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CONTACT STRUCTURES, DEFORMATIONS AND TAUT FOLIATIONS

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Abstract. Eynard has shown that any two taut foliations whose tangent distributions are homotopic as plane fields are also homotopic as foliations. By using Eliashberg and Thurston's deformations of foliations to contact structures we give examples of taut foliations that are not homotopic through taut foliations. Using similar methods we furthermore show that the space of representations of a hyperbolic surface with fixed Euler class is in general not path connected. We also consider the problem of which universally tight contact structures on Seifert fibered spaces are deformations of taut or Reebless foliations, giving a complete answer if the topological genus of the base is positive or if the twisting number of the contact structure is non-negative.

1. Introduction

In their book on confoliations Eliashberg and Thurston, [6] established a fundamental link between the theory and foliations and contact topology, by showing that any foliation that is not the product foliation on $S^2 \times S^1$ can be C^0-approximated by a contact structure. The proof of this result naturally leads to the study of confoliations, which are a generalisation of both contact structures and foliations. Recall that a smooth cooriented 2-plane field $\xi = \mathrm{Ker}(\alpha)$ on an oriented 3-manifold M is a confoliation if $\alpha \wedge d\alpha \geq 0$. For the most part interest has focussed on the contact case, where the study of deformations and isotopy are equivalent in view of Gray stability. On the other hand many questions in deformation theory of foliations or more generally confoliations remain to a large extent unexplored.

Rather than considering general confoliations, we will focus on questions concerning the topology of the space of foliations. In contact topology one has a tight vs. overtwisted dichotomy, which is in some sense mirrored in the theory of foliations by Reebless foliations and those with Reeb components. In analogy with 3-dimensional contact topology where one seeks to understand deformation classes of tight contact structures, we will be primarily concerned with studying the topology of the space of Reebless and taut foliations and the contact structures they approximate.

It is well known that every contact structure is a deformation of a foliation by Etnyre, [8], a result whose proof was implicit in Mori, [27]. The foliations that Etnyre considers use open books and by construction contain Reeb components. This led Etnyre to ask whether every universally tight contact structure on a manifold with infinite fundamental group is a deformation of a Reebless foliation. By considering the known criteria for the existence of Reebless foliations on small Seifert fibered spaces, it is easy to see that this is false in general. This was first observed by Lekili and Ozbagci, [20]. Nevertheless it is still an interesting problem to determine which contact structures can be realised as deformations of Reebless foliations, a problem which was already raised by Eliashberg and Thurston in [6]. Furthermore, the counter examples coming from small Seifert fibered spaces are not

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completely satisfactory, since the obvious necessary condition for a manifold to admit a Reebless foliation is that it admits universally tight contact structures in both orientations, and for small Seifert manifolds this is in fact equivalent to the existence of a Reebless foliation (cf. Proposition 6.5).

In contrast to the case of small Seifert manifolds Etnyre's original question has a positive answer for Seifert fibered spaces whose bases have positive topological genus. Before stating this result let us recall the notion of the twisting number $t(\xi)$ of a contact structure ξ on a Seifert fibered space. This is defined as the maximal Thurston-Bennequin number of a Legendrian knot that is isotopic to a regular fiber, where this is measured relative to the canonical framing coming from the base.

Theorem 1.1. *Let ξ be a universally tight contact structure on a Seifert fibered space with infinite fundamental group and $t(\xi) \geq 0$, then ξ is isotopic to a deformation of a Reebless foliation. If $g > 0$ and $t(\xi) < 0$, then ξ is isotopic to a deformation of a taut foliation.*

The proof of Theorem 1.1 involves examining the Giroux-type normal forms for universally tight contact structures of Massot and considering foliations that are well adapted to these normal forms. The cases of negative and non-negative twisting are treated separately, with the former being reduced to the $t(\xi) = -1$ case via a covering trick.

Other examples of tight contact structures on hyperbolic manifolds that admit no Reebless foliations were given by Etgü, [7]. However, these examples are neither known to be universally tight, nor is it shown that there are tight contact structures for both orientations. This then suggests the following topological version of Etnyre's original question, which then has an affirmative answer for Seifert fibered spaces:

Question 1.2. *Does every irreducible 3-manifold with infinite fundamental group that admits both positive and negative universally tight contact structures necessarily admit a (smooth) Reebless foliation?*

Until recently there was little known about the topology of the space of foliations on a 3-manifold. For the class of horizontal foliations on S^1-bundles Larchancé, [19] showed that the inclusion of the space of horizontal integrable plane fields into the space of all integrable plane fields is homotopic to a point and in particular its image is contained in a single path component in the space of all integrable plane fields. In her PhD thesis Eynard showed that a much more general result holds. In particular, she proved the following theorem, which mirrors Eliashberg's h-principal for overtwisted contact structures.

Theorem 1.3 (Eynard, [9])**.** *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be smooth oriented taut foliations on a 3-manifold M whose tangent distributions are homotopic as (oriented) plane fields. Then $T\mathcal{F}_0$ and $T\mathcal{F}_1$ are smoothly homotopic through integrable plane fields.*

The foliations that Eynard constructs use a parametric version of a construction of Thurston, first exploited by Larchancé, that allows foliations to be extended over solid tori. This then introduces an uncontrolled number of Reeb components. In view of this it is natural to ask whether any two horizontal foliations are in fact homotopic through horizontal foliations or more generally whether any two taut foliations whose tangent plane fields are homotopic are homotopic through taut or even Reebless foliations. Since any horizontal foliation on an S^1-bundle is essentially determined by its holonomy representation, the former question is then related to the topology of the representation space $\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$ considered with its natural C^∞-topology. Concerning the topology of this space we prove the following:

Theorem 1.4. *Let $\#Comp(e)$ denote the number of path components of $Rep(\pi_1(\Sigma_g), Diff_+(S^1))$ with fixed Euler class $e \neq 0$ such that e divides $2g-2 \neq 0$ and write $2g-2 = n\,e$. Then the following holds:*

$$\#Comp(e) \geq \sum_{d\,|\,n} d^{2g}.$$

The idea behind the proof of this theorem is very simple: a smooth family of representations ρ_t corresponds to a smooth family of foliations $\mathcal{F}_t$ via the suspension construction and one then deforms this family to a family of contact structures using a parametric version of Eliashberg and Thurston's perturbation theorem. Deformations of contact structures correspond to isotopies via Gray stability and this then gives an isotopy of contact structures, which then distinguish path components in the representation space.

In general, however, there is no parametric version of Eliashberg and Thurston's perturbation theorem, since in general the contact structure approximating a foliation is not unique. On the other hand under certain additional assumptions, that are for instance true for horizontal foliations on non-trivial S^1-bundles, Vogel, [31] has shown a remarkable uniqueness result for the isotopy class of a contact structure approximating a foliation, which implies in particular that this isotopy class is in fact a deformation invariant. Instead of using Vogel's results we give a simpler argument which uses linear perturbations to deform families of foliations to contact structures in a smooth manner and this suffices for our purposes.

We also present a second independent proof of Theorem 1.4, which uses the rich structure theory of Anosov foliations instead of contact topology. In the case of representations with maximal Euler class Matsumoto, [26] showed that any representation is topologically conjugate to the suspension of a Fuchsian representation and Ghys, [11] showed that this conjugacy can be assumed to be smooth. Such foliations correspond to the weak unstable foliation of the Anosov flow given by the geodesic flow of a hyperbolic metric on the unit cotangent bundle $ST^*\Sigma_g$. Consequently any smooth representation with maximal Euler class is in fact conjugate to a Fuchsian representation and the space of representations with maximal Euler class is path connected by results of Goldman, [14]. By considering fiberwise coverings it is easy to construct Anosov representations with non-maximal Euler classes. In general not every horizontal foliation lies in the same component as an Anosov foliation. We do however obtain the following analogue of Ghys' result, which answers a question posed to us by Y. Mitsumatsu.

Theorem 1.5. *Any representation $\phi \in Rep(\pi_1(\Sigma_g), Diff_+(S^1))$ that lies in the path component of an Anosov representation ϕ_{An} is itself Anosov. In particular, it is conjugate to a discrete subgroup of a finite covering of $PSL(2,\mathbb{R})$ and is injective.*

Since there always exist non injective representations in the case of non-maximal Euler class this immediately implies the existence of more than one path component in the representation space for any non-maximal Euler class that admits Anosov representations. By using certain conjugacy invariants (cf. Theorem 9.3) it is then easy to recover the precise estimates of Theorem 1.4.

Of course not every taut foliation on an S^1-bundle is horizontal so this theorem still leaves open the question of whether taut foliations are always deformable through taut foliations. By considering certain small Seifert fibered spaces, the deformation method in fact yields examples of this as well.

Theorem 1.6. *There exist taut foliations $\mathcal{F}_1, \mathcal{F}_2$ that are homotopic as foliations but not as taut foliations. Furthermore, any homotopy through foliations must contain at least one Reeb component.*

Further examples of taut foliations that cannot be joined by a path in the space of Reebless foliations are given by using the special structure of foliations on the unit cotangent bundle over a closed surface. In particular, we show that the weak unstable foliation of the geodesic flow $\mathcal{F}_{hor}$ on $ST^*\Sigma_g$ cannot be smoothly deformed to any taut foliation with a torus leaf $\mathcal{F}_T$ without introducing Reeb components (Corollary 8.11). One can view this fact as a generalisation of the result of Ghys and Matsumoto concerning horizontal foliations of $ST^*\Sigma_g$, in that it shows that the path component of an Anosov foliation in the space of all Reebless foliations contains only Anosov foliations. This is perhaps slightly surprising since for the product foliation on $\Sigma_g \times S^1$ one can spiral along any vertical torus $\gamma \times S^1$ to obtain smooth deformations that introduce incompressible torus leaves. On the other hand, although there exists no smooth deformation through taut foliations, it is not hard to construct a taut deformation between $\mathcal{F}_{hor}$ and $\mathcal{F}_T$ through foliations that are only of class C^0. Thus these examples exhibit further the difference between foliations of class C^0 and those of higher regularity.

Outline of paper: In Section 2 we recall some basic definitions and constructions of foliations and contact structures and in Section 3 we review some basic facts about Seifert fibered spaces and horizontal foliations. Section 4 contains the relevant versions of Eliashberg and Thurston's results on deforming foliations to contact structures and Section 5 contains background on horizontal contact structures and normal forms. In Sections 6 and 7 we prove Theorem 1.1 first for negative twisting numbers and then in the non-negative case. Section 8 contains our main results concerning deformations of taut foliations and finally in Section 9 we analyse components of the representation space of a surface group that contain Anosov representations, yielding an alternative proof of Theorem 1.4.

