A NON-COMMUTATIVE GEOMETRY APPROACH TO THE REPRESENTATION THEORY OF REDUCTIVE p-ADIC GROUPS: HOMOLOGY OF HECKE ALGEBRAS, A SURVEY AND SOME NEW RESULTS

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ABSTRACT. We survey some of the known results on the relation between the homology of the *full* Hecke algebra of a reductive p-adic group G, and the representation theory of G. Let us denote by $\mathcal{C}_c^\infty(G)$ the full Hecke algebra of G and by $\mathrm{HP}_*(\mathcal{C}_c^\infty(G))$ its periodic cyclic homology groups. Let $\hat{G}$ denote the admissible dual of G. One of the main points of this paper is that the groups $\mathrm{HP}_*(\mathcal{C}_c^\infty(G))$ are, on the one hand, directly related to the topology of $\hat{G}$ and, on the other hand, the groups $\mathrm{HP}_*(\mathcal{C}_c^\infty(G))$ are explicitly computable in terms of G (essentially, in terms of the conjugacy classes of G and the cohomology of their stabilizers). The relation between $\mathrm{HP}_*(\mathcal{C}_c^\infty(G))$ and the topology of $\hat{G}$ is established as part of a more general principle relating $\mathrm{HP}_*(A)$ to the topology of $\mathrm{Prim}(A)$, the primitive ideal spectrum of A, for any finite typee algebra A. We provide several new examples illustrating in detail this principle. We also prove in this paper a few new results, mostly in order to better explain and tie together the results that are presented here. For example, we compute the Hochschild homology of $\mathcal{O}(X) \rtimes \Gamma$, the crossed product of the ring of regular functions on a smooth, complex algebraic variety X by a finite group Γ. We also outline a very tentative program to use these results to construct and classify the cuspidal representations of G. At the end of the paper, we also recall the definitions of Hochschild and cyclic homology.

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INTRODUCTION

To motivate the results surveyed in this paper, let us look at the following simple example. Precise definitions will be given below. Let G be a finite group and $A := \mathbb{C}[G]$ be its complex group algebra. Then A is a finite dimensional, semi-simple complex algebra, and hence $A \simeq \oplus_{j=1}^{d} M_{n_j}(\mathbb{C})$. (This is an elementary result that can be found in [31]; see also [50]). The Hochschild homology of A is then, on the one hand,

$$\mathrm{HH}^0(A) \simeq \oplus_{j=1}^{d} \mathrm{HH}^0(M_{n_j}(\mathbb{C})) \simeq \mathbb{C}^d. \tag{1}$$

On the other hand, $\mathrm{HH}^0(A)$ is the space of traces on A, and hence it identifies with the space of class functions on G. Let $\langle G\rangle$ denote the set of conjugacy classes of G and $\#S$ denote the number of elements in a set S. Then $\mathrm{HH}^0(A)$ has dimension $\#\langle G\rangle$. In other words,

Proposition 0.1 (Classical). *Let G be a finite group. Then G has as many (equivalence classes of) irreducible, complex representations as conjugacy classes.*

One of our goals was to investigate to what extent Proposition 0.1 extends to other groups. It is clear that the formulation of any possible extension of Proposition 0.1 will depend on the class of groups considered and will not be as simple as in the finite group case. Moreover, this question will not be answered in a few papers and is more of a program (going back to Gelfand, Langlands, Manin, and other people) than an explicit question. Nevertheless, something from Proposition 0.1 does remain true in certain cases. An example is the theory of characters for compact Lie groups.

In this paper, we will investigate a possible analog of Proposition 0.1 for the case of a reductive p–adic group G. Recall that a p–adic group $G = \mathbb{G}(\mathbb{F})$ is the set of $\mathbb{F}$–rational points of a linear algebraic group $\mathbb{G}$ defined over a non-archemedean, non-discrete, locally compact field $\mathbb{F}$ of characteristic zero (so $\mathbb{F}$ is a finite algebraic extension of the field $\mathbb{Q}_l$ of l-adic numbers, for some prime l). A vague formulation of our main result is as follows.

Let $\mathrm{HP}_k(A)$ denote the periodic cyclic homology groups of the algebra A. Also, let $\mathcal{C}_c^\infty(G)$ denote the space of compactly supported, locally constant functions on G and let $\hat{G}$ denote the admissible dual of G with the Jacobson topology. Then the results of [2, 3, 29, 47] give the following result that will be made more precise below.

Theorem 0.2. *The groups $\mathrm{HP}_j(\mathcal{C}_c^\infty(G))$ are explicitly determined by the geometry of the conjugacy classes of G and the cohomology of their stabilizers and they are (essentially) isomorphic to the singular cohomology groups of $\hat{G}$.*

One of the main purposes of this paper is to explain the above theorem. This theorem is useful especially because it is much easier to determine the groups $\mathrm{HP}_j(\mathcal{C}_c^\infty(G))$ (and hence, to a large extent, the algebraic cohomology of $\hat{G}$) than it is to determine $\hat{G}$ itself. Moreover, we will briefly sketch a plan to say more about the actual structure of $\hat{G}$ using the knowledge of the topology of $\hat{G}$ acquired from the determination of $\mathrm{HP}_j(\mathcal{C}_c^\infty(G))$ in terms of the geometry of the conjugacy classes of G and the cohomology of their stabilizers. See also [1] for a survey of the applications of non-commutative geometry to the representation theory of reductive p-adic groups.

The paper is divided into two parts. The first part, consisting of Sections 1–4 is more advanced, whereas the last three sections review some basic material. In Section 1 we review the basic result relating the cohomology of the maximal spectrum of a commutative algebra A to its periodic cyclic homology groups $\mathrm{HP}_*(A)$. The relation between forms on $\mathrm{Max}(A)$ and the Hochschild homology groups $\mathrm{HH}_*(A)$ are also discussed here. These are basic results due to Connes [16], Feigin and Tsygan [20], and Loday and Quillen [33]. We also discuss the Excisition principle in periodic cyclic homology [19] and it's relation with K-theory. In Section 2 we discuss generalizations of these results to finite type algebras, a class of algebras directly relevant to the representation theory of p-adic groups that was introduced in [29]. We also use these results to compute the periodic cyclic homology and the Hochschild homology of several typical examples of finite type algebras. In the following section, Section 3, we introduce spectrum preserving morphisms, which were shown in [3] to induce isomorphisms on periodic cyclic homology. This then led to a determination of the periodic cyclic homology of Iwahori-Hecke algebras in that paper. In Section 4, we recall the explicit calculation of the Hochschild and periodic cyclic homology groups of the full Hecke algebra $\mathcal{C}_c^\infty(G)$. The last three sections briefly review for the benefit of the reader the definitions of Hochschild homology, cyclic and periodic cyclic homology, and, respectively, the Chern character.

This first part of the paper follows fairly closely the structure of my talk given at the conference "Non-commutative geometry and number theory" organized by Yuri Manin and Matilde Marcolli, whom I thank for their great work and for the opportunity to present my results. I have included, however, some new results, mostly to better explain and illustrate the results surveyed.

1. PERIODIC CYCLIC HOMOLOGY VERSUS SINGULAR COHOMOLOGY

Let us discuss first to what extent the periodic cyclic homology groups $\mathrm{HP}(\mathcal{C}_c^\infty(G))$ are related to the topology – more precisely to the singular cohomology – of $\hat{G}$. In the next section, we will discuss this again in the more general framework of "finite type algebras" (Definition 2.1). The definitions of the homology groups considered in this paper and of the Connes-Karoubi character are recalled in the last three sections of this paper.

One of the main goals of non-commutative geometry is to generalize the correspondence (more precisely, contravariant equivalence of categories)

$$\text{“Space” } X \quad \leftrightarrow \mathcal{F}(X) := \text{ “the algebra of functions on } X\text{”} \tag{2}$$

to allow for non-commutative algebras on the right–hand side of this correspondence. This philosophy was developed in many papers, including [5, 13, 4, 15, 16, 17, 25, 30, 32, 33, 40, 41, 42, 43, 44], to mention only some of the more recent ones. The study of the K-theory of C^*-algebras, a field on its own, certainly fits into this philosophy. The extension of the correspondence in Equation 2 would lead to methods to study (possibly) non-commutative algebras using our geometric intuition. In Algebraic geometry, this philosophy is illustrated by the correspondence (*i.e.* contravariant equivalence of categories) between affine algebraic varieties over a field $\mathfrak{k}$ and commutative, reduced, finitely generated algebras over $\mathfrak{k}$. In Functional analysis, this principle is illustrated by the Gelfand–Naimark equivalence between the category of compact topological spaces and the category of commutative, unital C^*-algebras. In all these cases, the study of the "space" then proceeds through the study of the "algebra of functions on that space." For this approach to be useful, one should be able to define many invariants of X in terms of $\mathcal{F}(X)$ alone, preferably without using the commutativity of $\mathcal{F}(X)$.

