p–ADIC BETTI LATTICES

by

Yves André

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

MPI/90 – 8

p–ADIC BETTI LATTICES

Yves André

Under the label "p–adic Betti lattices", we shall discuss two kinds of objects.

The first type of lattices arises via Artin's embedding of integral Betti cohomology into p–adic étale cohomology for complex algebraic varieties; there are comparison theorems with algebraic De Rham cohomology both over the complex numbers (Grothendieck) and p–adically (Fontaine–Messing–Faltings). The second type of lattices, which we believe be new, arises in connection with p–adic tori. Although its definition is purely p–adic, it is closely tied to the classical Betti lattice of some related complex torus, and can be viewed as a bridge between the Dwork and Fontaine theories of p–adic periods; "half" of this lattice is provided by the cohomology of the rigid analytic constant sheaf $\mathbb{Z}$. In fact, both themes of this paper are motivated by a question of Fontaine about the p–adic analog of the Grothendieck period conjecture, as follows.

1. Let X be a proper smooth variety over the field of rational numbers $\mathbb{Q}$. The singular rational cohomology space $H^n_B := H^n(X_{\mathbb{C}}, \mathbb{Q})$ carries a rational Hodge structure (for any n); this structure is defined by a complex one–parameter subgroup of $GL(H^n_B \otimes \mathbb{C})$, whose rational Zariski closure in $GL(H^n_B)$ is the so–called <u>Mumford–Tate group</u> of H^n_B.

Let H^n_{DR} enote the $n^{\underline{th}}$ algebraic De Rham cohomology group of X. There is a canonical isomorphism

$$\mathcal{P} : H^n_{DR} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^n_B \otimes_{\mathbb{Q}} \mathbb{C}$$

provided by the functor GAGA and the analytic Poincaré lemma. The entries in $\mathbb{C}$ of a matrix of $\mathcal{P}$ w.r.t. some bases of H^n_{DR}, H^n_B, are usually called periods.
One variant of the Grothendieck period conjecture [G1] predicts that the transcendence degree of the extension of $\mathbb{Q}$ generated by the periods is the dimension of the Mumford–Tate group.

2. On the other hand, let $H^n_{et} := H^n_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ denote the $n^{\underline{th}}$ p–adic étale cohomology group of

$X_{\overline{\mathbb{Q}}}$, where $\overline{\mathbb{Q}}$ stands for the complex algebraic closure of $\mathbb{Q}$.

Let us choose an embedding γ of $\overline{\mathbb{Q}}$ into the field $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$. The successive works of Fontaine, Messing and Faltings [FM] [Fa] managed to construct a canonical isomorphism of filtered $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$–modules:

$$H^n_{DR} \otimes_{\mathbb{Q}} B_{DR} \xrightarrow{\sim} H^n_{et}(X_{\mathbb{C}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{DR} ,$$

where B_{DR} denotes the quotient field of the universal pro–infinitesimal thickening of $\mathbb{C}_p$. Via Artin's comparison theorem and the theorem of proper base change for étale cohomology (applied to γ) [SGA 4] III, this supplies us with an isomorphism

$$\mathcal{P}_\gamma : H^n_{DR} \otimes_{\mathbb{Q}} B_{DR} \xrightarrow{\sim} H^n_{et} \otimes_{\mathbb{Q}_p} B_{DR} \xrightarrow{\sim} H^n_B \otimes_{\mathbb{Q}} B_{DR} .$$

The entries in B_{DR} of a matrix of $\mathcal{P}_\gamma$ w.r.t. some bases of H^n_{DR}, H^n_B are called (γ)–p–adic periods.

Fontaine asked whether the analog of Grothendieck's conjecture for p–adic periods holds true. The answer turns out to be negative; indeed, we shall prove:

Proposition 1. Let X be the elliptic modular curve $X_0(11)$, and $n = 1$, $p = 11$. There are two choices of γ for which the transcendence degree of the extension of $\mathbb{Q}$ generated by the respective p–adic periods differ.

Nevertheless, one can still ask in general whether the property holds true for "sufficiently general" γ. This would be a consequence of a standard conjecture on "geometric p–adic representations":

Proposition 2. Let G be the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(H^n_{et}) \simeq GL(H^n_B) | \mathbb{Q}_p$. Assume that the rational Zariski closure of G in $GL(H^n_B)$ contains the Mumford–Tate group. Then for "sufficiently general" γ, the transcendence degree of the extension of $\mathbb{Q}$ generated by the p–adic periods is not smaller than the dimension of the Mumford–Tate group; if moreover $n = 1$, there is equality.

3. Let us next turn to p–adic Betti lattices of the second kind, the construction of which it modelled on the following pattern. Let us assume that over some finite extension E of $\mathbb{Q}$ in $\mathbb{C}_p$, X_E acquires semi–stable reduction, i.e. admits locally a model over the valuation ring of the p–adic completion K of E, which is smooth over the scheme defined by an equation $x_1 x_2 \ldots x_n$ = some uniformizing parameter of K. In this situation Hyodo and Kato showed the

existence of a semi–stable structure on H^n_{DR} (as was conjectured by Jannsen and Fontaine): namely an isocrystal (H_0,φ) endowed with a nilpotent endomorphism N satisfying $N\varphi = p\varphi N$, together with an isomorphism $H^n_{DR} \otimes_E K \xrightarrow{\sim} H_0 \otimes_{K^0} K$ depending on the choice of a branch β of the p–adic logarithm on $K^\times$ (here K^0 denotes the maximal absolutely unramified subfield of K).

On the other side, one can sometimes use the combinatorics of the intersection graph of the reduction to provide lattices, well–behaved under φ, in suitable twited graded (w.r.t. the "p–adic monodromy" N) forms of H^n_{DR}, and then use φ in order to lift them to H^n_{DR}. For instance, this works pretty well when $X_E = A$ is an Abelian variety with multiplicative reduction at p .

4. Before we describe this situation, let us remind the classical situation $(E \subset \mathbb{C}) : A(\mathbb{C})$ is a complex torus $\mathbb{C}^g/L$, where L is a lattice of rank 2g ; furthermore $H^1_{DR} \otimes \mathbb{C} \simeq \mathrm{Hom}(L,\mathbb{C})$. Composition with a suitably normalized exponential map yields the Jacobi parametrization:

$A(\mathbb{C}) \simeq \mathbb{C}^{\times g}/M$ where M is a lattice of rank g ; thus L appears as an extension of M by $2i\pi M'^{\vee}$, where M' denotes the character group of $\mathbb{C}^{\times g}$. The bilinear map on M , say q , obtained by composing any "polarization" $M' \longrightarrow M$ with the bilinear map $M \times M' \longrightarrow \mathbb{G}_m$ (the multiplicative group) describing $M \longrightarrow \mathbb{C}^{\times g}$, enjoys the following property: $-\log|q|$ is a scalar product.

Similarly, at any place of multiplicative reduction above p , there is the Tate parametrization:

$A(\mathbb{C}_p) \simeq \mathbb{C}_p^{\times g}/M$ where M is again a lattice of rank g ; there is an analogous bilinear map q' on M such that $-\log|q|_p$ is a scalar product.