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Notation and Conventions: All manifolds, contact structures and foliations are smooth and oriented.

2. Foliations and contact structures

In this section we recall some basic definitions and constructions for foliations and contact structures. For a more in depth discussion of foliations on 3-manifolds we refer to the book of Calegari, [2].

2.1. **Foliations:** A codimension-1 foliation $\mathcal{F}$ on a 3-manifold M is a decomposition of M into immersed surfaces called *leaves* that is locally diffeomorphic to the foliation of $\mathbb{R}^3$ given by projection to the z-axis. We will always assume that all foliations are smooth and are cooriented. One can then define a global non-vanishing 1-form α by requiring that

$$\mathrm{Ker}(\alpha) = T\mathcal{F} = \xi \subset TM.$$

By Frobenius' Theorem such a cooriented distribution is tangent to a foliation if and only if

$$\alpha \wedge d\alpha = 0$$

and in this case ξ is called *integrable*. An important example of a foliation is the Reeb foliation.

Example 2.1 (Reeb foliation). Consider $D^2 \times S^1$ with coordinates (r, θ, ϕ). Choose a non-negative step function $\gamma(r)$ that has support in $[0, 1]$ and is decreasing on the interior. Then $\mathcal{F}_{Reeb}$ is defined as the kernel of the following form

$$\alpha = \gamma(r)\, d\phi + (1 - \gamma(r))dr.$$

This foliation has a unique compact leaf given by $\partial D^2 \times S^1$ and the foliation on $int(D^2) \times S^1$ is by parabolic planes. A solid torus with such a foliation will be called a *Reeb component*.

General foliations are very flexible - they satisfy an h-principal - and in particular every plane field is homotopic to the tangent distribution of a foliation. A more geometrically significant class of foliations are those that are *taut*. Here a foliation is taut if every leaf admits a closed transversal. Note that any foliation that contains a Reeb component is not taut, since the boundary leaf of the Reeb component is separating and compact. Thus taut foliations fall into the more general class of *Reebless* foliations, i.e. those that contain no Reeb component. The existence of a Reebless foliation puts restrictions on the topology of M due to the following theorem of Novikov.

Theorem 2.2 (Novikov). *Let $\mathcal{F}$ be a Reebless foliation on a 3-manifold. Then all leaves of $\mathcal{F}$ are incompressible, $\pi_2(M) = 0$ and all transverse loops are essential in $\pi_1(M)$. In particular, $\pi_1(M)$ is infinite.*

It follows from Novikov's theorem that a foliation is Reebless if and only if all its torus leaves are incompressible. We also have the following criterion for tautness due to Goodman.

Theorem 2.3 (Goodman). *Let $\mathcal{F}$ be a foliation on a 3-manifold M. If no oriented combination of torus leaves of $\mathcal{F}$ is null-homologous in $H_2(M)$, then $\mathcal{F}$ is taut.*

It will be important to modify foliations in various situations below and we will repeatedly make use of a spinning construction which introduces torus leaves into foliations that are transverse to an embedded torus.

Construction 2.4 (Spiralling along a torus). Let $\mathcal{F}$ be a foliation on a manifold obtained by cutting a closed manifold open along an embedded torus

$$\overline{M} = M \setminus T^2 \times (-\epsilon, \epsilon)$$

and assume that $\mathcal{F}$ is transverse on the boundary components T_-, T_+ of $\overline{M}$. We furthermore assume that $\mathcal{F}$ is linear on the boundary so that it is given as the kernel of closed 1-forms α_- and α_+ respectively. Letting z be the normal coordinate on $T^2 \times (-\epsilon, \epsilon)$ we then define a foliation as the kernel of the following form

$$\alpha = \rho(-z)\alpha_- + \rho(z)\alpha_+ + dz.$$

Here ρ is a step function that is positive for $z > 0$ and identically zero otherwise so that ρ vanishes to infinite order at the origin.

Note that spiralling along an embedded torus T has the effect of introducing a closed torus leaf. Furthermore, this modification can be achieved through a 1-parameter deformation of foliations. Finally observe that if T is a compressible torus given as the boundary of a closed transversal, then spinning along T has the effect of introducing a Reeb component having T as a closed leaf. In this case spinning along T corresponds to *turbulisation*. This in particular shows that Reeblessness and hence tautness are not deformation invariants of foliations.

2.2. **Contact structures:** In addition to foliations we will also consider totally non-integrable plane fields or *contact structures*. Here a contact structure ξ is a distribution such that $\alpha \wedge d\alpha$ is nowhere zero for any defining 1-form with $\xi = \mathrm{Ker}(\alpha)$. Unless specified our contact structures will always be *positive* with respect to the orientation on M so that $\alpha \wedge d\alpha > 0$. If α only satisfies the weaker inequality $\alpha \wedge d\alpha \geq 0$, then ξ is called a (positive) *confoliation*.

There is a fundamental classification of contact structures into those that are tight and those that are not.

Definition 2.5 (Overtwistedness)**.** *A contact structure ξ on manifold M is called **overtwisted** if it admits an embedded disc $D \hookrightarrow M$ such that*

$$TD|_{\partial D} = \xi|_{\partial D}.$$

If a contact structure ξ admits no such disc then it is called *tight*. A contact structure is *universally tight* if its pullback to the universal cover $\widetilde{M} \to M$ is tight.

3. Seifert manifolds and horizontal foliations

3.1. **Seifert manifolds:** A Seifert manifold is a closed 3-manifold that admits a locally free S^1-action. These manifolds are well understood and can all be built using the following recipe: Let R be an oriented, compact, connected surface with boundary of genus g and let $R_i = \partial_i R$ for $0 \leq i \leq r$ denote its oriented boundary components. We then obtain a Seifert manifold by gluing $W_i = D^2 \times S^1$ to the i-th boundary component of $R \times S^1$ in such a way that the oriented meridian $m_i = \partial D^2$ maps to $-\alpha_i[R_i] + \beta_i[S^1]$ in homology, where S^1 is oriented to intersect R positively.

The resulting manifold M admits a locally free S^1-action in a natural way and the numbers $(g, \frac{\beta_0}{\alpha_0}, ..., \frac{\beta_r}{\alpha_r})$ are called the Seifert invariants of M. This action has a finite number of orbits that have non-trivial stabilisers, which are called *exceptional fibers*. These exceptional fibers correspond to the cores of those solid tori W_i for which the attaching slope $\frac{\beta_i}{\alpha_i}$ is not integral. The Seifert invariants are not unique, as one can add and subtract integers so that the sum $\sum \frac{\beta_i}{\alpha_i}$ remains unchanged to obtain equivalent manifolds. This then corresponds to a different choice of section on $R \times S^1$ with respect to which the Seifert invariants were defined. However, the Seifert invariants can be put in a normal form by requiring that $b = \frac{\beta_0}{\alpha_0} \in \mathbb{Z}$ and that

$$0 < \frac{\beta_1}{\alpha_1} \leq \frac{\beta_2}{\alpha_2} \leq ... \leq \frac{\beta_r}{\alpha_r} < 1.$$

This normal form is then unique, except for a small list of manifolds (see [15]). Note that according to our conventions a Seifert fibered space M with normalised Seifert invariants $(g, b, \frac{\beta_1}{\alpha_1}, ..., \frac{\beta_r}{\alpha_r})$ is an oriented manifold. The Seifert fibered space $\overline{M}$ considered with the opposite orientation has Seifert invariants $(g, -b-r, 1 - \frac{\beta_1}{\alpha_1}, ..., 1 - \frac{\beta_r}{\alpha_r})$.

Warning: The conventions for Seifert manifolds differ greatly in the literature. Here we follow the convention of [23] and [5], which differs from [16] and [21].

Given a Seifert manifold with a decomposition $M = (R \times S^1) \cup W_0 \cup ... \cup W_r$ and Seifert invariants $(g, b, \frac{\beta_1}{\alpha_1}, ..., \frac{\beta_r}{\alpha_r})$, there is a natural fiberwise n-fold covering $M \xrightarrow{p} M'$ branched over the exceptional fibers, where the manifold M' has Seifert invariants $(g, nb, \frac{n\beta_1}{\alpha_1}, ..., \frac{n\beta_r}{\alpha_r})$. The best way to see this is to let $\mathbb{Z}_n \subset S^1$ be the n-th roots of unity and to set $M' = M/\mathbb{Z}_n$ with p being the quotient map. On the subset $R \times S^1$ the map p is just the product of the standard n-fold cover $S^1 \to S^1$ with the identity on $R \times S^1$, which extends in a unique way to the tori W_i in a fiber preserving way. Moreover, by considering standard fibered neighbourhoods of the exceptional fibers we see that the branching order around an exceptional fiber is $gcd(n, \alpha_i)$. We note this in the following proposition for future reference.

Proposition 3.1 (Fibrewise branched covers)**.** *Let M be a Seifert manifold with Seifert invariants $(g, b, \frac{\beta_1}{\alpha_1}, ..., \frac{\beta_r}{\alpha_r})$. Then there is a fiber preserving branched n-fold covering map $M \xrightarrow{p} M'$, where M' has (unnormalised) Seifert invariants $(g, nb, \frac{n\beta_1}{\alpha_1}, ..., \frac{n\beta_r}{\alpha_r})$. The branching locus of p is a (possibly empty) subset of the exceptional fibers and the branching order around the k-th singular fiber is $gcd(n, \alpha_k)$.*

3.2. **Horizontal Foliations:** We next discuss horizontal foliations on Seifert manifolds referring to [5] for details. Here a foliation on a Seifert space is called *horizontal*, if it is everywhere transverse to the fibers of the Seifert fibration. A horizontal foliation $\mathcal{F}$ on a Seifert fibered space is equivalent to a representation $\widetilde{\rho} : \pi_1(M) \to \widetilde{\mathrm{Diff}}_+(S^1)$, such that the homotopy class of the fiber is mapped to a generator of the centre of $\widetilde{\mathrm{Diff}}_+(S^1)$. One then has $M = (\widetilde{B} \times \mathbb{R})/\widetilde{\rho}$, where $\widetilde{B}$ denotes the universal cover of the quotient orbifold of M, and the horizontal foliation on the product descends to $\mathcal{F}$. The representation $\widetilde{\rho}$ then descends to a representation of the orbifold fundamental group of the base to the ordinary diffeomorphism group $\rho : \pi_1^{orb}(B) \to \mathrm{Diff}_+(S^1)$.