It is a remarkable fact that one can give completely algebraic definitions for $\Omega^q(X)$, the space of differential forms on X (for suitable X) just in terms of $\mathcal{F}(X)$. Even more remarkable is that the singular cohomology of X (again for suitable X) can be defined in purely algebraic terms using only $\mathcal{F}(X)$. In these definitions, we can then replace $\mathcal{F}(X)$ with a non-commutative algebra A. Let us now recall these results.

We denote by $\mathrm{HH}_j(A)$ the Hochschild homology groups of an algebra A (see Section 5 for the definition). As we will see below, for applications to representation theory we are mostly interested in the algebraic case, so we state those first and then we state the results on smooth manifolds. We begin with a result of Loday-Quillen [33], in this result $\mathcal{F}(X) = \mathcal{O}(X)$, the ring of regular (*i.e.* polynomial) functions on the algebraic variety X.

Theorem 1.1 (Loday-Quillen). *Let X be a smooth, complex, affine algebraic variety. Then*

$$\mathrm{HH}_j(\mathcal{O}(X)) \simeq \Omega^j(X),$$

the space of algebraic forms on X.

A similar results holds when X is a smooth compact manifold and $\mathcal{F}(X) = \mathcal{C}^\infty(X)$ is the algebra of smooth functions on X [16]. See also the Hochschild–Kostant–Rosenberg paper [23].

Theorem 1.2 (Connes). *Let X be a compact, smooth manifold. Then*

$$\mathrm{HH}_j(\mathcal{C}^\infty(X)) \simeq \Omega^j(X),$$

the space of smooth forms on X.

These results extend to recover the singular cohomology of (suitable) spaces, as seen in the following two results due to Feigin-Tsygan [20] and Connes [16]. For any functor F_j [respectively, F^j], we shall denote by $F_{[j]} = \oplus_k F_{j+2k}$ [respectively, $F^{[j]} = \oplus_k F^{j+2k}$]. This will mostly be used for $F^j(X) = \mathrm{H}^j(X)$, the singular cohomology of X.

Theorem 1.3 (Feigin-Tsygan). *Let X be a complex, affine algebraic variety and $\mathcal{O}(X)$ be the ring of regular (*i.e. *polynomial) functions on X. Then*

$$\mathrm{HP}_j(\mathcal{O}(X)) \simeq \mathrm{H}^{[j]}(X).$$

For smooth algebraic varieties, this result follows from the Loday-Quillen result on Hochschild homology mentioned above. See [29] for a proof of this theorem that proceed by reducing it to the case of smooth varieties. For smooth manifolds, the result again follows from the corresponding result on Hoschschild homology.

Theorem 1.4 (Connes). *Let X be a compact, smooth manifold and $\mathcal{C}^\infty(X)$ be the algebra of smooth functions on X. Then*

$$\mathrm{HP}_j(\mathcal{C}^\infty(X)) \simeq \mathrm{H}^{[j]}(X).$$

These results are already enough justification for declaring periodic cyclic homology to be the "right" extension of singular cohomology for the category (suitable) spaces to suitable categories of algebras. However, the most remarkable result justifying this is the "Excision property" in periodic cyclic homology, a breakthrough result of Cuntz and Quillen [19].

Theorem 1.5 (Cuntz-Quillen). *Any two-sided ideal J of an algebra A over a characteristic 0 field gives rise to a periodic six-term exact sequence*

(3)
$$\begin{tikzcd} \mathrm{HP}_0(J) \arrow[r] & \mathrm{HP}_0(A) \arrow[r] & \mathrm{HP}_0(A/J) \arrow[d, "\partial"] \\ \mathrm{HP}_1(A/J) \arrow[u, "\partial"] & \mathrm{HP}_1(A) \arrow[l] & \mathrm{HP}_1(J). \arrow[l] \end{tikzcd}$$

A similar results holds for Hochschild and cyclic homology, provided that the ideal J is an H-unital algebra in the sense of Wodzicki [53], see Section 5. We shall refer to the following result of Wodzicki as the "Excision principle in Hochschild homology."

Theorem 1.6 (Wodzicki). *Let $J \subset A$ be a H-unital ideal of a complex algebra A. Then there exists a long exact sequence*

$$\begin{aligned} 0 \leftarrow \mathrm{HH}_0(A/J) \leftarrow HH_0(A) \leftarrow \mathrm{HH}_0(J) \stackrel{\partial}{\leftarrow} \mathrm{HH}_1(A/J) \\ \leftarrow \mathrm{HH}_k(A/J) \leftarrow HH_k(A) \leftarrow \mathrm{HH}_k(J) \stackrel{\partial}{\leftarrow} \mathrm{HH}_{k+1}(A/J) \leftarrow \ldots \end{aligned}$$

The same result remains valid if we replace Hochschild homology with cyclic homology.

Also, there exist excision results for topological algebras [12, 18]. An important part of the proof of the Excision property is to provide a different definition of cyclic homology in terms of X-complexes. Then the proof is ingeniously reduced to Wodzicki's result on the excision in Hochschild homology, using also an important theorem of Goodwillie [22] that we now recall.

Theorem 1.7 (Goodwillie). *If $I \subset A$ is a nilpotent two-sided ideal, then the quotient morphism $A \to A/I$ induces an isomorphism $\mathrm{HP}_*(A) \to \mathrm{HP}_*(A/I)$. In particular, $\mathrm{HP}_*(I) = 0$ whenever I is nilpotent.*

One of the main original motivations for the study of cyclic homology was the need for a generalization of the classical Chern character $Ch : K^j(X) \to \mathrm{H}^{[j]}(X)$ [16, 17, 25, 32]. Indeed, an extension is obtained in the form of the Connes-Karoubi character

$$Ch : K_j(A) \to \mathrm{HP}_j(A). \tag{4}$$

It is interesting then to notice that excision in periodic cyclic homology is compatible with excision in K-theory, which is seen from the following result [46], originally motivated by questions in the analysis of elliptic operators (more precisely, Index theory).

Theorem 1.8 (Nistor). *Let $I \subset A$ the a two-sided ideal of a complex algebra A. Then the following diagram commutes*

$$\begin{tikzcd} K_1(I) \arrow[r] \arrow[d] & K_1(A) \arrow[r] \arrow[d] & K_1(A/I) \arrow[r, "\partial"] \arrow[d] & K_0(I) \arrow[r] \arrow[d] & K_0(A) \arrow[r] \arrow[d] & K_0(A/I) \arrow[d] \\ \mathrm{HP}_1(I) \arrow[r] & \mathrm{HP}_1(A) \arrow[r] & \mathrm{HP}_1(A/I) \arrow[r, "\partial"] & \mathrm{HP}_0(I) \arrow[r] & \mathrm{HP}_0(A) \arrow[r] & \mathrm{HP}_0(A/I). \end{tikzcd}$$

Let X be a complex, affine algebraic variety, $\mathcal{O}(X)$ the ring of polynomial functions on X, X^{an} the underlying locally compact topological space, and $Y \subset X$ be a subvariety. Let $I \subset \mathcal{O}(X)$ be the ideal of functions vanishing on Y. Then the above theorem shows, in particular, that the periodic six term exact sequence of periodic cyclic homology groups associated to the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{O}(Y) \longrightarrow 0, \tag{5}$$

of algebras by the Excision principle is obtained from the long exact sequence in singular cohomology of the pair $(X^{\mathrm{an}}, Y^{\mathrm{an}})$ by making the groups periodic of period two [29, 46]. The same result holds true for the exact sequence of algebras associated to a closed submanifold Y of a smooth manifold X.

2. Periodic cyclic homology and $\hat{G}$

Let A be an arbitrary complex algebra. The kernel of an irreducible representation of A is called a *primitive* ideal of A. We shall denote by $\mathrm{Prim}(A)$ the *primitive ideal spectrum* of A, consisting of all primitive ideals of A. We endow $\mathrm{Prim}(A)$ with the Jacobson topology. Thus, a set $V \subset \mathrm{Prim}(A)$ is open if, and only if, V is the set of primitive ideals not containing some fixed ideal I of A. We have

$$\mathrm{Prim}(\mathcal{O}(X)) =: \mathrm{Max}(\mathcal{O}(X)) = X,$$

the set of maximal ideals of $A = \mathcal{O}(X)$ with the Zariski topology. If $A = \mathcal{C}^\infty(X)$, where X is a smooth compact manifold, then again $\mathrm{Prim}(A) = \mathrm{Max}(A) = X$ with the usual (*i.e.* locally compact, Hausdorff) topology on X. We are interested in the primitive ideal spectra of algebras because

$$\mathrm{Prim}(\mathcal{C}_c^\infty(G)) = \hat{G}, \tag{6}$$

a deep result due to Bernstein [7]. For the purpose of this paper, we could as well take $\mathrm{Prim}(\mathcal{C}_c^\infty(G))$ to be the actual definition of $\hat{G}$.