Using the semi–stable structure, we construct the "p–adic" lattice L_β of rank 2g , formed of φ–invariants and depending on β, which sits in an exact sequence like L (in this new context, $2i\pi$ has to be understood as a generator of $\mathbb{Z}_p(1)$ inside B_{DR}).

Setting $K_{HT} := K[2i\pi,(2i\pi)^{-1}]$, we have moreover a canonical isomorphism: $\mathcal{P}_\beta : H^1_{DR} \otimes_E K_{HT} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(L_\beta, K_{HT})$.

5. We call the entries in K_{HT} of a matrix of $\mathcal{P}_\beta$ w.r.t. some basis of H^1_{DR}, L_β, "(β)–p–adic periods". We may now state a more rigid p–adic transcendence conjecture:

Conjecture 1: for suitable choice of β, the transcendence degree of the extension of E generated by the β– p–adic periods equals the dimension of the Mumford–Tate group of H^1_B .

This conjecture splits into two parts:
We first prove the inequality $\text{tr.deg}_E\, E[\mathcal{P}_\beta] \leq \dim \text{M.T.}$ under some extra hypothesis (*) (theorem 3); this amounts to showing the rationality of Hodge classes w.r.t. L_β. (The hypothesis (*) concerns the Shimura variety associated to A, but we think it is unnecessary, or even always satisfied). On the other side, we use G–function methods to prove inequalities of the type "boundary $\text{tr.deg}_E\, E[\mathcal{P}_\beta] \geq \dim \text{M.T.}$" refering to polynomial relations of bounded degree between periods (theorems 4 and 5).
Roughly speaking, this is made possible because, when A varies in a degenerating family defined over E, the β–p–adic periods involve the β–logarithm and p–adic evaluations of Taylor series with coefficients in E, whose complex evaluations give the usual periods (theorem 2).

6. The previous considerations suggest the possibility of a purely p–adic definition of (absolute) Hodge classes on A.

Conjecture 2: Let E' be any extension of E, and let ξ be a mixed tensor on $H^1_{DR} \otimes E'$ lying in the 0–step of the Hodge filtration. Then ξ is an absolute Hodge class (i.e. rational w.r.t. L for every $E' \hookrightarrow \mathbb{C}$) if and only if ξ is rational w.r.t. L_β for every $E' \hookrightarrow \mathbb{C}_p$ and every branch β of the p–adic logarithm.

I The p–adic comparison isomorphic
II Hodge classes
III Covanishing cycles and the monodromy filtration
IV Frobenius and the p–adic Betti lattice
V p–Adic lattice and Hodge classes

Convention: In this text, a smooth separated commutative group scheme will be called semi–abelian if each each fiber is an extension of an abelian variety by a torus.

Acknowledgements: I first thank the organizers of the Trento conference for inviting me to such a nice congress. This article was written out during a stay at the Max–Planck–Institut at Bon, partially supported by the Humboldt Stiftung; I thank both institutions heartily. The paper benefited very much from several conversations with W. Messing, J.M. Fontaine and especially J.P. Wintenberger.

I. The p–adic comparison isomorphism

1. The "Barsotti rings" B_{DR} and B_{cris}

Let K be a p–adic field, i.e. a finite extension of $\mathbb{Q}_p$ with valuation $v|p$. Let K^0, resp. $\overline{K}$, $\mathbb{C}_p$ denote the maximal nonramified extension of $\mathbb{Q}_p$ inside K, resp. an algebraic closure of K, and its completion. Let $R, R^0, \overline{R}, \mathbb{R}_p$ denote the respective rings of integers, and let $\mathcal{G}$ denote the Galois group $\mathrm{Gal}(\overline{K}/K)$. Fontaine has constructed a universal p–adic pro–infinitesimal thickening of $\mathbb{C}_p$, see e.g. [F1] [F2].

It is denoted by B^+_{DR} and can be constructed as follows.

Let us consider the Witt ring W of the perfection $\varprojlim_{x\mapsto x^p} \overline{R}/p\overline{R}$ of the residual ring $\overline{R}/p\overline{R}$. It sits in an exact sequence $0 \longrightarrow F^1 \longrightarrow W \longrightarrow \mathbb{R}_p \longrightarrow 0$, where the ring homomorphism is defined by the diagram:

$$\begin{array}{ccccccc}
 & & W & \longrightarrow & \varprojlim \overline{R}/p\overline{R} & & \\
 & & \uparrow\wr\wr & & \uparrow\wr\wr & & \\
[v_n] & \in & \mathrm{Witt}(\varprojlim \overline{R}) & \longrightarrow & \varprojlim \overline{R} & \ni & v_n = (v_n^0, v_n^1, \ldots) \\
\text{(Teichmüller lifting of } v_n\text{)} & & \Big\downarrow & & \Sigma\, p^n [v_n] & & \\
 & & & & \Big\downarrow & & \\
 & & \mathbb{R}_p & & \Sigma\, p^n v_n^0 & &
\end{array}$$

This provides a continuous surjective homomorphism $B^+_{DR} \longrightarrow \mathbb{C}_p$, where B^+_{DR} denotes the F^1–adic completion of $W[\frac{1}{p}]$. The fraction field B_{DR} of B^+_{DR} is a $\overline{K}[\,\mathcal{G}\,]$–module, endowed with the F^1–adic (called Hodge) filtration F, and $\mathrm{Gr}^{\cdot}_F B_{DR} \simeq \bigoplus_{r\in\mathbb{Z}} \mathbb{C}_p(r)$ (Tate twists).

On the other hand, there is a universal PD–thickening of $\mathbb{C}_p$, denoted by B^+_{cris}. It is obtained by inverting p in the p–adic completion of the subalgebra of $W[\frac{1}{p}]$ generated by the $\frac{p^n}{n!}$'s. For instance, if $\epsilon = (\epsilon_0, \epsilon_1, \ldots)$ is a generator of $\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}(\overline{K})$, $t_p := \log[\epsilon] = \sum \frac{(-)^{n-1}([\epsilon]-1)^n}{n} \in B^+_{cris}$. The Frobenius φ of W then extends to

$B_{cris} = B^+_{cris}[\frac{1}{t_p}]$ $(\varphi t_p = pt_p)$ and commutes with the $\mathcal{G}$–action. Moreover $B_{cris} \otimes_{K_0} K$ imbeds into B_{DR}.

2. The comparison theorem for Abelian varieties

Let $A = A_K$ be an Abelian variety over K. According to Fontaine–Messing [F1] [FM], there is a canonical isomorphism of filtered $\mathcal{G}$–modules:

$$\text{F.M.}: H^*_{DR}(A) \underset{K}{\otimes} B_{DR} \xrightarrow{\sim} H^*_{et}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{DR} .$$

In particular H^*_{DR} can be recovered from H^*_{et} as the space of $\mathcal{G}$–invariants in the R.H.S. This isomorphism can be reformulated as a pairing:

$$H^1_{DR}(A) \otimes T_p(A_{\overline{K}}) \longrightarrow B_{DR} .$$

[Faltings and Wintenberger have generalized this pairing to the relative case [W]; the relative H^*_{DR} and B_{DR} are endowed with connections and the relative comparison isomorphism is horizontal.]

