \in X_{\nu} : \varphi(x) \leq c\}.$$

It is easy to see that we may assume that there exists a positive constant c_0 such that the boundary ∂K_c is smooth for every $c \geq c_0$. We fix such c and set

$$K := K_c$$

for simplicity.

In the estimate of u, we try to make the estimate independent of $c \geq c_0$ for the later use.

Since $X_{\nu} - D$ is canonically biholomorphic to a Zariski open subset of X, we may consider K as a compact subset of X. We consider the following Dirichlet problem for a parabolic Monge-Ampère equation.

$$\begin{cases} \frac{\partial u}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u \quad \text{on } K \times [0,T) \\ u &= (1 - e^{-t^4})\xi_{\nu} \quad \text{on } \partial K \times [0,T) \\ u &= 0 \quad \text{on } K \times \{0\}, \end{cases} \tag{10}$$

where ω_t,Ω are the same as in the last section and T is a maximal existence time for the smooth solution on $\bar{K}$(the closure in the usual topology). We shall assume that

$$\omega_0^n = \Omega$$

holds. It is easy to find such ω_0 and Ω by using the solution of Calabi's conjecture ([28]). By multiplying a common sufficiently large positive number to ω_0 and Ω, if neceessary, we may assume that

$$\omega_0 + \operatorname{Ric}\Omega = \omega_0 - \omega_{\infty} > 0$$

holds. Please do not confuse u with the one in the last section. We use the same notation for simplicity. For the first we shall show

Theorem 4.2 *T is infinite and*

$$u_\infty = \lim_{t\to\infty} u$$

exists in C^∞-topology on $\bar{K}$.

4.5 C^0-estimate

We note that by the above choice of ω_0,Ω and the Dirichlet condition, the Dirichlet problem (10) is compatible up to 3-rd order on the corner $\partial K \times \{0\}$.

Hence by the standard implicit function theorem, we see that T is positive. Suppose T is finite. Then if $\lim_{t\to T} u$ exist on K in C^∞-topology, then this is a contradiction. Because again by the implicit function theorem, we can continue the solution a little bit more. Hence to prove Theorem 4.2, it is sufficient to obtain an estimate of C^k-norm of u on K which is independent of t.

We begin with C^0-estimate.

Lemma 4.3 *There exists a constant C_0^+ such that*

$$\frac{\partial u}{\partial t} \leq C_0^+ e^{-t} \quad \text{on } K \times [0,T)$$

holds.

Proof We set

$$\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}u$$

and

$$\tilde{\Delta} = tr_\omega \sqrt{-1}\partial\bar{\partial}.$$

As in the proof of Lemma 3.1, we have

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \tilde{\Delta}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - e^{-t} tr_\omega(\omega_0 - \omega_\infty)$$

holds on $K \times [0,T)$. Since

$$u = (1 - e^{-t^4})\xi_\nu \quad \text{on } \partial K \times [0,T)$$

and $\omega_0 - \omega_\infty$ is a Kähler form on X, by maximal principle

$$\frac{\partial u}{\partial t} \leq C_0^+ e^{-t} \text{ on } K \times [0,T)$$

holds for

$$C_0^+ = \max\{\max_{x\in K} \log\frac{\omega_0^n}{\Omega}(x), \max_{x\in\partial K} \xi_\nu(x)\}.$$

Q.E.D.

We set

$$v = u - (1 - e^{-t^4})\xi_\nu,$$
$$\Omega_\nu = \exp(\xi_\nu)\Omega,$$

and

$$\hat{\omega}_t = \omega_t + (1 - e^{-t^4})\sqrt{-1}\partial\bar{\partial}\xi_\nu.$$

Then v satisfies the equation:

$$\begin{cases} \frac{\partial v}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}v)^n}{\Omega_\nu} - v & \text{on } K \times [0,T) \\ v = 0 & \text{on } \partial K \times [0,T) \\ v = 0 & \text{on } K \times \{0\} \end{cases} \quad (11)$$

We note that $\hat{\omega}_t$ is a Kähler form on $(X_\nu - D) \times [0, \infty]$ and $\hat{\omega}_\infty = \omega_\infty^\nu$. Then since

$$\log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}v)^n}{\hat{\omega}_t^n} = \int_{a=0}^1 \tilde{\Delta}_a v,$$

where $\tilde{\Delta}_a$ is the Laplacian with respect to the Kähler form

$$\hat{\omega}_t + a\sqrt{-1}\partial\bar{\partial}v,$$

by maximum principle and Lemma 4.2, we have that

$$v \geq \min\{\min_{x \in K} \log \frac{\hat{\omega}_t^n}{\Omega_\nu}(x) - C_0^+ e^{-t}, 0\} \text{ on } K \times [0,T)$$

holds. Hence we have

Lemma 4.4

$$u \geq C_0^- + (1 - e^{-t^4})\xi_\nu \ \textit{on} \ K \times [0,T),$$

where

$$C_0^- = \min\{\inf_{(x,t) \in K \times [0,\infty)} \log \frac{\hat{\omega}_t^n}{\Omega_\nu}(x,t), 0\} - C_0$$

We note that C_0^- may depend on K because $\log(\hat{\omega}^n/\Omega_\nu)$ may not be bounded from below on $X_\nu - D$. To obtain the C^0-estimate from below which is independent of K, we shall consider for $\varepsilon \in (0,1]$,

$$v_\varepsilon = u - (1 - e^{-t^4})(\xi_\nu + \varepsilon\delta_\nu).$$

Then by the same argument, we have

Lemma 4.5 *Let $\varepsilon \in (0,1]$. Then there exists a constant $C_0^-(\varepsilon)$ which is independent of K such that*

$$u \geq C_0^-(\varepsilon) + (1 - e^{-t^4})(\xi_\nu + \varepsilon\delta_\nu) \text{ on } K \times [0,T)$$

The reason why $C_0^-(\varepsilon)$ is independent of K is simply because

$$\hat{\omega}_\infty + \varepsilon\sqrt{-1}\partial\bar{\partial}\delta_\nu$$

on $X_\nu - D$ extends to a smooth Kähler form on X_ν and

$$\exp(\xi_\nu + \varepsilon\delta_\nu)f_\nu^*\Omega$$

is a smooth semipositive (n,n) form on X_ν.

4.6 C^1-estimate on ∂K

Hereafter we estimate derivatives of v basically by using the method in [4]. But our estimates is a little bit more complicated because we are working on a quasi-projective variety which cannot admits a global flat Kähler metric. We set

$$\psi_c = b(\varphi - c),$$

where b is a positive constant. We note that

$$\psi_c = 0 \text{ on } \partial K$$

and

$$\psi_c < 0 \text{ on } K.$$

Then since ω_φ is a complete Kähler form of Poincareé growth, if we take b sufficiently large

$$\log\frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\psi_c)^n}{\Omega_\nu} - \psi_c \geq 0 \text{ on } K \times [0,\infty)$$

holds. It is easy to see that we can take b independent of c and t. Then we have

$$\begin{cases} \frac{\partial(v-\psi_c)}{\partial t} & \geq \log\frac{(\hat{\omega}_t+\sqrt{-1}\partial\bar{\partial}v)^n}{(\hat{\omega}_t+\sqrt{-1}\partial\bar{\partial}\psi_c)^n} - (v-\psi_c) \quad \text{on } K \times [0,T) \\ v - \psi_c & = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial K \times [0,T) \\ v - \psi_c & = -\psi_c \qquad\qquad\qquad\qquad\quad \text{on } K \times \{0\} \end{cases}$$

Since

$$\log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}v)^n}{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\psi_c)^n} = \int_0^1 \dot{\Delta}_a(v - \psi_c)da,$$

where $\dot{\Delta}_a$ is the Laplacian with respect to the Kähler form

$$\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\{(1-a)\phi_c + av\}$$

, by the maximum principle, we obtain

$$v \geq \psi_c \quad \text{on } K \times [0, T).$$

On the other hand, trivially

$$\hat{\Delta}_t v \geq -n \text{ on } K \times [0, T)$$

holds, where $\hat{\Delta}_t$ is the Laplacian with respect to the Kähler form $\hat{\omega}_t$. Let h be the C^∞-function on $\bar{K} \times [0, T)$ such that

$$\begin{cases} \hat{\Delta}_t h &= -n \quad \text{on } K \times [0, T) \\ h &= 0 \quad \text{on } \partial K \times [0, T). \end{cases}$$

Then by the maximum principle, we have

$$v \leq h \text{ on } K \times [0, T).$$

Hence we have

$$\psi_c \leq v \leq h \text{ on } K \times [0, T). \tag{12}$$

Now to fix C^k-norms on $X_\nu - D$, we shall construct a complete Kähler-Einstein form on $X_\nu - D$.

We quote the following theorem.

Theorem 4.3 *([18]) Let $\bar{M}$ be a nonsingular projective manifold and let B be an effective divisor with only simple normal crossings. If $K_{\bar{M}} + B$ is ample, then there exists a unique (up to constant multiple) complete Kähler-Einstein form on $M = \bar{M} - B$ with negative Ricci curvaure.*

By the construction of D, D is a divisor with simple normal crossings and $K_{X_\nu} + D$ is ample. Hence by Theorem 4.3, there exists a complete Kähler-Einstein form ω_D on $X_\nu - D$ such that

$$\omega_D = -\mathrm{Ric}_{\omega_D}.$$

Then we have

$$\| dv \| \leq \max\{\| dh \|, \| d\psi_c \|\} \text{ on } \partial K \times [0, T),$$

where $\| \ \|$ is the pointwise norm with respect to ω_D. To make this estimate independent of K, we need to use special properties of ω_D.

Definition 4.3 *Let V be an open set in $\mathbf{C}^n$. A holomorphic map from V into a complex manifold M of dimension n is called a* quasi-coordinate map *iff it is of maximal rank everywhere on V. (V; Euclidean coordinate of $\mathbf{C}^n$) is called a* local quasi-coordinate *of M.*

Lemma 4.6 *(cf. [18, p.405, Lemma 2 and pp. 406-409]) There exists a family of local quasi-coordinates $\mathcal{V} = \{(V; v^1, \ldots, v^n)\}$ of $X_\nu - D$ with the following properties.*

1. *$X_\nu - D$ is covered by the images of $(V; v^1, \ldots, v^n)$'s.*

2. *The completment of some open neighbourhood of D is covered by a finite number of $(V, v^1, \ldots, v^n)$'s which are local coordinate in the usual sense.*

3. *Each V, as an open subset of the complex Euclidean space $\mathbf{C}^n$, contains a ball of radius $1/2$.*

4. *There exists positive constants c_D and $\mathcal{A}_k (k = 0, 1, 2, \ldots)$ independent of V's such that at each $(V, v^1, \ldots v^n)$, the inequalities:*

   $\frac{1}{c_D}(\delta_{ij}) < (g^D_{i\bar{j}}) < c_D(\delta_{ij}),$

   $\mid (\partial^{|p|+|q|}/\partial v^p \partial \bar{v}^q) g^D_{i\bar{j}} \mid < \mathcal{A}_{|p|+|q|},$ *for any multiindices p and q*

   hold, where $g^D_{i\bar{j}}$ denote the components of ω_D with respect to v^i's.

Definition 4.4 *$(\bar{M}, B)$ be a pair of smooth projective variety of dimension n and a divisor with simple normal crossings on it. A complete Kähler metric ω_M on $M = \bar{M} - B$ is said to have bounded Poincaré growth on (M, B) if for any polydisk $\Delta^n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid\mid z_i \mid < 1 (1 \le i \le n)\}$ in $\bar{M}$ such that*

$$\Delta^n \cap B = \{(z_1, \ldots, z_n) \in \Delta^n \mid \ z_1 \cdots z_k = 0\} (k \le n),$$

$\omega_M \mid \Delta^n$ is quasi-isometric to

$$\omega_P = \sum_{i=1}^{k} \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{\mid z_i \mid^2 (\log \mid z_i \mid)^2} + \sum_{i=k+1}^{n} \sqrt{-1} dz_i \wedge d\bar{z}_i$$

on every compact subset of Δ^n and every covariant derivative of $\omega_M \mid \Delta^n$ is bounded on every compact subset of Δ^n.

Then by the construction of ω_D, we have:

Lemma 4.7 *([18, pp.400-409]) ω_D has bounded Poincaré growth. on (X_ν, B).*

Remember the definition of φ in Lemma 4.2. Then the following lemma is trivial.

Lemma 4.8 *$\varphi^{-1} \| d\varphi \|$ is uniformly bounded on $X_\nu - D$.*

We note that v satisfies the following differential inequality.

$$\Delta_D v \geq -tr_{\omega_D}\hat{\omega}_t \text{ on } K \times [0, T),$$

where Δ_D is the Laplacian with respect to ω_D. Let h_D be the solution of the Dirichlet problem

$$\begin{cases} \Delta_D h_D & = -tr_{\omega_D}\hat{\omega}_t \quad \text{on } K \times [0, T) \\ h_D & = 0 \qquad\qquad\quad \text{on } \partial K \times [0, T) \end{cases}$$

Then by the maximum principle, we have

$$v \leq h_D$$

holds on $K \times [0, T)$. Hence

$$\psi_c \leq v \leq h_D$$

holds on $K \times [0, T)$ by (12). Hence by the maximum principle, we have

$$\| dv \| \leq \max\{\| d\psi_c \|, \| dh_D \|\} \text{on } \partial K \times [0, T).$$

By the standard boundary estimate for the second order linear ellptic equations (cf. [10]), we see that h_D is smooth on $\bar{K}$. By using the standard elliptic estimate and Lemma 4.6, it is easy to obtain an estimate for $\| dh_D \|$ on K. But in this case, since $\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\varphi$ is a complete Kähler form of Poincar/'e growth, we can find a negative constant b' independent of c and t such that

$$b'\Delta_D(\varphi - c) \leq -tr_{\omega_D}\hat{\omega}_t$$

holds. Then by the maximum principle, we see that

$$h_D \leq b'(\varphi - c) \quad \text{on } K$$

holds.

Then since b and b' are independent of c and t, by Lemma 4.8, we have:

Lemma 4.9 *There exists a positive constant C_1' independent of $c \geq c_0$ such that*

$$\| dv \| \leq C_1' c \quad \text{on } \partial K \times [0, T)$$

where $\| \ \|$ is the norm with respect to the Kähler form ω_D.