In view of the results presented in the previous section, it is reasonable to assume that the determination of the groups $\mathrm{HP}_j(\mathcal{C}_c^\infty(G))$ would give us some insight into the topology of $\hat{G}$. I do not know any general result relating the singular cohomology of $\hat{G}$ to $\mathrm{HP}_j(\mathcal{C}_c^\infty(G))$, although it is likely that they coincide. Anyway, due to the fact that the topology on $\hat{G}$ is highly non-Hausdorff topology, it is not clear that the knowledge of the groups $\mathrm{H}^j(\hat{G})$ would be more helpful than the knowledge of the groups $\mathrm{HP}_j(\mathcal{C}_c^\infty(G))$.

For reasons that we will explain below, it will be convenient in what follows to work in the framework of "finite typee algebras" [29]. All our rings have a unit (*i.e.* they are unital), but the algebras are not required to have a unit.

Definition 2.1. Let $\mathfrak{k}$ be a finitely generated commutative, complex ring. A finite typee $\mathfrak{k}$-algebra is a $\mathfrak{k}$-algebra that is a finitely generated $\mathfrak{k}$-module.

The study of $\hat{G}$ as well as that of $A = \mathcal{C}_c^\infty(G)$ reduces to the study of finite typee algebras by considering the connected components of $\hat{G}$ and their commuting algebras, in view of some results of Bernstein [7] that we now recall. Let $D \subset \hat{G}$ be a connected component of $\hat{G}$. Then D corresponds to a cuspidal representation σ of a Levi subgroup $M \subset G$. Let M_0 be the subgroup of M generated by the compact subgroups of M and H_D be the representation of G induced from the restriction of σ to M_0. The space H_D can be thought of as the holomorphic family of induced representations of $\mathrm{ind}_M^G(\sigma\chi)$, where χ ranges through the caracters of M/M_0. Let A_D be the algebra of G-endomorphisms of H_D. This is Bernstein's celebrated "commuting algebra." The annihilator of H_D turns out to be a direct summand of $\mathcal{C}_c^\infty(G)$ with complement the two-sided ideal $\mathcal{C}_c^\infty(G)_D \subset \mathcal{C}_c^\infty(G)$. Then the category of modules over $\mathcal{C}_c^\infty(G)_D$ is equivalent to the category of modules over A_D. Our main reason for introducing finite typee algebras is that the algebra A_D is a unital finite typee algebra and

$$D = \mathrm{Prim}(A_D). \tag{7}$$

Moreover,

$$\mathcal{C}_c^\infty(G)) = \oplus_D \mathcal{C}_c^\infty(G)_D. \tag{8}$$

To get consequences for the periodic cyclic homology, we shall need the following result.

Proposition 2.2. *Let B be a complex algebra such that $B \simeq \oplus_{n\in\mathbb{N}} B_n$ for some two-sided ideals $B_n \subset B$. Assume that $B = \cup e_k B e_k$, for a sequence of idempotents $e_k \in B$ and that $\mathrm{HH}_q(B) = 0$ for $q > N$, for some given N. Then*

$$\mathrm{HP}_q(B) \simeq \oplus_{n\in\mathbb{N}} \mathrm{HP}_q(B_n).$$

Proof. The algebra B and each of the two-sided ideals B_n are H-unital. Then

$$\mathrm{HH}_q(B) \simeq \oplus_{n\in\mathbb{N}} \mathrm{HH}_q(B_n),$$

by Wodzicki's excision theorem and the continuity of Hochschild homology (*i.e.* the compatibility of Hochschild homology with inductive limits). Then $\mathrm{HH}_q(B_n) = 0$ for $q > N$ and any n. Therefore $\mathrm{HP}_k(B_n) = \mathrm{HC}_k(B_n)$ for $k \geq N$, by the SBI-long exact sequence (this exact sequence is recalled in Section 6). Unlike periodic cyclic homology, cyclic homology is continuous (*i.e.* it is compatible with inductive limits). Using again Wodzicki's excision theorem, we obtain

$$\mathrm{HP}_k = \mathrm{HC}_k(B) \simeq \oplus_{n\in\mathbb{N}} \mathrm{HC}_k(B_n) = \oplus_{n\in\mathbb{N}} \mathrm{HP}_k(B_n),$$

for $k \geq N$. This completes the proof. □

The above discussion gives the following result mentioned in [29] without proof. For the proof, we shal also use Theorem 4.1, which implies, in particular, that $\mathrm{HH}_q(\mathcal{C}_c^\infty(G))$ vanishes for q greater than the split rank of G. This is, in fact, a quite non-trivial property of $\mathcal{C}_c^\infty(G)$ and of the finite type algebras A_D, as we shall see below in Example 2.14.

Theorem 2.3. *Let D be the set of connected components of $\hat{G}$, then*

$$\mathrm{HP}_q(\mathcal{C}_c^\infty(G)) = \oplus_D \mathrm{HP}_q(\mathcal{C}_c^\infty(G)_D) \simeq \oplus_D \mathrm{HP}_q(A_D).$$

Proof. The first part follows directly from the results above, namely from Equation (8), Proposition 2.2, and Theorem 4.1 (which implies that $\mathrm{HH}_q(\mathcal{C}_c^\infty(G)) = 0$ for $q > N$, for N large).

To complete the proof, we need to check that

$$\mathrm{HP}_q(\mathcal{C}_c^\infty(G)_D) \simeq \mathrm{HP}_q(A_D). \tag{9}$$

Let e_k be a sequence of idempotents of $\mathcal{C}_c^\infty(G)$ corresponding to a basis of neighborhood of the identity of G consisting of compact open subgroups. Then, for k large, the unital algebra $e_k\mathcal{C}_c^\infty(G)e_k$ is Morita equivalent to A_D, an imprimitivity module being given by $e_k H_D$. (Recall from above that H_D is the induced representation from a cuspidal representation of M_0, where M is a Levi subfactor defining the connected component D and M_0 is the subgroup of M generated by its compact subgroups.)

In particular, $\mathrm{HH}_q(e_k\mathcal{C}_c^\infty(G)_d e_k)$ vanish for q large. The same argument as above (using Theorem 4.1 below) shows that $\mathrm{HP}_q(\mathcal{C}_c^\infty(G)_D) \simeq \mathrm{HP}_q(e_k\mathcal{C}_c^\infty(G)_D e_k)$, for k large. The isomorphism $\mathrm{HP}_q(\mathcal{C}_c^\infty(G)_D) \simeq \mathrm{HP}_q(A_D)$ then follows from the invariance of Hochschild homology with respect to Morita equivalence. □

For suitable G and D,

$$A_D = H_q, \tag{10}$$

that is, the commuting algebra of D is the Iwahori-Hecke algebra associated to G (or to its extended affine Weyl group), [9]. The periodic cyclic homology groups of H_q were determined in [2, 3], and will be recalled in Section 3.

In view of Equations (7) and (8) and of the Theorem 2.3, we see that in order to relate the groups $\mathrm{HP}_*(\mathcal{C}_c^\infty(G))$ to the topology of $\hat{G}$, it is enough to relate $\mathrm{HP}_*(A)$ to the topology of $\mathrm{Prim}(A)$ for an arbitrary finite typee algebra. For the rest of this and the following section, we shall therefore concentrate on finite typee algebras and their periodic cyclic homology.

If $\mathfrak{I} \subset A$ is a primitive ideal of the finite typee $\mathfrak{k}$-algebra A, then the intersection $\mathfrak{I} \cap Z(A)$ is a maximal ideal of $Z(A)$. The resulting map

$$\Theta : \mathrm{Prim}(A) \to \mathrm{Max}(Z(A))$$

is called the *central character* map. We similarly obtain a map $\Theta : \mathrm{Prim}(A) \to \mathrm{Max}(\mathfrak{k})$, also called the central character map.

The topology on $\mathrm{Prim}(A)$ and the groups $\mathrm{HP}_j(A)$ are related through a spectral sequence whose E^2 are given by the singular cohomology of various strata of $\mathrm{Prim}(A)$ that are better behaved than $\mathrm{Prim}(A)$ itself, Theorem 2.5 below. To state our next result, due to Kazhdan, Nistor, and Schneider [29], we need to introduce some notation and definitions.

We shall use the customary notation to denote by 0 the ideal $\{0\}$. Recall that an algebra B is called *semiprimitive* if the intersection of all its primitive ideals is 0. Also, we shall denote by $Z(B)$ the center of an algebra B. We shall need the following definition from [29]

Definition 2.4. A finite decreasing sequence

$$A = \mathfrak{I}_0 \supset \mathfrak{I}_1 \supset \ldots \supset \mathfrak{I}_{n-1} \supset \mathfrak{I}_n$$

of two-sided ideals of a unital finite type algebra A is an *abelian filtration* if and only if the following three conditions are satisfied for each k:

(i) The quotient $A/\mathfrak{I}_k$ is semiprimitive.

