On the Heat Kernel Comparison Theorems for Minimal Submanifolds

by

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Abstract. In [3] Cheng, Li and Yau proved comparison theorems (upper bounds) for the heat kernels on minimal submanifolds of space forms. In the present note we show, that these comparison theorems together with a series of corollaries remain true for minimal submanifolds in ambient spaces with just an upper bound on the sectional curvature.

1. Introduction. Let M^m be a minimally immersed submanifold of N^n . For a given point $p \in M$ we define the normal range $U(p)$ to be the complement of the cut locus of p in N . Let $r_p(\cdot)$ denote the distance function from p in N . The ball $B_R(p) = \{x \in N \mid r_p(x) \leq R\}$ is then regular if $B_R(p) \subset U(p)$ and (when $\sup K_N = b > 0$) $R \leq \pi/2\sqrt{b}$.

Let $D \subset M^m$ be a compact domain of M containing p . Following [3] we denote the p-centered heat kernels on D by $H(p,y,t)$ (with Dirichlet boundary condition) and $K(p,y,t)$ (with Neumann boundary condition) respectively.

If $\tilde{D}_R^b(\tilde{p})$ denotes the totally geodesic disc with center $\tilde{p}$, radius R and dimension m in a space form $\tilde{N}_b^n$ of constant

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curvature $b \in \mathbb{R}$, then the $\tilde{p}$-centered heat kernels on $\tilde{D}$ only depend on $r_{\tilde{p}}(\cdot)$ and t; Hence we may, and do, write them as $\tilde{H}^b_R(r_{\tilde{p}}(y),t)$ and $\tilde{K}^b_R(r_{\tilde{p}}(y),t)$ respectively.

We can now formulate the extended comparison theorems for H and K as follows.

<u>Theorem 1.</u> Let M^m be a minimally immersed submanifold of N^n with $K_N \leq b$. Let D be a compact domain in M.

i) If D is contained in a regular ball $B_R(p) \subset N$ for some $p \in D$, then the Dirichlet heat kernel on D satisfies

(1.1) $$H(p,y,t) \leq \tilde{H}^b_R(r_p(y),t) \quad \text{for all} \quad y \in D \quad \text{and} \quad t \in \mathbb{R}_+ .$$

ii) If $D = B_R(p) \cap M$ for some (not necessarily regular) ball $B_R(p) \subset U(p) \subset N$, $p \in D$, and (if $b > 0$) $R \leq \pi/\sqrt{b}$, then the Neumann heat kernel on D satisfies

(1.2) $$K(p,y,t) \leq \tilde{K}^b_R(r_p(y),t) \quad \text{for all} \quad y \in D \quad \text{and} \quad t \in \mathbb{R}_+ .$$

This type of theorem was first proved by Cheng, Li and Yau for space form ambient spaces N^n (cf. [3] Theorems 1 to 5). The extension to ambient spaces with variable curvature is essentially a consequence of the following result, which may be proved by standard index comparison theory (cf. [4] Propos. 8).

<u>Proposition 2.</u> We make the same assumptions as in Theorem 1. Let $F : [0,R] \longrightarrow \mathbb{R}$ be a smooth function with $F'(r) \geq 0$ for all $r \in [0,R]$, and let $\{X_j\}$, $1 \leq j \leq m$, be an orthonormal basis of $T_q D$. Then the Laplacian Δ_D of D satisfies the following inequality at q:

$$(1.3) \quad \Delta_D(F \circ r_p|_D) \geq \left(F''(r) - F'(r)h_b(r)\right)\sum_{j=1}^{m} \langle \mathrm{grad}\ r, X_j\rangle^2 + m\,F'(r)h_b(r) ,$$

where $h_b(r)$ is the constant mean curvature of any distance sphere of radius r in a space form of constant curvature b.

For the proof of Theorem 1 we also need the following special version of a result due to Cheeger and Yau ([1] Lemma 2.3).

<u>Proposition 3.</u>

$$(1.4) \quad \frac{\partial}{\partial r}\, \tilde{H}_R^b(r,t) < 0 ,$$

and

$$(1.5) \quad \frac{\partial}{\partial r}\, \tilde{K}_R^b(r,t) < 0$$

for all $t > 0$ and $r \in [0,R]$
$\left(\text{with } R < (\text{rsp. } \leq)\ \pi/\sqrt{b} \text{ if } b > 0\right)$

2. Proof of Theorem 1 and some consequences. Following [3] closely throughout, we only have to show that the transplanted heat kernels $\tilde{H}^b_R(r_p(y),t)$ and $\tilde{K}^b_R(r_p(y),t)$: $\{p\} \times D \times [0,\infty[\longrightarrow [0,\infty]$ satisfy $\Box_y \tilde{H} \leq 0$ and $\Box_y \tilde{K} \leq 0$ respectively. Here $\Box_y = \Delta_y - \frac{\partial}{\partial t}$, where Δ_y is the Laplacian operating on functions on the second factor in the domain $\{p\} \times D \times [0,\infty[$.