Remark 4.3 *As you have seen above, in the proof of Lemma 4.9, the use of h_D is not unnecessary. We can use a $b'\psi_c$ instead of h_D from the first. The reason why we have used h_D here is that the method can be applicable more general situations.*

4.7 C^1-estimate on K

Let $\pi_\alpha : X_\nu \longrightarrow \mathbf{P}^n$ be the generic projection constructed in 4.3. And let

$$Z = \mathrm{Re}(\sum_i \beta_i \frac{\partial}{\partial(z_i/z_0)}),$$

where $(\beta_1, \ldots, \beta_n) \in \mathbf{C}^n - \{O\}$. Then

$$\theta = \pi_\alpha^*(Z)$$

is a holomorphic differential operator on $X_\nu - D$ which is meromorphic on X_ν. By operating θ to (11), we have

$$\begin{cases} \frac{\partial(\theta v)}{\partial t} &= \tilde{\Delta}(\theta v) - \theta v + \theta \log \frac{\hat{\omega}^n}{\Omega_\nu} \quad \text{on } K \times [0,T) \\ \theta v &= \theta v \quad \text{on } \partial K \times [0,T) \\ \theta v &= 0 \quad \text{on } \partial K \times \{0\}, \end{cases}$$

where $\tilde{\Delta}$ is the Laplacian with repect to

$$\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial} v.$$

Then by the maximam principle and Lemma 4.8, we get

$$\| \theta v \| \leq C_1(\theta, K),$$

where

$$C_1(\theta, K) = \max\{ \sup_{\partial K \times [0,T)} \| \theta v \|, \sup_{K \times [0,T)} \| \theta \log \frac{\hat{\omega}^n}{\Omega_\nu} \| \}.$$

Then since $\| d \log(\hat{\omega}^n / \Omega_\nu) \|$ is bounded on $X_\nu - D$, if we take m sufficiently large and $\pi_\alpha (1 \leq \alpha \leq m)$ properly, we get :

Lemma 4.10 *There exists a positive constant $C_1(K)$ which depends on $c \geq c_0$ such that*

$$\| dv \| \leq C_1(K) \quad \text{on } K \times [0,T)$$

holds.

The estimate is getting worse if the point goes far from the boundary because θ has a pole along D.

We set

$$K_c(\varepsilon) = K_c - K_{c-\varepsilon}.$$

Then by the above argument and the construction of D in 4.3, we obtain the following estimate.

Lemma 4.11 *There exists positive constants C_1 and A_1 independent of $c \geq c_0$ such that*

$$\| dv \| \leq C_1 c \quad \text{on } K(e^{-A_1 c}) \times [0,T).$$

Remark 4.4 *This idea is inspired by the idea in [8].*

4.8 C^2-estimate on ∂K

In this subsection, we follow the argument in [4, pp. 218-223] and prove:

Lemma 4.12 *There exists a positive constant C_2' independent of $c \geq c_0$ such that*

$$\| \sqrt{-1}\partial\bar{\partial}v \| \leq \exp(C_2'c) \quad on\ \partial K \times [0, T)$$

Let P be a point on ∂K. Choose a coordinates $z_1, \ldots, z_n$ with origin at P such that

1. $dg_{i\bar{j}}^D(P) = 0$ and $g_{i\bar{j}}^D(P) = \delta_{ij}$.

2. There exists a positive number $\hat{b}$ such that

   $$r = \hat{b}(\varphi - c)$$

   satisfies $r_{z_\alpha}(0) = 0$ for $\alpha < n$, $r_{y_n}(0) = 0, r_{x_n} = -1$, where

   $$z_\alpha = x_\alpha + \sqrt{-1}y_\alpha$$

   and

   $$r_{z_\alpha} = \frac{\partial r}{\partial z_\alpha}$$

   and so on.

We set $s_1 = x_1, s_2 = y_1, \ldots, s_{2n-3} = x_{n-1}, s_{2n-2} = y_{n-1}, s_{2n-1} = y_n = s, s' = (s_1, \ldots, s_{2n-1})$. By $\partial\bar{\partial}$-Poincaré lemma, we choose a smooth function ϕ defined on a open neighbourhood U of P such that

$$\hat{\omega}_t = \sqrt{-1}\partial\bar{\partial}\phi$$

holds on U. Let g be a function defined by

$$g = \phi + v.$$

It is clear that to estimate $\sqrt{-1}\partial\bar{\partial}v(P)$ is equivalent to $\sqrt{-1}\partial\bar{\partial}g(P)$ because $\hat{\omega}_t$ is uniformly bounded with respect to ω_D on $X_\nu - D$ by a constant independent of t. . Moreover by Lemma 4.7, the convariant derivatives of $\hat{\omega}_t$ of any order with respect to ω_D is uniformly bounded with respect to the norm defined by ω_D on $X_\nu - D$ by a constant independent of t. Then by Lemma 4.6 , we may assume that U contains a ball of radius $1/2c_D$ with center P and any derivatives of ϕ of a fixed order with respect to $(z_1, \ldots, z_n)$ is bounded by a constant independent of $c \geq c_0$, if we allow $(U, z_1, \ldots, z_n)$ to be a quasi-coordinate. Since the estimate is completely local, this does not cause any trouble in our estimate in this subsection. Hence the C^2-estimate of v on ∂K is reduced completely to the C^2-estimate of g on ∂K.

Sublemma 4.1 *There exists a positive constant $\dot{C}_2$ independent of $c \geq c_0$ such that*

$$| g_{s_i s_j}(0) | \leq \dot{C}_2 \quad i,j \leq 2n-1.$$

For r near 0 we may represent g as

$$g = \phi + \sigma r.$$

Then

$$g_{x_n}(0) = \phi_{x_n} - \sigma(0),$$

so that by Lemma 4.9, $\sigma(0) \leq C_1' c$. Hence

$$g_{s_i s_j}(0) = \phi_{s_i s_j} + \sigma(0) r_{s_i s_j}$$

holds. We note that $r_{s_i s_j} = O(1/c)$ because of the normalization. Hence we get the sublemma.

Sublemma 4.2 *There exists a positive constant $\ddot{C}_2$independent of $c \geq c_0$ such that*

$$| g_{s_i x_n}(0) | \leq \exp(\ddot{C}_2 c)$$

holds.

The proof of Sublemma 4.2 is a little bit technical.

Writing the Taylor expansion of r up to second order we obtain:

$$r = \mathrm{Re}(-z_n + \sum a_{ij} z_i z_j) + \sum b_{i\bar{j}} z_i \bar{z}_j + O(| z |^3).$$

Introducing new coordinates of the form

$$\begin{aligned} z_n' &= z_n - \sum a_{ij} z_i z_j, \\ z_k' &= z_k \text{ for } k \leq n-1, \end{aligned}$$

we can write

$$r = -\mathrm{Re}(z_n') + \sum c_{i\bar{j}} z_i' \bar{z}_j' + O(| z |^3). \tag{13}$$

It is clear that $(c_{i\bar{j}})$ is positive definite.

We define T_i in a neighbourhood of 0 by

$$T_i = \frac{\partial}{\partial s_i} - \frac{r_{s_i}}{r_{x_n}} \frac{\partial}{\partial x_n}, \text{for } i = 1, \ldots, 2n-1;$$

then $T_i r = 0$ nad we have $T_i(g - \phi) = 0$ on $r = 0$.

We show that for suitable $\varepsilon > 0$, in the region

$$S_\varepsilon = \{x \in U \mid r(x) \leq 0, x_n \leq \varepsilon\},$$

where U is a neighbourhood of the origin, we set

$$w = \pm T_i(g-\phi) + (g_s - \phi_s)^2 - Ax_n + B \mid z \mid^2$$

We claim :

(a) For B sufficiently large, $\tilde{L}w \geq 0$;
(b)On ∂S_ε, if A is sufficiently large, $w \leq 0$ holds.

To prove (a) set

$$a = -r_{s_i}/r_{x_n}$$

and consider (we use summation convention)

$$\tilde{L}(T_i g) = T_i \log \Psi(z, g(z)) + g^{p\bar{q}} a_p g_{x_n \bar{q}} + g^{p\bar{q}} a_{\bar{q}} g_{x_n, p} + g^{p\bar{q}} a_{p\bar{q}} g_{x_n}, \tag{14}$$

where

$$\tilde{L} = \tilde{\Delta} - \frac{\partial}{\partial t},$$

and

$$\Psi(z, g(z)) = \frac{\Omega_\nu}{(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n} \exp(-g\phi).$$

Obeserve that $g^{p\bar{q}} g_{n\bar{q}} = \delta^p_n$ and that

$$\frac{\partial}{\partial x_n} = 2\frac{\partial}{\partial z_n} + \sqrt{-1}\frac{\partial}{\partial s}$$

so that

$$g_{x_n \bar{q}} = 2g_{n\bar{q}} + \sqrt{-1} g_{s\bar{q}}.$$

Thus the second term on the right-hand side of (14) is of the form

$$a_n + g^{p\bar{q}} a_p g_{t\bar{q}} = O(1 + (\sum g^{i\bar{i}})^{1/2} (g^{p\bar{q}} g_{ps} g_{\bar{q}s})^{1/2}).$$

A similar estimate holds for the third term on the right of (14) while the forth term is $O(\sum g^{i\bar{i}})$. Thus by Lemma 4.3 and the arithmetic-geometric mean inequality, we have

$$\pm \tilde{L} T_i g \leq -C\Psi^{-1/n} - g^{p\bar{q}} g_{ps} g_{\bar{q}s}.$$

Further

$$\begin{aligned} \tilde{L}(g_s - \phi_s)^2 &= 2g^{p\bar{q}} g_{ps} g_{\bar{q}s} + 2(g_s - \phi_s)(\partial_s \log \Psi - \tilde{L}\phi_s) \\ &\geq 2g^{p\bar{q}} g_{ps} g_{\bar{q}s} - C\Psi^{-1/n} \end{aligned}$$

holds on S_ε by the C^1-estimate (Lemma 4.11) and the arithmetic-geometric mean inequality (if we take ε sufficiently small). Hence we find

$$\begin{aligned} \tilde{L}w &\geq B \sum g^{i\bar{i}} - C\Psi^{-1/n} \\ &\geq 0 \quad \text{on } S_\varepsilon \times [0, T), \end{aligned}$$

if B is sufficiently large. B depends on the C^1-estimate of v on $S_\varepsilon \times [0,T)$ and which is uniform with the weight φ by Lemma 4.10. This completes the proof of (a).

To prove (b), consider first $\partial S_\varepsilon \cap \partial K$. Here we write $x_n = \rho(s_1, \dots, s_{2n-1})$ and from (13), we deduce that

$$\rho(t) = \sum_{ij<2n} b_{i\bar{j}} s_i s_j + O(|\, s' \,|^3) \tag{15}$$

with $(b_{i\bar{j}})$ positive definite. Thus on ∂K near O we have

$$x_n \geq a \,|\, z \,|^2,$$

where a is uniformly bounded from below by a positive constant times $1/c$ where by the construction of K. Also,

$$g(s, \rho(s)) = \phi(s, \rho(s)),$$

so that

$$|\, g_s - \phi_s \,|^2 \leq C \,|\, s \,|^2 \leq C\rho.$$

Taking A large we obtain (b). By Lemma 4.11, if we take a positive constant $\ddot{C}_2$ sufficiently large, we may assume that A is bounded from above by a constant times $\exp(\ddot{C}_2 c)$ for some positive constant $\ddot{C}_2$ independent of $c \geq c_0$. By the maximum principle and (a),(b),

$$w \leq 0 \qquad \text{on } S_\varepsilon$$

holds.

In view of the maximum principle, we have

$$|\, (T_i g)_{x_n}(0) \,| \leq A.$$

This completes the proof of Sublemma 4.2.

Using, still the special coordinate above we see that to finish our proof of Lemma 4.12 , we have only to establish the estimate

$$|\, g_{x_n x_n}(O) \,| \leq \exp(\acute{C}_2 c) \tag{16}$$

for some constant $\acute{C}_2$ independent of $c \geq c_0$. By the previous estimates:

$$|\, g_{s_i s_j}(O) \,| \leq \dot{C}_2 \quad (1 \leq i, j \leq 2n-1),$$
$$|\, g_{s_i x_n} \,| \leq \exp(\ddot{C}_2 c) \quad (1 \leq i \leq 2n-1),$$

it suffices to prove

$$|\, g_{n\bar{n}}(O) \,| \leq \exp(C_2' c)$$

for some $\hat{C}_2$ independent of $c \geq c_0$. We may solve the equation

$$\det(g_{i\bar{j}})(O) = \frac{\Omega_\nu}{\omega_D^n}(O)$$

for $g_{n\bar{n}}(O)$. Then since there exists a positive constant $C^\flat$ such that

$$\exp(-C^\flat c) \leq \frac{\omega_D^n}{\Omega_\nu} \leq \exp(C^\flat c)$$

holds on $X_\nu - D$, we see that (16) follows from (17) provided we know the following sublemma.

Sublemma 4.3 *([4, pp. 221-223]) The $(n-1)$ by $(n-1)$ matrix*

$$(g_{z_\alpha \bar{z}_\beta}(O))_{\alpha,\beta<n} \geq C_3 (\frac{\Omega_\nu}{\omega_D^n})^{\frac{1}{n}} \frac{1}{c} I$$

for some C_3; here I is the $(n-1)$ by $(n-1)$ identity matrix. C_3 is independent of $c \geq c_0$

The proof of this sublemma is very technical.

After subtraction of a linear function we may assume that $\phi_{s_j}(O) = 0, j \leq 2n-1$. To prove Sublemma 4.3, it suffices to prove

$$\sum_{\alpha,\beta<n} \gamma_\alpha \bar{\gamma}_\beta g_{z_\alpha \bar{z}_\beta}(O) \geq C_3 \mid \gamma \mid^2$$

which we shall do for $\gamma = (1, 0, \ldots, 0)$. We shall show that

$$g_{1\bar{1}}(O) \geq C_4, \tag{17}$$

where C_4 is a positive constant. Let $\tilde{g} = g - \lambda x_n$ with λ so chosen that

$$(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2})\tilde{g}(s_1, \ldots, s_{2n-1}, \rho(s_1, \ldots, s_{2n-1})) = 0 \quad \text{at } O,$$

i.e.

$$0 = g_{1\bar{1}}(O) + \tilde{g}_{x_n}(O)\rho_{1\bar{1}}(O) = g_{1\bar{1}}(O) + (g_{x_n}(O) - \lambda)\rho_{1\bar{1}}(O). \tag{18}$$

Using the fact that any real homogeneous cubic polynomial in (s_1, s_2) admits the unique decomposition

$$\mathrm{Re}(\alpha(s_1 + \sqrt{-1}s_2)^3 + \beta(s_1 + \sqrt{-1}s_2)(s_1 + \sqrt{-1}s_2)^2),$$

we find on expanding $\tilde{g} \mid_{\partial K \cap U}$ in a Taylor series, in $s_1, \ldots, s_{2n-1}$,

$$\begin{aligned}\tilde{g} \mid_{\partial K \cap U} = \mathrm{Re} \textstyle\sum_2^{n-1} a_j z_1 \bar{z}_j + \mathrm{Re}(a z_1 s) \quad & + \mathrm{Re}(p(z_1, \ldots, z_{n-1}) + \beta z_1 \mid z_1 \mid^2) \\ & + O(s_3^2 + \ldots + s_{2n-1}^2),\end{aligned}$$

where p is a holomorphic cubic polynomial.