In all but a few cases a Seifert manifold admits a horizontal foliation if and only if it admits one with holonomy in $\mathrm{PSL}(2, \mathbb{R})$, in the sense that the image of the holonomy map in ρ lies in $\mathrm{PSL}(2, \mathbb{R})$. Moreover, an examination of the proof of ([5], Theorem 3.2) and its analogue for $\mathrm{PSL}(2, \mathbb{R})$-foliations shows that it is always possible to ensure that the holonomy around some embedded curve in the base is hyperbolic provided that the base has positive genus. We note this in the following proposition.

Proposition 3.2 (Existence of horizontal foliations, [5])**.** *Let M be a Seifert fibered space whose base has topological genus g, then M admits a horizontal foliation if*

$$2 - 2g - r \leq -b - r \leq 2g - 2.$$

In this case the horizontal foliation can be taken to have holonomy in $PSL(2, \mathbb{R})$ and the holonomy around some embedded curve in the base can be chosen to be hyperbolic. If $g > 0$ then the converse also holds.

Thus in most cases the existence of a horizontal foliation on M is the same as the existence of a flat connection on M thought of as an orbifold $\mathrm{PSL}(2, \mathbb{R})$-bundle. In the case of genus zero, one has slightly more elaborate criteria for the existence of a $\mathrm{PSL}(2, \mathbb{R})$-foliation.

Theorem 3.3 ([18], Theorem 1). *Let M be a Seifert manifold with normalised invariants $(0, b, \frac{\beta_1}{\alpha_1}, , ..., \frac{\beta_r}{\alpha_r})$. Then M admits a horizontal foliation with holonomy in $PSL(2,\mathbb{R})$ if and only if one of the following holds:*

- $2 - r \leq -b - r \leq -2$
- $b = -1$ *and* $\sum_{i=1}^{r} \frac{\beta_i}{\alpha_i} \leq 1$ *or* $b = 1 - r$ *and* $\sum_{i=1}^{r} \frac{\beta_i}{\alpha_i} \geq r - 1$.

4. Perturbing foliations

In their book on confoliations, Thurston and Eliashberg showed how to perturb foliations to contact structures. In its most general form, their theorem shows that any 2-dimensional foliation $\mathcal{F}$ that is not the product foliation on $S^2 \times S^1$ can be C^0-approximated by a positive and a negative contact structure. However, under additional assumptions on the holonomy of the foliation this perturbation can actually be realised as a deformation. That is, there is a smooth family ξ_t of plane fields, such that ξ_0 is the tangent plane field of $\mathcal{F}$ and ξ_t is contact for all $t > 0$. Moreover, if every closed leaf has linear holonomy or if the foliation is minimal with some holonomy, then $\mathcal{F}$ can be linearly perturbed to a contact structure. Here a linear perturbation is a family of 1-forms α_t such that $Ker(\alpha_0) = T\mathcal{F}$ and

$$\frac{d}{dt}\alpha_t \wedge d\alpha_t \Big|_{t=0} > 0.$$

This latter condition is then equivalent to the existence of a 1-form β such that

$$\langle \alpha, \beta \rangle = \alpha \wedge d\beta + \beta \wedge d\alpha > 0.$$

Note further that

$$\langle f\alpha, f\beta \rangle = f^2 \langle \alpha, \beta \rangle$$

so that the condition of being linearly deformable depends only on the foliation and not on the particular choice of defining 1-form.

Theorem 4.1 (Eliashberg-Thurston, [6]). *Let $\mathcal{F}$ be a C^2-foliation that is not without holonomy.*

1. *If all closed leaves admit some curve with attracting holonomy. Then $T\mathcal{F}$ can be smoothly deformed to a positive resp. negative contact structure.*
2. *If all closed leaves have linear holonomy, then this deformation can be chosen to be linear.*

Remark 4.2. Foliations without holonomy are very special and can be C^0-approximated by surface fibrations over S^1. Thus the assumption that the foliation has some holonomy can be replaced by the topological assumption that the underlying manifold does not fiber over S^1. Examples of manifolds which cannot fiber are non-trivial S^1-bundles, or more generally Seifert fibered spaces with non-trivial Euler class, and rational homology spheres.

In general it is not possible to deform families of foliations to contact structures in a smooth manner. However, if a family of foliations $\mathcal{F}_\tau$ admits linear deformations for all τ in some compact parameter space K, then the fact that $\langle \alpha, \beta \rangle > 0$ is a convex condition, means that one can use a partition of unity to smoothly deform the entire family. We note this in the following proposition, which will be mainly applied when the family has no closed leaves at all.

Proposition 4.3 (Deformation of families)**.** *Let $\mathcal{F}_\tau$ be a smooth family a foliations that is parametrised by some compact space K and suppose that each foliation in the family admits a linear deformation. Then $\mathcal{F}_\tau$ can be smoothly deformed to a family of positive rep. negative contact structures $\xi_\tau^\pm$.*

Another consequence of the convexity of the linear deformation condition is that any two positive linear deformations a foliation are isotopic by Gray stability.

Proposition 4.4. *Any two positive, resp. negative linear deformations of a foliation are isotopic.*

5. Horizontal contact structures

Horizontal contact structures on Seifert manifolds, like horizontal foliations, may be thought of as connections with a certain curvature condition. As opposed to the flat case where the horizontal distribution is a foliation, the distribution in question is contact if and only if the holonomy around the boundary of any small disc is less than the identity. This then puts topological restrictions on the topology of Seifert manifolds that admit horizontal contact structures and one has the following necessary and sufficient conditions.

Theorem 5.1 ([16], [21])**.** *A Seifert manifold with normalised invariants $(g, b, \frac{\beta_1}{\alpha_1}, ..., \frac{\beta_r}{\alpha_r})$ carries a (positive) contact structure transverse to the Seifert fibration if and only if one of the following holds:*

- $-b - r \leq 2g - 2$
- $g = 0$, $r \leq 2$ *and* $-b - \sum \frac{\beta_i}{\alpha_i} < 0$
- $g = 0$ *and there are relatively prime integers* $0 < a < m$ *such that*

$$\frac{\beta_1}{\alpha_1} > \frac{m-a}{m}, \frac{\beta_2}{\alpha_2} > \frac{a}{m} \text{ and } \frac{\beta_i}{\alpha_i} > \frac{m-1}{m}, \text{ for } i \geq 3.$$

Remark 5.2. The final condition is the realisability condition of [5], which is equivalent to the existence of a horizontal foliation by Naimi, [28]. For $g > 0$ the condition for the existence of a horizontal contact structure is the upper bound of the double sided inequalities that determine the existence of horizontal foliations (cf. Proposition 3.2).

A given Seifert manifold can admit several isotopy classes of horizontal contact structures. Massot, [23] showed that a contact structure can be isotoped to a horizontal one if and only if it is universally tight and has negative *twisting number*. Recall that the twisting number $t(\xi)$ of ξ is the maximal Thurston-Bennequin number of a knot that is smoothly isotopic to a regular fiber, where the Thurston-Bennequin invariant is measured relative to the canonical framing coming from the base.

Furthermore, any horizontal contact structure ξ admits a *normal form*. This means that ξ can be isotoped to a contact structure which is vertical on $\widehat{M} \cong R \times S^1$, where $\widehat{M}$ denotes the total space with open neighbourhoods of the exceptional fibers removed. The contact structure on $\widehat{M}$ is vertical and has twisting number $-n$ so that ξ is given as the pullback of the canonical contact structure under a fiberwise n-fold covering $\widehat{M} \to ST^*R$ that we denote p_ξ. By ([23], Proposition 6.1) such a contact structure admits at most one extension to M which is universally tight, up to isotopy and orientation reversal of plane fields.

Theorem 5.3 (Normal form, [23]). *Let ξ be a universally tight contact structure on a Seifert manifold with $t(\xi) < 0$. Then ξ admits a normal form.*

Moreover, if $g > 0$, then this normal form is unique, that is the covering homotopy class of the covering map p_ξ determines ξ completely up to isotopy, unless $-b - r < 2g - 2$ and $n = -1$, in which case there is only one isotopy class up to changing the orientation of ξ without any assumption on the genus.

Remark 5.4. The statement in [23] does not use the map p_ξ, but rather the homotopy class of a non-vanishing 1-form λ. For this one notes that the choice of section $\widehat{s}$ in $\widehat{M}$ used to compute the normalised Seifert invariants gives a section in ST^*R via the covering map p_ξ. These sections then give identifications of ST^*R and $\widehat{M}$ with $R \times S^1$ and with respect to these identifications the map p_ξ is up to fiberwise isotopy the product of the identity with the standard n-fold cover of S^1.

Under the identification of $R \times S^1$ with ST^*R the canonical contact structure is isotopic to the kernel of some 1-form

$$\alpha_\lambda = \cos(\theta)\lambda + \sin(\theta)\lambda \circ J,$$

where λ is a non-vanishing 1-form on R and J is an almost complex structure. The contact structure on $\widehat{M}$ is then given by the kernel of the 1-form

$$\alpha_{\lambda,n} = \cos(n\theta)\lambda + \sin(n\theta)\lambda \circ J.$$

Now if M admits a contact structure ξ with twisting number $-n$, then one can see that ξ is isotopic to the pullback of a contact structure ξ' with twisting number -1 under an n-fold fiberwise branched cover. We note this in the following proposition.