In order to describe part of this pairing in down–to–earth terms, let us assume that A has semi–stable reduction, i.e. extends to a semi–abelian scheme A_R over R. (By a fundamental result of Grothendieck this always happens after replacing K by a finite extension). Let $\hat{A}_R$ be the formal group attached to A_R; then $T_p(\hat{A}_R)(\overline{K})$ is the "fixed part" of $T_p(A_{\overline{K}})$ [G2]. Now the restricted pairing $H^1_{DR}(A) \otimes T_p(\hat{A}_R)(\overline{K}) \longrightarrow B_{cris} \otimes_{K_0} K$ is easily described as follows:

a) It factorizes through the quotient $H^1_{DR}(\hat{A}_R)_K \otimes T_p(\hat{A}_R)(\overline{K})$.

b) Using the formal Poincaré lemma, write any $\omega \in H^1_{DR}(\hat{A}_R)_K$ in the form $\omega = df$, $f \in \mathcal{O}_{\hat{A}_K}$.

c) For any $\gamma = (\gamma_0, \gamma_1, \ldots) \in T_p(\hat{A}_R)(\overline{R}) = T_p(\hat{A}_R)(\overline{K})$, lift every $\gamma_n \in \overline{R}$ to $\tilde{\gamma}_n \in B_{cris}$.

d) The coupling constant $<\omega,\gamma> \in B_{cris} \otimes_{K_0} K$ is then given by $\lim_n p^n f(\tilde{\gamma}_n)$. See [Co].

3. The crystalline and semi–stable structures

a) Let us first assume that A has good reduction, i.e. extends to an Abelian scheme A_R over

R , and let us denote the special fiber of A_R by $\tilde{A}$. In this case $H^*_{DR}(A)$ carries a natural K^0–structure, namely $H^*_0 := H^*_{cris}(\tilde{A}/R^0) \otimes_{R^0} K^0$; moreover this K^0–space is a crystal: it is canonically endowed with a semi–linear "Frobenius" isomorphism φ . The Fontaine–Messing isomorphism is then induced by an isomorphism of filtered φ– and $\mathcal{G}$– modules:

$$H^*_0 \otimes_{K^0} B_{cris} \xrightarrow{\sim} H^*_{et}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} .$$

In particular H^*_{et} can be recovered from H^*_0 as the space of φ–invariants in the F^0–subspace of the L.H.S.

b) If contrawise A has bad reduction, let us use Grothendieck's theorem to reduce to the case of semi–stable reduction. [Jannsen had the idea that there is still a fine structure on H^*_{DR} , involving some "monodromy operator", and such that H^*_{et} could be recovered in a similar way as in the good reduction case [J]. Fontaine then formulated a precise conjecture and proved it in the case of Abelian varieties]. The result is [F2]:

Choose a branch β of the v–adic logarithm. Then

$b_1)$ there exists a canonical K^0–structure H^*_0 on $H^*_{DR}(A)$, endowed with a nilpotent endomorphism N ; N = 0 iff A has good reduction.

$b_2)$ H^*_0 is naturally endowed with a semi–linear "Frobenius" $\varphi = \varphi_\beta$, related to N by means of the formula: $N\varphi = p\varphi N$.

$b_3)$ there exists $u_\beta \in B_{DR}$ such that $B_{ss} := B_{cris}[u_\beta]$ is $\mathcal{G}$–stable, and such that $N = d/du_\beta$ and the extension of φ to B_{ss} given by $\varphi u_\beta = pu_\beta$ commute with the $\mathcal{G}$–action.

$b_4)$ the p–adic comparison isomorphism is induced by an isomorphism of filtered $K^0(\mathcal{G})$–modules compatible with φ and N :

$$H^*_0 \otimes_{K^0} B_{ss} \xrightarrow{\sim} H^*_{et}(A_{\overline{K}}) \otimes_{\mathbb{Q}_p} B_{ss} .$$

In particular H^*_{et} can be recovered from H^*_0 as the space $[F^0(H^*_{et} \otimes B_{ss})]^{\varphi=1,N=0}$.
For a concrete description of the semi–stable structure due to Raynaud, see below III 4, IV 1 and [R2].

4. Rigid 1–motives and Fontaine's LOG

In the study of the comparison isomorphism, it is useful to embed Abelian varieties into the bigger category of 1–motives [D1].

a) Recall that a smooth 1–motive $[\underline{M} \xrightarrow{\psi} \underline{G}]$ on a scheme S consists in

i) an étale sheaf $\underline{M}$ locally defined by a free abelian group of finite rank

ii) a semi–abelian scheme $\underline{G}$ over S

iii) a morphism $\psi : \underline{M} \longrightarrow \underline{G}$.

For each prime p, one attaches to $[\underline{M} \xrightarrow{\psi} \underline{G}]$ a (Barsotti–Tate) p–divisible group, and its étale cohomology (= étale realization of $[\underline{M} \xrightarrow{\psi} \underline{G}]$).

On the other hand, the universal vectorial extension $\underline{M} \longrightarrow \underline{G}^{\natural}$ of $\underline{M} \longrightarrow \underline{G}$ provides the De Rham realization $H^1_{DR}[\underline{M} \longrightarrow \underline{G}] := \underline{\text{Colie}}\ \underline{G}^{\natural}$, with its Hodge filtration $F^1H^1_{DR} = \underline{\text{Colie}}\ \underline{G}$.

b) There is a notion of duality for 1–motives. We shall only consider symmetrizable 1–motives, i.e. 1–motives isogeneous to their duals (the isogeny inducing a polarization of the Abelian quotient of $\underline{G}$). This amounts to giving

i) a polarized Abelian scheme $(\underline{A},\lambda)$ over S

ii) a morphism $\chi ; \underline{M} \longrightarrow \underline{A}$, where $\underline{M}$ is an étale sheaf of lattices; let $\chi^{\vee} = \lambda \circ \chi$

iii) a symmetric trivialization of the inverse image by $(\chi,\chi^{\vee})$ of the Poincaré biextension of $\underline{A} \times \underline{A}'$.

c) It is convenient to view 1–motives as complexes in degree $(-1,0) : \overset{-1}{\underline{M}} \longrightarrow \overset{0}{\underline{G}}$. When $S = \text{Spec}\, K$, $K =$ p–adic field, it is more convenient, according to Raynaud [R2], to identify 1–motives which are quasi–isomorphic in the rigid analytic category; for instance, if A is isomorphic to the rigid quotient G/M, we consider A (or $[0 \longrightarrow A]$) and $[M \longrightarrow G]$ as two incarnations of the same rigid 1–motive.