With the aid of (13), we may replace the term $\beta z_1 \mid z_1 \mid^2$ to $(\rho_{z_1\bar{z}_1}(O))^{-1}\beta z_1 x_n$, if we change the a_j, a and p. Thus by changing the a_j, a and p appropriately we may obtain the inequality:

$$\tilde{g} \mid_{\partial K \cap U} \leq \text{Re } p(z) + \text{Re} \sum_2^n a_j z_1 \bar{z}_j + C \sum_{j=2}^n \mid z_j \mid^2 .$$

Let $\check{g} = \tilde{g} - \text{Re}\, p(z)$ and observe that $\check{g}$ satisfies

$$\det(\check{g}_{j\bar{k}}) = \det(\tilde{g}_{j\bar{k}}) = \Psi(z, g(z)).$$

Recall that $\Psi(z, g(z)) \geq \delta > 0$ on a neighbourhood of O, where δ depends on $\Omega_\nu / \omega_D^n(O)$ and the C^1-estimate of v on the neighbourhood. With ε small we see that in the set S_ε, we have $\Psi(z, g(z)) \geq \delta$. Let

$$h = -\delta_0 x_n + \delta_1 \mid z \mid^2 + \frac{1}{B} \sum_2^n \mid a_j z_1 + B z_j \mid^2 .$$

We wish to show that with the suitable choice of $\delta_0, \delta_1, B > 0$ we have $h \geq \check{g}$ on ∂S_ε. First observe that if B is sufficiently large and δ_0 so small that $-\delta_0 x_n + \delta_1 \mid z \mid^2 \geq 0$ on $\partial S_\varepsilon \cap \partial K$ (the dependence of δ_0 and δ_1 is controlled by the Levi form of ∂K). By Lemma 4.11, if we take B sufficiently large, we have

$$\check{g} \leq h \quad \text{on } \partial S_\varepsilon.$$

The function h is plurisubharmonic and the lowest eigenvalues of the complex Hessian $(h_{i\bar{j}})$ are bounded independently by δ_1 while the other eigenvalues are bounded independently of δ_1.

Hence choosing δ_1 equal to small const. times $\delta^{1/n}$

$$\det(h_{i\bar{j}}) \leq \delta \ \text{ in } S_\varepsilon$$

holds. By the mximum principle

$$\check{g} \leq h \quad \text{on } S_\varepsilon$$

holds. Hence by the maximum principle

$$\check{g}_{x_n}(O) \leq h_{x_n}(O) = -\delta_0.$$

The desired inequality follows from (19). This completes the proof of Lemma

4.12.

4.9 C^2-estimate on K

Using the C^2-estimate on ∂K, we shall obtain a C^2-estimate inside K. The method here is the same as in [25].

Let H be a smooth function on $X_\nu - D$ defined by

$$H = \exp(\delta_\nu)(\prod_k \| \lambda_k \| \prod_k (\log \frac{1}{\| \lambda_k \|})^{-1})^\varepsilon$$

where δ_ν is the one in 4.2 and $\| \lambda_k \|$'s are the ones in 4.6 and ε is a sufficiently small positive number such that

$$\omega_H = \omega_t + \sqrt{-1}\partial\bar{\partial}\log H$$

is a complete Kähler form on $(X_\nu - D) \times [0, \infty]$ which is quasi-isometric to ω_D on $X_\nu - D$, i.e., there exists a positive constant $C(D, H) > 1$ such that

$$\frac{1}{C(D,H)}\omega_H \leq \omega_D \leq C(D,H)\omega_H$$

holds on $X_\nu - D$. We note that ω_H have bounded Poincaré growth so that the bisectional curvature of ω_H is bounded between two constants uniformly on $(X_\nu - D) \times [0, \infty]$.

We set

$$v_H = v - \log H = u - (1 - e^{-t^4})\xi_\nu - \log H,$$

$$\Omega_H = H \cdot \Omega_\nu.$$

Then v_H satisfies the equation

$$\frac{\partial v_H}{\partial t} = \log \frac{(\omega_H + \sqrt{-1}\partial\bar{\partial}v_H)^n}{\Omega_H} - v_H \quad \text{on } K \times [0, T).$$

By Lemma 4.3 and Lemma 4.5 v_H satisfies the C^0-estimate:

Lemma 4.13 *For every sufficiently small positive number* ε

$$\begin{aligned} v_H \geq\ & C_0^-(\varepsilon) - \log H + (1 - e^{-t^4})\varepsilon\delta_\nu \\ v_H \leq\ & C_0^+(1 - e^{-t}) - (1 - e^{-t^4})\xi_\nu - \log H \end{aligned}$$

holds on $K \times [0, T)$*, where* $C_0^+, C_0^-(\varepsilon)$ *are constants in Lemma 4.3 and 4.5 respectively.*

We have the following lemma.

Lemma 4.14 *([25, Lemma 3.2])*

$$H^{-C}e^{Cv}(\tilde{\Delta} - \frac{\partial}{\partial t})(e^{-Cv}H^{C}tr_{\omega_H}\omega) \geq$$
$$(-\Delta_H \log\frac{\omega_H^n}{\Omega_H} - n^2 \inf_{i\neq j} R^H_{i\bar{i}j\bar{j}} - n) + C(n - \frac{1}{C} - \frac{\partial v_H}{\partial t})tr_{\omega_H}\omega -$$
$$e^{-t}tr_{\omega_H}((\omega_0 - \omega_\infty)\cdot\omega) + (C + \inf_{i\neq j} R^H_{i\bar{i}j\bar{j}})\exp(\frac{1}{n-1}(-\frac{\partial v_H}{\partial t} - v_H + \log\frac{\omega_H^n}{\Omega}))(tr_{\omega_H}\omega)^{\frac{n}{n-1}},$$

holds on $K \times [0,T)$, where $tr_{\omega_H}((\omega_0 - \omega_\infty)\cdot\omega)$ is defined as in Lemma 3.4, $\inf_{i\neq j} R^H_{i\bar{i}j\bar{j}}$ denotes the infimum of the bisectional curvarue of ω_H on $(X_\nu - D)\times[0,\infty]$ and C is a positive constant such that

$$C + \inf_{i\neq j} R^H_{i\bar{i}j\bar{j}} > 1$$

holds.

The proof of this lemma is the same as one of Lemma 3.2 in [25]. Hence we omit it.

Lemma 4.15 *If we take C sufficiently large, then there exists a positive constant C_2 independent of $c \geq c_0$ such that*

$$H^C tr_{\omega_D}\omega \leq C_2 \quad on\ K\times[0,T)$$

holds.

Proof. By the definition H has zero of order at least $r_j^\nu/2$ along R_j^ν (cf. 4.2). Then by Lemma 4.12, if we take C sufficiently large, there exists a constant $\tilde{C}_2$ independent of $c \geq c_0$ such that

$$H^C tr_{\omega_D}\omega \leq \tilde{C}_2 \text{ on } \partial K\times[0,T).$$

holds. Suppose $e^{-Cv}H^C tr_{\omega_H}\omega$ takes its maximum at $P_0 \in K\times\{t_0\}(t_0 \in [0,T))$ then by Lemma 4.14, we have

$$(tr_{\omega_H}\omega)(P_0) \leq \hat{C}_2$$

for a positive constant $\hat{C}_2$ independent of $c \geq c_0$ and C(if it is sufficiently large). Hence in this case we have

$$(H^C tr_{\omega_H}\omega)(P) \leq H^C(P_0)\exp(-C(v(P_0) - v(P)))\hat{C}_2 \quad \text{on } K\times[0,T)$$

holds. By Lemma 4.13 (since in Lemma 4.13, we can take ε arbitrarily small), $H\exp(-v) = \exp(-v_H)$ is uniformly bounded from above on $X_\nu - D$.

Hence if we change $\hat{C}_2$, if necessary, by the maximum principle for parabolic equations, we may assume that

$$(H^C tr_{\omega_H}\omega)(P) \leq \exp(Cv(P))\hat{C}_2$$

holds on $K \times [0,T)$. We note that ω_D and ω_H are quasi-isometric on $X_\nu - D$. Hence by Lemma 4.5, if we take C sufficiently large, there exists a positive constant C_2 independent of $c \geq c_0$ such that

$$H^C tr_{\omega_D}\omega \leq C_2 \ \text{ on } K \times [0,T)$$

holds. Q.E.D.

By [26], the higher order interior estimate on $K \times [0,T)$ follows. As for the boundary estimate of u, we just need to follow the argument in [4]. This completes the proof of Theorem 4.2.

4.10 Costruction of Kähler-Einstein currents

Let u_∞ be as in Theorem 4.2. Then by the construction

$$\omega_K = \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty$$

is a Kähler-Einstein form on $\bar{K} = \bar{K}_c$. We may assume that $c > 1$. Let us take the exahustion $\{K_{lc}\}_{l=1}^\infty$ of $X_\nu - D$ and let

$$\omega_l^\nu = \omega_{K_{lc}}.$$

Then by Lemma 4.3 and Lemma 4.15 and the regularity theorem in [26], we have the following lemma.

Lemma 4.16 *There exists a subsequence of $\{\omega_l^\nu\}_{l=1}^\infty$ which converges uniformly on every compact subset of $X_\nu - D$ in C^∞-topology to a Kähler-Einstein form ω^ν on $X_\nu - D$.*

Although ω^ν is a Kähler-Einstein form, it is not enough good for our purpose.

Let us consider the linear system $\mid m!\nu K_X \mid$ and construct $\xi_{m!\nu}$ as before. We denote $\xi_{m!\nu}$ by $\xi^{(m)}$ for simplicity. Since $X_{m!\nu} - F^{m!\nu}$ are all biholomorphic to $X - \mathrm{SBs}(K_X)$, we may consider $\{\xi^{(m)}\}$ as a family of functions on $X_\nu - D$. Let us denote $X_{m!\nu}$ by $X^{(m)}$ for simplicity and define $X_{(1)} = X$.

Lemma 4.17 *We can construct $\{\xi^{(m)}\}$ so that*

$$\xi^{(1)} \leq \xi^{(2)} \leq \ldots \leq \xi^{(m)} \leq \xi^{(m+1)} \leq \ldots$$

holds on $X_\nu - D$.

Proof. Let $\mathcal{I}_\mu$ denote the ideal sheaf of the base scheme Bs $|\ \mu K_X\ |$. Then clearly

$$\mathcal{I}_{\mu_1+\mu_2} \hookrightarrow \mathcal{I}_{\mu_1} \otimes_{\mathcal{O}_X} \mathcal{I}_{\mu_2}$$

holds. Hence inductively we can construct a morphism

$$\mu_m : X_{(m)} \longrightarrow X_{(m-1)}\ \ (m \geq 2)$$

such that

1. $f_{(m)} := \mu_m \circ \cdots \circ \mu_1 : X_{(m)} \longrightarrow X_{(0)}(= X)$ is a resolution of Bs $|\ m!\nu K_X\ |$.

2. The fixed part of $|\ f^*_{(m)}(m!K_X)\ |$ is a divisor with normal crossings on $X_{(m)}$.

An explicit construction of $\{\xi^{(m)}\}_{m=1}^{\infty}$ is as follows. Let $V^{(1)} = \{\eta_i^{(1)}\}_{i=0}^{N(1)}$ be a basis of $H^0(X, \mathcal{O}_X(\nu K_X))$. By induction for each $m \geq 1$, we can construct a finite subset

$$V^{(m)} = \{\eta_1^{(m)}, \ldots, \eta_{N(m)}^{(m)}\}$$

in $H^0(X, \mathcal{O}_X(m!\nu K_X))$ with the following properties.

1. $V^{(m)}$ spans $H^0(X, \mathcal{O}_X(m!\nu K_X))$.

2. $V^{(m)}$ contains all the elements of the form:

$$(\eta_{i_1}^{(m-1)})^{\otimes a_1} \otimes \cdots \otimes (\eta_{i_m}^{(m-1)})^{\otimes a_m}$$

$$\sum_{i=1}^{m} a_i = m, a_i \geq 0, 0 \leq i_1 < \ldots < i_m \leq N(m-1).$$

Now we set

$$\xi^{(m)} = \frac{1}{m!\nu} \log\left(\sum_{i=0}^{N(m)} \frac{(\sqrt{-1})^{m!\nu n^2} \eta_i^{(m)} \wedge \bar{\eta}_i^{(m)}}{\Omega^{m!\nu}}\right).$$

We may consider $\xi^{(m)}$ as a function on $X_\nu - D$. Then by the construction

$$\xi^{(1)} \leq \xi^{(2)} \leq \ldots \leq \xi^{(m)} \leq \ldots$$

holds and

$$\omega_\infty + \sqrt{-1}\partial\bar{\partial}\xi^{(m)}$$

is a smooth semipositive form on $X_{(m)}$ and positive on $X_\nu - D$. This completes the proof of the lemma. Q.E.D.

Now we consider the following Dirichlet problem.

$$\begin{cases} \frac{\partial u^{(m)}}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u^{(m)})^n}{\Omega} - u^{(m)} \quad \text{on } K \times [0, T_m) \\ u^{(m)} &= (1 - e^{-t^4})\xi^{(m)} \quad \text{on } \partial K \times [0, T_m) \\ u^{(m)} &= 0 \quad \text{on } K \times \{0\}, \end{cases}$$

where T_m is the maximal existence time for smooth solution on $\bar{K}$.

Lemma 4.18 *The followings are true.*

1. *T_m is infinite and $u^{(m)}_\infty = \lim_{t\to\infty} u^{(m)}$ exists in C^∞-topology on $\bar{K}$.*
2. *$\omega^{(m)}_1 := \omega_\infty + \sqrt{-1}\partial\bar{\partial}u^{(m)}_\infty$ is a Kähler-Einstein form on K.*
3. *If we define a sequence of Kähler-Einstein forms $\{\omega^{(m)}_l\}^\infty_{l=1}$ in the same manner as the definition of $\{\omega^\nu_l\}^\infty_{l=1}$ above, then there exists a subsequence of $\{\omega^{(m)}_l\}$ which converges uniformly on every compact subset of $X_\nu - D$ uniformly in C^∞-topology.*

Proof. The only difference between the above equation and the equation (10) is that $\nabla^k\xi^{(m)}(k \geq 1)$ is bounded with respect to ω_D with weights different from before. It is easy to find such weights. In fact, there exists a positive constant $C(m,k)$ depending only on m and k such that $H^{C(m,k)} \parallel \nabla^k\xi^{(m)} \parallel$ is bounded by a positive constant on $X_\nu - D$. Hence the previous argument is valid with some weight with respect to H. Hence the first assertion is trivial.