Proposition 5.5. *Let M be a Seifert manifold admitting a contact structure with twisting number $t(\xi) = -n < -1$, then there is a fiberwise branched covering $M \xrightarrow{p} M'$ and a contact structure ξ' on M' with twisting -1 such that ξ is isotopic to $p^*\xi'$.*

Proof. Let $(g, b, \frac{\beta_1}{\alpha_1}, ..., \frac{\beta_r}{\alpha_r})$ be the normalised Seifert invariants of M and let $M \xrightarrow{p} M'$ be the n-fold fiberwise branched cover given by Proposition 3.1. Since $t(\xi) = -n$ by assumption, the contact structure ξ admits a normal form with associated 1-form

$$\alpha_{\lambda,n} = \cos(n\theta)\,\lambda + \sin(n\theta)\,\lambda \circ J.$$

By ([23], Proposition 8.2) the indices of λ are $(nb, \lceil \frac{n\beta_1}{\alpha_1} \rceil, ..., \lceil \frac{n\beta_r}{\alpha_r} \rceil)$ and Poincaré-Hopf implies

$$nb + \sum_{i=1}^{r} \left\lceil \frac{n\beta_i}{\alpha_i} \right\rceil = 2 - 2g.$$

It then follows by ([23], Theorem B) that the Seifert manifold M', which has Seifert invariants $(g, nb, \frac{n\beta_1}{\alpha_1}, ..., \frac{n\beta_r}{\alpha_r})$, admits a contact structure ξ' with normal form given by

$$\alpha_\lambda = \cos(\theta)\,\lambda + \sin(\theta)\,\lambda \circ J,$$

which is in particular transverse to the branching locus of the map p. The pullback of the contact structure ξ' can then be perturbed in a C^∞-small fashion to obtain a contact structure, which is denoted by $p^*\xi'$. Since there is a unique way to extend the pullback $p^*\xi'|_{\widehat{M}}$ to a contact structure on all of M, which can be made positively transverse ([23], Proposition 6.1) and $p^*\xi'$ is positively transverse to the fibration, we conclude that ξ is isotopic to $p^*\xi'$. $\square$

We next note that any two contact structures with twisting number -1 are necessarily contactomorphic modulo orientation reversal of plane fields.

Proposition 5.6. *Let ξ, ξ' be contact structures on a Seifert fibered space which are universally tight and satisfy $t(\xi) = t(\xi') = -1$. Then ξ and ξ' are contactomorphic as unoriented contact structures.*

Proof. We first observe that if $t(\xi) = t(\xi') = -1$ and $-b - r \neq 2g - 2$ then ξ' and ξ are isotopic as unoriented contact structures by Theorem 5.3. If $-b - r = 2g - 2$ then we let α_λ and $\alpha_{\lambda'}$ be the 1-forms associated to the normal form of ξ and ξ' respectively on $\widehat{M} = R \times S^1 \subset M$. After possibly replacing λ with $-\lambda$, we may assume that both ξ and ξ' are isotopic to positively transverse contact structures. By ([23], Proposition 8.2) the indices of both λ and λ' must then agree on ∂R. This is equivalent to the restrictions of the maps

$$p_\xi, p_{\xi'} : \widehat{M} \to ST^*R$$

being fiberwise isotopic on the boundary of $\widehat{M}$. Furthermore, since $t(\xi) = t(\xi') = -1$ the maps above are in fact diffeomorphisms so that after an initial isotopy we may assume that p_ξ and $p_{\xi'}$ agree near $\partial\widehat{M}$. It follows that $p_\xi \circ p_{\xi'}^{-1}$ is a diffeomorphism of $\widehat{M}$ that extends to all of M so that ξ and ξ' are contactomorphic. □

Remark 5.7. If M admits an orientation preserving diffeomorphism that reverses the orientation on the fibers, then any oriented horizontal contact structure is contactomorphic to the contact structure given by reversing the orientation of the plane field. In this case the above proposition in fact holds for contactomorphism classes of *oriented* contact structures. Examples of such manifolds are given by Brieskorn spheres $\Sigma(p, q, r) \subset \mathbb{C}^3$, in which case the conjugation map on $\mathbb{C}^3$ yields the desired map.

6. Deformations of taut foliations on Seifert manifolds

In this section we consider the problem of determining which contact structures on Seifert manifolds are deformations of taut foliations. The obvious necessary condition for a contact structure to be a perturbation of a taut foliation is that it is universally tight. We will show that in most cases a universally tight contact structure ξ with negative twisting is a deformation of a taut foliation. By this we mean that there is a smooth family of plane fields ξ_t, so that ξ_0 is the tangent distribution of a taut foliation and ξ_t is a contact structure for $t > 0$ that is isotopic to ξ.

In fact by Proposition 5.5 it suffices to consider contact structures with twisting number -1, in which case it is fairly easy to construct the necessary foliations at least when the genus of the base is at least one. The genus zero case is more subtle as not every contact structure with negative twisting can be a perturbation of a taut foliation. We first note some preliminary lemmas.

Lemma 6.1. *Let $\xi = p^*\xi'$ be a horizontal contact structure on a Seifert fibered space which is the pullback of a horizontal contact structure ξ' under a fibered branched cover $M \xrightarrow{p} M'$. Assume that $t(\xi') = -1$ and that ξ' is isotopic to a deformation of a taut foliation through a deformation that is transverse to the branching locus of p. Then ξ is also a deformation of a taut foliation.*

Proof. Let α' be a defining form for ξ' and let α_t be a smooth family of non-vanishing 1-forms so that $\mathrm{Ker}(\alpha_0)$ is integrable and tangent to a taut foliation and α_t is contact for $t > 0$. After applying a further isotopy, we may also assume that $\mathrm{Ker}(\alpha_1) = \xi'$ and that the entire family is transverse to the branching locus L of p. Then $p^*\alpha_t$ is a deformation of a taut foliation that is contact away from L, where it is closed. We let β be any exact 1-form so that $\alpha_0 \wedge \beta|_L > 0$. Then $\widetilde{\alpha}_t = \alpha_t + \epsilon\beta$ provides the desired deformation for any ϵ that is sufficiently small. □

We shall also need a slightly more precise version of Theorem 5.3.

Lemma 6.2. *Let ξ be a horizontal contact structure on a Seifert manifold M and let F_0 be a regular fiber that is Legendrian and satisfies $tb(F_0) = t(\xi)$. Then ξ can be brought into normal form by an isotopy that fixes neighbourhoods of the exceptional fibers.*

We now come to the main result of this section.

Theorem 6.3. *Let ξ be a universally tight contact structure with negative twisting number $-n$ on a Seifert manifold and assume that the base orbifold has genus $g > 0$. Then ξ is a deformation of a taut foliation. Moreover, if $n > 1$ or $-b - r = 2g - 2$ then this foliation can be taken to be horizontal and the deformation linear.*

Proof. By Proposition 5.5 there is a fiberwise branched covering $M \xrightarrow{p} M'$ and a horizontal contact structure ξ' so that ξ is isotopic to $p^*\xi'$ and $t(\xi') = -1$. For convenience we assume that both ξ and ξ' are in normal form and that $p^*\xi' = \xi$. We let $(g, nb, n\frac{\beta_1}{\alpha_1}, , ..., n\frac{\beta_r}{\alpha_r})$ denote the unnormalised Seifert invariant of M'. By Theorem 5.1 we have that $-b - r \leq 2g - 2$. If $n > 1$ then according to ([23], Proposition 8.2), we also have

$$nb + \sum_{i=1}^{r} \left\lceil \frac{n\beta_i}{\alpha_i} \right\rceil = 2 - 2g \tag{1}$$

so that the normalised invariants $(g, b', \frac{\beta'_1}{\alpha'_1}, ..., \frac{\beta'_r}{\alpha'_r})$ of M' satisfy $-b' - r = 2g - 2$. In particular, if $n > 1$ and $g > 0$, then we must have that $b \leq 0$.

Case 1: We first assume in addition that $2 - 2g \leq -b$, in the case that $n = 1$. Proposition 3.2 then gives a horizontal $\mathrm{PSL}(2, \mathbb{R})$-foliation $\mathcal{F}$ on M' with hyperbolic holonomy around some embedded curve γ. We may then apply Theorem 4.1 part (2) to deform the foliation linearly to a horizontal contact structure ξ_{hor}. The characteristic foliation on the torus T_γ corresponding to γ is Morse-Smale and has two closed orbits each intersecting a fiber in a point. This is then stable under a suitably small linear deformation and by Giroux's Flexibility Theorem there is an isotopy with support in a neighbourhood of T_γ so that $tb(F_0) = -1$ for some regular fiber F_0. In particular, we deduce that $t(\xi_{hor}) \geq -1$. The opposite inequality holds for all horizontal contact structures by ([23], Proposition 4.5) and we conclude that $t(\xi_{hor}) = -1$. We may then isotope ξ_{hor} into normal form through an isotopy that is fixed near the exceptional fibers of M' by Lemma 6.2. Since all contact structures with twisting number -1 are contactomorphic by Proposition 5.6, we may assume that the normal form of ξ_{hor} agrees with that of ξ' after applying a suitable diffeomorphism. Note that this diffeomorphism can also be chosen with support disjoint from the singular locus. It follows that ξ' is isotopic to a deformation of taut foliation. Since this deformation was chosen to satisfy the hypotheses of Lemma 6.1 the result follows in this case.

Case 2: We next assume that $-b < 2-2g$. In this case the twisting number of ξ must be -1, since $b > 0$ so that equation (1) cannot have any solutions with $g > 0$. Moreover, there is only one such contact structure on M up to changing the orientations of the plane field by ([23], Theorem D). Thus it will suffice to show that some horizontal contact structure is a deformation of a taut foliation. To this end we let γ be a homologically essential simple closed curve in the base orbifold B, which exists by our assumption that $g > 0$. We cut M open along the torus T_γ which is the preimage of γ in M and take any horizontal foliation on the complement of T_γ whose holonomy is conjugate to a rotation on the two boundary components of $B \setminus \gamma$. We may assume that the rotation angles are distinct, unless $M = T^3$, in which case all tight contact structures are deformations of some product foliation.