Indeed, the associated p–divisible groups, resp. filtered De Rham realizations, are isomorphic; furthermore this isomorphism is compatible with the Fontaine–Messing comparison isomorphism, which extends to the case of 1–motives (its semi–stable refinement also extends to this case (Fontaine–Raynaud)).

d) Let us illustrate this in the simple case $[\mathbb{Z} \xrightarrow{\psi} \mathbb{G}_m]$ $\begin{smallmatrix} 1 \longmapsto q \end{smallmatrix}$ (when q is not a unit in K, this is the 1–motive attached to the Tate curve $K^{\times}/q^{\mathbb{Z}}$). The Tate module sits in an exact sequence

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow T_p \longrightarrow q^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow 0$$

Let t_p be a generator of $\mathbb{Z}_p(1)$, and let $u_p \in T_p$ lift q. Let moreover μ be a generator of the character group $X(\mathbb{G}_m)$, so that $d\mu/1+\mu$ generates the K–space $\Omega^1_{\mathbb{G}_m}$. At last let us represent u_p by a sequence $(q,q_1,\ldots)$ with $q_{n+1}^p = q_n$, and let $\tilde{q}_n$ lift q_n in B_{DR}. The p–adic periods of $[\mathbb{Z} \xrightarrow{\psi} \mathbb{G}_m]$ are given by:

$$<t_p, d\mu/1+\mu> = \pm t_p \text{ in } B_{DR} ,$$

$$<u_p, d\mu/1+\mu> = \lim_n \log \tilde{q}_n^{p^n}/q .$$

By abuse language, one denotes this limit by LOG q ; its class mod $\mathbb{Z}_p(1)$ depends only on q . If one requires more rigidity, one may embed K into $\mathbb{C}$ somehow, and choose u_p in the $\mathbb{Z}$–lattice given by the Betti realization of the corresponding complex 1–motive; LOG q is then defined up to addition by $\mathbb{Z}t_p$, as in the classical case.

e) More generally, let us consider a 1–motive $[\underline{M} \xrightarrow{\psi} T]$, where T is a torus. In this case the universal extension splits canonically: $G^\natural = T \times \mathrm{Hom}(M,\mathbb{G}_a)^\vee$; this induces a canonical splitting of the Hodge filtration: $H^1_{DR}[\underline{M} \longrightarrow T] = F^1 \oplus \mathrm{Hom}(\underline{M},K)$. On the other hand, let M' denote the character group of T and $q : M \times M' \longrightarrow \mathbb{G}_m$ the bilinear form induced by ψ. Again the étale cohomology sits in an extension

$$0 \longrightarrow \mathrm{Hom}(M,\mathbb{Q}_p) \longrightarrow H^1_{et}[M \longrightarrow T] \longrightarrow M' \otimes_{\mathbb{Z}} \mathbb{Q}_p(-1) \longrightarrow 0 .$$

Now assume that M and M' are constant.

Let $(m_i^\vee)$ denote a basis of $\mathrm{Hom}(M,\mathbb{Z})$ as well as its images in H^1_{et} and H^1_{DR} resp.; let (μ_j) denote a basis of M', let $d\mu_j/1+\mu_j$ be the corresponding basis in F^1, and let $\tilde{\mu}_j$ lift μ_j/t_p inside H^1_{et}. At last, let (m_i) denote the basis of M dual to $(m_i^\vee)$, and set $q_{ij} = q(m_i,\mu_j)$. Then in the bases of H^1_{DR} (resp. H^1_{et}) given by $\{d\mu_j/1+\mu_j; m_j^\vee\}$ (resp. $\{\tilde{\mu}_i, m_i^\vee\}$), the matrix of the comparison isomorphism takes the shape:

$$\left[\begin{array}{c|c} t_p I & 0 \\ \hline (\mathrm{LOG}\ q_{ij}) & I \end{array}\right] .$$

This completes the description of this isomorphism for any Abelian variety with split multiplicative reduction.

II. Hodge classes.

1. The complex setting.

a) Let E be a field embeddable into $\mathbb{C}$, and let A_E be an Abelian variety over E . An element $\xi \in F^0\left[H^1_{DR}(A_E)^{\otimes n} \otimes H^1_{DR}(A_E)^{v\otimes n}\right] = F^0\left[\mathrm{End}\, H^1_{DR}(A_E)\right]^{\otimes n}$ (for any n) is called a Hodge class if its image in $\left[\mathrm{End}\, H^1_B(A_{\mathbb{C}},\mathbb{C})\right]^{\otimes n}$ lies in the rational subspace $\left[\mathrm{End}\, H^1_B(A_{\mathbb{C}},\mathbb{Q})\right]^{\otimes n}$. By Deligne's theorem on absolute Hodge cycles $[D_2]$, this definition does not depend on the chosen embedding $E \hookrightarrow \mathbb{C}$. Moreover, after a preliminary finite extension of E , one gets no more Hodge class by further extending E . It follows that the connected component of identity of the Hodge group of A_E (which is by definition the algebraic subgroup of $GL\left[H^1_{DR}(A_E)\right]$ which fixes the Hodge classes) is an E–form of the Mumford–Tate group of $H^1_B(A_{\mathbb{C}},\mathbb{Q})$. It is known that the Hodge group is a classical reductive group.

b) Let us fix an embedding $\iota : E \hookrightarrow \mathbb{C}$. For any E–algebra E′ , the E′–linear bijections $H^1_{DR}(A_E) \otimes_E E' \xrightarrow{\sim} H^1_B(A_E \otimes_\iota \mathbb{C},\mathbb{Q}) \otimes_{\mathbb{Q}} E'$ which preserve Hodge classes form the set of E′–valued points of a E–torsor P_ι under the Hodge group; for $E' = \mathbb{C}$, one has a canonical point $\mathcal{P}_\iota$ given by "integration of differential forms of second kind".

Lemma 1: the torsor P_ι is irreducible.

Indeed, there exists a finite Galois extension E′ of E such that the Hodge group of $A_{E'}$ is connected; hence the associated torsor P'_ι is geometrically irreducible. But via the isomorphism $H^*_{DR}(A_E) \otimes_E E' = H^*_{DR}(A_{E'})$, a Hodge class on A_E is just a Hodge class on $A_{E'}$ which is fixed by Gal(E′/E) . Therefore P_ι is the Zariski closure of P'_ι over E , and is irreducible.

Conjecture (Grothendieck): if E is algebraic over $\mathbb{Q}$, $\mathcal{P}_\iota$ is a (Weil) generic point of P_ι (over E) .

Thanks to the irreducibility lemma, this amounts to say that the transcendence degree over $\mathbb{Q}$ of the periods equals the dimension of the Mumford–Tate group (here, "periods" means entries of a matrix of $\mathcal{P}_\iota$ w.r.t. bases of $H^1_{DR}(A_E)$, $H^1_B(A_{\mathbb{C}},\mathbb{Q})$) . [This deep problem is solved only for Abelian varieties isogeneous to some power of an elliptic curve with complex multiplication (Chudnovsky).

The conjecture can also be formulated as follows: every polynomial relation between periods, with coefficients in E, comes from Hodge classes. A major result in transcendence theory establishes this for linear relations (Wüstholz); the only Hodge classes which appear in this context are classes of endomorphisms.]

2. Behaviour under the p–adic comparison isomorphism

Assume now that E is a number field; let $v|p$ be a finite place of E, and $K = E_v$ be the completion of E w.r.t. v; $\overline{E}$ denotes the algebraic closure of E in $\overline{K}$.

Let us choose an embedding $\gamma : \overline{K} \hookrightarrow \mathbb{C}$ and denote by ι its restriction to K.