Then by replacing K to K_{lc}, we get a sequence of Kähler-Einstein form $\{\omega^{(m)}_l\}^\infty_{l=1}$ which are defined on K_{lc} respectively ($\omega^{(1)}_l = \omega^\nu_l$).

We would like to find a subsequence of $\{\omega^{(m)}_l\}^\infty_{l=1}$ which converges in C^∞-topology on every compact subset of $X_\nu - D$.

For the first, replacing ξ_ν by $\xi^{(m)}$, completely analogous estimate as Lemma 4.5 holds for $u^{(m)}$ with the perturbation $\delta^{(m)}$ completely analogous to δ_ν. As for the C^2-estimate, by the proof of Lemma 4.18, Lemma 4.15 also holds for ω^m_l if we replace C and C_2 to appropriate constants independent of l. The rest of the proof is the same as in the proof of Lemma 4.16. Q.E.D.

Taking subsequence, if necessary, we obtain a Kähler-Einstein form $\omega^{(m)}$ as before, where $\omega^{(1)} = \omega^\nu$.

We shall consider $\omega^{(m)}$ as ' a d-closed positive $(1,1)$-current on X by

$$\omega^{(m)} := \omega_\infty + \sqrt{-1}\partial\bar{\partial}\log\frac{(\omega^{(m)})^n}{\Omega},$$

where $\partial\bar{\partial}$ is taken in the sense of a current. This definition is well defined by the C^0-estimate (Lemma 4.5) and clearly

$$[\omega^{(m)}] = 2\pi c_1(K_X)$$

holds. Now we want to show that

Proposition 4.1

$$\omega_E = \lim_{m\to\infty}\omega^{(m)}$$

exists in the sense of a d-closed positive $(1,1)$*-current on* X_ν.

Lemma 4.19

$$(\omega^{(m)})^n \le \omega_D^n \quad on\ X_\nu - D$$

holds.

Proof. Since both ω_D and $\omega^{(m)}$ are Kähler-Einstein forms on $X_\nu - D$ and ω_D is a *complete* Kähler-Einstein form on $X_\nu - D$, by applying Yau's Schwarz lemma ([29]) to the holomorphic map

$$id : (X_\nu - D, \omega^{(m)}) \le (X_\nu - D, \omega_D),$$

we obtain the lemma. Q.E.D.

Hence $\{(\omega^{(m)})^n\}$ is uniformly bounded from above. For the next we shall show:

Lemma 4.20 *For every* $m \ge 1$*, we have that*

$$(\omega^{(m)})^n \le (\omega^{(m+1)})^n \quad on\ X_\nu - D$$

holds.

Proof. $u^{(m+1)} - u^{(m)}$ satisfies the following equations:

$$\begin{cases} \frac{\partial(u^{(m+1)}-u^{(m)})}{\partial t} &= \log\frac{(\omega_t+\sqrt{-1}\partial\bar{\partial}u^{(m+1)})^n}{(\omega_t+\sqrt{-1}\partial\bar{\partial}u^{(m)})^n} - \left(u^{(m+1)} - u^{(m)}\right) \quad \text{on } K\times[0,\infty) \\ u^{(m+1)} - u^{(m)} &= (1-e^{-t})(\xi^{(m+1)} - \xi^{(m)}) \quad \text{on } \partial K\times[0,\infty) \\ u^{(m+1)} - u^{(m)} &= 0 \quad \text{on } K\times\{0\}. \end{cases}$$

Since

$$\log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u^{(m+1)})^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}u^{(m)})^n} = \int_0^1 \Delta_a^{(m,m+1)}(u^{(m+1)} - u^{(m)})da,$$

where $\Delta_a^{(m,m+1)}(a \in [0,1])$ is the Laplacian with respect to the Kähler form

$$\omega_t + \sqrt{-1}\partial\bar{\partial}\{(1-a)u^{(m)} + au^{(m+1)}\},$$

we see that this equation is of parabolic type. We note that $\xi^{(m+1)} \geq \xi^{(m)}$ on $X_\nu - D$ by the construction. Then by the maximum principle we obtain

$$u^{(m)} \leq u^{(m+1)} \text{on } K \times [0, \infty).$$

We set

$$u_\infty^{(m)} = \lim_{t\to\infty} u^{(m)}.$$

Then we have

$$\omega_1^{(m)} = \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty^{(m)}$$

and

$$(\omega_1^{(m)})^n = \exp(u_\infty^{(m)})\Omega$$

on K. Hence we see that

$$(\omega_1^{(m)})^n \leq (\omega_1^{(m+1)})^n$$

holds on K. By replacing K to K_{lc} and repeating the same argument, we see that

$$(\omega_l^{(m)})^n \leq (\omega_l^{(m+1)})^n$$

holds on K_{lc}. By letting l tend to infinifty, we completes the proof of the lemma. Q.E.D.

Hence $\{(\omega^{(m)})^n\}_{m=1}^\infty$ is monotone increasing and bounded from above uniformly on every compact subset of $X_\nu - D$

$$\omega_E^n := \lim_{m\to\infty}(\omega^{(m)})^n$$

exists.

We shall define a d-closed positive $(1,1)$ current ω_E on X by

$$\omega_E = \omega_\infty + \sqrt{-1}\partial\bar{\partial}\log\frac{\omega_E^n}{\Omega}.$$

ω_E is well defined by the C^0-estimates in the last subsection. Then it is clear that $[\omega_E] = 2\pi c_1(K_X)$.

4.11 Finite orderness of the Kähler-Einstein current ω_E

We shall prove that ω_E is smooth on a Zariski open subset of X and satisfies some growth condition near the singular set. The method used here is a modification of the one in [28].

Definition 4.5 *Let T be a d-closed positive $(1,1)$-current on a projective maifold M which is smooth on a Zariski open subset U of M. Let S denote $M-U$ and consider S as a reduced subvariety of M. Let $\mathcal{I}_S$ denote the ideal sheaf of S. T is said to be of finite order on M along S, if there exists a positive integer m such that for every point of $s \in S$ and $\sigma \in (\mathcal{I}_S^{\otimes m})_s$, there exists a neighbourhood V of s such that σ is defined on V and*

$$|\sigma|^2 (T \mid U \cap V)$$

extends locally as a bounded form on V.

Lemma 4.21 *ω_E is smooth on $X_\nu - D$ (remember we can identify $X_\nu - D$ with a Zariski open subset of X) and has finite order along D.*

To prove this lemma, by Lemma 4.15 it is sufficient to prove the following.

Lemma 4.22 *$\omega^{(m)} = \omega^{(1)}$ for all $m \geq 1$. Hence in particular $\omega_E = \omega^{(1)}$.*

Proof. Let m be a fixed positive integer. Let dV be the volume form on K defined by

$$dV = \frac{(\omega_1^{(1)})^n}{n!}$$

and let w be the fuction on K defined by

$$w = \log \frac{(\omega_1^{(m)})^n}{(\omega_1^{(1)})^n}.$$

By Lemma 4.20, $w \geq 0$ on K. Then by Stokes' theorem we have

$$\int_K \Delta_1^{(1)}(w^2)dV = \int_{\partial K} \frac{\partial}{\partial \mathrm{n}}(w^2)dS,$$

where $\Delta_1^{(1)}$ is the Laplacian with respect to $\omega_1^{(1)}$, $\partial/\partial\mathrm{n}$ is the derivation with respect to the unit outer normal vector field on ∂K and dS is the volume form on ∂K with respect to the Kähler form $\omega_1^{(1)}$. On the other hand noting that $w \geq 0$ on K, we have

$\frac{1}{2}\int_K \Delta_1^{(1)}(w^2)dV = \int_K |\nabla w|^2 dV + \int_K w\Delta_1^{(1)}w dV$

$\geq \int_K w \cdot n(\exp(\frac{1}{n}w) - 1)dV$ by algebro-geometric mean inequality,

$\geq \int_K w^2 dV$ by $e^x \geq x + 1$.

Hence we obtain the inequality:

$$\int_{\partial K} \frac{\partial}{\partial \mathbf{n}}(w^2)dS \geq 2\int_K w^2 dV. \tag{19}$$

By Lemma 4.3 and its proof, we obtain the following sublemma.

Sublemma 4.4 *There exists a positive constant $C_0^{(m)}$ independent of c such that*

$$(\omega_1^{(m)})^n \leq C_0^{(m)}\Omega \quad on\ K$$

holds.

We define a C^∞-function $u^{(m)}$ on $\bar{K} \times [0, \infty]$ by

$$\begin{cases} \frac{\partial u^{(m)}}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial} u^{(m)})^n}{\Omega} - u^{(m)} \quad \text{on } K \times [0, \infty), \\ u_t^{(m)} &= (1 - e^{-t^4})\xi^{(m)} \quad \text{on } \partial K \times [0, \infty), \\ u^{(m)} &= 0 \quad \text{on } K \times \{0\} \end{cases}$$

as before. We note that by Lemma 4.18,

$$u_\infty^{(m)} = \lim_{t \to \infty} u^{(m)}$$

exists in C^∞-topology on $\bar{K}$.

Sublemma 4.5 *Let C be a positive number. If we replace $\xi^{(m)}$ to $\xi^{(m)} + C$, $\omega^{(m)}$ will be unchanged.*

Proof. Let $u_C^{(m)}$ be the solution of the Dirichlet problem:

$$\begin{cases} \frac{\partial u_C^{(m)}}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial} u_C^{(m)})^n}{\Omega} - u_C^{(m)} \quad \text{on } K \times [0, \infty), \\ u_C^{(m)} &= (1 - e^{-t^4})(\xi^{(m)} + C) \quad \text{on } \partial K \times [0, \infty) \\ u_C^{(m)} &= 0 \quad \text{on } K \times \{0\} \end{cases}$$

We set

$$w_C^{(m)} = \lim_{t \to \infty}(u_C^{(m)} - u^{(m)}).$$

Then as in the proof of Lemma 4.20, we have

$$w_C^{(m)} \geq 0 \quad \text{on } K.$$

As before, we have

$$\int_{\partial K}\frac{\partial}{\partial \mathrm{n}(m)}(w_C^{(m)})^2 dS_{(m)} = C\int_{\partial K}\frac{\partial}{\partial \mathrm{n}(m)} w_C^{(m)} dS_{(m)}$$
$$\geq 2\int_K (w_C^{(m)})^2\frac{(\omega^{(m)})^n}{n!}, \tag{20}$$

where $\mathrm{n}(m)$ is the unit outer normal vector field on ∂K with respect to $\omega_1^{(m)}$ and $dS_{(m)}$ is the volume form of ∂K induced from $\omega_1^{(m)}$. Then we have

$$\begin{cases} \log\frac{(\omega_1^{(m)}+\sqrt{-1}\partial\bar{\partial} w_C^{(m)})^n}{(\omega_1^{(m)})^n} = w_C^{(m)} & \text{on } K, \\ w_C^{(m)} = C & \text{on } \partial K. \end{cases}$$

Let us take a positve number ε such that

$$\log\frac{(\omega_1^{(m)}+\varepsilon\sqrt{-1}\partial\bar{\partial}\varphi)^n}{(\omega_1^{(m)})^n} - \varepsilon(\varphi - c) - C \geq 0 \ \text{ on } K$$

holds. Then as in 4.6 by the maximum principle, we have

$$w_C^{(m)} \geq \varepsilon(\varphi - c) + C. \tag{21}$$

Since $\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\varphi$ is a complete Kähler form of Poincaré growth, by Sublemma 4.4, it is easy to see that we can take ε such that

$$\varepsilon = O(\frac{1}{c}). \tag{22}$$

By the completely analogous estimate as Sublemma 4.1 we have the following sublemma.

Sublemma 4.6 *Let $T(\partial K)_{\mathbb{C}}$ be the complex tangent bundle of the CR-manifold ∂K. Let us fix a complete Kähler form $\omega_D^{(m)}$ of Poincaré growth on $X_{(m)} - F^{m!\nu}$. Then the norm of the restriction of $\omega_1^{(m)}$ to $T(\partial K)_{\mathbb{C}}$ with respect to $\omega_D^{(m)}$ is bounded by a constant independent of c.*

Then by Sublemma 4.6, we see that

$$\| \frac{\partial}{\partial \mathrm{n}(m)} \otimes dS_{(m)} \| = O(c) \tag{23}$$

holds, where $\| \ \|$ is the norm with respect to $\omega_D^{(m)}$. Combining (21),(22),(23) and (24), letting c tend to infinity, we see that

$$\lim_{c\to\infty} w_C^{(m)} = 0$$

holds on K. This completes the proof of Sublemma 4.5. Q.E.D.

Sublemma 4.7 *Let $\tilde{\xi}^{(m)}$ be a function such that*

1.
$$\omega_\infty + \sqrt{-1}\partial\bar{\partial}\tilde{\xi}^{(m)}$$
is a C^∞-semipositive form on $X_{(m)}$ and positive on $X_\nu - D$.

2. *There exists a positive constant C such that*
$$\xi^{(m)} - C \leq \tilde{\xi}^{(m)} \leq \xi^{(m)} + C$$
holds.

Then if we replace $\xi^{(m)}$ to $\tilde{\xi}^{(m)}$, $\omega^{(m)}$ will be unchanged.

Proof. Let $\tilde{u}^{(m)}$ be the solution of the Dirichlet problem :

$$\begin{cases} \frac{\partial \tilde{u}^{(m)}}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\tilde{u}^{(m)})^n}{\Omega} - \tilde{u}^{(m)} & \text{on } K \times [0,\infty), \\ \tilde{u}^{(m)} = (1 - e^{t^4})\tilde{\xi}^{(m)} & \text{on } \partial K \times [0,\infty), \\ \tilde{u}^{(m)} = 0 & \text{on } K \times \{0\}. \end{cases}$$

Then by the maximum principle, we see that

$$u^{(m)}_{-C} \leq \tilde{u}^{(m)} \leq u^{(m)}_C \quad \text{on } K \times [0,\infty)$$

holds. Then by Sublemma 4.5, this completes the proof. Q.E.D.

We set for $\varepsilon > 0$,

$$\xi^{(m)}_\varepsilon = \frac{1}{m!\nu} \log\left(\exp(m!\nu\xi^{(1)}) + \varepsilon \exp(m!\nu\xi^{(m)})\right).$$

Then $\xi^{(m)}_\varepsilon$ has the following properties.

1.
$$\omega_\infty + \sqrt{-1}\partial\bar{\partial}\xi^{(m)}_\varepsilon$$
is a C^∞-semipositive form on $X_{(m)}$ and positive on $X_\nu - D$.

