

# Hodge theory of Teichmüller curves

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## Introduction

Let  $M_g$  be the moduli space of curves of genus  $g \geq 2$  and let  $\Omega M_g \rightarrow M_g$  be the total space of the Hodge bundle. There is a natural  $\mathrm{GL}_2^+(\mathbb{R})$ -action on the complement  $\Omega M_g^*$  of the zero section in  $\Omega M_g$ : For  $A \in \mathrm{GL}_2^+(\mathbb{R})$  and  $(X, \omega) \in \Omega M_g^*$  postcompose the charts given by integration of  $\omega$  with the linear action of  $A$  on  $\mathbb{C} \cong \mathbb{R}^2$ . This defines a new complex structure on  $X$  and a one-form that is holomorphic for this complex structure. Moreover the action preserves the stratification of  $\Omega M_g^*$  according to the number and multiplicities of the zeros of  $\omega$ .

Some motivation why it seems worth studying this action:

First, the  $\mathrm{GL}_2^+(\mathbb{R})$ -action lifts in a natural way to the Hodge bundle over the Teichmüller space  $T_g$ . The images of orbits in  $T_g$  are complex geodesics for the Teichmüller (equivalently: Kobayashi-) metric.

In the -rare- case that the image  $C$  of such a geodesic in  $M_g$  is closed,  $C$  is called a *Teichmüller curve*. This happens if and only if the  $\mathrm{GL}_2^+(\mathbb{R})$ -orbit is closed in  $\Omega M_g$ .

Second, the action sheds much light in the behaviour of trajectories on rational-angled billiard tables. See [MaTa] for a survey and the references therein.

Third, how do closures of  $\mathrm{GL}_2^+(\mathbb{R})$ -orbits look like?  $M_g$  seems to behave as nicely as a symmetric domain with respect to this action. Compare the following two results:

**Theorem 1** (*Ratner, special case*) *Let  $M = \mathbb{H}^n/\Gamma$  be the quotient of hyperbolic  $n$ -space by a discrete group. Then the closure of an  $\mathrm{SL}_2(\mathbb{R})$ -orbit in the frame bundle  $FM = \mathrm{SO}^0(1, n)/\Gamma$  is the orbit of a closed subgroup  $H < \mathrm{SO}^0(1, n)$ , whose intersection with some conjugate of  $\Gamma$  is a lattice.*

**Theorem 2** (*McMullen, [Mc1]*) *In  $\Omega M_2$  an orbit closure is one of the following possibilities: i) the lift of a Teichmüller curve to  $\Omega M_g$ ; ii) the locus of eigenforms for real multiplication by a fixed order; iii) a stratum  $\Omega M_2(2)$  or  $\Omega M_2(1, 1)$ .*

The arguments used to prove this are particular to genus 2. In higher genus the locus of eigenforms for real multiplication is no longer  $\mathrm{GL}_2^+(\mathbb{R})$ -invariant.

In the sequel we restrict ourselves to the case of closed orbits i.e. to Teichmüller curves and give a characterization valid in all genera as well as applications towards the classification of Teichmüller curves.

## Teichmüller curves

Given a point  $(X, \omega) \in \Omega M_g^*$ , we denote by  $\text{Aff}^+(X, \omega)$  the group of orientation preserving diffeomorphism of  $X$ , that are affine with respect to the charts given by  $\omega$ . There is a well-defined map  $D : \text{Aff}^+(X, \omega) \rightarrow \text{SL}_2(\mathbb{R})$  by taking the matrix part of the diffeomorphism. Let  $\Gamma$  be the image of  $D$ . Then  $(X, \omega)$  generates a Teichmüller curve if and only if  $\Gamma$  is a lattice in  $\text{SL}_2(\mathbb{R})$ . In this case, up to conjugation,  $C = \mathbb{H}/\Gamma$ .

This observation makes it possible to detect Teichmüller curves. Examples are torus coverings ramified over only one point, or regular  $2n$ -gons with opposite sides glued by translation. The latter belong to the first series of examples, discovered by Veech ([Ve]), where the trace field  $K = \mathbb{Q}(\text{tr}(\gamma), \gamma \in \Gamma)$  is different from  $\mathbb{Q}$ .  $K$  is always a number field. Let  $L$  be the Galois closure of  $K/\mathbb{Q}$ .

**Theorem 3** ([Mo1]) *If  $C \rightarrow M_g$  is a Teichmüller curve, then there exists a finite unramified covering  $C_1 \rightarrow C$ , such that the variation of Hodge structures (VHS) of the family  $f : \mathcal{X} \rightarrow C_1$  decomposes as follows:*

$$R^1 f_* L = (\oplus_{\sigma \in \text{Gal}(L/\mathbb{Q})/\text{Gal}(L/K)} \mathbb{L}^\sigma) \oplus \mathbb{M}.$$

Here  $\mathbb{L}^{\text{id}}$  is 'maximal Higgs', i.e. the corresponding representation  $\pi_1(C) \rightarrow \text{Aut}(\text{Fibre of } \mathbb{L}^{\text{id}})$  has image  $\Gamma$ .  $\mathbb{M}$  splits off over  $\mathbb{Q}$ .

Conversely a family  $f$  whose VHS contains a maximal Higgs local subsystem of rank 2 defined over  $\mathbb{R}$  comes from a finite unramified covering of a Teichmüller curve.

Hence Teichmüller curves 'behave' a little like Shimura curves, for which the VHS is built up only of local subsystems, that are maximal Higgs, and unitary local systems, see [ViZu] for more details.

Some consequences:

**Corollary 4** ([Mo1]) *The family of Jacobians over a Teichmüller curve contains a family of  $r$ -dimensional abelian subvarieties  $A_r$  with RM by  $K$ . Teichmüller curves are defined over number fields and the absolute Galois group of  $\mathbb{Q}$  acts on the set of Teichmüller curves.*

Suppose  $(X, \omega)$  generates a Teichmüller curve. A point  $P$  on  $X$  is called *periodic*, if the orbit  $\text{Aff}^+(X, \omega) \cdot P$  is finite. Examples of periodic points are the zeros of  $\omega$ , preimages of torsion points for torus coverings and Weierstraß points for hyperelliptic Teichmüller curves. Periodic points give rise to a section of  $f$  for a suitable unramified covering  $C_1 \rightarrow C$ .

**Theorem 5** ([Mo2]) *For each unramified covering  $C_1 \rightarrow C$  the Mordell-Weil group of the family of abelian varieties  $A_r/C_1$  is finite. In particular if  $r = g$  the difference of two periodic points is torsion.*

A Teichmüller curve generated by  $(X, \omega)$  is called primitive, if  $(X, \omega)$  does not arise via a covering from lower genus. If  $r = g$ , a Teichmüller curve is obviously primitive, but the converse does not hold.

Since there are 'few' torsion points lying on the image of a curve in its Jacobian, Theorem 5 'explains' why -contrary to the intuition coming from dimension counting- primitive Teichmüller curves in strata with many zeros are 'rare'. In fact the above result is used in [Mc2] to complete the classification of Teichmüller curves in genus 2.

## References

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