At last, let $\mathcal{P}_\gamma : H^1_{DR}(A_E) \otimes_E B_{DR} \xrightarrow{\sim} H^1_B(A_E \otimes_\iota \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{DR}$ denote the composed isomorphism:

$$H^1_{DR}(A_E) \otimes_E B_{DR} \xrightarrow{\sim} H^1_{DR}(A_K) \otimes_K B_{DR} \xrightarrow[\sim]{F.M.} H^1_{et}(A_{\overline{E}}) \otimes_{\mathbb{Q}_p} B_{DR}$$

$$\xrightarrow{\sim} H^1_{et}(A_{\overline{E}}) \otimes_{\mathbb{Q}_p} B_{DR} \xrightarrow{\sim} H^1_B(A_{\overline{E}} \otimes_\gamma \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{DR}$$

$$\xrightarrow{\sim} H^1_B(A_E \otimes_\iota \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{DR}$$

Blasius–Ogus [Bl] and independently Wintenberger have recently proved the following striking result:

Theorem 1. For every γ above ι, $\mathcal{P}_\gamma$ is a B_{DR}–valued point of P_ι.

[The Wintenberger proof uses the relative comparison isomorphism while the Blasius–Ogus proof uses Faltings's comparison theorem applied to smooth compactifications of "total spaces" of Abelian schemes]. With the notation of I 3, it follows formally that Hodge classes lie in $(\mathrm{End}\, H^1_0)^{\otimes n}$, are Frobenius–invariant and killed by N.

In view of this theorem, it is natural to ask whether the p–adic analog of Grothendieck's conjecture holds, namely whether $\mathcal{P}_\gamma$ is a (Weil) generic point of P_ι over K. [After I communicated the counterexample in prop. 3 to Fontaine, he suggested the following:]

Conjecture 4: for "sufficiently general" γ above ι, $\mathcal{P}_\gamma$ is a (Weil) generic point of P_ι over K.

See below, § 4.

3. Proof of proposition 1.

In this example $E = \mathbb{Q}$, and $A_{\mathbb{Q}}$ is the elliptic curve $X_0(11)$. For $p = 11$, $A_{\mathbb{Q}_p}$ is a Tate curve $\mathbb{Q}_p^\times / q^{\mathbb{Z}}$, $q \in p\mathbb{Z}_p$. With the notations of I 4b, consider the exact sequence $0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow T_p(A_{\overline{\mathbb{Q}}}) \longrightarrow q^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow 0$, and let t_p be a $\mathbb{Z}_p$–generator of $\mathbb{Z}_p(1)$ such that $t_p \wedge u_p$ is a $\mathbb{Z}$–generator of the image of $\overset{2}{\wedge} H_1(A_{\overline{\mathbb{Q}}} \otimes_\gamma \mathbb{C}, \mathbb{Z})$ in $\overset{2}{\wedge} T_p(A_{\overline{\mathbb{Q}}})$ for some fixed $\gamma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$; this determines t_p up to sign. Let moreover ν be a unit in $\mathbb{Z}_p$ such that $\omega := \frac{1}{\nu} \frac{d\mu}{1+\mu}$ belongs to the rational subspace $\Omega^1_{A_{\mathbb{Q}}}$ of $\Omega^1_{A_{\mathbb{Q}_p}}$. According to I 4b, we then have:

$$< \nu\, t_p, \omega > = \pm\, t_p \,.$$

Now let $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; changing γ into $\gamma \circ g$ modifies the Betti lattice inside T_p via the formula:

$$T_p(A_{\overline{\mathbb{Q}}}) \xrightarrow[g^*]{\sim} T_p(A_{\overline{\mathbb{Q}}}) \simeq H_1(A_{\overline{\mathbb{Q}}} \otimes_\gamma \mathbb{C}, \mathbb{Z}) \otimes \mathbb{Z}_p \,,$$

where g^* denotes the image of g under the group homomorphism

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(T_p) \,.$$

But in our case, this homomorphism is surjective, according to Serre [S 1]. In particular, there exists some $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, with $\det g^* = 1$, and such that $\nu\, t_p \in T_p$ lies in the Betti lattice $H_1(A_{\overline{\mathbb{Q}}} \otimes_{\gamma \circ g} \mathbb{C}, \mathbb{Z})$; since $\det g^* = 1$, changing γ to $\gamma \circ g$ preserves t_p.

It then follows from the relation $< \nu t_p, \omega > = \pm\, t_p$ that the Zariski closure of $\mathcal{P}_{\gamma \circ g}$ over $\mathbb{Q}$ is contained in a hypersurface of P. On the other hand, it follows from Serre's result and the next lemma that for some other $\gamma' : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, the Zariski closure of $\mathcal{P}_{\gamma'}$ over $\mathbb{Q}$ is the full torsor P.

4. Proof of proposition 2 (Abelian case).

We prove the following variant for an Abelian variety A_E over a number field E [Proposition 2 itself is proved in the same way with only minor modifications involving simple general facts about absolute Hodge cycles contained in the beginning of [ ] J.

Let us fix $\gamma_0 : \overline{E} \hookrightarrow \mathbb{C}$ and denote by $H^1_{\gamma_0}$ the rational structure $H^1_B(A_{\overline{E}} \otimes_{\gamma_0} \mathbb{C}, \mathbb{Q})$ inside $H^1_{et}(A_{\overline{E}}, \mathbb{Q}_p) = H^1_{et}(A_{\overline{K}}, \mathbb{Q}_p)$ (for $\overline{E}$ = algebraic closure of E in $\overline{K}$, where $K = E_v$, $v|p$). The Galois representation $H^1_{et}(A_{\overline{E}}, \mathbb{Q}_p)$ is described by a homomorphism :
$Gal(\overline{E}/E) \longrightarrow GL(H^1_{\gamma_0})(\mathbb{Q}_p)$.

Let us denote by G_{γ_0} the Zariski closure of the image of $Gal(\overline{E}/E)$ over $\mathbb{Q}$, which is the smallest algebraic subgroup of $GL(H^1_{\gamma_0})$ whose group of p–adic points contains the image of $Gal(\overline{E}/E)$.

<u>Conjecture 5</u>: the Mumford–Tate group of $H^1_{\gamma_0}$ is the connected component of identity in G_{γ_0}.

[One easily checks that the truth of this conjecture does not depend on the choice of γ_0; on the other side, the fact that the Mumford–Tate group contains $G^0_{\gamma_0}$ is a theorem of Borovoi [Bo]]. This conjecture is a weak form of the well–known conjecture of Mumford–Serre–Tate (replace $\mathbb{Q}$ by $\mathbb{Q}_p$ in the statement).

<u>Proposition 2'</u>: Conjecture 5 implies conjecture 4.

<u>Proof</u>: let $\mathcal{P}^E_{\gamma_0}$ denote the Zariski closure of $\mathcal{P}_{\gamma_0}$ over E, inside the torsor $P = P_\iota (\iota = \gamma_{0|E})$; let $G^\alpha_{\gamma_0}$ denote any connected component of G_{γ_0}. For any $g_\alpha \in G^\alpha_{\gamma_0}(\mathbb{Q}_p)$, let

$$\psi_{g_\alpha} : \operatorname{Spec} B_{DR} \longrightarrow \operatorname{Spec} E[\mathcal{P}_{\gamma_0}] \times \operatorname{Spec} \mathbb{Q}_p \longrightarrow \mathcal{P}^E_\gamma \times G^\alpha_{\gamma_0|E}$$

be the composed morphism of affine schemes given by $(\mathcal{P}_{\gamma_0}, g_\alpha)$.