2.
$$\lim_{\varepsilon \downarrow 0} \xi^{(m)}_\varepsilon = \xi^{(1)} \quad \text{on } X_\nu - D$$
holds.

3.

$$\xi_\varepsilon^{(m)} \geq \xi^{(1)} \quad \text{on } X_\nu - D$$

holds.

4. There exists a positive constant C_ε depending on ε such that

$$\xi^{(m)} - C_\varepsilon \leq \xi_\varepsilon^{(m)} \leq \xi^{(m)} + C_\varepsilon \quad \text{on } X_\nu - D$$

holds.

Let $u_\varepsilon^{(m)}$ be the solution of Dirichlet problem:

$$\begin{cases} \frac{\partial u_\varepsilon^{(m)}}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial} u_\varepsilon^{(m)})^n}{\Omega} - u_\varepsilon^{(m)} \quad \text{on } K \times [0,\infty), \\ u^{(m)} &= (1 - e^{t^4})\xi_\varepsilon^{(m)} \quad \text{on } \partial K \times [0,\infty), \\ u_\varepsilon^{(m)} &= 0 \quad \text{on } K \times \{0\}. \end{cases}$$

We set

$$w_\varepsilon = \lim_{t \to \infty}(u_\varepsilon^{(m)} - u^{(m)}).$$

By Lemma 4.18 and the maximum principle, we see that w_ε exists and C^∞ on $\bar{K}$ and is nonnegative on K. Then as (20), we have the inequality:

$$\int_{\partial K} \frac{\partial}{\partial \mathbf{n}}(w_\varepsilon)^2 \cdot dS \geq 2\int_K w_\varepsilon^2 dV \quad \text{on } K. \tag{24}$$

Sublemma 4.8 *Let dS_D be the volume form of ∂K induced from the Kähler form ω_D. Then there exists a positive constant $\hat{C}_m$ independent of c and ε such that*

$$\frac{\partial w_\varepsilon^{(m)}}{\partial \mathbf{n}} dS \leq \hat{C}_m c \cdot dS_D$$

holds.

Proof. We note that by Sublemma 4.7 (taking m to be 1)

$$\| \frac{\partial}{\partial \mathbf{n}} \otimes dS \|$$

is bounded by a constant independent of c, where $\| \ \|$ denote the the norm with respect to ω_D. Then by Lemma 4.9, it is suficient to prove that there exists a positive constant $\tilde{C}_m$ independent of c such that

$$\frac{\partial u_{\varepsilon\infty}^{(m)}}{\partial \mathbf{n}} dS \leq \tilde{C}_m c \cdot dS_D$$

holds, where

$$u_{\varepsilon\infty}^{(m)} = \lim_{t\to\infty} u_\varepsilon^{(m)}$$

($u_{\varepsilon\infty}^{(m)}$ exists and C^∞ on $\bar{K}$ by Lemma 4.18). We set

$$\hat{\omega}_{\varepsilon\infty}^{(m)} = \omega_\infty + \sqrt{-1}\partial\bar{\partial}\xi_\varepsilon^{(m)}$$

and

$$v_{\varepsilon\infty}^{(m)} = u^{(m)} - \xi_\varepsilon^{(m)}.$$

We note that by the construction $\hat{\omega}_{\varepsilon\infty}^{(m)}$ is a smooth semipositive form on $X_{(m)}$. Since $\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\varphi$ is a complete Käjhler form of Poincaré growth on $X_\nu - D$, it is easy to find a positive number α independent of c and ε such that

$$\log \frac{(\hat{\omega}_{\varepsilon\infty}^{(m)} + \sqrt{-1}\alpha\partial\bar{\partial}\varphi)^n}{\Omega} - \alpha(\varphi - c) \geq 0 \quad \text{on } K$$

holds. Then as in the proof of Lemma 4.9, by the maximum principle, we see that

$$v_{\varepsilon\infty}^{(m)} \geq \alpha(\varphi - c) \quad \text{on } K$$

holds. By Lemma 4.10, we see that there exists a positive constant $\dot{C}_m$ independent of c and ε such that

$$\frac{\partial v_{\varepsilon\infty}^{(m)}}{\partial \mathbf{n}} dS \leq \dot{C}_m c \cdot dS_D \quad \text{on } \partial K$$

holds. On the other hand by direct calculation, (by identifying $X_\nu - D$ as a Zariski open subset of $X_{(m)}$), using Sublemma 4.7, we see that there exists a postive constant $\dot{C}'_m$ independent of c and ε such that

$$\frac{\partial \xi_\varepsilon^{(m)}}{\partial \mathbf{n}} dS \leq \dot{C}'_m c \cdot dS_D \quad \text{on } \partial K$$

holds. Combining the above estimates, we obtain the sublemma. Q.E.D.

We note that $\xi_\varepsilon^{(m)}$ and $\xi^{(1)}$ have very slow growthes ("logarithmic growth") along D respectively and the volume of ∂K with respect to ω_D is bounded by a constant times $1/c$ because ω_D is a complete Kähler form of Poincaré growth on $X_\nu - D$. Then since there exists a subvariety $V_{(m)}$ of $X_{(m)}$ of codimension ≥ 2 such that $\xi_\varepsilon^{(m)} - \xi^{(1)}$ is locally bounded on $X_{(m)} - V_{(m)}$, by Sublemma 4.8, we see that

$$\int_{\partial K} (\xi_\varepsilon^{(m)} - \xi^{(1)}) \frac{\partial w_\varepsilon}{\partial \mathbf{n}} dS$$

is bounded from above by a positive constant which is independent of c. Since $V_{(m)}$ is of codimension at least 2 in X_ν, by Sublemma 4.8, we see that

$$\limsup_{\varepsilon\downarrow 0, c\to\infty} \int_{\partial K} (\xi_\varepsilon^{(m)} - \xi^{(1)}) \frac{\partial w_\varepsilon}{\partial \mathrm{n}} dS = 0 \tag{25}$$

holds. By (25),(26) and Sublemma 4.7, we obtain the equality:

$$(\omega^{(1)})^n = (\omega^{(m)})^n \quad \text{on } X_\nu - D.$$

Since $\omega^{(1)}$ and $\omega^{(m)}$ are both Kähler-Einstein form with the same constant Ricci curvature, we have

$$\omega^{(1)} = \omega^{(m)} \quad \text{on } X_\nu - D.$$

This completes the proof of Lemma 4.22. Q.E.D.

5 L^2-vanishing theorem

In the last section we constructed a Kähler-Einstein current ω_E on X. In this section we shall prove the following theorem.

Theorem 5.1 *Let $f : Y \longrightarrow X$ be an arbitrary modification from a smooth projective variety. Then $\{\Theta(f^*\omega_E, y) \mid y \in Y\}$ is a finite set.*

5.1 L^2-estimate on a complete Kähler manifold

We shall briefly review the L^2-estimate on a complete Kähler manifold.

Let (M, ω) be a complete Kähler manifold of dimension m and let (L, h) be a hermitian line bundle on M. Let $A_c^{0,p}(M, L)(0 \leq p \leq n)$ denote the space of L-valued smooth $(0,p)$ form on M with compact support. $A_c^{0,p}(M, L)$ has a natural pre-Hilbert space structure with respect to the hermitian metric h and the Kähler form ω. Let

$$\bar\partial : A_c^{0,p}(M, L) \longrightarrow A_c^{0,p+1}(M, L)$$

be the natural $\bar\partial$ operator and let

$$\vartheta : A_c^{0,p}(M, L) \longrightarrow A_c^{0,p-1}(M, L)$$

be the formal adjoint of $\bar\partial$ Let $\mathcal{L}^{0,p}(M, L)$ denote the space obtained by taking the form closure with respect to the graph norm

$$A_c^{0,p}(M, L) \ni f \mapsto \| f \|^2 + \| \bar\partial f \|^2 + \| \vartheta f \|^2 .$$

We define the L^2-cohomology group $H^p_2(M, \mathcal{O}_M(L))$ by

$$H^p_{(2)}(M.\mathcal{O}_M(L), h) = \frac{\ker \bar{\partial} \mid \mathcal{L}^{0,p}(M, L)}{\overline{\bar{\partial} A^{0,p-1}_c(M.L)}},$$

where the closure is taken with respect to the graph norm. By Hörmander's L^2-estimate, we obtain:

Theorem 5.2 *([12]) Assume that there exists a positive constant c such that*

$$Ric_\omega - \sqrt{-1}\partial\bar{\partial}\log h \geq c\omega.$$

Then we have

$$H^p_{(2)}(M, \mathcal{O}_M(L), h) = 0$$

holds for all $p \geq 1$.

The following trivial cororally is important for our purpose.

Cororally 5.1 *Assume that there exists a volume form* Ω *on* M *and a positive constant c such that*

$$Ric\,\Omega - \sqrt{-1}\partial\bar{\partial}\log h \geq c\omega.$$

Then we have

$$H^p_{(2)}(M, \mathcal{O}_M(L), \tilde{h}) = 0 \text{ for } p > 0,$$

where the L^2*-cohomology is taken with respect to the twisted hermitian metric*

$$\tilde{h} = h \otimes \frac{\Omega}{\omega^n}.$$

As an example of the most typical application of Theorem 5.1 to algebraic geometry, we shall prove the well known Kawamata-Viehweg vanishing theorem for ample $\mathbf{R}$-divisors.

Theorem 5.3 *([27, 16]) Let* M *be a smooth projective variety and let* D *be an ample* $\mathbf{R}$*-divisor on* M *such that* $Supp\{D\}$ *is a divisor with normal crossings. Then*

$$H^p(M, \mathcal{O}_M(K_M + \lceil D \rceil)) = 0$$

holds for $p \geq 1$.

Let $\{D\} = \sum a_i D_i$ be the irreducible decomposition of $\{D\}$ and let σ_i be the section of $\mathcal{O}_M(D_i)$ with divisor D_i. Let $\parallel\ \parallel = \parallel\ \parallel_i$ be a hermitian metric on $\mathcal{O}_X(D_i)$ such that $\parallel \sigma_i \parallel < 1$ on M. Then by the assumption, there exists a hermitian metric h on $\mathcal{O}_M([D])$ such that

$$-\sqrt{-1}\partial\bar{\partial}\log h - \sum a_i \sqrt{-1}\partial\bar{\partial}\log \parallel \sigma_i \parallel^2$$

is a smooth Kähler form on M. Let us define a singular hermitian metric $\hat{h}$ on $\mathcal{O}_X([D])$ by

$$\hat{h} = (\prod_i \| \sigma_i \|^{2a_i} (\log \frac{1}{\| \sigma_i \|})^{\varepsilon})h,$$

where ε is a positive constant. Then by easy direct calculation, we see that if we take ' ε sufficiently small,

$$\omega = -\sqrt{-1}\partial\bar{\partial}\log\hat{h}$$

is a complete Kähler form on $N = M - \mathrm{Supp}\{D\}$. Let Ω be a smooth volume form on M. Then by Theorem 5.1, we have

$$H^p_{(2)}(N, \mathcal{O}_N(K_M + [D] \mid N)) = 0$$

holds for $p \geq 1$, where the L^2 cohomology is taken with respect to the volume form Ω and the hermitian metric $\hat{h} \otimes \Omega^{-1}$. We note that

$$H^p(M, \mathcal{O}_M(K_M + \lceil D \rceil))$$

is isomorphic to the corresponding Dolbeault cohomology for all p. Then by the L^2-regularity theorem for $\bar{\partial}$-operator, we get the canonical injection

$$H^p(M, \mathcal{O}_M(K_M + \lceil D \rceil)) \hookrightarrow H^p_{(2)}(N, \mathcal{O}_N(K_M + [D]) \mid N))$$

for every p. Since the right-hand side vanishes for $p \geq 1$. This completes the proof of the theorem. Q.E.D.

5.2 Relation between the asymptotic behavior of the base locus of the pluricanonical system and the Lelong number of the Kähler-Einstein current ω_E

Now we shall relate the Lelong number of ω_E and the multiplicity of the base scheme $\mathrm{Bs} \mid \nu K_X \mid$.

Lemma 5.1 *There exists a positive integer ν_0 such that for every positive integer ν and $x \in X$,*

$$mult_x Bs \mid \nu K_X \mid \geq \nu\Theta(\omega_E, x)$$

holds.

Proof. Let $\omega^{(m)}$ be as in 4.10. We set

$$\hat{u}^{(m)} = \log \frac{(\omega^{(m)})^n}{\Omega}.$$

Then for every sufficiently small positive number ε, there exists a constant $C^{-}_{0,\varepsilon}(m)$ such that

$$\hat{u}^{(m)} \geq C^{-}_{0,\varepsilon}(m) + \xi^{(m)} + \varepsilon\delta^{(m)}$$

holds , where $\delta^{(m)} = \delta_{m!\nu}$ be a function on $X^{(m)} = X_{m!\nu}$ defined as in 4.2. Then by Lemma 2.3 and 2.5 and the definition of $\xi^{(m)} = \xi^{m!\nu}$, letting ε tend to 0, we have

$$m!\nu\Theta(\omega^{(m)}, x) \leq \mathrm{mult}_x \mathrm{Bs} \mid m!\nu K_X \mid$$

holds for all $x \in X$. On the other hand by the construction, $\{\hat{u}^{(m)}\}_{m=1}^{\infty}$ is monotone increasing by Lemma 4.20 (as we have seen Lemma 4.22 the sequence is stable actually). Hence we have

$$\Theta(\omega_E, x) \leq \Theta(\omega^{(m)}, x)$$

and

$$m!\nu\Theta(\omega_E, x) \leq \mathrm{mult}_x \mathrm{Bs} \mid m!\nu K_X \mid$$

holds for all $x \in X$. Since $\mathrm{mult}_x \mathrm{Bs} \mid abK_X \mid \leq a\, \mathrm{mult}_x \mathrm{Bs} \mid bK_X \mid$ holds for all positive integers a, b, this completes the proof of the lemma. Q.E.D.

More generally we have:

Lemma 5.2 *Let $f : Y \longrightarrow X$ be an arbitrary modification from a smooth projective variety. Then for a sufficiently large ν*

$$mult_y Bs \mid f^*(\nu K_X) \mid \geq \nu\Theta(f^*\omega_E, y)$$

holds for all $y \in Y$.

Proof. Let $Y^{(m)}$ be a resolution of the fibre product $Y \times_X X^m$ and let $g_{(m)} : Y^{(m)} \longrightarrow X$ be the natural morphism. Then by the same argument as above, we have

$$m!\nu\Theta(g^*_{(m)}\omega_E, y) \leq \mathrm{mult}_y \mathrm{Bs} \mid g^*_{(m)}(m!\nu K_X) \mid$$

holds for all $y \in Y^{(m)}$. Since m is an arbitrary positive integer, by Lemma 2.5 and the same argument as in the proof of Lemma 5.1, we obtain the lemma. Q.E.D.