We rewrite $\tilde{H}^b_R(r_p(y),t)$ as a function of s and t , i.e. $\tilde{H}^b_R(r,t) = \tilde{H}(s(r),t) = \tilde{H}(s,t)$, where

$$\text{(2.1)} \quad s(r) = \begin{cases} 1-\cos(\sqrt{b}\,r) & \text{if } b > 0 \\ r^2/2 & \text{if } b = 0 \\ \cosh(\sqrt{-b}\,r) - 1 & \text{if } b < 0 . \end{cases}$$

Now consider the following identity

$$\text{(2.2)} \quad \Delta_y \tilde{H}(s,t) = \tilde{H}'' \, \| \operatorname{grad}_D s \|^2 + \tilde{H}' \Delta_D s \, , \quad \text{where}$$
$$\tilde{H}' = \frac{\partial}{\partial s} \tilde{H}(s,t) \ .$$

From Proposition 2 with $F(r) = s(r)$ and $s''(r) - h_b(r)s'(r) \equiv 0$ we get $\Delta_D s \geq m \frac{ds}{dr} h_b(r) = \tilde{\Delta}_{\tilde{D}} s$, where $\tilde{\Delta}_{\tilde{D}}$ is the Laplacian on the space form disc $\tilde{D}^b_R(\tilde{p})$. Proposition 3 implies $\frac{ds}{dr}\tilde{H}' \leq 0$, and since $\frac{ds}{dr} \geq 0$ we get $\tilde{H}'\Delta_D s \leq \tilde{H}'\tilde{\Delta}_{\tilde{D}} s$. Furthermore $\| \operatorname{grad}_D s \| \leq \| \operatorname{grad}_N s \| = \| \widetilde{\operatorname{grad}}_{\tilde{D}} s \|$, and finally also

$\tilde{H}'' \geq 0$ (by [3] pp. 1038-1043 and 1045-1049). In total we therefore have from (2.2)

(2.3) $\Delta_y \tilde{H}(s,t) \leq \tilde{H}'' \, \| \widetilde{\mathrm{grad}_{\tilde{D}}\, s} \|^2 + \tilde{H}' \tilde{\Delta}_{\tilde{D}} = \tilde{\Delta}_y \tilde{H}(s,t)$, so that $\Box_y \tilde{H} \leq \tilde{\Box}_y \tilde{H} = 0$.

The inequality $\Box_y \tilde{K} \leq \tilde{\Box}_y \tilde{K} = 0$ for the transplanted Neumann heat kernel follows similarly from $\tilde{K}'' \geq 0$ ([3] pp. 1044 and 1049-1050). The proof may now be completed by Proposition 1 of [3].

$\Box$

Once Theorem 1 is in hand we may now consider the series of corollaries given in [3] for similar extensions. Since the proofs of the generalized versions follow almost verbatim the space from proofs we will omit them.

<u>Corollary A.</u> Let M^m be a minimally immersed submanifold of N^n . Suppose $K_N \leq b$ and let f be a nonnegative subharmonic function on M . If $p \in M$ and $\Omega_R = B_R(p) \cap M$ for a regular ball $B_R(p)$, then

(2.4) $$f(p) \leq C^{-1}(m,b,R) \int_{\partial\Omega_R} f * 1 \,,$$

where

$$C(m,b,R) = m\,\omega_m \cdot \begin{cases} \left(\frac{1}{\sqrt{b}} \sin \sqrt{b}\, R\right)^{m-1} & \text{if } b > 0 \\ R^{m-1} & \text{if } b = 0 \\ \left(\frac{1}{\sqrt{-b}} \sin \sqrt{-b} R\right)^{m-1} & \text{if } b < 0 \,, \end{cases}$$

and ω_m is the volume of the unit m-ball in $\mathbb{R}^m$.

<u>Corollary B.</u> Let M^m be a minimally immersed submanifold of N^n, $K_N \leq b$. Let $B_R(p)$ be a ball in the normal range of $p \in M$. Then

$$\text{(2.5)} \qquad \mathrm{vol}(B_R(p) \cap M) \geq \mathrm{vol}\left(\tilde{D}^b_{\min\{R,\pi/\sqrt{b}\}}\right).$$

In particular, if M is compact and contained in $B_R(p)$ with $R \leq \pi/\sqrt{b}$, then $b > 0$ and

(2.6) $\mathrm{vol}(M^m) \geq \mathrm{vol}(S^m_b)$, where S^m_b is the round sphere of dimension m and constant curvature b. If equality occurs in (2.6) and if $M \subset B_{\pi/\sqrt{b}}(p) \subset U(p)$ for <u>all</u> $p \in M$, then

$$\text{(2.7)} \qquad \#\{i \mid 0 < \lambda_i(M) \leq mb\} \leq m+1,$$

where $\{\lambda_i(M)\}$ is the ordered set of eigenvalues (with multiplicities) of Δ_M.

<u>Remarks.</u> The last statement follows from the generalization of Theorem 6 in [3]. The inequality (2.6) generalizes a result of B.-Y. Chen who proved it for compact minimal submanifolds of spheres (cf. [2]).

<u>Corollary C.</u> Let M^m be a minimally immersed submanifold of N with $K_N \leq b$. Let D be a compact domain in M which is contained in a regular ball $B_R(p)$ for some $p \in M$. Then the 1.st Dirichlet eigenvalue $\lambda_1(D)$ of Δ_D satisfies

$$\text{(2.8)} \qquad \lambda_1(D) \geq \lambda_1(\tilde{D}^b_R) \geq \frac{m\,\pi^2}{4\,R^2}.$$

If the first inequality is an equality, then D is radial, i.e. D is generated by geodesics of length R from p.

Furthermore, if $b \leq 0$, then the k.th Dirichlet eigenvalue for D satisfies

(2.9) $$\left(\lambda_k(D)\right)^{m/2} \geq \frac{k\cdot(4\pi)^{m/2}}{e\cdot \mathrm{vol}(D)}$$

Remark. The inequality $\lambda_1(D) \geq m\,\pi^2/4\,R^2$ was proved in [4].

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