(ii) For each maximal ideal $\mathfrak{p} \subset Z_k := Z(A/\mathfrak{I}_k)$ not containing $I_{k-1} := Z_k \cap (\mathfrak{I}_{k-1}/\mathfrak{I}_k)$, the localization $(A/\mathfrak{I}_k)_{\mathfrak{p}} = (Z_k \setminus \mathfrak{p})^{-1}(A/\mathfrak{I}_k)$ is an Azumaya algebra over $(Z_k)_{\mathfrak{p}}$,

(iii) The quotient $(\mathfrak{I}_{k-1}/\mathfrak{I}_k)/I_{k-1}(A/\mathfrak{I}_k)$ is nilpotent and $\mathfrak{I}_n$ is the intersection of all primitive ideals of A.

Consider an abelian filtration $(\mathfrak{I}_k)$, $k = 0, \ldots, n$ of a finite type $\mathfrak{k}$-algebra A. Then, for each k, the center Z_k of $A/\mathfrak{I}_k$ is a finitely generated complex algebra, and hence it is isomorphic to the ring of regular functions on an affine, complex algebraic variety X_k. Let $Y_k \subset X_k$ be the subvariety defined by $I_{k-1} := Z_k \cap (\mathfrak{I}_{k-1}/\mathfrak{I}_k)$. For any complex algebraic variety X, we shall denote by X^{an} the topological space obtained by endowing the set X with the locally compact topology induced by some embedding $X \subset \mathbb{C}^N$, N large. We shall call X^{an} the analytic space underlying X. For instance, we shall refer to X_k^{an} and Y_k^{an} as the *analytic spaces* associated to the filtration $(\mathfrak{I}_k)$ of A. Then we have the following result from [29]

Theorem 2.5 (Kazhdan-Nistor-Schneider)**.** *If $Y_p^{\mathrm{an}} \subset X_p^{\mathrm{an}}$ are the analytic spaces associated to an abelian filtration of a finite type algebra A, then there exists a natural spectral sequence with*

$$E^1_{-p,q} = \mathrm{H}^{[q-p]}(X_p^{\mathrm{an}}, Y_p^{\mathrm{an}})$$

convergent to $\mathrm{HP}_{q-p}(A)$.

If the algebra A in the above theorem is commutative, then any decreasing filtration of A (by radical ideals) is abelian, which explains the terminology "abelian filtration." Moroever, in this case our spectral sequence reduces to the spectral sequence in singular cohomology associated to the filtration of a space by closed subsets.

Every finite type algebra has an abelian filtration. Indeed, we can take $\mathfrak{I}_k$ to be the intersection of the kernels of all irreducible representations of dimension at most k. This filtration is called the *standard filtration*.

Example 2.6. Let $a_1, a_2, \ldots, a_l \in \mathbb{C}$ be distinct points and $v_1, v_2, \ldots, v_l \in \mathbb{C}^2$ be non-zero (column) vectors. Let $\mathbb{C}[x] = \mathcal{O}(\mathbb{C})$ denote the algebra of polynomials in one variable and define

$$A_1 := \{P \in M_2(\mathbb{C}[x]),\ F(a_j)v_j \in \mathbb{C}v_j\}. \tag{11}$$

Then A_1 is a finite type algebra with center $Z = Z(A_1)$, the subalgebra of matrices of the form PE_2, where $P \in \mathbb{C}[x]$ and E_2 is the identity matrix in $M_2(\mathbb{C})$.

We filter A_1 by the ideals

$$\mathfrak{I}_1 = \{P \in A_1,\ F(a_j) = 0\}$$

and $\mathfrak{I}_2 = 0$. We have

$$A_1/\mathfrak{I}_1 \simeq \mathbb{C}^{2k}$$

a semi-primitive (*i.e.* reduced) commutative algebra with center

$$Z_1 := Z(A_1/\mathfrak{I}_1) = A_1/\mathfrak{I}_1 = \mathbb{C}^{2k}.$$

Then $I_0 := Z(A_1/\mathfrak{I}_1) \cap \mathfrak{I}_0 = A_1$ and hence no maximal ideal $\mathfrak{p}$ contains I_0. On the other hand, the algebra $A_1/\mathfrak{I}_1$ is an Azumaya algebra, so Condition (ii) of Definition 2.4 is automatically satisfied for $k = 1$. Similarly, the quotient $(\mathfrak{I}_0/\mathfrak{I}_1)/I_0(A_1/\mathfrak{I}_1) = 0$, and hence it is nilpotent. Hence Condition (iii) of Definition 2.4 is also satisfied for $k = 1$. The space X_1 consists of $2l$ points (two copies of each a_j) and Y_1 is empty.

Next, $Z_2 = Z \simeq \mathbb{C}[x]$ and $A_1 = A_1/\mathfrak{I}_2$ is semi-primitive. The ideal $I_1 := Z_2 \cap \mathfrak{I}_1$ consists of the polynomials vanishing at the given values a_j. In this case, $X_2 = \mathbb{C}$ and $Y_2 = \{a_1, \ldots, a_l\}$. The Conditions (i–iii) of Definition 2.4 are easily checked for $k = 2$. This also follows from the fact that $\mathcal{I}_j$ is the standard filtration of A_1.

The spectral sequence of Theorem 2.5 then becomes $E^1_{-p,q} = 0$, unless $p = 1$ or $p = 2$, in which case we get

$$E^1_{-1,q} = \mathrm{H}^{[q-1]}(X_1, Y_1) = \begin{cases} \mathbb{C}^{2l} & \text{if } q \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

and

$$E^1_{-2,q} = \mathrm{H}^{[q-2]}(X_2, Y_2) = \begin{cases} \mathbb{C}^{l-1} & \text{if } q \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

The differential $d^1 : E^1_{-p,q} \to E^1_{-p-1,q}$ turns out to be surjective for $p = 1$ and q odd. For some obvious geometric reasons, the spectral sequence $E^r_{p,q}$ at $r = 2$ then collapses and gives

$$\mathrm{HP}_0(A_1) \simeq E^2_{-1,1} \simeq \mathbb{C}^{l+1} \tag{12}$$

and $\mathrm{HP}_1(A_1) = 0$. We shall look again at the algebra A_2, from a different point of view, in Example 3.5.

Let us consider now the following related example.

Example 2.7. Let $A_2 = M_2(\mathbb{C}[x]) \oplus \mathbb{C}^l$. Then the standard filtration of A_2 is $\mathfrak{I}_0 = A_2$, $\mathfrak{I}_1 = M_2(\mathbb{C}[x])$, and $\mathfrak{I}_2 = 0$. The center of $A_2/\mathfrak{I}_1$ is $Z_1 \simeq \mathbb{C}^l$. We have that X_1 consists of l points and $Y_1 = 0$. The center of $A_2/\mathfrak{I}_2$ is $Z_2 \simeq \mathbb{C}[x] \oplus \mathbb{C}^l$. Then $X_2 = \mathbb{C} \cup X_1$ and $Y_2 = X_1$, where $X_1 \cap \mathbb{C} = \emptyset$.

The spectral sequence of Theorem 2.5 then becomes $E^1_{-p,q} = 0$, unless $p = 1$ or $p = 2$, in which case we get

$$E^1_{-1,q} = \mathrm{H}^{[q-1]}(X_1, Y_1) = \begin{cases} \mathbb{C}^l & \text{if } q \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

and

$$E^1_{-2,q} = \mathrm{H}^{[q-2]}(X_2, Y_2) = \begin{cases} 0 & \text{if } q \text{ is odd} \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

The spectral sequence collapses at the E^1 term for geometric reasons and hence

$$\mathrm{HP}_0(A_2) \simeq E^1_{-1,1} \oplus E^1_{-2,2} \simeq \mathbb{C}^{l+1} \tag{13}$$

and $\mathrm{HP}_1(A_1) = 0$.

The algebras A_1 and A_2 in the above examples turned out to have the same periodic cyclic homology groups. These algebras are simple, but representative, of the finite type algebras arising in the representation theory of reductive p-adic groups. Clearly, the periodic cyclic homology groups of these algebras provide important information on the structure of these algebras, but fails to distinguish them. At a heuristical level, distinguishing between A_1 and A_2 is the same problem as distinguishing between square integrable representations and supercuspidal representations. This issue arises because both these types of representations provide similar homology classes in HP_0 (through the Chern characters of the idempotents defining them). It is then an important question to distinguish between these homology classes.

Let us complete our discussion with some related results on the Hochschild homology of finite type algebras. We begin with the Hochschild homology of certain cross product algebras.