Definition 5.1 *Let Y be a projective variety and let D be a $\mathbf{R}$-Cartier divisor on Y. We set*

$$\Xi(D, y) = \liminf_{\nu \to \infty} \nu^{-1} mult_y Bs \mid [\nu D] \mid$$

and call it the limit base multiplicity of D.

As a cororally of the above lemma, we have

Cororally 5.2 *Let $f : Y \longrightarrow X$ be an arbitrary modification from a smooth projective variety. Then*

$$\Theta(f^*\omega_E, y) \leq \Xi(f^*K_X, y)$$

holds for all $y \in Y$.

But the following theorem holds.

Theorem 5.4 *Let $f : Y \longrightarrow X$ be an arbitrary modification from a smooth projective variety. Then*

$$\Theta(f^*\omega_E, y) = \Xi(f^*K_X, y)$$

holds for all $y \in Y$.

Proof. Let ν_0 be a positive integer such that $\mid \nu K_X \mid$ gives a birational rational map from X into a projective space. Let $f : Y \longrightarrow X$ be an arbitrary resolution of Bs $\mid \nu K_X \mid$. Let

$$f^*(\nu K_X) = L + F$$

be the decompositin into the free part L and the fixed part F and let $F = \sum a_i F_i$ be the irreducible decomposition of F. We set

$$e_i = \inf_{y \in F_i} \Theta(f^*\omega_E, y).$$

Since L is big by the construction, by Lemma 4.1, there exists an effective $\mathbf{Q}$-divisor $R = \sum r_j R_j$ such that $L - R$ is an ample $\mathbf{Q}$-divisor on Y. Let y_0 be a point on F_{i_0} such that $\Theta(f^*\omega_E, y_0) = e_{i_0}$. Let ω be a Kähler form on Y and let r_0 denote the distance function from y_0 with respect to the Kähler form ω. Then there exists an open neighbourhood U such that $\log r_0 \mid U$ is a plurisubharmonic function on U. Let ρ be a nonnegative smooth function with compact support on U such that ρ is identically 1 on a neighbourhood of y_0. We set

$$\psi = (2n + 2 + 2e_{i_0})\rho \log r_0.$$

Let m be a positive integer such that

1. $H = m(L - R)$ is an ample Cartier divisor on Y.

2. There exists a hermitian metric h on $\mathcal{O}_Y(H)$ and a positive constant c_0 such that
$$-\sqrt{-1}\partial\bar{\partial}\log(e^{-\psi}h) \geq c_0\omega.$$

Let M be the complement of Supp $F \cup$ Supp R in Y. Let us define a hermitian metric h_ν on $\mathcal{O}_M(f^*(\nu K_X) \mid M)(\nu \geq m\nu_0)$ by

$$h_\nu = h \otimes (f^*\omega_E^n)^{(\nu - m\nu_0)}.$$

Let σ_i be section of $\mathcal{O}_Y(F_i)$ with divisor F_i and let τ_j be a section of $\mathcal{O}_Y(R_j)$ with divisor R_j. Let $\| \ \|$ denote hermitian metrics on $\mathcal{O}_Y(F_i)$'s and $\mathcal{O}_Y(R_j)$'s. Then for a sufficiently small positive number ε

$$\hat{\omega} = \omega - \varepsilon\sqrt{-1}\partial\bar{\partial}\sum_i \log\log\frac{1}{\| \sigma_i \|} - \varepsilon\sqrt{-1}\partial\bar{\partial}\sum_j \log\log\frac{1}{\| \tau_j \|}$$

and

$$\breve{\omega} = -\sqrt{-1}\partial\bar{\partial}\log h_\nu - \varepsilon\sqrt{-1}\partial\bar{\partial}\sum_i \log\log\frac{1}{\| \sigma_i \|} - \varepsilon\sqrt{-1}\partial\bar{\partial}\sum_j \log\log\frac{1}{\| \tau_j \|}$$

are both complete Kähler forms on M. And by the construction there exists a positive constant c such that

$$\breve{\omega} \geq c\hat{\omega}.$$

Let us define a hermitian metric on $\mathcal{O}_M(f^*(\nu K_X \mid M))$ by

$$\hat{h}_\nu = (\prod_i \log\frac{1}{\| \sigma_i \|}\prod_j \log\frac{1}{\| \tau_j \|})^\varepsilon h_\nu.$$

Then the curvature form of $\hat{h}_\nu$ is nothing but $\breve{\omega}$. Let Ω denote the volume form $f * \omega_E^n$ on M. Then by Cororally 5.1 , we have

$$H^p_{(2)}(M, \mathcal{O}_M(f^*(\nu K_M)) = 0,$$

for $\nu \geq m\nu_0 + 1$ and $p \geq 1$, where the L^2-cohomology is taken with respect to the twisted hermitian metric

$$\begin{aligned} \check{h}_\nu &= \hat{h}_\nu \otimes (\frac{\Omega}{\hat{\omega}^n}) \\ &= \hat{h}_\nu \otimes \frac{f^*\omega_E^n}{\hat{\omega}^n} \end{aligned}$$

and the complete Kähler form $\hat{\omega}$. Let σ be a local section of

$$\mathcal{L} = \mathcal{O}_Y(H + f^*((\nu - m\nu_0)K_X) - \sum_i[(\nu - m\nu_0)e_i]F_i)$$

around y_0 such that $\sigma(y_0) \neq 0$. We may assume without loss generality that σ is defined on U. Then since $\Theta(f^*\omega_E, y_0) = e_{i_0}$, there exists an open neighbourhood V of y in U and a nonnegative smooth function $\tilde{\rho}$ of compact support on U such that

1. $\tilde{\rho}$ is identically 1 on V.

2. $G := \bar{\partial}(\tilde{\rho}\sigma) \mid U \cap M$ is a L^2-form on M with respect to the hermitian metric $\check{h}_\nu$ and the complete Kähler form $\hat{\omega}$.

Then by Theorem 5.1, we see that there exists $u \in L^2_{(0,0)}(M, f^*(\nu K_X), \check{h}_\nu)$ such that

$$\bar{\partial}u = G.$$

We set

$$\mu = \tilde{\rho}\sigma - u.$$

Then since $\mu \in H^0_{(2)}(M, \mathcal{O}_M(f^*(\nu K_X))$, by Lemma 2.2 and Lemma 2.3, we have

$$\mu \in H^0(Y, \mathcal{L}).$$

We note that by the definition of $\check{h}_\nu$(see the definition of ψ),

$$\mu(y_0) = \sigma(y_0) \neq 0.$$

Since there exists a canonical injection $\mathcal{O}_Y(H) \hookrightarrow \mathcal{O}_Y(f^*(\nu_0 K_X))$, by letting ν tend to infinity, we have

$$\Xi(f^*K_X, y_0) \geq \Theta(f^*\omega_E, y_0).$$

Hence by Cororally 5.1, we have

$$\Xi(f^*K_X, y_0) = \Theta(f^*\omega_E, y_0).$$

Let y be an arbitray point on Y. Let us with center y. Then repeating the same argument with repect to a general point on the exceptional divisor, by Lemma 2.5 we obtain

$$\Xi(f^*K_X, y) = \Theta(f^*\omega_E, y).$$

This completes the proof of the theorem for this special $f : Y \rightarrow X$.

To prove the theorem for a general $f : Y \longrightarrow X$, we shall take the resolution $g : Y' \longrightarrow X$ of Bs $\mid \nu K_X \mid$ such that g factors through f. We have already known that the theorem holds for $g : Y' \longrightarrow X$. Then we see that the theorem also holds also for $f : Y \longrightarrow X$ by Lemma 2.5. Q.E.D.

5.3 Use of the argument of Benveniste-Kawamata-Shokurov

Combining L^2-vanihing theorem and the argument of Benveniste-Kawamata-Shokurov ([3, 15, 22]), we shall prove the following theorem.

Theorem 5.5 *Let $\mu : Z \longrightarrow X$ be a modification from a nonsingular projective variety. Then for every point $z \in Z$ there exists a positive integer $\nu(z)$ d epending on z) such that*

$$mult_z Bs \mid \nu(z)\mu^* K_X \mid = \nu(y)\Theta(\mu^*\omega_E, z)$$

holds. In particular $\Theta(\mu^*\omega_E, z)$ is a rational number for every $z \in Z$.

Proof. Let ν be a positive integer such that

1. $\mid \nu K_X \mid$ defines a birational rational map from X into a projective space.

2. Supp Bs $\mid \nu K_X \mid = \mathrm{SB}(K_X)$.

We construct a birational morphism

$$f : Y \longrightarrow X$$

such that

1. There exists a divisor $F = \sum F_i$ with only normal crossings.

2. $K_Y = f^* K_X + \sum a_i F_i$ for some $a_i \in \mathbf{Z}$ with $a_i \geq 0$ for all i.

3. We set $e_i = \inf_{y \in F_i} \Theta(f^* \omega_E, y)$.

$$H = f^* K_X - \sum e_i F_i.$$

Then

$$\mid \lceil \nu H \rceil \mid = \mid L \mid + \sum r_i F_i,$$

where $\mid L \mid$ is the free and $\sum r_i F_i$ is effective (We do not assume $\mid L \mid$ the "*full*" free part).

4. $L - \sum \delta_i F_i$ is ample for some $\delta_i \in \mathbf{Q}$ with $0 < \delta_i << 1$.

The existence of such f follows from Hironaka resolution and Cororally 4.1.

Let us choose an arbitrary irreducible component say F_0 of F and fix it. We set

$$c = \frac{(a_0 + e_0 + 1 - \delta_0)}{r_0} + \delta,$$

where δ is a sufficiently small positive number which we will specify later. If δ is sufficiently small,

$$\lceil -cr_0 + a_0 + e_0 - \delta_0 \rceil = -1$$

holds. We set

$$A = \sum_{i \neq 0} (-cr_i + a_i + e_i - \delta_i) F_i$$

and

$$B = -(-cr_0 + a_0 + e_0 - \delta_0)F_0.$$

We consider

$$\begin{aligned} M &= b'H + A - B - K_Y \\ &= (cL - \sum \delta_i F_i) + (b'H - c\lceil \nu H \rceil - f^* K_X + \sum e_i F_i) \\ &= (cL - \sum \delta_i F_i) + c\{-\nu H\} + (b' - c\nu - 1)H, \end{aligned}$$

where b' is a positive integer which will be specified later. If we replace $\{\delta_i\}$ to $\{\varepsilon\delta_i\}$ for a suffuciently small positive number ε, we may assume that $cL - \sum \delta_i F_i$ is ample. And multiplying a suitable positive integer to ν (to make $\{-\nu H\}$ *small*) if necessary, we may assume that

$$cL - \sum \delta_i F_i + c\{-\nu H\}$$

is ample (to do this operation we do not assume that $\mid L \mid$ is the full free part of $\mid \lceil \nu H \rceil \mid$). Let h be a hermitian metric on $\mathbf{R}$-line bundle associated with the $\mathbf{R}$-divisor $cL - \sum \delta_i F_i + c\{-\nu H\}$ such that the curvarure form $-\sqrt{-1}\partial\bar{\partial}\log h$ is a smooth Kähler form on Y (we note that the $\mathbf{R}$-line bundle does not make actual sense but the hermitian metric makes sense). Let σ_i be a global section of $\mathcal{O}_Y(F_i)$ with divisor F_i. Let $\parallel \ \parallel$ be a hermitian metric of $\mathcal{O}_Y(F_i)$ such that $\parallel \sigma_i \parallel < 1$ holds on Y respectively. Then for a sufficiently small positive number ε

$$\omega = -\sqrt{-1}\partial\bar{\partial}\log h + \varepsilon \sum_i \sqrt{-1}\partial\bar{\partial}\log\frac{1}{\parallel \sigma_i \parallel}$$

is a complete Kähler form (of Poincaré growth) on $Y - F$. We consider $Y - F$ as a complete Kähler manifold $(Y - F, \omega)$ hereafter. Let us define a singular hermitian metric h_M on $\mathcal{O}_Y(b'K_Y)$ by

$$h_M = h \otimes (f^*\omega_E^n)^{-\otimes(b'-c\nu)} \otimes \prod_i (\log\frac{1}{\parallel \sigma_i \parallel})^{\varepsilon}$$

Then we have by Theorem 5.1,

$$H^p(Y - F, \mathcal{O}_{Y-F}(b'K_Y \mid Y - F), \hat{h}) = 0 \quad \text{for } p \geq 1,$$

where $\hat{h}$ is a hermitian metric on $\mathcal{O}_Y(b'K_Y)$ defined by

$$\hat{h} = h_M \otimes \frac{f^*\omega_E^n}{\omega^n}.$$

Let us define a presheaf $\mathcal{F}$ on Y by

$$\mathcal{F}(U) = \{\sigma \in \Gamma(U, \mathcal{O}_Y(b'K_Y)) \mid \int_U \hat{h}(\sigma,\sigma)\omega^n < \infty\},$$

where U is an arbitrary open subset of Y. We denote the sheaf associated with this presheaf also by $\mathcal{F}$. $\mathcal{F}$ is the sheaf of germs of local L^2 holomorphic sections of $\mathcal{O}_Y(b'K_Y)$. By Hörmander's L^2-estimate for $\bar{\partial}$-operator, using the isomorphism between the Dolbeaut cohomology and the Cěch cohomology we get the canonical injection :

$$H^p(Y, \mathcal{F}) \hookrightarrow H^p_{(2)}(Y - F, \mathcal{O}_{Y-F}(b'K_Y \mid Y - F), \hat{h})$$

for all p. Hence we get:

Sublemma 5.1

$$H^p(Y, \mathcal{F}) = 0$$

holds for $p \geq 1$.

To prove Theorem 5.5, it is necessary to modify $f : Y \longrightarrow X$. We need the following sublemma.

Sublemma 5.2 *There exists a point y_i on F_i such that*

1. $y_i \in F_{reg}$ *(F_{reg} denote the regualar part of F).*
2. $\Theta(f^*\omega_E, y_i) = e_i$.
3. *Let $\pi_i : Y_i \longrightarrow Y$ be the blowing up with center y_i and let E_i be the exceptional divisor. Then for every $y \in E_i$,*

$$\Theta(\pi_i^*(f^*\omega_E - \sum_k e_k F_k), y) = 0$$

holds.

We note that $f^*\omega_E - \sum_k e_k F_k$ is a d-closed positive $(1,1)$current on Y by Lemma 2.6.