From lemma 1 and conjecture 5, it follows that $G_{\gamma_0|E} = \bigcup G^\alpha_{\gamma_0|E}$ acts <u>transitively</u> on P, and that $Q \cdot G^\alpha_{\gamma_0|E} = P$ for any non–empty E–subscheme Q of P. We can now make the expression "sufficiently general γ" (in conjecture 4) precise: it means "any γ of the form $\gamma = \gamma_0 \circ g_\alpha$ where $g_\alpha \in \operatorname{Im} Gal(E/E)$ is such that ψ_{g_α} maps to the generic point"; indeed for these embeddings γ,

$$\overline{\mathcal{P}_\gamma}^{E} = \overline{\mathcal{P}_{\gamma_0} \cdot g_\alpha}^{E} = \overline{\mathcal{P}_{\gamma_0}}^{E} \cdot (\overline{g_\alpha}^{\mathbb{Q}})_{|E} = \overline{\mathcal{P}_{\gamma_0}}^{E} \cdot G^\alpha_{\gamma_0 | E} = P .$$

It remains to prove the existence of (uncountably many) such g_α. To this aim, let us remark that there are only countably many subvarieties of $G^\alpha_{\gamma_0 | E(\mathcal{P}_{\gamma_0})}$; we denote them by Q_n, $n \in \mathbb{N}$. Hence there exist linear subspaces $\prod$ of $\operatorname{End} H^1_{\gamma_0} \otimes \mathbb{Q}_p$, of codimension $\dim P - 1$, such that $\prod \cap G^\alpha_{\gamma_0}(\mathbb{Q}_p) \cap Q_n \neq \prod \cap G^\alpha_{\gamma_0}(\mathbb{Q}_p)$ for every n. Any $g_\alpha \in \prod \cap G^\alpha_{\gamma_0}(\mathbb{Q}_p)$ being outside the countable subset $\bigcup_n \prod \cap G^\alpha_{\gamma_0}(\mathbb{Q}_p) \cap Q_n$ then satisfies the required property $\overline{g_\alpha}^{E(\mathcal{P}_{\gamma_0})} = G^\alpha_{\gamma_0 | E(\mathcal{P}_{\gamma_0})}$.

III. Covanishing cycles and the monodromy filtration.

1. Covanishing cycles.

a) Let again A be an Abelian variety of dimension g over the p–adic field K, with semi–stable reduction. For any finite extension K' of K, let $A_{K'}^{rig}$ denote the associated rigid analytic variety ("Abeloid variety") over K'.

The (Čech) cohomology $H^1(A_{K'}^{rig},\mathbb{Z})$ of the constant sheaf $\mathbb{Z}$ on $A_{K'}^{rig}$ can be interpreted as the group of Galois covers of $A_{K'}^{rig}$ with group $\mathbb{Z}$ [R_1] [U].

For reasons which will soon be clear, we denote this group by $\underline{M}^{\vee}(K')$. One defines this way an etale sheaf $\underline{M}$ on Spec K, described by the $\mathcal{G}$– module $M^{\vee} := \underline{M}^{\vee}(\overline{K})$; points of the lattice $M^{\vee}$ will be called (integral) covanishing cycles.

b) In order to understand the geometrical meaning of $M^{\vee}$, let us consider the Raynaud extension G (resp. G') of A (resp. of the dual Abelian variety A') :

$$
\begin{array}{ccccc}
 & & \underline{M} & & \\
 & & \downarrow & & \\
T & \longrightarrow & G & \longrightarrow & B \\
 & & \downarrow & & \\
 & & A & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & \underline{M}' & & \\
 & & \downarrow & & \\
T' & \longrightarrow & G & \longrightarrow & B' \\
 & & \downarrow & & \\
 & & A' & &
\end{array}
$$

G is an extension of an Abelian variety B by an unramified torus T of dimension $r \le g$ (lifting the torus part of the semi–stable reduction of A), and A (resp. $A_{\mathbb{C}_p}$) is the rigid analytic quotient of G (resp. $G_{\mathbb{C}_p}$) by the lattice $\underline{M}(K)$ of K–characters (resp. the lattice $M := \underline{M}(\overline{K})$ of characters) of T' ; and symmetrically for G' ...

This description of $A_{\mathbb{C}_p}$ shows that $M^{\vee}$ is the dual of M ; in particular the (finite) $\mathcal{G}$– action is unramified (since T' is). [In Berkovich's astonishing theory of analytic spaces, one associates with A some pathwise connected locally simply connected topological space A^{an} ; $\underline{M}(K)$ should then appear as its fundamental group in the ordinary topological sense [Be]].

c) Composing the morphisms

$$M^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \varprojlim_n H^1(A^{rig}_{\mathbb{C}_p}, \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\text{GAGR}} \varprojlim_n H^1_{et}(A_{\mathbb{C}_p}, \mathbb{Z}/p^n\mathbb{Z})$$

[where GAGR denotes the functor studied by Kiehl [K]], yields a natural injection of $\mathbb{Z}_p[\mathcal{G}]$–modules:

$$\iota_{et} : M^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow H^1_{et}(A_{\bar{K}}, \mathbb{Z}_p) .$$

d) On the other side, the lattice $\underline{M}^\vee(K)$ is naturally isomorphic to the group of rigid analytic homomorphisms from $\mathbb{G}_m$ to A' [R_1], see also [BL] for the variant over $\mathbb{C}_p$.

Composing the morphisms

$$\mathrm{Hom}_{rig}(\mathbb{G}_m, A') \xrightarrow{\text{pull-back}} \mathrm{Hom}(H^1_{DR}(A'^{rig}), H^1_{DR}(\mathbb{G}_m^{rig}))$$

$$\xrightarrow{\text{duality}} H^1_{DR}(A''^{rig}) \xrightarrow{\text{GAGR}} H^1_{DR}(A'')$$

yields a natural embedding:

$$\iota_{DR} : \underline{M}^\vee(K) \otimes_{\mathbb{Z}} K \hookrightarrow H^1_{DR}(A) .$$

[Le Stum [lS] interprets the image of ι_{DR} as follows. By means of some compactification $\bar{A}$ of the semi–abelian group scheme A_R over R extending A, there is the notion of strict neighborhood in $\bar{A}^{rig}_K$ of the formal completion $\hat{A}$. For any $\mathcal{O}_A$rig–module $\mathcal{F}$, set $j^+\mathcal{F} = \varinjlim_\lambda j_{\lambda *} j_\lambda^* \mathcal{F}$, where j_λ runs over all embeddings of strict neighborhoods of $\hat{A}$ inside A^{rig}; j^+ is an exact functor, and there is a canonical epimorphism $\mathcal{F} \twoheadrightarrow j^+ F$ [B]. Define the <u>covanishing complex</u> by $\phi := \mathrm{Ker}(\Omega^\cdot_{A\ rig} \longrightarrow j^+\Omega^\cdot_{A\ rig})$, which gives rise to a long exact sequence

$$\longrightarrow \mathbb{H}^n(A^{rig}, \phi) \longrightarrow H^n_{DR}(A) \longrightarrow H^n_{rig}(\tilde{A}) \longrightarrow$$

involving Berthelot's rigid cohomology of the special fiber $\tilde{A}$. The group $\mathbb{H}^1(A^{rig}, \phi)$ can then be identified with $\mathrm{Im}\ \iota_{DR}$; this justifies the label "covanishing cycles" by analogy with the complex case.]