We then spiral this foliation along the torus T_γ (cf. Section 2) to obtain a foliation $\mathcal{F}_\gamma$ with a unique torus leaf that is non-separating. If α_{-1}, α_1 denote closed forms defining the foliation on the boundary components of a tubular neighbourhood $T \times [-1,1]$ of T_γ, then $\mathcal{F}_\gamma$ is given as the kernel of the following 1-form:

$$\alpha_0 = \rho(-z)\alpha_{-1} + \rho(z)\alpha_1 + dz,$$

where ρ is a suitably chosen step function and z denotes the second coordinate in $T \times [-1,1]$. Since the only torus leaf of $\mathcal{F}$ is non-separating, it follows that $\mathcal{F}_\gamma$ is taut. Moreover, we may deform $\mathcal{F}_\gamma$ to a confoliation that is contact near T_γ. This is given by the following explicit deformation

$$\alpha_t = \begin{cases} \rho(|z| + t^2)\left(\cos(f(t^{-1}z))\alpha_{-1} + \sin(f(t^{-1}z))\alpha_1\right) + dz, & \text{if } t \neq 0 \\ \rho(-z)\alpha_{-1} + \rho(z)\alpha_1 + dz, & \text{if } t = 0 \end{cases}$$

for a non-decreasing function $f : \mathbb{R} \to [0, \frac{\pi}{2}]$ that is constant outside of $[-1,1]$, has positive derivative on $(-1,1)$ and satisfies

$$f(z) = \begin{cases} 0, & \text{if } z \leq -1 \\ \frac{\pi}{2}, & \text{if } z \geq 1. \end{cases}$$

Since $\rho(z)$ is infinitely tangent to the identity at the origin the family α_t is in fact smooth. After possibly changing orientations we may assume that the slope of α_{-1} is smaller than that of α_1. Note that since ρ vanishes to infinite order at 0 this deformation is smooth and by construction $\xi_t = \mathrm{Ker}(\alpha_t)$ is horizontal for $t \in (0,1]$. Moreover, the assumption that $-b < 2-2g$ means that there can be a horizontal foliation for one and only one orientation on M, so the change of orientation does not affect anything. The resulting confoliation is then transitive and can thus be C^∞-perturbed to a contact structure which is by construction horizontal. By ([6], Proposition 2.8.3), this perturbation can then be altered to a deformation. □

For the case that $g = 0$ we have the following:

Theorem 6.4. *Let ξ be a universally tight contact structure with negative twisting number $-n$ on a Seifert manifold M with normalised Seifert invariants $(g, b, \frac{\beta_1}{\alpha_1}, , ..., \frac{\beta_r}{\alpha_r})$ and $r \geq 3$. Then ξ is a linear deformation of a horizontal (and hence taut) foliation if one of the following holds:*

- *$g = 0$ and $2 - r \leq -nb - \sum \lfloor \frac{n\beta_i}{\alpha_i} \rfloor - r \leq -2$.*
- *ξ is isotopic to a vertical contact structure and the base orbifold is hyperbolic.*

Proof. We let $M \xrightarrow{p} M'$ be the n-fold branched cover given by Proposition 5.5. The assumption in the first case means that M' admits a $\mathrm{PSL}(2,\mathbb{R})$ foliation that has hyperbolic holonomy around some embedded curve by Proposition 3.2. The proof is then identical to case 1 of the proof of Theorem 6.3 above and the first claim follows.

In the second case the vertical contact structure gives a natural n-fold covering

$$M \xrightarrow{p_\xi} ST^*B,$$

where ST^*B is the unit cotangent bundle of the base orbifold of M, which is in turn a compact quotient of $\mathrm{PSL}(2,\mathbb{R})$. The cotangent bundle ST^*B carries a canonical contact structure ξ_{can} which descends from a left-invariant one on $\mathrm{PSL}(2,\mathbb{R})$ and $p_\xi^*\xi_{can} = \xi$. It is easy to see that this contact structure is a linear deformation of a taut foliation by considering the linking form on the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$ and thus the same holds on the quotient ST^*B (cf. [1], Example 3.1). The proposition then follows by pulling back under p_ξ. □

Note that there can be no general statement in the genus zero case. For as a consequence of Theorem 5.1, there are Seifert manifolds that admit horizontal contact structures, but no taut foliations. A particularly interesting case is that of small Seifert fibered spaces which are those having 3 exceptional fibers and whose base orbifold has genus 0. In this case any universally tight contact structure must have negative twisting number, which is equivalent to being isotopic to a horizontal contact structure. Furthermore, swapping the orientation of M has the effect of changing b to $-b+3$. Thus inspection of the criteria of Theorem 5.1 shows that M admits a horizontal contact structure in both orientations if and only if its invariants are realisable and hence this is equivalent to the existence of a horizontal foliation. For Seifert fibered spaces whose bases are of genus $g = 0$ the existence of a taut foliation is equivalent to that of a horizontal foliation. We summarise in the following proposition, which is proved by Lisca and Stipsicz, [22] using Heegard-Floer homology, rather than using Theorem 5.1 which can be proven by completely elementary methods (cf. [16]).

Proposition 6.5. *Let M be a Seifert fibered space over a base of genus $g = 0$. Then the following are equivalent:*

(1) *M admits a universally tight contact structure in both orientations with negative twisting number*
(2) *M admits a horizontal contact structure in both orientations*
(3) *M admits a horizontal foliation*
(4) *M admits a taut foliation.*

If M is small, the assumption on the twisting number can be removed in (1) and one can replace taut by Reebless in (4).

Note that it is not clear whether any given horizontal contact structure on a small Seifert fibered space is the deformation of a horizontal foliation in the case that both exist. However, in all likelihood this ought to be the case.

7. Deformations of Reebless foliations on Seifert manifolds

In this section we show that all universally tight contact structures ξ with $t(\xi) \geq 0$ are deformations of Reebless foliations. This follows from the existence of a normal form for such contact structures given in [24] which generalises Giroux's normal form for tight contact structures with non-negative maximal twisting on S^1-bundles.

In the following a *very small* Seifert fibered space is a Seifert fibered space that admits a Seifert fibering with at most 2 exceptional fibers. Note that a very small Seifert fibered space is either a Lens space (including S^3) or $S^1 \times S^2$. The Lens spaces do not admit Reebless foliations by Novikov's Theorem and the only Reebless foliation on $S^1 \times S^2$ is the product foliation, which cannot be perturbed to any contact structure. Thus it is natural to rule out such spaces when showing that certain contact structures are deformations of Reebless foliations. Furthermore, a folklore result of Eliashberg and Thurston, [6] states that a perturbation of a Reebless foliation is universally tight. Unfortunately, as pointed out by V. Colin, [4] the proof given *loc. cit.* contains a gap, and one only knows that there exists some perturbation that is universally tight. But in any case it is a reasonable assumption to make when considering which contact structures are deformations of Reebless foliations.

We first note the existence of normal forms for universally tight contact structures with non-negative twisting.

Theorem 7.1 ([24], Theorem 3)**.** *Let ξ be a universally tight contact structure on a Seifert manifold M that is not very small and is not a T^2-bundle with finite order monodromy. If $t(\xi) \geq 0$, then ξ is isotopic to a contact structure that is horizontal outside a finite collection of incompressible embedded vertical tori $T = \sqcup\, T_i$.*

Conversely, any two contact structures ξ_0, ξ_1 that are vertical on a fixed collection of vertical tori T and horizontal elsewhere are isotopic as unoriented contact structures.

With the aid of the normal form described above it is now a simple matter to show the following.

Theorem 7.2. *Let ξ be a universally tight contact structure on a Seifert fibered space M with $t(\xi) \geq 0$ and assume that M is not very small. Then ξ is a deformation of a Reebless foliation.*

Proof. First assume that M is not a torus bundle with finite order monodromy. Then by Theorem 7.1 we may assume after a suitable isotopy that ξ is horizontal away from a collection of tori $T = \sqcup T_i$ where ξ is vertical. Now let $\mathcal{F}$ be any foliation which has the incompressible tori T_i as closed leaves and is horizontal otherwise. We also require that the sign of the intersection of any fiber with $\mathcal{F}$ agrees with that of ξ on $M \setminus T$. Such foliations can easily be constructed by taking any horizontal foliation on the components of $M \setminus T$ that has the correct co-orientation and then spiralling into the torus leaves. Note that all torus leaves are incompressible so that $\mathcal{F}$ is Reebless.

We first deform $\mathcal{F}$ near the torus leaves as in the proof of Theorem 6.3 to obtain a transitive confoliation ξ' which is contact near the closed leaves and has vertical tori precisely corresponding to the T_i. The confoliation ξ' can then be deformed to a contact structure which is horizontal on $M \setminus T$. By Theorem 7.1 this contact structure, suitably oriented, is then isotopic to ξ. Finally the two step deformation of $\mathcal{F}$ can be achieved via a single deformation in view of ([6], Proposition 2.8.3).

If M is a torus bundle with finite order holonomy, then the universally tight contact structures are classified (see [17]) and it is easy to see that they are all deformations of some T^2-fibration. □

8. Topology of the space of taut and horizontal foliations

The topology of the space of representations $\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{PSL}(2,\mathbb{R}))$ for a closed surface group of genus $g \geq 2$ has been well studied and its connected components were determined by Goldman, [14]. Recall that for any topological group the representation space of a surface group is

$$\{(a_1, b_1, ..., a_g, b_g) \in G^{2g} \mid \prod_{i=1}^{g} [a_i, b_i] = 1\}.$$

In the case of $\mathrm{PSL}(2,\mathbb{R})$ the connected components of the representation space are given by preimages under the map given by the Euler class

$$\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{PSL}(2,\mathbb{R})) \xrightarrow{e} [2-2g, 2g-2].$$

Moreover, the quotient of the connected component with maximal Euler class under the natural conjugation action is homeomorphic to Teichmüller space and is hence contractible. On the other hand the topology of the representation space $\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$ endowed with the natural C^∞-topology, which can be interpreted as the space of foliated S^1-bundles after quotienting out by conjugation, is not as well understood. It would perhaps be natural to conjecture that map induced by the inclusion

$$G = \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{Diff}_+(S^1)$$

induces a weak homotopy equivalence on representation spaces or at least a bijection on path components. It is known that both representation spaces are path connected in the case of the maximal component (cf. [11], [26]). Indeed, results of Matsumoto and Ghys show that any maximal representation is smoothly conjugate to one that is Fuchsian.

On the other hand, we will show that this is not the case for the space of representations with non-maximal Euler class. The basic observation is that the cyclic d-fold cover G_d of $G = \mathrm{PSL}(2,\mathbb{R})$ also acts smoothly on the circle via $\mathbb{Z}_d$-equivariant diffeomorphisms so that there is a natural map

$$\mathrm{Rep}(\pi_1(\Sigma_g), G_d) \longrightarrow \mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1)).$$

In general the images of these maps lie in different path components for different values of d and fixed Euler class. More precisely, we have the following.