Proof. The existence of such y_i follows from Theorem 5.4. In fact by Theorem 2.2, for every $\varepsilon > 0$

$$F_i(\varepsilon) := \{y \in F_i \mid \Theta(f^*\omega_E, y) \geq e_i + \varepsilon\}$$

is a proper subvariety of F_i. In particular $\cup_{m\geq 1} F_i(1/2^m)$ is a countable uniton of proper subvarieties of F_i. Let y_i be a point in $F_i - \cup_{m\geq 1} F(1/2^m) - \cup_{j\neq i} F_j$ such that we do not need any blowing up with center containing y_i to resolve the base locus of $\mid m!\nu f^*K_X \mid$ for all m. Such y_i form a complement of a countable union of proper subvarieties of F_i. Then by the costruction $\Theta(f^*\omega_E, y_i) = e_i$ and by Theorem 5.4, for every point $y \in E_i$,
$\Theta(\pi_i*(f^*\omega_E - e_i F_i), y) = 0$ holds. Q.E.D.

Let $y_0 \in F_0$ be as in Sublemma 5.2 and let $\pi_0 : Y_0 \longrightarrow Y$ be the blowing up with centre y_0. We choose y_0 *as in the proof* of Sublemma 5.2. Let E_0 denote the exceptional divisor.

Let F_0' be the strict transform of F_0 in Y_0. Let $\pi_0^1 : Y_0^1 \longrightarrow Y_0$ be the blowing up with centre $E_0 \cap F_0'$ and let F_0^1 be the exceptional divisor. Let E_0^1 be the strict transform of E_0 in Y_0^1. Let $\pi_0^2 : Y_0^2 \longrightarrow Y_0^1$ be the blowing up with centre $E_0^1 \cap F_0^1$ and let F_0^2 be the exceptional divisor. Let E_0^2 be the strict transform of E_0^1 in Y_0^2. Repeating this process m-times we get successive blowing ups

$$\tilde{\pi}_0 : Y_0^m \to Y_0^{m-1} \to \cdots \to Y_0^1 \to Y_0 \to Y.$$

Let J denote the exceptional divisor of $\tilde{\pi}_0$. Let F' be the strict transform of F in Y_0^n.

We shall replace $f : Y \longrightarrow X$ to $f \circ \tilde{\pi}_0 : Y_0^m \longrightarrow X$ and F to $J + F'$. Then it is easy to see $J + F'$ has the same properties as F. Hence we may assume that $Y = Y_0^m$ and $F = J + F'$ for the first. By changing the order, if necessary we may assume that $F_0 = E_0^m$, $F_1 = F_0^m$, and $F_{m+1} = F_0'$ (the strict transform of "*the old*" F_0). Then since $r_1 = (m+1)r_0$ and $e_1 = (m+1)e_0$, we see that if we take m sufficiently large and $\{\delta_i\}$'s sufficiently small,

$$c = \frac{a_0 + e_0 + 1 - \delta_0}{r_0} < \frac{a_1 + e_1 + 1 - \delta_1}{r_1} \tag{26}$$

holds by the construction. In fact

$$\lim_{m \to \infty} a_1/m = a_0 + 1$$

holds.

We note that F_0 is biholomorphic to the one point blowing up of $\mathbf{P}^{n-1}$ and $F_0 \cap F_1$ is the exceptional divisor in F_0. It is easy to see that if $\{b'e_0\}$ is sufficiently small, then the restriction $\mathcal{O}_Y(\lceil b'H \rceil) \mid F_0$ is trivial. We assume that $\{b'e_0\}$ is sufficiently small. Let σ be a nontrivial section of

$$H^0(F_0, \mathcal{O}_{F_0}(\lceil b'H \rceil)) \simeq H^0(F_0, \mathcal{O}_{F_0}) \simeq \mathbf{C}$$

Let U be a sufficiently small open neighbourhood of F_0 in Y and let

$$\tilde{\sigma} \in \Gamma(U, \mathcal{O}_Y(\lceil b'H \rceil))$$

be a holomorphic extension of σ to U. Let V be an open neighbourhood of F_0 and let ρ be a nonnegative C^∞-function on Y such that

1. ρ has a compact support in U.

2. ρ is identically 1 on V.

And we set

$$G = \bar{\partial}(\rho\tilde{\sigma}).$$

Then G is a $\mathcal{O}_Y(\lceil b'H\rceil)$-valued C^∞-$(0,1)$-form. By (27) if we take δ sufficiently small and m large,

$$\lceil -cr_1 + a_1 + e_1 - \delta_1 \rceil \geq 0$$

holds. Hence by the construction of F_0 (cf. Sublemma 5.2), Lemma 2.3 and the upper semicontinuity of Lelong numbers, if we take U sufficiently small we see that

$$G \in L^2_{(0,1)}(Y, \mathcal{O}_Y(b'K_Y), \hat{h})$$

holds. Hence G represents a cohomology class in $H^1(Y, \mathcal{F})$ which is zero by Sublemma 5.1. This implies that if we take δ sufficiently small and take b' so that $\{b'e_0\}$ is sufficiently small and $b' - c\nu - 1 > 0$, there exists a section $\hat{\sigma} \in H^0(Y, \mathcal{O}_Y(\lceil b'H\rceil + \lceil A\rceil))$ such that

$$\hat{\sigma} \mid F_0 = \sigma$$

(we note that there is a canonical injection $\mathcal{O}_Y(\lceil b'H + A\rceil) \hookrightarrow \mathcal{O}_Y(\lceil b'H\rceil + \lceil A\rceil)$). Let I denote the union of the index i such that $e_i = 0$ and define a divisor $\dot{A}$ by

$$\begin{aligned} \dot{A} &= \textstyle\sum_{i\in I}(-cr_i + a_i + e_i - \delta_i)F_i \\ &= \textstyle\sum_{i\in I}(-cr_i + a_i - \delta_i)F_i. \end{aligned}$$

Since $f_*\lceil \dot{A}\rceil = 0$, if we take b' sufficiently large, by the birational invariance of plurigenra, we see that $\hat{\sigma}$ defines a element $\tau \in H^0(Y, \mathcal{O}_Y(f^*(b'K_X)))$ such that

$$\mathrm{mult}_y D(\tau) \leq b'e_0 \quad \text{for } y \in F_0 \cap F_{reg},$$

where $D(\tau)$ denotes the member of $\mid f^*(b'K_X) \mid$ defined by τ. On the other hand, by Lemma 5.2, we have

$$\mathrm{mult}_y D(\tau) \geq b'e_0 \quad \text{for } y \in F_0.$$

Hence we have the equality:

$$\mathrm{mult}_y D(\tau) = b'e_0 \quad \text{for } y \in F_0 \cap F_{reg}.$$

Hence e_0 is a rational number and there exists a nonempty Zariski open subset $U_0 = F_0 \cap F_{reg}$ of F_0 such that

$$\Theta(f^*\omega_E, y) = e_0 \quad \text{for } y \in U_0$$

and

$$\mathrm{SBs}(H) \cap U_0 = \phi$$

hold. By the construction of F_0, there exists a nonempty Zariski open subset U_{m+1} of F_{m+1} such that

$$\Theta(f^*\omega_E, y) = e_{m+1}(= e_0) \quad \text{for } y \in U_{m+1}$$

and

$$\mathrm{SBs}(H) \cap U_{m+1} = \phi$$

hold. Hence by continuing this process, we see that for every i, e_i is a rational number and there exists a nonempty Zariski open subset U_i of F_i such that

$$\Theta(f^*\omega_E, y) = e_i \quad \text{for every } y \in U_i$$

and

$$\mathrm{SBs}(H) \cap U_i = \phi.$$

Let $\mu : Z \longrightarrow X$ be an arbitrary modification from a nonsingular projective variety and let $z \in Z$ be an arbitrary point. Let $\pi_z : Z_z \longrightarrow Z$ be the blowing up at z and let E be the excepsional divisor. Let $g : Z_z \longrightarrow X$ denote the composition $\mu \circ \pi_z$. By Lemma 2.5, for almost every point p on E

$$\Theta(g^*\omega_E, p) = \Theta(\mu^*\omega_E, z).$$

holds as in the proof of Sublemma 5.2. Hence by taking $f : Y \longrightarrow X$ so that f factors through $g : Z_z \longrightarrow X$, we see that there exists a nonempty Zariski open subset U of E such that

$$\Theta(g^*\omega_E, p) = \Theta(\mu^*\omega_E, z) \quad \text{for every } p \in U$$

and

$$\mathrm{SBs}(g^*K_X - \Theta(\mu^*\omega_E, z)E) \cap U = \phi.$$

This implies that there exists a positive integer $\nu(z)$ depending on z such that

$$\mathrm{mult}_z \mid \nu(z)\mu^*K_X \mid = \nu(z)\Theta(\mu^*\omega_E, z)$$

holds. This completes the proof of Theorem 5.5. Q.E.D.

Cororally 5.3 *Let $f; Y \longrightarrow X$ be an arbitrary modification from a smooth projective variety Y. We set*

$$S(f^*\omega_E) = \{y \in Y \mid \Theta(\omega_E, x) > 0\}.$$

Then $S(f^*\omega_E) = SBs(f^*K_X)$.

Proof. By Lemma 5.2, we have that $S(f^*\omega_E) \subseteq \mathrm{SBs}(f^*K_X)$. On the other hand, by Theorem 5.5, we have that $\mathrm{SBs}(f^*K_X) \subseteq S(f^*\omega_E)$. Q.E.D.

Now we shall prove Theorem 5.1.

Proof of Theorem 5.1. Let $f : Y \longrightarrow X$ be as in Theorem 5.1 and let $S(f^*\omega_E)$ be as in Cororally 5.3. Let

$$S(f^*\omega_E) = \sum_\alpha S_\alpha$$

be the decomposition of $S(\omega_E)$ into the irreducible components. Let

$$e_\alpha = \inf_{y \in S_\alpha} \Theta(\omega_E, y).$$

By Cororally 5.3, we see that $e_\alpha > 0$ for every α.

Theorem 5.5 implies that there exists a nonempty Zariski open subset U_α of S_α such that

$$\Theta(f^*\omega_E, y) = e_\alpha$$

for every $y \in U_\alpha$. Then by Noetherian induction, we conclude that

$$\{\Theta(f^*\omega_E, y) \mid y \in Y\}$$

is a finite set. This completes the proof of Theorem 5.1. Q.E.D.

6 Zariski Decomposition of Canonical Divisor

In this section we shall prove Conjecture 2.1.

6.1 An algorithm to construct a Zariski decomposition

By the results of the previous sections, we can define an algorithm to construct a Zariski decomposition of K_X.

We set $Y_0 = X, P_0 = \omega_E$. We shall construct inductively sequences of smooth projective varieties $\{Y_m\}$, d-closed positive $(1,1)$-current $\{P_m\}$, and divisors N_m on Y_m for $m \geq 0$ as follows.

1. Set
$$S_m = \{y \in Y_m \mid \Theta(P_m, y) > 0\}.$$
S_m is a subvariety of Y_m by Theorem 5.1 and Theorem 2.2. We decompose S_m into the irreducible components:
$$S_m = S_m^1 \cup \ldots \cup S_m^{l_m}.$$
We set
$$n_m^i = \inf_{y \in S_m^i} \Theta(P_m, y).$$

By changing the order, if necessary, we may assume that

$$0 < n_m^1 \leq \ldots \leq n_m^{l_m}$$

holds.

2. Take an embedded resolution $p_m^1 : \tilde{Y}_m^1 \longrightarrow Y_m$ of S_m^1 and let $\hat{S}_m^1$ be the strict transform of S_m^1.

3. Let $\pi_m^1 : Y_m^1 \longrightarrow \tilde{Y}_m^1$ be the blow up with center $\hat{S}_m^1$ and set

$$N_m^1 = (\pi_{m+1}^1)^{-1}(\hat{S}_m^1)$$

(π_m^1 is the identity morphism, if $\hat{S}_m^1$ is a divisor) and $f_m^1 = p_m \circ \pi_m$.

4. Let $\hat{S}_m^2$ be the strict transform of S_m^2 in Y_m^1 and repeat the same process by replacing S_m^1 by $\hat{S}_m^2$. Then we obtain smooth projective variety Y_m^2, a morphism $f_m^2 : Y_m^2 \longrightarrow Y_m^1$ and a divisor N_m^2 as in 3.

5. Repeating this process we obtain sequences of smooth projective varieties $\{Y_m^l\}_{l=1}^{l_m}$ and morphisms

$$f_m^l : Y_m^l \longrightarrow Y_m^{l-1} \ (1 \leq l \leq l_m)$$

(where we have defined $Y_m^0 := Y_m$) and divisors $\{N_m^l\}(1 \leq l \leq l_m)$ on Y_m^l.

6. We set $Y_{m+1} := Y_m^{l_m}$ and

$$f_{m+1} := f_m^{l_m} \circ \cdots \circ f_m^1 : Y_{m+1} \longrightarrow Y_m.$$

7. We define a divisor on Y_{m+1} by

$$N_{m+1} = \sum n_m^l \hat{N}_m^l + \sum_k e_m^k E_m^k,$$

where
$\hat{N}_m^l$ the strict transform of N_m^l in Y_{m+1},
$\{E_m^k\}$: irreducible components of the strict transforms of the exceptional divisors which appear in the embedded resolutions $p_m^l : \tilde{Y}_m^l \longrightarrow Y_m^l$'s,
$e_m^k := \inf_{y \in E_m^k} \Theta(f_{m+1}^* P_m, y)$.

We define a d-closed positive $(1,1)$-current P_{m+1} by

$$P_{m+1} = f_{m+1}^*(P_m) - N_{m+1}$$

(P_{m+1} is a d-closed positive current by Lemma 2.6).

Lemma 6.1 *Suppose that there exists a positive integer m such that S_m is empty. Then if we set*

$$N := \sum_{i=0}^{m} \tilde{N}_i,$$

where $\tilde{N}_i$ is the total transform of N_i in Y_m,

$$P := F^*K_X - N$$

and

$$F := f_1 \circ \cdots \circ f_m,$$

the expression

$$F^*K_X = P + N \quad (P, N \in Div(\tilde{X}) \otimes \mathbf{Q})$$

is a Zariski decomposition of F^*K_X. Hence Conjecture 2.1 is true under the assumption.

Proof. We set $Y = Y_m$ for notational simplicity. By the definition of N, N is effective. For the next, we shall prove that P is numerically effective.

We have already seen that $P_m = F^*\omega_E - N$ is a d-closed positive $(1,1)$-current. Since S_m is empty, the Lelong number of P_m is 0 everywehere on Y. Hence P is numerically effective by Cororally 2.1.