Let Γ be a finite group acting on a smooth complex algebraic variety X. For any $\gamma \in \Gamma$, let us denote by $X^\gamma \subset X$ the points of X fixed by γ. Let

$$\Gamma_\gamma := \{g \in \Gamma, g\gamma = \gamma g\}$$

denote the centralizer of γ in Γ. Let C_γ be the (finite, cyclic) subgroup generated by γ. There exists a natural Γ-invariant map

$$\hat{X} := \bigcup_{\gamma \in \Gamma} \{\gamma\} \times X^\gamma \times (\Gamma_\gamma / C_\gamma) \to X, \tag{14}$$

given simply by the projection onto the second component. This gives then rise to a Γ-equivariant morphism $\mathcal{O}(X) \to \mathcal{O}(\hat{X})$. Choose a representative $x \in \Gamma$ from each conjugacy class $\langle x \rangle$ of Γ and denote by m_x the number of elements in the conjugacy class of m_x. Denote by C^*_γ the dual of C_γ, that is the set of multiplicative maps $\pi : C_\gamma \to \mathbb{C}^*$. Recall that $\langle \Gamma \rangle$ denotes the set of conjugacy classes of Γ. We are finally ready to define the morphisms

$$\psi_{\gamma,\pi} : \mathcal{O}(X) \rtimes \Gamma \to M_{m_x}(\mathcal{O}(X^\gamma)),$$

where γ runs through a system of representatives of the conjugacy classes of Γ and $\pi \in C^*_\gamma$

$$\begin{aligned} \psi = \bigoplus_{\langle\gamma\rangle \in \langle\Gamma\rangle, \pi \in C^*_\gamma} \psi_{\gamma,\pi} : \mathcal{O}(X) \rtimes \Gamma \to \mathcal{O}(\hat{X}) \rtimes \Gamma \\ \simeq \bigoplus_{\langle\gamma\rangle \in \langle\Gamma\rangle} M_{m_x}(\mathcal{O}(X^\gamma) \otimes \mathbb{C}[C_\gamma]) \simeq \bigoplus_{\langle\gamma\rangle \in \langle\Gamma\rangle, \pi \in C^*_\gamma} M_{m_x}(\mathcal{O}(X^\gamma)). \end{aligned} \tag{15}$$

The equations above give rise to a map

$$\phi = \bigoplus_{\langle\gamma\rangle\in\langle\Gamma\rangle} \phi_\gamma : \mathrm{HH}_q(\mathcal{O}(X)\rtimes\Gamma) \to \bigoplus_{\langle\gamma\rangle\in\langle\Gamma\rangle} \mathrm{H}^q(X^\gamma), \tag{16}$$

defined using $\mathrm{HH}_q(M_m(\mathcal{O}(X^\gamma))) \simeq \mathrm{HH}_q(\mathcal{O}(X^\gamma)) \simeq \mathrm{H}^q(X^\gamma)$ and

$$\phi_\gamma = \sum_{\pi\in C_\gamma^*} \frac{\overline{\pi(\gamma)}}{\#C_\gamma} \mathrm{HH}_q(\psi_{\gamma,\pi}) : \mathrm{HH}_q(\mathcal{O}(X)\rtimes\Gamma) \to \Omega^q(X^\gamma). \tag{17}$$

(This map was defined in a joint work in progress with J. Brodzki [11].)

Then we have the following lemma, which is a particular case of the Theorem 2.11 below.

Lemma 2.8. *Assume that $X = \mathbb{C}^n$ and that Γ acts linearly on $\mathbb{C}^n$. Then the map ϕ of Equation 16 defines an isomorphism*

$$\phi : \mathrm{HH}_q(\mathcal{O}(X)\rtimes\Gamma) \to \bigoplus_{\langle\gamma\rangle\in\langle\Gamma\rangle} \Omega^q(X^\gamma)^{\Gamma_\gamma}.$$

Proof. Let A be a complex algebra acted upon by automorphisms by a group Γ. Let us recall [21, 44] that the groups $\mathrm{HH}_q(A\rtimes\Gamma)$ decompose naturally as a direct sum

$$\mathrm{HH}_q(A\rtimes\Gamma) \simeq \oplus_{\langle\gamma\rangle\in\langle\Gamma\rangle} \mathrm{HH}_q(A\rtimes\Gamma)_\gamma. \tag{18}$$

The components $\mathrm{HH}_q(A\rtimes\Gamma)_\gamma$ are then identified as follows. Let A_γ be the A–A bimodule with action $a\cdot b\cdot c = ab\gamma(c)$. Let $\mathrm{HH}_q(A, M)$ denote the Hochschild homology groups of an A–bimodule M [38, 39, 52]. Then

$$\mathrm{HH}_q(A\rtimes\Gamma)_\gamma \simeq \mathrm{HH}_q(A, A_\gamma)^{\Gamma_\gamma}. \tag{19}$$

This follows from example from [44][Lemma 3.3] (take $G = \Gamma$ in that lemma). This will also be discussed in [11, 49].

Let now $A = \mathcal{O}(X)$, with $X = \mathbb{C}^n$ and Γ acting linearly on X. The same method as the one used in the proof of [15][Lemma 5.2] shows that

$$\mathrm{HH}_q(A, A_\gamma) \simeq \mathrm{HH}_q(\mathcal{O}(X^\gamma), \mathcal{O}(X^\gamma)) = \mathrm{HH}_q(\mathcal{O}(X^\gamma)) = \Omega^q(X^\gamma), \tag{20}$$

the isomorphism being given by the restriction morphism $\mathcal{O}(X) \to \mathcal{O}(X^\gamma)$.

A direct calculation based on the formula for the map J in [44][Lemma 3.3] shows that the composition of the morphisms of Equations (19) and (20) is the map ϕ_γ of Equation (17). This completes the proof. $\square$

We now extend the above result to X an arbitrary smooth, complex algebraic variety. To do this, we shall need two results on Hochschild homology. We begin with the following result of Brylinski from [14].

Proposition 2.9 (Brylinski)**.** *Let S be a multiplicative subset of the center Z of the algebra A. Then $\mathrm{HH}_*(S^{-1}A) \simeq S^{-1}\mathrm{HH}_*(A)$.*

A related result, from [29], studies the completion of Hochschild homology. In the following result, we shall use *topological Hochschild homology*, whose definition is similar to that of the usual Hochschild homology, except that one completes with respect to the powers of an ideal (below, we shall use this for the ideals I and IA).

Theorem 2.10. *Suppose that A is a unital finite type $\mathfrak{k}$–algebra and $I \subset \mathfrak{k}$ is an ideal. Then the natural map $\mathrm{HH}_*(A) \to \mathrm{HH}_*^{\mathrm{top}}(\hat{A})$ and the $\mathfrak{k}$–module structure on $\mathrm{HH}_*(A)$ define an isomorphism*

$$\mathrm{HH}_*(A) \otimes_{\mathfrak{k}} \hat{\mathfrak{k}} \simeq \mathrm{HH}_*^{\mathrm{top}}(\hat{A})$$

of $\hat{\mathfrak{k}}$–modules.

We are ready now to prove the following theorem.

Theorem 2.11. *Assume that X is a smooth, complex algebraic variety and that Γ acts on X by algebraic automorphisms. Then the map ϕ of Equation 16 defines an isomorphism*

$$\phi : \mathrm{HH}_q(\mathcal{O}(X) \rtimes \Gamma) \to \bigoplus_{\langle\gamma\rangle \in \langle\Gamma\rangle} \Omega^q(X^\gamma)^{\Gamma_\gamma}.$$

Proof. Let

$$Z := \mathcal{O}(X)^\Gamma = \mathcal{O}(X/\Gamma).$$

Then the morphism ψ of Equation (15) is Z-linear. It follows that the map ϕ of Equation (16) is also Z-linear. It is enough hence to prove that the localization of this map to any maximal ideal of Z is an isomorphism. It is also enough to prove that the completion of any of these localizations with respect to that maximal ideal is an isomorphism. Since X is smooth (and hence the completion of the local ring of X at any point is a power series ring) this reduces to the case of $X = \mathbb{C}^n$ acted upon linearly by Γ. The result hence follows from Lemma 2.8. □

Let us include the following corollary of the above proof.

Corollary 2.12. *The map $\phi_\gamma : \mathrm{HH}_q(\mathcal{O}(X) \rtimes \Gamma) \to \Omega^q(X^\gamma)$ of (17) is such that $\phi_\gamma = 0$ on $\mathrm{HH}_q(\mathcal{O}(X) \rtimes \Gamma)_g$ if g and γ are not in the same conjugacy class and induces an isomorphism*

$$\phi_\gamma : \mathrm{HH}_q(\mathcal{O}(X) \rtimes \Gamma)_\gamma \to \Omega^q(X^\gamma)^{\Gamma_\gamma}.$$

Let us apply these results to the algebra A_1 of Example 2.6.

Example 2.13. Let $\Gamma = \mathbb{Z}/2\mathbb{Z}$ act by $z \to -z$ on $\mathbb{C}$. Chose $l = 1$ and $a_1 = 0$ in Example 2.6. Then $\mathcal{O}(\mathbb{C}) \rtimes \Gamma \simeq A_1$, and hence

$$\mathrm{HH}_q(A_1) \simeq \begin{cases} \mathcal{O}(\mathbb{C})^\Gamma \oplus \mathbb{C} & \text{if } q = 0 \\ \mathcal{O}(\mathbb{C})^\Gamma & \text{if } q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathrm{HH}_q(A_1)$ vanishes for q large. Also, note that

$$\mathcal{O}(\mathbb{C})^\Gamma = \mathbb{C}[x]^\Gamma = \mathbb{C}[x^2] \simeq \mathbb{C}[x].$$

Let us see now an example of a finite type algebra for which Hochschild homology does not vanish in all degrees.