e) It turns out that the maps ι_{et} and ι_{DR} are compatible with the Fontaine–Messing isomorphism; More precisely:

<u>Proposition 3</u>: the following triangle is commutative:

$$\begin{array}{ccc} & \underline{M}^{\vee}(K) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p & \\ \iota_{DR} \swarrow & & \searrow \iota_{et} \\ H^1_{DR}(A) \otimes_K B_{DR} & \xrightarrow{\text{F.M.}} & H^1_{et}(A_{\overline{K}}) \otimes_{\mathbb{Q}_p} B_{DR} \end{array}$$

<u>Proof</u>: let us introduce the Raynaud realization $[\underline{M} \longrightarrow G]$ of the (rigid) 1–motive A .

The map ι_{et} can be identified with the natural injection of $\mathcal{G}$–modules : $\mathrm{Hom}(\underline{M}(K),\mathbb{Q}_p) \hookrightarrow H^1_{et}[\underline{M} \longrightarrow G]$.

On the other side, getting rid of double duality, one easily sees that ι_{DR} can be identified with the natural embedding $\mathrm{Hom}(\underline{M}(K),K) \hookrightarrow H^1_{DR}[\underline{M} \longrightarrow G]$, see also [1S] 6.7. The required commutativity then follows from the fact that F.M. is tautological for the quotient 1–motive $[\underline{M}(K) \longrightarrow 1]$ (whose associated p–divisible group is $\cong (\mathbb{Q}_p/\mathbb{Z}_p)^n$) .

f) An <u>orientation</u> of $\mathbb{C}_p$ is an embedding of $\mu_{p^\infty}(\mathbb{C}_p) = \mathbb{Z}_p(1)$ into $\mathbb{C}^\times$; this amounts to the choice of a generator t_p of the $\mathbb{Z}_p$–module $\mathbb{Z}_p(1)$ up to sign, [a further orientation of $\mathbb{C}$ itself would fix the sign], or else to the choice of an embedding of Abelian groups $X_*(\mathbb{G}_m) \longrightarrow T_p(\mathbb{G}_m)(= \mathbb{Z}_p(1))$.

By using an orientation of $\mathbb{C}_p$ and duality, we get from c) an injection:

$$j_{et} : M'^{\vee}(1) := M'^{\vee} \otimes X_*(\mathbb{G}_m) \longrightarrow H^1_{et}(A'_{\overline{K}},\mathbb{Z}_p) \otimes T_p(\mathbb{G}_m) \simeq T_p(A_{\overline{K}}) .$$

Using the Raynaud 1–motive $[M \longrightarrow G]$ over $\overline{K}$, it is then clear that the Fontaine–Messing pairing between H^1_{DR} and $M'^{\vee}(1)$ takes its values in $K't_p$ for some finite unramified extension K' of K (even in Kt_p if the torus part of the semi–stable reduction $\mathcal{A}$ splits).

2. Raynaud extensions and the q–matrix.

Let $f: \underline{A} \longrightarrow S$ be a semi–abelian scheme with proper generic fiber, S being an affine normal connected noetherian scheme; we put $S = \mathrm{Spec}\ \mathcal{R}$, $\mathcal{K} = \mathrm{Frac}\ \mathcal{R}$.

a) Let us first assume that $\mathcal{R}$ is complete w.r.t. some ideal I (we set $S_0 := \mathrm{Spec}\ \mathcal{R}/I$), and that the rank r of the toric part T_0 of $A_0 = \underline{A} \times_S S_0$ is constant.

One constructs the Raynaud extension over $\mathcal{R}$ [CF] II, $0 \longrightarrow T \longrightarrow \underline{G} \longrightarrow \underline{B} \longrightarrow 0$, where T lifts T_0 and $\underline{B}$ is an Abelian scheme. There is also the Raynaud extension $0 \longrightarrow T' \longrightarrow \underline{G}' \longrightarrow \underline{B}' \longrightarrow 0$ attached to the dual Abelian scheme $\underline{A}'$, and $\underline{B}'$ is the dual of $\underline{B}$; moreover $\mathrm{rk}\ T = \mathrm{rk}\ T' = r$. These extensions arise via push–out from morphisms of fppf sheaves

$$\underline{M} \longrightarrow \underline{B}, \text{ where } \underline{M} = \underline{X}^*(T') \text{ (character groups).}$$

$$\underline{M}' \longrightarrow \underline{B}' \qquad \underline{M}' = \underline{X}^*(T)$$

The objects $\underline{G}$, T, $\underline{M}$, $\underline{B}$ (resp. $\underline{G}'$, ...) are functorial in $\underline{A}$ (resp. $\underline{A}'$).

b) Replacing S by some open dense subset U, the Faltings construction (using an auxiliary ample line bundle $\mathcal{L}$ on $G_{\mathcal{K}}$ [CF] II 5.1), or methods of rigid analytic geometry ([BL_1] with less generality), provide a trivialization q (independent of $\mathcal{L}$ [CF] III 7.2) of the $\mathbb{G}_m$–biextension of $\underline{M} \times \underline{M}'$ obtained as inverse image of the Poincaré biextension of $\underline{B} \times \underline{B}'$; this amounts to giving a lifting $\underline{M}_U \longrightarrow \underline{G}_U$ of $\underline{M} \longrightarrow \underline{B}$ (whence a smooth 1–motive $[\underline{M} \xrightarrow{\psi} \underline{G}]$ on U). When T_0 splits, so that $\underline{M} = M$ and $\underline{M}' = M'$ are constant, one can use some basis $\{(m_i, \mu_j)\}$ of $M \times M'$ in order to express the bilinear form $q: M \times M' \longrightarrow \mathbb{G}_{m,U}$ by a matrix with entries $q_{ij} \in \mathcal{K}^{\times}$. [If moreover $\underline{A}$ is principally polarizable, such a polarization induces an isomorphism $M \cong M'$, and then $q: M \otimes M \longrightarrow \mathcal{K}^{\times}$ is symmetric. In the literature on Abeloid varieties, the associated q–matrix is often referred to as the "period matrix"; however this terminology conflicts with the Fontaine–Messing theory, but some precise relation will be exhibited in IV].

c) In order to understand the complex counterpart, we replace S by Δ^n, where Δ denotes the unit disk in $\mathbb{C}$. Assume that the restriction of f to the inverse image of $S^* = \Delta^{*n}$ is proper, where Δ^* stands for the punctured Δ.