Theorem 8.1. *Let $\#Comp(e)$ denote the number of path components of $Rep(\pi_1(\Sigma_g), Diff_+(S^1))$ with fixed Euler class $e \neq 0$ such that e divides $2g-2 \neq 0$ and write $2g-2 = n\,e$. Then the following holds:*

$$\#Comp(e) \geq \sum_{d \mid n} d^{2g}.$$

Proof. By ([13], Theorem 3.1) for each divisor d of $n = |\frac{2g-2}{\chi}|$ there is a contact structure ξ_d with twisting number $-d$ on the S^1-bundle E with Euler class e. If $d > 1$, then any such contact structure is vertical and is thus a linear deformation of a horizontal foliation $\mathcal{F}_d$ by Proposition 6.4. In the case $d = 1$ the contact structure ξ_1 is also a linear deformation of a horizontal foliation by the proof of Theorem 6.3. In either case, we let ρ_d be the associated representation in $\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$. Note that the image of ρ_d can be assumed to lie in G_d. Assume that ρ_t is a smooth family of representations joining ρ_d to $\rho_{d'}$ for $d \neq d'$. We let $\mathcal{F}_t$ denote the smooth family of associated suspension foliations joining $\mathcal{F}_d$ and $\mathcal{F}_{d'}$.

Then since each foliation in the family $\mathcal{F}_t$ cannot have any closed leaves and E does not fiber over S^1, we may perturb the family linearly to a 1-parameter family of contact structures by Proposition 4.3. It then follows from Proposition 4.4 that ξ_d is isotopic to $\xi_{d'}$, which is a contradiction. Thus both ρ_d and $\rho_{d'}$ lie in distinct components of $\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$.

For a fixed d the vertical contact structure ξ_d determines a fiberwise d-fold cover of the unit cotangent bundle $ST^*\Sigma_g$. By ([13], Lemme 3.9) the isotopy class of the associated d-fold covering is a deformation invariant of ξ_d and hence of ρ_d. Isotopy classes of fiberwise d-fold coverings are in one to one correspondence with elements in $H^1(\Sigma_g, \mathbb{Z}_d)$ and it follows that the numbers of path components of representations whose perturbations have twisting number d is at least d^{2g}. From this we conclude that

$$\#Comp(e) \geq \sum_{d\,|\,n} d^{2g}. \quad \square$$

Remark 8.2. For the sake of concreteness let us consider the representation ρ_{2d} given by a $(2d)$-fold fiberwise cover of the suspension of a Fuchsian representation determined by the inclusion of a discrete cocompact lattice in $\mathrm{PSL}(2, \mathbb{R})$ and ρ_{st} the stabilisation of a Fuchsian representation of a surface of Euler characteristic $\frac{1}{2d}(2-2g)$. These representations have the same Euler class but lie in different components of $\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$, which answers a question raised by Y. Mitsumatsu and E. Vogt in studying certain turbulisation constructions for 2-dimensional foliations on 4-manifolds.

Larchancé, [19] also considered the problem of deforming taut foliations through certain restricted classes of foliations. She noted that on T^2-bundles over S^1 with Anosov monodromy of a certain kind, the stable and unstable foliations $\mathcal{F}_s, \mathcal{F}_u$ cannot be deformed to one another through foliations without torus leaves. This uses Ghys and Sergiescu's classification results, [12] for foliations without closed leaves on such manifolds. However, $\mathcal{F}_s$ and $\mathcal{F}_u$ can be deformed to one another through taut foliations: one first spirals both foliations along a fixed torus fiber to obtain foliations $\mathcal{F}'_s, \mathcal{F}'_u$ with precisely one closed torus leaf T. On the complement of T one has a foliation by cylinders on $T^2 \times (0,1)$ intersecting each fiber in a linear foliation. It is then easy to construct a deformation between $\mathcal{F}'_s$ and $\mathcal{F}'_u$ through foliations with one homologically non-trivial torus leaf. Thus we conclude that one can deform $\mathcal{F}_s$ to $\mathcal{F}_u$ through taut foliations.

In view of this, it remains to find taut foliations that cannot be deformed to one another through taut foliations, although their tangent distributions are homotopic. We give two types of examples of this phenomenon: the first uses deformations and contact topology and the other uses the special structure of taut foliations on cotangent bundles.

Theorem 8.3. *The space of taut foliations is in general not path connected on small Seifert fibered spaces.*

Proof. We let $M = -\Sigma(2,3,6k-1)$ be the link of the complex singularity $z_1^2 + z_2^3 + z_3^{6n+5} = 0$ taken with the opposite orientation, which has Seifert invariants $(0, -2, \frac{1}{2}, \frac{2}{3}, \frac{5k-1}{6k-1})$. As noted on ([23], p. 1746) the Seifert manifold M admits a vertical contact structure ξ_{vert} that has twisting number $-(6k-7)$ if $k > 1$, which is then a linear deformation of a taut foliation $\mathcal{F}$ by Proposition 6.4. One further checks that the following holds

$$-2n + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{2n}{3} \right\rceil + \left\lceil \frac{n(5k-1)}{6k-1} \right\rceil = 2 \tag{2}$$

if and only if $n = 6l - 1$ and $1 \leq l \leq k-1$. By ([23], Proposition 8.2) this is a necessary condition for the existence of a horizontal contact structure on M with twisting number $-n$. Moreover, the quotient space of the $(6l-1)$-fold cover $M \xrightarrow{p} M'_l$ given by Proposition 3.1 has normalised invariants $(0, -1, \frac{1}{2}, \frac{1}{3}, \frac{k-l}{6k-1})$ and thus admits a horizontal foliation $\mathcal{F}_l$ by Theorem 3.3. Since $\mathcal{F}_l$ cannot have any closed leaves and M does not fiber over S^1, the foliation $\mathcal{F}_l$ can be linearly deformed to a horizontal contact structure ξ_l. Now the corresponding necessary condition for the existence of a horizontal contact structure on M'_l with twisting number $t(\xi_l)$ is obtained by substituting $n = -(6l-1)t(\xi_l)$ into equation (2) and it follows that

$$-(6l-1)t(\xi_l) = 6l' - 1, \text{ for some } l \leq l' \leq k-1.$$

Note that the (negative) twisting number of a contact structure is sub-multiplicative under covering maps. Thus, if $6l - 1$ is coprime to $6k - 7$, then we deduce that

$$-t(p^*\xi_l) \leq -(6l-1)t(\xi_l) < 6k - 7$$

so that $\xi' = p^*\xi_l$ cannot be isotopic to ξ_{vert}. Note that $6l - 1$ will be coprime to $6k - 7$ for all values of l such that $l > \frac{1}{6}(\sqrt{6k-7} - 1)$ with at most one exception.

Since M is non-Haken all taut foliations are without closed leaves. Thus any path of taut foliations joining $\mathcal{F}$ to $\mathcal{F}' = p^*\mathcal{F}_l$ can be deformed to an isotopy between ξ_{vert} and ξ' by Propositions 4.3 and 4.4, which yields a contradiction if $6l - 1$ is coprime to $6k - 7$. □

Remark 8.4. A better understanding of the classification of Ghiggini and Van-Horn-Morris should yield that the space of taut foliations on $-\Sigma(2, 3, 6k-1)$ has at least $k-1$ components.

Furthermore, Vogel, [31] has shown that the isotopy class of a contact structure approximating a foliation without torus leaves is a deformation invariant for such foliations, provided the manifold is not a T^2-bundle. This give alternative proofs of Theorems 8.1 and 8.3. As his results only assume C^0-closeness they yield that the conclusions about path components also hold with respect to the weaker C^0-topology. All results also remain true for foliations that are only of class C^2 as this suffices for Theorem 4.1 and its various consequences.

Since the foliations in Theorem 8.3 are by construction horizontal, their tangent distributions are homotopic as oriented plane fields. Thus by [19], they are homotopic as foliations (cf. also [9]). The construction of such a homotopy of integrable plane fields introduces many Reeb components, so it is natural to ask whether this is necessary. Since the manifolds $\Sigma(2, 3, 6k-1)$ used in Theorem 8.3 are non-Haken the notions of tautness and Reeblessness coincide and we deduce the following.

Corollary 8.5. *There exist taut foliations $\mathcal{F}_1, \mathcal{F}_2$ that are homotopic as foliations but not as taut foliations. Furthermore, any homotopy through foliations must contain at least one Reeb component.*

Note that since the twisting number of a contact structure is a contactomorphism invariant, the proof of Theorem 8.3 shows that the space of taut foliations is also disconnected even if one only considers *diffeomorphism classes* of foliations. If instead one only considers deformation classes of taut foliations, then it is possible to give examples where the numbers of components is infinite. This uses not only the special structure of foliations on the unit cotangent bundle but also the structure of a foliation near torus leaves.

8.1. **Torus leaves and Kopell's Lemma:** The fundamental result concerning the way a foliation behaves near a torus leaf is the following lemma of Kopell.

Lemma 8.6 (Kopell)**.** *Let f, g be commuting C^2-diffeomorphisms mapping $[0, 1)$ into itself (not necessarily surjectively) and assume that f is contracting. Then either g has no fixed point in $(0, 1)$ or $g = Id$.*

Furthermore, torus leaves occur in a finite number of stacks in the following sense (cf. [9], [30]).

Lemma 8.7. *Let $\mathcal{F}$ be a 2-dimensional foliation on a 3-manifold M. Then there is a finite collection of disjoint embeddings $N_i = T^2 \times [0, c_i]$ in M with $c_i \geq 0$ so that $T_i \times \{0\}$ and $T_i \times \{c_i\}$ are leaves and such that $M \setminus \cup N_i$ contains no torus leaves. Furthermore, these neighbourhoods may be chosen such that $\mathcal{F}$ is transverse to foliation given by the intervals $\{pt\} \times [0, c_i]$.*

Now for any stack N_i as in Lemma 8.7, one has an induced holonomy homomorphism defined on a slightly larger neighbourhood $T^2 \times [-\epsilon, c_i + \epsilon]$ of N_i. This then induces germinal holonomy maps near each of the boundary components $T_i \times \{0\}$ and $T_i \times \{c_i\}$ respectively. We let f, g be representatives of the commuting holonomy germs around generators $\alpha, \beta \in \pi_1(T^2)$ both having domain $[c_i, c_i + \epsilon]$. After possibly replacing f with its inverse, we may assume that the germ of f at c_i is non-trivial and that $f(c_i + \epsilon) < c_i + \epsilon$ by the assumption that $M \setminus \cup N_i$ contains no torus leaves. Now suppose g has a fixed point y in the interval $(c_i, c_i + \epsilon]$. Again y cannot be a fixed point of any f^n as there are no torus leaves in the complement of the N_i. Since f and g commute the sequence $y_n = f^n(y)$ consists of fixed points of g. Moreover, y_n is bounded and monotone increasing or decreasing, depending on whether $f(y)$ is greater than or less than y. Thus the sequence y_n has a limit which is a common fixed point of both f and g. This common fixed point must be c_i, since $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \{c_i\}$ on $[c_i, c_i + \epsilon]$. It follows that f has no fixed points on (c_i, y) and is thus a contraction on $[c_i, y)$. Kopell's Lemma then implies that g is either trivial or has no fixed points on (c_i, y).