Example 2.14. Let $A_3 \subset M_2(\mathcal{O}(\mathbb{C})) = M_2(\mathbb{C}[x])$ be the subalgebra of those matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} =: P_{11}e_{11} + P_{12}e_{12}P_{21}e_{21} + P_{22},$$

with the property that $P_{21}(0) = P_{21}'(0) = 0$. For a suitable choice of v_1, this is a subalgebra of the algebra A_1 considered in Example 2.13. Let $V_1 := A_3e_{11}$ and $V_2 := A_3e_{22}$.

Then $M := V_2/V_1 \simeq \mathbb{C}[x]/(x^2)$. The modules V_1 and $V_2^\tau := e_{22}A_3$ can be used to produced a projective resolution of A_3 with free A_3–A_3 bimodules that gives

$$\mathrm{HH}_q(A_3) \simeq \begin{cases} \mathbb{C}[x] \oplus M & \text{if } q = 0 \\ \mathbb{C}[x] \oplus \mathbb{C} & \text{if } q = 1 \\ \mathbb{C} & \text{otherwise.} \end{cases} \tag{21}$$

3. Spectrum preserving morphisms

We shall now give more evidence for the close relationship between the topology of $\mathrm{Prim}(A)$ and $\mathrm{HP}_*(A)$ by studying a class of morphisms implicitly appearing in Lusztig's work on the representation of Iwahori-Hecke algebras, see [34, 35, 36, 37].

Let L and J be two finite type $\mathfrak{k}$-algebras. If $\phi : L \to J$ is a $\mathfrak{k}$-linear morphism, we define

$$\mathcal{R}_\phi := \{(\mathfrak{P}', \mathfrak{P}) \subset \mathrm{Prim}(J) \times \mathrm{Prim}(L),\, \phi^{-1}(\mathfrak{P}') \subset \mathfrak{P}\}. \tag{22}$$

We now introduce the class of morphisms we are interested in.

Definition 3.1. Let $\phi : L \to J$ be a $\mathfrak{k}$-linear morphism of unital, finite type $\mathfrak{k}$-algebras. We say that ϕ is a *spectrum preserving morphism* if, and only if, the set $\mathcal{R}_\phi$ defined in Equation (22) is the graph of a bijective function

$$\phi^* : \mathrm{Prim}(J) \to \mathrm{Prim}(L).$$

More concretely, we see that $\phi : L \to J$ is spectrum preserving if, and only if, the following two conditions are satisfied:

(1) *For any primitive ideal $\mathfrak{P}$ of J, the ideal $\phi^{-1}(\mathfrak{P})$ is contained in a unique primitive ideal of L, namely $\phi^*(\mathfrak{P})$*, and
(2) *The resulting map $\phi^* : \mathrm{Prim}(J) \to \mathrm{Prim}(L)$ is a bijection.*

We have the following result combining two theorems from [3].

Theorem 3.2 (Baum-Nistor)**.** *Let L and J be finite type $\mathfrak{k}$–algebras and $\phi : L \to J$ be a $\mathfrak{k}$–linear spectrum preserving morphism. Then the induced map $\phi^* : \mathrm{Prim}(J) \to \mathrm{Prim}(L)$ between primitive ideal spectra is a homeomorphism and the induced map $\phi_* : \mathrm{HP}_*(L) \to \mathrm{HP}_*(J)$ between periodic cyclic homology groups is an isomorphism.*

We also obtain an isomorphism on periodic cyclic homology for a slightly more general class of algebra morphisms.

Definition 3.3. A morphism $\phi : L \to J$ of finite type algebras is called *weakly spectrum preserving* if, and only if, there exist increasing filtrations

$$(0) = L_0 \subset L_1 \subset \ldots \subset L_n = L \quad \text{and} \quad (0) = J_0 \subset J_1 \subset \ldots \subset J_n = J$$

of two-sided ideals such that $\phi(L_k) \subset J_k$ and the induced morphisms $L_k/L_{k-1} \to J_k/J_{k-1}$ are spectrum preserving.

Combining the above theorem with the excision property, we obtain the following result from [3].

Theorem 3.4 (Baum-Nistor)**.** *Let L and J be finite type $\mathfrak{k}$–algebras and $\phi : L \to J$ be a $\mathfrak{k}$–linear weakly spectrum preserving morphism. Then the induced map $\phi_* : \mathrm{HP}_*(L) \to \mathrm{HP}_*(J)$ between periodic cyclic homology groups is an isomorphism.*

The main application of this theorem is the determination of the periodic cyclic homology of Iwahori-Hecke algebras in [2, 3]. Let H_q be an Iwahori-Hecke algebra and J the corresponding asymptotic Hecke algebra associated to an extended affine Weyl group $\widehat{W}$ [36] (their definition is recalled in [3], where more details and more complete references are given). Then there exists a morphism $\phi : H_q \to J$ of $\mathfrak{k}$–finite type algebras, $\mathfrak{k} = Z(H_q)$ that is weakly spectrum-preserving provided that q is not a root of unity or $q = 1$. Therefore

$$\phi_* : \mathrm{HP}_*(H_q) \to \mathrm{HP}_*(J) \tag{23}$$

is an isomorphism. The algebra H_1 is a group algebra, and hence its periodic cyclic homology can be calculated directly.

The above theorem also helps us clarify the Examples 2.6 and 2.7.

Example 3.5. Let us assume that $v_j = v \in \mathbb{C}^2$ in Example 2.6. Let $e \in M_2(\mathbb{C})$ be the projection onto the vector e. The morphism $\phi : \mathbb{C}[x] \ni P \to Pe \in A_1 \subset M_2(\mathbb{C}[x])$ is not weakly spectrum preserving. However, $\phi : \mathbb{C}[x] \to \mathfrak{I}_1$ is a spectrum preserving morphism of $\mathbb{C}[x]$–algebras. Combining with the inclusion $\mathfrak{I}_1 \subset M_2(\mathbb{C}[x])$, we see that $\phi_* : \mathrm{HP}_*(\mathbb{C}[x]) \to \mathrm{HP}_*(\mathfrak{I}_1)$ is an isomorphism and $\mathrm{HP}_*(\mathfrak{I}_1)$ is a direct summand of $\mathrm{HP}_*(A_1)$. The excision theorem then gives

$$\mathrm{HP}_q(A_1) \simeq \mathrm{HP}_q(\mathfrak{I}_1) \oplus \mathrm{HP}_q(A/\mathfrak{I}_1) \simeq \mathrm{HP}_q(\mathbb{C}[x]) \oplus \mathrm{HP}_q(\mathbb{C}^l) = \begin{cases} \mathbb{C}^{l+1} & \text{if } q \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

The case of the algebra A_2 of example is even simpler.

Example 3.6. The inclusion $Z(A_2) \to A_2$ is a spectrum preserving morphism of $Z(A_2)$–algebras. Consequently,

$$\mathrm{HP}_q(A_2) \simeq \mathrm{HP}_q(\mathbb{C}[x]) \oplus \mathrm{HP}_q(\mathbb{C}^l) = \begin{cases} \mathbb{C}^{l+1} & \text{if } q \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Let us notices that by considering the action of the natural morphisms

$$\mathbb{C}[x] \to Z(A_1) \subset A_1,\ A_1 \to M_2(\mathbb{C}[x]),\ \mathbb{C}[x] \to Z(A_2) \subset A_2,\ or A_2 \to M_2(\mathbb{C}[x])$$

on periodic cyclic homology, we will still not be able to distinguish between A_1 and A_2. However, the natural products $\mathrm{HP}_i(\mathbb{C}[x]) \otimes \mathrm{HP}_j(A_k) \to \mathrm{HP}_*(A_k)$ (see [26, 27]) *will* distinguish between these algebras.

4. The periodic cyclic homology of $\mathcal{C}_c^\infty(G)$

Having discussed the relation between $\mathrm{HP}_*(\mathcal{C}_c^\infty(G))$ and the admissible spectrum $\hat{G} = \mathrm{Prim}(\mathcal{C}_c^\infty(G))$, let us recall the explicit calculation of $\mathrm{HP}_*(\mathcal{C}_c^\infty(G))$ from [47]. The calculation of $\mathrm{HP}_*(\mathcal{C}_c^\infty(G))$ in [47] follows right away from the calculation of the Hochschild homology groups of $\mathcal{C}_c^\infty(G)$. The calculations of presented in this section complement the results on the cyclic homology of p–adic groups in [45, 48].