On the other hand by Lemma 5.2, we have a natural inclusion:

$$0 \to H^0(Y, \mathcal{O}_Y(\nu F^*K_X)) \to H^0(Y, \mathcal{O}_Y([\nu P]))$$

for all $\nu \geq 0$. Because of the converse inclusion is trivial, we see that

$$H^0(Y, \mathcal{O}_Y([\nu P])) \simeq H^0(Y, \mathcal{O}_Y(\nu F^*(K_X)))$$

holds for all positive integer ν. This completes the proof of Lemma 6.1. Q.E.D.

6.2 Termination of the algorithm

We shall prove Conjecture 2.1 by using the algorithm in 6.1.

Theorem 6.1 *Let X be a smooth projective variety of general type. Then there exists a modification $f : Y \longrightarrow X$ such that f^*K_X has a Zariski decomposition.*

Proof. By Lemma 6.1, it is sufficient to prove that there exists m such that S_m is empty. By Lemma 4.21, ω_E has finite order along $S = \mathrm{SBs}(K_X) \cup SE(K_X)$. We need the following lemma.

Lemma 6.2 *Let T be a d-closed positive $(1,1)$-current on the unit polydisk $\Delta^k = \{(z_1, \ldots, z_k) \in \mathbf{C}^k \mid \mid z_i \mid < 1 (1 \leq i \leq k)\}$ in $\mathbf{C}^k (k \geq 2)$. Suppose that the following conditions are satisfied.*

1. *$T \mid \Delta^* \times \Delta^{k-1}$ is smooth ($\Delta^* = \Delta - \{0\}$).*

2. *We shall express $T \mid \Delta^* \times \Delta^{k-1}$ as*

$$T \mid \Delta^* \times \Delta^{k-1} = \sqrt{-1} \sum_{ij} a_{ij} dz_i \wedge d\bar{z}_j.$$

Then a_{ij}'s are uniformly bounded on $\Delta^* \times \Delta^{k-1}$ for all $i, j \geq 2$.

Then $\Theta(T)$ is constant along $\{0\} \times \Delta^{k-1}$.

Proof. By Theorem 2.2, $\Theta(T, x)$ is constant say a almost everywhere on $\{O\} \times \Delta^{k-1}$ with respect to the usual Lebesgue measure on Δ^{k-1}. By Lemma 2.1, there exists a plurisubharmonic function φ such that

$$T = \sqrt{-1}\partial\bar{\partial}\varphi \quad \text{on } \Delta^k.$$

Let $c \in \Delta^*$ be an arbitrary number and let $B(x, r)$ denote the ball of radius r with center x in $\mathbf{C}^{k-1}$ such that $B(x, r) \subset \Delta^{k-1}$. Then by Stokes's theorem

$$\int_{\partial B(x,r)} \frac{\partial\varphi}{\partial\mathbf{n}}(c, y) dS_y = \int_{B(x,r)} \Delta\varphi(c, y) dV_y,$$

where
dV_y : the Euclidean volume form on $B(x, r)$,
dS_y : the usual volume form on $\partial B(x, r)$,
$\mathbf{n}$: the outer unit normal vector field on $\partial B(x, r)$, and
Δ : the Laplacian with respect to the Euclidean metric of Δ^{k-1}.
Since $\Delta\varphi$ is uniformly bounded on Δ^{k-1} by a constant independent of c by the assumption, we have for every $r \in (0, 1 - \parallel x \parallel)$,

$$\varphi(c, x) = (\mathrm{vol}\, \partial B(x, r))^{-1} \int_{\partial B(x,r)} \varphi(c, y) dS_y + O(1)$$

and

$$\varphi(c, x) = (\mathrm{vol}\, B(x, r))^{-1} \int_{B(x,r)} \varphi(c, y) dV_y + O(1) \tag{27}$$

hold, where $O(1)$'s denote functions which are uniformly bounded with respect to x and c respectively.

We note that if we add some positive multiple of $\log \mid z_1 \mid$ to φ, we may assume that φ is negative on Δ^k. Let us fix $x \in B(O, 1/2)$ and let ε be a small positive number such that $B(x, 1/2 + \varepsilon) \subset \Delta^{k-1}$. Let x' be a point in $B(x, \varepsilon)$. Then the trivial inequality

$$\int_{B(x,1/2)} \varphi(c, y) dV_y \geq \int_{B(x',1/2+\varepsilon)} \varphi(c, y) dV_y \tag{28}$$

holds. By Lemma 2.4 we can find parallel lines L and L' passsing through $(0,x)$ and $(0,x')$ respectively such that $T \mid L$ and $T \mid L'$ are well defined and

$$\begin{aligned} \Theta(T \mid L,(0,x)) &\geq \Theta(T,(0,x)), \\ \Theta(T \mid L',(0,x')) &= \Theta(T,(0,x')) \end{aligned}$$

hold. Then by (27),(28) and Lemma 2.3, we see that

$$\Theta(T \mid L,(0,x)) \leq (1+2\varepsilon)^{2k-2}\Theta(T \mid L',(0,x')) = (1+2\varepsilon)^{2k-2}\Theta(T,x')$$

holds. Since $\Theta(T) = a$ almost everywhere on $\{0\} \times \Delta^{k-1}$, letting ε tend to 0, we obtain

$$\Theta(T,(0,x)) \leq \Theta(T \mid L,(0,x)) \leq a.$$

Hence $\Theta(T)$ is identically a on $\{0\} \times B(O,1/2)$.

Now it is clear that $\Theta(T,(0,x)) = a$ holds for every $x \in \Delta^{k-1}$. Q.E.D.

The proof of the following cororally is completely analogous to the proof of Lemma 6.2. Hence we omit it.

Cororally 6.1 *Let T be a d-closed positive $(1,1)$-current on Δ^2. Suppose that the following conditions are satisfied.*

1. *T is smooth on $\Delta^* \times \Delta^*$.*
2. *For every $c \in \Delta^*$, we consider $T \mid \{c\} \times \Delta$ as a d-closed positive $(1,1)$-current T_c on Δ. Then the Lelong number of T_c is everywhere 0.*
3. *$\lim_{c\to 0} T_c = 0$ holds.*

Then $\Theta(T)$ is constant along $\{0\} \times \Delta$.

Remark 6.1 *It is easy to generalize Cororally 6.1 to the higher dimensional case.*

Lemma 6.3 *There exists a positive integer m such that $S_m = \phi$.*

Proof. We claim that for a fixed k, there exists some $m \geq k$ such that

$$F_m(\mathrm{Supp}\, N_m) \neq F_k(\mathrm{Supp}\, N_k)$$

holds. Suppose the contrary. We note that the inclusion

$$F_m(\mathrm{Supp}\, N_m) \subset F_k(\mathrm{Supp}\, N_k)$$

is clear by the construction of the algorithm.

Hereafter for simplicity we denote the divisor and its support by the same notation, if without fear of confusion. Suppose that for all $m \geq k$

$$F_m(\operatorname{Supp} N_m) = F_k(\operatorname{Supp} N_k)$$

hold.

For simplicity, for the first we shall consider the case $\dim X = 2$. Let us look at the asymptotic behavior of the pull-back of ω_E along the fibre of $f_m^l : Y_m^l \longrightarrow Y_m^{l-1}$ $(1 \leq l \leq l_m)$ as a form. Then by Lemma 4.21, if m is sufficiently large, for every point $p \in N_m^l$, there exists a unit polydisk $\Delta^2 = \{(z_1, z_2) \mid | z_i | < 1 (i = 1, 2)\}$ such that

1.

$$\Delta^2 \cap N_m^l = \{(z_1, z_2) \in \Delta^2 \mid z_1 = 0\}$$

2. If we express the restriction T of the pullback of ω_E to $\Delta^* \times \Delta^*$ as

$$T := \sqrt{-1} \sum_{i,j=1}^{2} a_{ij} dz_i \wedge d\bar{z}_j$$

then T is smooth and extends to a d-closed positive $(1, 1)$-current on Δ^2 by

$$T(\eta) = T(\eta \mid \Delta^* \times \Delta^*) \quad (\eta \in A_c^{n-1,n-1}(\Delta^2)).$$

And T satisfies the conditions of Lemma 6.2 or Cororally 6.1.

Hence we see that if m is sufficiently large, there exists an irreducible component A of N_m and an irreducible component B of the strict transform of N_{m-1} such that

1. $A \cap B \neq \phi$,
2. $\inf_{y \in A} \Theta(F_m^* \omega_E, y) = \inf_{y \in B} \Theta(F_m^* \omega_E, y)$

hold by Lemma 6.2 and Cororally 6.1. This contradicts the definition of the algorithm (note that n_m^l's are all positive). Hence there exists some $m \geq k$ such that

$$F_m(\operatorname{Supp} N_m) \neq F_k(\operatorname{Supp} N_k)$$

holds. In the case of $\dim X \geq 3$, by slicing N_{m-1} by a suitable surface and considering the strict transform of it in Y_m, one can get the same contradiction by using Lemma 2.4.

Then by the definition of the algorithm, using Noetherian induction, we see that there exists a positive integer m such that S_m is empty. Q.E.D.

By Lemma 6.1, this completes the proof of Theorem 6.1. Q.E.D.

Now Theorem 1.1 follows from the following theorem.

Theorem 6.2 *([14, p.425, Theorem 1]) Let $f : X \longrightarrow S$ be a proper surjective morphism of normal algebraic varieties, let Δ be a $\mathbf{Q}$-divisor on X such that the pair (X, Δ) is log-terminal. Assume that $K_X + \Delta$ is f-big, i.e. $\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) = \dim X_\eta$, where X_η is the generic fibre of f and $\Delta_\eta = \Delta \mid X_\eta$, and that there exists the Zariski decomposition*

$$K_X + \Delta = P + N \quad \text{in } Div(X) \otimes \mathbf{R}$$

of $K_X + \Delta$ relative to f. Then the positive part P is f-semiample. i.e., $mP \in Div(X)$ and the natural homomorphism

$$f^* f_* \mathcal{O}_X(mP) \to \mathcal{O}_X(mP)$$

is surjective for some positive integer m. Thus the relative log-canonical ring

$$R(X/S, K_X + \Delta) = \sum_{m \geq 0} f_* \mathcal{O}_X([m(K_X + \Delta)])$$

is finitely generated as an $\mathcal{O}_S$-algebra.

Theorem 1.2 follows from Theorem 1.1 easily because the problem is completely local (for the proof see [21, p.479,Proposition 4.4]). Since the termination of flips is known up to dimension 4 ([16, p.337,Theorem 5.15]), we have Theorem 1.3.

References

[1] Andreotti, A.- Tomassini, G., Some remarks on pseudoconcave manifolds, Essays on Topology and Related Topics, dedicated to G. de Rham, ed. by Haefliger and R. Narasimhan, 85-104, springer-Verlag (1970).

[2] Aubin, T., Equations du type Monge-Ampère sur les variétés kählerienne compactes, Bull. Sc. Math. France 102 (1978), 63-95.

[3] Benveniste, X., Variétés de type général de dimension 3, Thesis, Univ. Paris Sud, (1984).

[4] Caffarelli, L.-Kohn J.J.-Nirenberg L.-Spruck J., The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic equations, Comm. Pure and Appl. Math. 38, 209-252 (1985).

[5] Carlson, J.-Griffith, P.H., A defect relation for equidimensional holomorphic mappings between algebraic varieties, Ann. of Math. 95 (1972), 557-584.

[6] Demailly, J.-P., Singular hermitian metrics on positive line bundles, preprint (1990).

[7] Demailly, J.-P., A numerical criterion for very ample line bundes, preprint (1990).

[8] Faltings, G., Diophantine approximation on abelian varieties, preprint (1989).

[9] Griffith, P.A.-Harris, J., Priciples of Algebraic Geometry, John Wiley and Sons (1980).

[10] Gilberg -Trudinger, N.S., Elliptic partial differential equations of 2nd order, 2nd edition, Springer-Verlag.

[11] Hamilton, R.S., Three manifolds with positive Ricci curvature, J. Diff. Geom. 17, 255-306 (1982).

[12] Hörmander, L., An Introduction to Complex Analysis in Several Variables 3-rd ed., North-Holland (1990).

[13] Kawamata, Y., A generalization of Kodaira-Ramanujam's vanishing theorem, Math. Ann. 261 (1982), 43-46.

[14] Kawamata, Y., The Zariski Decomposition of Log Canonical Divisors, Proc. of Symp. in Pure Math. 46 (1987), 425-433.

[15] Kawamata, Y., The cone of curves of algebraic varieties, Ann. of Math. 119, 603-633 (1984).

[16] Kawamata, Y.-Matsuda, K.-Matsuki, K., Introduction to the Minimal Model Problem, Adv. Studies in Pure Math. 10, Algebraic Geometry Sedai 1985, 283-360 (1987).

[17] Kleiman, S., Toward a numerical theory of ampleness, Ann. of Math. 84 (1966), 293-344.

[18] Kobayashi R., Einstein Kähler metrics on an open algebraic manifold, Osaka J of Math. 21 (1984), 399-418.

[19] Mori, S., Flip conjecture and the existence of minimal models for 3-folds, Jour. of Amer. Math. Soc. 1 (1988), 117-253.

[20] Mori, S., Three folds whose canonical bundles are not numerically effective, Ann. of Math. 116 (1982),133-176.

[21] Moriwaki, A., Semi-ampleness of the numerically effective part of Zariski decomposition, J. Math. Kyoto Univ. 26 (1986), 465-481.

[22] Shokurov, V.V., The nonvanishing theorem, Izv. Akad. Nauk SSSR Izv. 26, 591-604 (1986).

[23] Siu, Y.-T., Analyticity of sets associted to Lelong numbers and the extension of closed positive currents, Invent. Math. 27, 53-156 (1974).

[24] K. Sugiyama, Einstein-Kähler metrics on projevtive varieties of general type with canonical singularities, tp appear.

[25] Tsuji. H.: Existence and Degeneration of Kähler-Einstein Metrics on Minimal Algebraic Varieties of General Type, Math. Ann. 281, 123-133 (1988).

[26] Trudinger, N.S.: Fully nonlinear elliptic equation under natural structure conditions, Trans. Am. Math. Soc. 272, 751-769 (1983).

[27] Viehweg, E., Vanishing theorems, J. Reine Angew. Math. 335, 1-8 (1982).

[28] Yau, S.-T., On the Ricci curvature of a compact Kähler manifold and the complex Monge Ampere equation I. Comm. Pure Appl. Math. 31, 339-411 (1978).

[29] Yau, S.-T., A general Schwarz lemma for Kähler manifolds, Amer. J. of Math. 100, 197-203 (1978).

[30] Zariski, O. The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math. 76, 560-615 (1962).

Author's address : Hajime Tsuji
Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26,
5300 Bonn 3, Germany
and
Department of Mathematics, Tokyo Metropolitan University
Minami Ohsawa, Hachiôji, Tokyo, Japan