From the analysis of the previous paragraph we see that the foliation $\mathcal{F}$ is transverse to each torus $T_{\epsilon'} = T^2 \times \{c_i + \epsilon'\}$ for all sufficiently small $\epsilon' > 0$. Moreover, the induced foliation $\mathcal{F}|_{T_{\epsilon'}}$ on $T_{\epsilon'}$ is transverse to the foliation on T^2 given by circles parallel to the generator $\alpha \in \pi_1(T^2)$ and inherits a natural orientation. Thus with respect to this S^1-foliation $\mathcal{F}$ induces a return map $\phi \in \mathrm{Diff}_+(S^1)$. This return map has an associated rotation number

$$\mathrm{rot}(\phi) = \lim_{n \to \infty} \frac{\phi^n(x)}{n} \in S^1.$$

Note that for all ϵ' sufficiently small the induced foliations on $T_{\epsilon'}$ are conjugate. Thus the rotation number being conjugation invariant is well-defined for all ϵ' sufficiently small. We let $\lambda_+ = \mathrm{rot}(\phi)$ be the asymptotic slope of the foliation on the top of N_i, i.e. near $T^2 \times \{c_i\}$. There is also an asymptotic slope λ_- on the bottom of N_i. Note that fixed points of holonomy maps correspond to closed orbits of $\mathcal{F}|_{T_{\epsilon'}}$. Note further that if the asymptotic slope is rational, then ϕ must have a periodic point and it follows from Kopell's Lemma that ϕ has finite order. Thus in the case of rational asymptotic slope the foliation on $T_{\epsilon'}$ is conjugate to a fibration by circles of slope $\mathrm{rot}(\phi)$.

Now if the slopes λ_- and λ_+ do not coincide then the stack of leaves is *stable* in the sense that any foliation in a C^0-neighbourhood of $\mathcal{F}$ has a closed torus leaf in a neighbourhood of N_i. For if the asymptotic slopes are different, then there are generators γ, δ in $\pi_1(T^2)$

so that the holonomy maps $h = \mathrm{Hol}(\gamma)$ and $g = \mathrm{Hol}(\delta)$ respectively are defined on the interval $I = (-\epsilon, c_i + \epsilon)$ and are contracting near 0 and c_i. Since h is contracting near the end points of $[0, c_i]$, it follows that $h(x) > x$ and $h(y) < y$ for some $x, y \in I$ with $x < y$ and consequently h has a C^0-stable fixed point in the interval I by the Intermediate Value Theorem. The sequence $z_n = g^n(y)$ is then monotone and bounded and its limit is a common fixed point of g and h which corresponds to a torus leaf contained in some neighbourhood of N_i. If a stack of tori has arbitrarily small perturbations that are without closed leaves then the stack is called *unstable*. We summarise this discussion in the following lemma.

Lemma 8.8. *Let $N_i = T^2 \times [0, c_i]$ be an unstable stack of torus leaves of a 2-dimensional foliation and assume that the asymptotic slope at one end is rational. Then on a slightly enlarged neighbourhood $T^2 \times (-\epsilon, c_i+\epsilon)$ the induced foliations on $T^2 \times \{-\epsilon'\}$ and $T^2 \times \{c_i+\epsilon'\}$ are conjugate fibrations by circles of the same slope for any $0 < \epsilon' < \epsilon$.*

The final ingredient is Thurston's straightening procedure for foliations on S^1-bundles (see also [2]).

Theorem 8.9 (Thurston, [30])**.** *Let $\mathcal{F}$ be a foliation on an S^1-bundle without closed leaves. Then $\mathcal{F}$ is isotopic to a horizontal foliation. Furthermore, if $\mathcal{F}$ is already horizontal on a vertical torus T, then this isotopic can be made relative to T.*

We are now ready to prove the following theorem.

Theorem 8.10. *The space of taut foliations on $ST^*\Sigma_g$ has at least $\mathbb{Z}^{2g}$ components if $g \geq 2$, all of which are homotopic as foliations.*

Proof. By [13], [17] all horizontal contact structures on $ST^*\Sigma_g$ are contactomorphic and can be made vertical. Furthermore, their isotopy classes are parametrised by $H^1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, where $H^1(\Sigma_g, \mathbb{Z}) = [\Sigma_g, S^1]$ acts via gauge transformations. Choose ξ_{vert}, ξ'_{vert} non-isotopic vertical contact structures. Such contact structures are deformations of foliations $\mathcal{F}, \mathcal{F}'$ by Theorem 6.3. In fact, identifying a vertical contact structure with the canonical contact structure on $ST^*\Sigma_g$ it is easy to see that they are linear deformations of foliations that are descended from left invariant foliations on $\mathrm{PSL}(2, \mathbb{R})$ (cf. [1]).

Now suppose $\mathcal{F}_t$ is a deformation of taut foliations joining $\mathcal{F}$ and $\mathcal{F}'$. Then since both foliations are without closed leaves there is a smallest t_0 such that $\mathcal{F}_{t_0}$ has closed leaves. Otherwise we could linearly perturb the deformation by Proposition 4.3 to obtain a contradiction. Note that all the closed leaves of $\mathcal{F}_{t_0}$ are unstable incompressible tori. There is then a finite collection of embeddings $N_i = T_i \times [0, c_i]$ so that the foliation contains no closed leaves outside the union of the N_i and both $T_i \times \{0\}$ and $T_i \times \{c_i\}$ are closed leaves by Lemma 8.7. After an isotopy we may assume that the T_i are vertical tori and we let γ_i denote their image curves in Σ_g.

We claim that the asymptotic slopes of all tori $T_{c_i+\epsilon'}$ with $0 < \epsilon' < \epsilon$ near top end of N_i are rational. For note that the foliation $\mathcal{F}_t$ has no closed leaves and remains transverse to the torus $T_{c_i+\epsilon'}$ for t close to t_0 and ϵ' fixed. Thus applying Thurston's straightening procedure we may isotope $\mathcal{F}_t$ to be made horizontal whilst keeping $T_{c_i+\epsilon'}$ fixed. The induced foliation $\mathcal{F}_{t_0}|_{T_{c_i+\epsilon'}}$ is just the suspension of the holonomy diffeomorphism about $\gamma_i \in \pi_1(\Sigma_g)$. But by [26] the holonomies of a horizontal foliation on $ST^*\Sigma_g$ are always conjugate to hyperbolic elements in $\mathrm{PSL}(2, \mathbb{R})$ and thus must have fixed points. It follows that the asymptotic slope of $\mathcal{F}_t|_{T_{c_i+\epsilon'}}$ is rational for all $t < t_0$ and by continuity this also holds for t_0. Thus since N_i is unstable, Lemma 8.8 implies that $T_{c_i+\epsilon'}$ and $T_{\epsilon'}$ must be foliated by circles of the same slope.

We then cut open the manifold and reglue along $T_{c_i+\epsilon'}$ and $T_{-\epsilon'}$ to obtain a smooth foliation $\mathcal{F}''$ without closed leaves whose restriction to some vertical tori is a foliation by circles. After again applying Thurston's straightening procedure to $\mathcal{F}''$ we obtain a horizontal foliation, whose holonomy around γ_i is conjugate to a rational rotation. After taking the pullback of a suitable covering of the base this holonomy can be assumed to be trivial. Since the Euler class is multiplicative under coverings, the associated pullback foliation is also a horizontal foliation on an S^1-bundle with maximal Euler class.

Furthermore, since the holonomy around γ_i is trivial, we can cut along γ_i and glue in discs to obtain a representation of a surface group that contradicts the Milnor-Wood inequality. Thus we conclude that no foliation in the family can have closed leaves. It follows that the family cannot exist and that $\mathcal{F}$ and $\mathcal{F}'$ cannot be deformed to one another through taut foliations. Since there are $\mathbb{Z}^{2g}$ different isotopy classes of contact structures there are at least this many deformation classes of taut foliations. Finally since all foliations are horizontal their tangent distributions are homotopic as plane fields and thus by [9], [19] they are homotopic as integrable plane fields. □

In fact, the proof of Theorem 8.10 shows that if a family of taut foliations $\mathcal{F}_t$ on $ST^*\Sigma_g$ contains a foliation which does not have closed leaves, then the same is true for the entire family. This observation also applies to families of Reebless foliations. Furthermore, since a foliation on $ST^*\Sigma_g$ without closed leaves is isotopic the suspension foliation given by a Fuchsian representation in view of [11] we deduce the following corollary.

Corollary 8.11. *Let $\mathcal{F}_{hor}$ be a horizontal foliation on the unit cotangent bundle of a hyperbolic surface $ST^*\Sigma_g$. Then any foliation in the path component of $\mathcal{F}_{hor}$ in the space of Reebless foliations is isotopic to the foliation given by the suspension of a Fuchsian representation.*

It is easy to construct taut foliations $\mathcal{F}_T$ with a single vertical torus leaf on any S^1-bundle as long as the base has positive genus and we may assume that the tangent distribution of such a foliation is homotopic to a horizontal distribution. Moreover, in view of Corollary 8.11 there can be no Reebless deformation between $\mathcal{F}_T$ and any horizontal foliation, even if one allows diffeomorphisms of either foliation.

We conclude that any deformation of foliations joining $\mathcal{F}_T$ to a horizontal foliation must in fact have Reeb components. The same applies to any diffeomorphic horizontal foliations whose contact perturbations are not isotopic. That is to say there are foliations on the toroidal manifold $ST^*\Sigma_g$, such that any deformation between them must have Reeb components.

Note, however, that the foliations $\mathcal{F}_T$ and $\mathcal{F}_{hor}$ can in fact be deformed to one another through taut foliations that are only of class C^0. This follows by first spiralling the horizontal foliation $\mathcal{F}_{hor}$ along the vertical torus T, which can be done in a C^0 manner. The remainder of the foliation is determined by a representation of a free group to $\widetilde{\mathrm{Diff}}_+(S^1)$. Joining any two such representations arbitrarily and spiralling into T then gives the desired deformation.