To state the main result of [47] on the Hochschild homology of the algebra $\mathcal{C}_c^\infty(G)$, we need to introduce first the concepts of a "standard subgroup" and of a "relatively regular element" of a standard subgroup. For any group G and any subset $A \subset G$, we shall denote

$$C_G(A) := \{g \in G, ga = ag,\ \forall a \in A\}\,, \quad N_G(A) := \{g \in G, gA = Ag\}\,,$$

$W_G(A) := N_G(A)/C_G(A)$, and $Z(A) := A \cap C_G(A)$. This latter notation will be used only when A is a subgroup of G. The subscript G will be dropped from the notation whenever the group G is understood. A commutative subgroup S of G is called *standard* if S is the group of semisimple elements of the center of $C(s)$ for some semi-simple element $s \in G$. An element $s \in S$ with this property will be called *regular relative to* S, or S*-regular*. The set of S-regular elements will be denoted by $S^{\rm reg}$.

We fix from now on a p-adic group G. (Recall that a p–adic group $G = \mathbb{G}(\mathbb{F})$ is the set of $\mathbb{F}$–rational points of a linear algebraic group $\mathbb{G}$ defined over a non-archemedean, non-discrete, locally compact field $\mathbb{F}$ of characteristic zero.) Our results will be stated in terms of standard subgroups of G. We shall denote by H_u the set of unipotent elements of a subgroup H. Sometimes, the set $C(S)_u$ is also denoted by $\mathcal{U}_S$, in order to avoid having to many paranthesis in our formulae. Let $\Delta_{C(S)}$ denote the modular function of the group $C(S)$ and let

$$\mathcal{C}_c^\infty(\mathcal{U}_S)_\delta := \mathcal{C}_c^\infty(C(S)_u) \otimes \Delta_{C(S)},$$

be $\mathcal{C}_c^\infty(\mathcal{U}_S)$ as a vector space, but with the product $C(S)$-module structure, that is

$$\gamma(f)(u) = \Delta_{C(S)}(\gamma) f(\gamma^{-1} u \gamma),$$

for all $\gamma \in C(S)$, $f \in \mathcal{C}_c^\infty(\mathcal{U}_S)_\delta$ and $u \in \mathcal{U}_S$.

The groups $\mathrm{HH}_*(\mathcal{C}_c^\infty(G))$ are determined in terms of the following data:

(1) the set Σ of conjugacy classes of standard subgroups S of G;
(2) the subsets $S^{\rm reg} \subset S$ of S-regular elements;
(3) the actions of the Weyl groups $W(S)$ on $\mathcal{C}_c^\infty(S)$; and
(4) the continuous cohomology of the $C(S)$–modules $\mathcal{C}_c^\infty(\mathcal{U}_S)_\delta$.

Combining this proposition with Corollary 2.3, we obtain the main result of this section. Also, recall that $\mathcal{U}_S$ is the set of unipotent elements commuting with the standard subgroup S, and that the action of $C(S)$ on $\mathcal{C}_c^\infty(\mathcal{U}_S)$ is twisted by the modular function of $C(S)$, yielding the module $\mathcal{C}_c^\infty(\mathcal{U}_S)_\delta = \mathcal{C}_c^\infty(\mathcal{U}_S) \otimes \Delta_{C(S)}$.

Theorem 4.1. *Let G be a p–adic group. Let Σ be a set of representative of conjugacy classes of standard subgroups of $S \subset G$ and $W(S) = N(S)/C(S)$, then we have an isomorphism*

$$\mathrm{HH}_q(\mathcal{C}_c^\infty(G)) \simeq \bigoplus_{S \in \Sigma} \mathcal{C}_c^\infty(S^{\rm reg})^{W(S)} \otimes \mathrm{H}_q(C(S), \mathcal{C}_c^\infty(\mathcal{U}_S)_\delta).$$

The isomorphism of this theorem was obtained by identifying the E^∞-term of a spectral sequence convergent to $\mathrm{HH}_q(\mathcal{C}_c^\infty(G))$, and hence it is not natural. This isomorphism can be made natural by using a generalization of the Shalika germs [47] .

The periodic cyclic homology groups of $\mathcal{C}_c^\infty(G)$ are then determined as follows. Recall that our convention is that $\mathrm{HH}_{[q]} := \oplus_{k \in \mathbb{Z}} \mathrm{HH}_{q+2k}$. An element $\gamma \in G$ is called *compact* if the closure of the set $\{\gamma^n\}$ is compact. The set G_{comp} of compact elements of G is open and closed in G, if we endow G with the locally compact, Hausdorff topology obtained from an embedding $G \subset \mathbb{F}^N$. We clearly have $\gamma G_{comp} \gamma^{-1} = G_{comp}$, that is, G_{comp} is G-invariant for the action of G on itself by conjugation. Also, we shall denote by $\mathrm{HH}_{[q]}(\mathcal{C}_c^\infty(G))_{comp}$ the localization of the homology group $\mathrm{HH}_{[q]}(\mathcal{C}_c^\infty(G))$ to the set of compact elements of G (see [8] or [44]). This localization is defined as follows. Let $R^\infty(G)$ be the ring of locally constant Ad_G-invariant functions on G with the pointwise product. If $\omega = f_0 \otimes f_1 \otimes \dots f_n \in \mathcal{C}_c^\infty(G)^{\otimes(n+1)} = \mathcal{C}_c^\infty(G^{n+1})$ and $z \in R^\infty(G)$, then we define

$$[z\omega](g_0, g_1, \dots, g_n) = z(g_0 g_1 \dots g_n)\omega(g_0, g_1, \dots, g_n) \in \mathcal{C}_c^\infty(G^{n+1}).$$

Let χ be the characteristic function of G_{comp} (so $\chi = 1$ on G_{comp} and $\chi = 0$ otherwise). Then $\mathrm{HH}_{[q]}(\mathcal{C}_c^\infty(G))_{comp} = \chi\,\mathrm{HH}_{[q]}(\mathcal{C}_c^\infty(G))$.

Theorem 4.2. *We have*

$$\mathrm{HP}_q(\mathcal{C}_c^\infty(G)) \simeq \mathrm{HH}_{[q]}(\mathcal{C}_c^\infty(G))_{comp}. \tag{24}$$

Let S_{comp} be the set of compact elements of a standard subgroup S, then

$$\mathrm{HP}_q(\mathcal{C}_c^\infty(G)) \simeq \bigoplus_{S\in\Sigma} \mathcal{C}_c^\infty(S_{comp}^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_{[q]}(C(S), \mathcal{C}_c^\infty(\mathcal{U}_S)_\delta). \tag{25}$$

The Equation (24) follows also from the results in [45].

It is conceivable that a next step would be to study the "discrete parts" of the groups $\mathrm{HH}_q(\mathcal{C}_c^\infty(G))$, following the philosophy of [6, 28]. This can be defined as follows. In [47], we have defined "induction morphisms"

$$\phi_M^G : \mathrm{HH}_q(\mathcal{C}_c^\infty(G)) \to \mathrm{HH}_q(\mathcal{C}_c^\infty(M)) \tag{26}$$

for every Levi subgroup $M \subset G$. We define $\mathrm{HH}_q(\mathcal{C}_c^\infty(G))_0$ to be the intersection of all kernels of ϕ_M^G, for M a proper Levi subgroup of G and call this the discrete part of $\mathrm{HH}_q(\mathcal{C}_c^\infty(G))_0$.

Assuming that one has established, by induction, a procedure to construct the cuspidal representations of all p–adic groups of lower split rank, then one can study the action of the center of $\mathcal{C}_c^\infty(G)$ corresponding to the cuspidal associated to proper Levi subgroups on the discrete part of $\mathrm{HH}_q(\mathcal{C}_c^\infty(G))$, which would hopefully allow us to distinguish between the cuspidal part of $\mathrm{HH}_q(\mathcal{C}_c^\infty(G))_0$ and its part coming from square integrable representations that are not super-cuspidal.

5. HOCHSCHILD HOMOLOGY

We include now three short sections that recall some of the definitions used above. Nothing in this and the next section is new, and the reader interested in more details as well as precise references should consult the following standard references [10, 5, 16, 17, 25, 33, 32, 51].

We begin by recalling the definitions of Hochschild homology groups of a complex algebra A, not necessarily with unit. We define b and b' define two linear maps

$$b,\, b' \,:\, A^{\otimes n+1} \to A^{\otimes n}. \tag{27}$$

where $A^{\otimes n} := A \otimes A \otimes \ldots A$ (n times) by the formulas

$$\begin{aligned} b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \\ b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1}, \end{aligned} \tag{28}$$

where $a_0, a_1, \ldots, a_n \in A$. Let

$$\mathcal{H}_n(A) = \mathcal{H}'_n(A) := A \otimes A \otimes \ldots A \ (n+1 \text{ times }). \tag{29}$$

Also, let $\mathcal{H}(A) := (\mathcal{H}_n(A), b)$ and $\mathcal{H}'(A) := (\mathcal{H}/_n(A), b')$. The homology groups of $\mathcal{H}(A)$ are, by definition, the *Hochschild homology* groups of A and are non-zero. The nth Hochschild homology group of A is denoted $\mathrm{HH}_n(A)$. By dualizing, we obtain the Hochschild cohomology groups $\mathrm{HH}^n(A)$.

If A has a unit, then the complex $\mathcal{H}'(A)$ is acyclic (*i.e.* it has vanishing homology groups) because $b's + sb' = 1$, where

$$s(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n. \tag{30}$$

Therefore, if A has a unit, the complex $\mathcal{H}'(A)$ is a resolution of A by free A-bimodules.

Recall that *a trace* on A is a linear map $\tau : A \to \mathbb{C}$ such that $\tau(a_0a_1 - a_1a_0) = 0$ for all $a_0, a_1 \in A$. The space of all traces on A is then isomorphic to $\mathrm{HH}^0(A)$.

An algebra A such that $\mathcal{H}'(A)$ is acyclic is called H*-unital*, following Wodzicki [53].

Clearly the groups $\mathrm{HH}_n(A)$ are *covariant* functors in A, in the sense that any algebra morphism $\phi : A \to B$ induces a morphism

$$\phi_* = \mathrm{HH}_n(\phi) : \mathrm{HH}_n(A) \to \mathrm{HH}_n(B)$$

for any integer $n \geq 0$. Similarly, we also obtain a morphism $\phi^* = \mathrm{HH}^n(\phi) : \mathrm{HH}^n(B) \to \mathrm{HH}^n(A)$. In other words, Hochschild cohomology is a *contravariant* functor. It is interesting to note that if Z is the center of A, then $\mathrm{HH}_n(A)$ is also a Z-module, where, at the level of complexes the action is given by

$$z(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = za_0 \otimes a_1 \otimes \ldots \otimes a_n. \tag{31}$$

for all $z \in Z$. As z is in the center of A, this action will commute with the Hochschild differential b.

6. Cyclic homology

Let A be a unital algebra. We shall denote by t the (signed) generator of cyclic permutations:

$$t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1} \tag{32}$$

Using the operator t and the contracting homotopy s of the complex $\mathcal{H}'(A)$, Equation (30), we construct a new differential $B := (1-t)B_0$, of degree $+1$, where

$$B_0(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = s \sum_{k=0}^{n} t^k (a_0 \otimes a_1 \otimes \ldots \otimes a_n). \tag{33}$$

It is easy to check that $B^2 = 0$ and that $[b, B]_+ := bB + Bb = 0$.

The differentials b and B give rise to the following complex

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd}$$

FIGURE 1. The cyclic bicomplex of the algebra A.

We notice that columns the above complex are copies of the Hochschild complex $\mathcal{H}(A)$. The cyclic complex $\mathcal{C}(A)$ is by definition the total complex of this double complex. Thus the space of cyclic n-chains is defined by

$$\mathcal{C}(A)_n = \bigoplus_{k\geq 0} \mathcal{H}_{n-2k}(A), \tag{34}$$

we see that $(\mathcal{C}(A), b + B)$, is a complex, called *the cyclic complex* of A, whose homology is by definition the *cyclic homology* of A, denoted $\mathrm{HC}_q(A)$, $q \geq 0$.

There is a canonical operator $S : \mathcal{C}_n(A) \to \mathcal{C}_{n-2}(A)$, called the *Connes periodicity operator*, which shifts the cyclic complex left and down, explicitly defined by

$$S(\omega_n, \omega_{n-2}, \omega_{n-4}, \dots) \mapsto (\omega_{n-2}, \omega_{n-4}, \dots), \tag{35}$$

where $\omega_k \in \mathcal{H}_k(A)$, for all k. This operator induces the short exact sequence of complexes

$$0 \to \mathcal{H}(A) \xrightarrow{I} \mathcal{C}(A) \xrightarrow{S} \mathcal{C}(A)[2] \to 0,$$

where the map I is the inclusion of the Hochschild complex as the first column of the cyclic complex. The snake Lemma in homology [38, 39, 52] gives the following long exact sequence, called the *SBI–exact sequence*

$$\dots \to \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \xrightarrow{I} \dots , \tag{36}$$

where B is the differential defined above, see [16, 33] for more details. The *periodic cyclic complex* of an algebra A is the complex

$$\mathcal{C}^{\mathrm{per}}(A) := \lim_{\leftarrow} \mathcal{C}(A),$$

the inverse limit being taken with respect to the periodicity morphism S. It is a $\mathbb{Z}/2\mathbb{Z}$-graded complex, whose chains are (possibly infinite) sequences of Hochschild chains with degrees of the same parity. The homology groups of the periodic cyclic complex $\mathcal{C}^{\mathrm{per}}(A)$ are, by definition, the *periodic cyclic homology groups* of A. A simple consequence of the SBI–exact sequence is that if $\phi : A \to B$ is a morphism of algebras that induces an isomorphism on Hochschild homology, then ϕ induces an isomorphism on cyclic and periodic cyclic homology groups as well. Here is an application of this simple principle.

Here is an application of this lemma. Consider

$$Tr_* : \mathcal{H}_q(M_N(A)) \to \mathcal{H}_q(A), \quad q \in \mathbb{Z}_+, \tag{37}$$

the map defined by $Tr_*(b_0 \otimes \dots \otimes b_q) = Tr(m_0 m_1 \dots m_q) a_0 \otimes \dots a_q$, if $b_k = m_k \otimes a_k \in M_N(\mathbb{C}) \otimes A = M_N(A)$. Also consider the (unital) inclusion $\iota : A \to M_N(A)$ and ι_* be the morphism induced on the Hochschild complexes.

Proposition 6.1. *The map Tr_* commutes with b and B. Both ι_* and Tr_* induce isomorphisms on Hochschild, cyclic, and periodic cyclic homologies and cohomologies such that $(\iota_*)^{-1} = N^{-1} Tr_*$.*

The cyclic and periodic cyclic homology groups of a non-unital algebra A are defined as the cokernels of the maps $\mathrm{HC}_q(\mathbb{C}) \to \mathrm{HC}_q(A^+)$ and $\mathrm{HP}_q(\mathbb{C}) \to \mathrm{HP}_q(A^+)$, where $A^+ = A \oplus \mathbb{C}$ is the algebra with adjoined unit.

More generally, cyclic homology groups can be defined for "mixed complexes," [5, 24, 27, 29].

7. THE CHERN CHARACTER

We shall use these calculations to construct Chern characters. By taking X to be a point in Theorem 1.3, we obtain that $\mathrm{HC}_{2q}(\mathbb{C}) \simeq \mathbb{C}$ and $\mathrm{HC}_{2q+1}(\mathbb{C}) \simeq 0$. We can take these isomorphisms to be compatible with the periodicity operator S and such that for $q = 0$ it reduces to

$$\mathrm{HC}_0(\mathbb{C}) = \mathrm{HH}_0(\mathbb{C}) = \mathbb{C}/[\mathbb{C}, \mathbb{C}] = \mathbb{C}.$$

We shall denote by $\eta_q \in \mathrm{HC}_{2q}(\mathbb{C})$ the unique element such that

$$S^q \eta_q = 1 \in \mathbb{C} = \mathrm{HC}_0(\mathbb{C}). \tag{38}$$

For any projection $e \in M_N(A)$ is a projection we obtain a (non-unital) morphism $\psi : \mathbb{C} \to M_N(A)$ by $\lambda \mapsto \lambda e$. Then *Connes-Karoubi Chern character of e in cyclic homology* [16, 17, 25, 32] is defined by

$$Ch_q([e]) = Tr_*(\psi_*(\eta_q)) \in \mathrm{HC}_{2q}(A). \tag{39}$$

This map can be shown to depend only on the class of e in K-theory and to define a morphism

$$Ch_q : K_0(A) \to \mathrm{HC}_{2q}(A). \tag{40}$$

One can define similarly the Chern character in periodic cyclic homology and the Chern character on K_1 (algebraic K-theory). For the Connes-Karoubi Chern character on K_1, we use instead $Y = \mathbb{C}^*$, whose algebra of regular functions is $\mathcal{O}[Y] \simeq \mathbb{C}[z, z^{-1}]$, the algebra of Laurent polynomials in z and z^{-1} (this algebra, in turn, is isomorphic to the group algebra of $\mathbb{Z}$). Then $\mathrm{HC}_q(\mathcal{O}[Y]) \simeq \mathbb{C}$, for any $q \geq 1$. We are interested in the odd groups, which will be generated by elements $v_k \in \mathrm{HC}_{2k+1}(\mathcal{O}[Y])$, which can be chosen to satisfy $v_1 = z^{-1} \otimes z$ and $S^k v_{2k+1} = v_1$.

Then, if $u \in M_N(A)$ is an invertible element, it defines a morphism $\psi : \mathbb{C}[\mathbb{C}^*] \to M_N(A)$. The *Connes-Karoubi Chern character of u in cyclic homology* is thus defined by

$$Ch_q([u]) = Tr_*(\psi_*(v_q)) \in \mathrm{HC}_{2q+1}(A). \tag{41}$$

Again, this map can be shown to depend only on the class of u in K-theory and to define a morphism

$$Ch_q : K_1(A) \to \mathrm{HC}_{2q+1}(A). \tag{42}$$

Both the Chern character on K_0 and on K_1 are functorial, by construction.

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