9. Anosov foliations

In this section we give an alternative approach to the results obtained above that uses the classification of Anosov foliations of Ghys, [11]. We will call a representation Anosov if its associated suspension foliation is diffeomorphic to the weak unstable foliation of an

Anosov flow. Recall that a flow Φ^t_X generated by a vector field X on a closed 3-manifold M is Anosov if the tangent bundle splits as a sum of line bundles

$$TM = E^u \oplus E^s \oplus X$$

such that for some choice of metric and $C, \lambda > 0$

$$||(\Phi^t_X)_*(v_u)|| \geq C^{-1}e^{\lambda t}||v_u|| \text{ and } ||(\Phi^t_X)_*(v_s)|| \leq Ce^{-\lambda t}||v_s||,$$

where $v_u \in E^u, v_s \in E^s$. The line fields E^u, E^s are called the strong stable resp. unstable foliations of the flow and the foliations $\mathcal{F}_u, \mathcal{F}_s$ tangent to the integrable plane fields

$$E^u \oplus X \ , \ E^s \oplus X$$

are called the weak unstable resp. stable foliations of the flow. An important property of Anosov flows and foliations is their structurally stability.

Lemma 9.1 (C^1-stability of Anosov foliations)**.** *Let $\mathcal{F}$ be an Anosov foliation. Then any foliation in a C^1-small neighbourhood of $\mathcal{F}$ is also Anosov.*

Proof. Suppose $\mathcal{F} = \mathcal{F}_u$ is diffeomorphic to the weak unstable foliation of an Anosov flow X, which is normalised to have unit length with respect to the chosen metric and let $\mathcal{F}_s$ be the corresponding weak stable foliation. We rewrite the Anosov condition in a more convenient form:

$$\det(\Phi^t_X)_*|_{\mathcal{F}_u} \geq C^{-1}e^{\lambda t} \text{ and } \det(\Phi^t_X)_*|_{\mathcal{F}_s} \leq Ce^{-\lambda t}, \tag{3}$$

where the determinant is measured with respect to an orthonormal frame on the weak unstable and stable foliations. We let T be such that

$$\det(\Phi^T_X)_*|_{\mathcal{F}_u} > 2 \text{ and } \det(\Phi^T_X)_*|_{\mathcal{F}_s} < \frac{1}{2}. \tag{4}$$

Note that condition (4) is C^1-stable in X and we claim that it is equivalent to the conditions in (3). To see this note that after rescaling the flow and the metric we may assume that $T = 1$. We then write $t = n + r$ with $0 \leq r < 1$ so that

$$\det(\Phi^t_X)_*|_{\mathcal{F}_u} = \det(\Phi^{n+r}_X)_*|_{\mathcal{F}_u} > 2^n \det(\Phi^r_X)_*|_{\mathcal{F}_u} \geq 2^{n+r} \inf_{0\leq s\leq 1} \frac{1}{2} \det(\Phi^s_X)_*|_{\mathcal{F}_u}.$$

So the first inequality in (3) holds for a suitable choice of constant C and $\lambda = \ln 2$. Similarly the second condition also holds. Thus the Anosov condition on a flow is C^1-stable.

Now suppose $\mathcal{F}'$ is sufficiently C^1-close to $\mathcal{F}$ so that it is transversal to $\mathcal{F}_s$ and set $X' = \mathcal{F}' \cap \mathcal{F}_s$, again normalised to have unit length. Note that $\mathcal{F}'_u = \mathcal{F}'$ and $\mathcal{F}_s$ are invariant under the flow given by X', which is Anosov because of C^1-stability, and the lemma follows. □

We will need the following version of a result of Matsumoto, [25].

Lemma 9.2. *Let $\rho_1, \rho_2 \in Rep(\Gamma, Homeo_+(S^1))$ for an arbitrary finitely generated group Γ and assume that there are lifts $\overline{\rho}_1, \overline{\rho}_2$ to $\widetilde{Homeo}_+(S^1)$ such that the translation numbers satisfy*

$$tr(\overline{\rho}_1(g_1)\overline{\rho}_1(g_2)) - tr(\overline{\rho}_1(g_1)) - tr(\overline{\rho}_1(g_2)) = tr(\overline{\rho}_1(g_1)\overline{\rho}_1(g_2)) - tr(\overline{\rho}_1(g_1)) - tr(\overline{\rho}_1(g_2))$$

for all $g_1, g_2 \in \Gamma$. Assume further that the rotation numbers of $\rho_1(g_i)$ and $\rho_2(g_i)$ agree for generators g_i of Γ. Then $\overline{\rho}_1$ and $\overline{\rho}_2$ have the same bounded integral Euler class and are thus conjugate provided that the actions they induce are minimal.

With the aid of this lemma we obtain the following theorem, which gives an alternate proof of Theorem 8.1 and also answers a question posed to us by Y. Mitsumatsu.

Theorem 9.3. *The space of Anosov representations $R_{An} \subset Rep(\pi_1(\Sigma_g), Diff_+(S^1))$ is both open and closed with respect to the C^∞-topology. It has finitely many connected components which are distinguished by the rotation numbers of the images of a set of standard generators $a_i, b_i \in \pi_1(\Sigma_g)$.*

Proof. We first note that since the Anosov condition is C^1-stable, the set R_{An} is open in $\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$. Now let ρ_n be a sequence of Anosov representations converging to ρ. Since each ρ_n is Anosov it is conjugate to an embedding of a discrete subgroup in some k-fold cover G_k of $G = \mathrm{PSL}(2,\mathbb{R})$ by [11] where k is fixed. The number of components of $\mathrm{Rep}_{max}(\pi_1(\Sigma_g), G_k)$ is finite by [14] and are distinguished by elements in $H^1(\pi_1(\Sigma_g), \mathbb{Z}_k)$. Furthermore, all representations in a fixed component with maximal Euler class project to Fuchsian representations in $\mathrm{Rep}_{max}(\pi_1(\Sigma_g), G)$. By [26] all Fuchsian representations are topologically conjugate and thus the same holds for each component of $\mathrm{Rep}_{max}(\pi_1(\Sigma_g), G_k)$. It follows that the sequence ρ_n contains only finitely many topological conjugacy classes, so after choosing a subsequence we may assume that

$$\rho_n = \phi_n \, \rho_{An} \, \phi_n^{-1}$$

for a fixed Anosov representation ρ_{An} and some $\phi_n \in \mathrm{Homeo}_+(S^1)$. We choose lifts $\overline{\phi}_n$ and $\overline{\rho}_{An}$ to $\widetilde{\mathrm{Homeo}}_+(S^1)$ and set

$$\overline{\rho}_n = \overline{\phi}_n \, \overline{\rho}_{An} \, \overline{\phi}_n^{-1}.$$

We then set

$$\overline{\rho}(g) = \lim_{n\to\infty} \overline{\rho}_n(g).$$

This limit exists since by assumption $\rho_n(g)$ converges to $\rho(g)$ and the translation number of $\overline{\rho}_n(g)$ being conjugation invariant is constant. Since the rotation number is also conjugacy invariant we see that the hypotheses of Lemma 9.2 are satisfied for ρ and ρ_{An}. Furthermore, since ρ_{An} is Anosov it is automatically minimal.

The action on S^1 induced by ρ has either a periodic orbit, an exceptional minimal set or it is minimal. Note that the first case is ruled out since the Euler class is non-zero. If the action had an exceptional minimal set $K \subset S^1$, then the semi-proper leaves of the associated suspension foliation must have infinitely many ends by Duminy's Theorem (cf. [3]). Collapsing the complement of K would give a minimal (C^0)-action ρ_{min} whose associated suspension foliation has a leaf with infinitely many ends. Furthermore, ρ_{min} has the same bounded integral Euler class as ρ and ρ_{An}. It follows that ρ_{min} is topologically conjugate to ρ_{An}, which is Anosov and thus a covering of a Fuchsian representation. By [10] the suspension foliation of a Fuchsian group has only leaves with 1 or 2 ends and thus the same holds for ρ_{An}, giving a contradiction. Thus ρ is minimal and consequently by Lemma 9.2 it is topologically conjugate to ρ_{An}. It is then in fact smoothly conjugate to an Anosov representation by [11].

Finally the components of $\mathrm{Rep}_{max}(\pi_1(\Sigma_g), G_k)$ can be distinguished by the rotation numbers on generators a_i, b_i, since these correspond precisely to the elements $H^1(\pi_1(\Sigma_g), \mathbb{Z}_k)$ that distinguish components. Since the rotation number of an Anosov representation lies in the k-th roots of unity $\mathbb{Z}_k \subset S^1$ these rotation numbers are constant on components and this concludes the proof. □

As a consequence of Theorem 9.3 we obtain the following extension of Ghys and Matsumoto's global stability statement about conjugacy classes of representations in $\mathrm{Rep}(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$ for the case of maximal Euler class, [11], [26] to other topological components.

Corollary 9.4. *Any representation $\phi \in Rep(\pi_1(\Sigma_g), Diff_+(S^1))$ that lies in the path component of an Anosov representation ϕ_{An} is smoothly conjugate to an embedding of a discrete subgroup in some finite cover of $PSL(2,\mathbb{R})$ and is topologically conjugate to ϕ_{An}. In particular, it is injective.*

Remark 9.5. Theorem 9.3 and its corollary also remain true for representations of any hyperbolic orbifold group $\pi_1^{orb}(B_{hyp})$.

In the case of maximal Euler class the property of being an Anosov representation is in fact C^0-stable. This is just a consequence of the fact that any representation with maximal Euler class is conjugate to an Anosov one. It would be interesting to know whether such C^0-stability holds in for Anosov representations that are not maximal. Of course in this case it is no longer true that every horizontal foliation is diffeomorphic to one that is Anosov by Theorem 8.1. More generally, one might ask the following:

Question 9.6. *Is the property of being the weak unstable foliation of a smooth Anosov flow C^0-stable?*

This question cannot be answered by considering contact pertubations, since Vogel's results, [31] on the uniqueness of contact structures approximating foliations imply that the isotopy class of a contact structure C^0-approximating an Anosov foliation is unique. So the obvious obstruction coming from contact topology is not present.

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