The kernel $\underline{\Lambda}$ of the exponential map exp: $\underline{\mathrm{Lie}}\ \underline{A}/S \longrightarrow \underline{A}$ is a sheaf of lattices extending the local system $\{H_1(A_s,\mathbb{Z})\}_{s\in S^*}$. The (unique) extension in $\underline{\Lambda}$ of the fiber of $\underline{\Lambda}$ over 0 is a local system $\underline{N}$ of rank $2g-r$. Via exp (which factorizes through $\underline{N}$), $\underline{A}$ becomes a quotient of the semi–abelian family $\underline{G} = (\underline{\text{Lie}}\ \underline{A}/S)/\underline{N} : \underline{A} = \underline{G}/\underline{M}$, where $\underline{M}$ denotes the sheaf of lattices $\underline{\Lambda}/\underline{N}$ (which degenerates at 0).

This supplies us with a (complex analytic) smooth symmetrizable 1–motive $[\underline{M} \longrightarrow \underline{G}]$ over S^*. Both the Betti realizations H^1_B and the De Rham realizations H^1_{DR} (endowed with the Hodge filtration) of $\underline{A}$ and $[\underline{M} \longrightarrow \underline{G}]$ are canonically isomorphic. However, one may not identify these "1–motives" because the weight filtrations differ, see below § 4.

d) We now start with the following global situation:

S_1 is an affine variety over a field E of characteristic 0; 0 is a smooth rational point of S_1, and $x_1, \dots, x_n$ are local coordinates around 0;

$f_1 : \underline{A}_1 \longrightarrow S_1$ is a semi–Abelian scheme, proper outside the divisor $x_1x_2 \dots x_n = 0$, and the toric rank is constant on this divisor.

Because f_1 is of finite presentation, it arises by base change from a semi–abelian scheme $\tilde{f}_1 : \tilde{\underline{A}}_1 \underset{\tilde{E}}{\longrightarrow} \tilde{S}_1$ (where $\tilde{E}$ is a sub–$\mathbb{Z}$–algebra of E of finite presentation), with the same properties as f_1. If we put $\mathcal{R} = \tilde{E}[[x_1, \dots, x_n]]$, $S = \operatorname{Spec} \mathcal{R}$ (the completion of $\tilde{S}_1$ at 0), $I = (x_1x_2 \dots x_n)$, $f = \tilde{f}_{1/S}$, we are in the situation a) b). Moreover, the open subscheme U may be defined by the condition $x_1x_2 \dots x_n \neq 0$. It follows that the entries q_{ij} of the q–matrix belong to $\tilde{E}[[x_1, \dots, x_n]]\left[\dfrac{1}{x_1x_2 \dots x_n}\right]$.

e) Assume moreover that E is a number field, with ring of integers $\mathcal{O}_E$. Then $\tilde{E}$ can be chosen in the form $\mathcal{O}_E\left[\frac{1}{\nu}\right]$, where ν is a product of distinct prime numbers. Thus for every finite place v of E not dividing ν, the q_{ij} entries are meromorphic functions on Δ_v^n, analytic on Δ_v^{*n} (Δ_v, resp. Δ_v^* denotes the v–adic "open" unit disk, resp. punctured unit disk), and bounded away from 0. On the other hand, one can also see (using construction c)) that the q_{ij}'s define meromorphic functions on some complex polydisk centered at 0.

[Remark: following [C], an element y of $E[[x_1, \dots, x_n]]$ is said to be <u>globally bounded</u> if $y \in \mathcal{O}_E\left[\frac{1}{\nu}\right][[x_1, \dots, x_n]]$ for some ν, and if y has non–zero radius of convergence at every place of E. (Such series form a regular noetherian ring with residue field E, and the filtered union of these rings over all finite extensions of E , is strictly henselian). One can show that the $(x_1 \dots x_n)^m q_{ij}$'s are globally bounded series (for suitable m) . The problem is to show that the v–adic radius of converge is not 0 for any $v \mid \nu$. Using the compactification of Siegel modular stacks over $\mathbb{Z}$, one can find a semi–abelian extension of $\mathcal{Y}_1$ over an $\mathcal{O}_E$–model of some covering of $\mathfrak{S}_1$, and afterwards, one has to use the 2–step construction of [CF] III 10 to keep track of the possible variation of the torus rank of the reduction, after replacing the divisor $x_1 \dots x_n = 0$ by $\nu x_1 \dots x_n = 0$] .

b) Lemma 2. If $v \nmid \nu$, then the entries of the q–matrix are units w.r.t. the v–adic Gauss norm.

Proof (sketch): let $\mathcal{E}$ denote the completion of the quotient field of $\mathcal{R} = \tilde{E}[[x_1, \dots x_n]]$ w.r.t. the v–adic Gauss norm $|\ |_{Gauss}$ (= "sup" norm on $\mathcal{R}$) . Because v is discrete, so is $|\ |_{Gauss}$ by Gauss' lemma, hence $\mathcal{E}$ is a complete discretely valued field of unequal characteristics.

By construction of the Raynaud extension, the Barsotti–Tate groups associated to $\underline{A}/\mathcal{E}$ resp. to the 1–motive $[\underline{M}/\mathcal{E} \xrightarrow{\psi} \underline{G}/\mathcal{E}]$ coincide. It follows that Grothendieck's monodromy pairing associated to $\underline{A}/\mathcal{E}$ is induced by the pairing $M \times M' \longrightarrow \mathcal{E}^\times \longrightarrow \mathbb{Z}$ given by the valuation of the q–matrix w.r.t. $|\ |_{Gauss}$. Since $\underline{A}/\mathcal{E}$ has good reduction modulo the valuation ideal of $\mathcal{E}$ (indeed its reduction is the generic fiber of the reduction of $\underline{A}$ modulo v , which is proper when $v \nmid \nu$) , this pairing has to be trivial:

$$|q_{ij}|_{Gauss} = 1 .$$

g) An example: let us consider the Legendre elliptic pencil with parameter $x = \lambda$, given by the affine equation

$$v^2 = u(u-1)(u-x) .$$

Here one can choose $\tilde{E} = \mathbb{Z}\left[\frac{1}{2}\right]$, and one has the explicit formulae:

$$16q = x(1-x)^{-1}e^{-G/F}$$

$$x = 16q\Big(\prod_{m=1}^{\infty}(1+q^{2m})(1+q^{2m-1})^{-1}\Big)^8 ,$$

$$\text{where } F = \sum_{m=0}^{\infty} ((\tfrac{1}{2})_m/m!)^2 x^m$$

$$G = 2 \sum_{m=1}^{\infty} ((\tfrac{1}{2})_m/m!)^2 (\sum_{\ell=1}^{\infty} \tfrac{1}{\ell}) x^m .$$

This example is studied thoroughly in [Dw].

3. Vanishing periods.

a) Let us take up the situation 2d again, and assume that E is contained in the p–adic field K, with $\tilde{E} \subset R$. Assume also that the torus part of the semi–stable reduction splits.

As before, we then have our constant sheaves of lattices $\underline{M} = M$, $\underline{M}' = M$ on the v–adic unit polydisk Δ_v^n; let $\{\mu_i\}$ be a basis of M', and let $\{\mu_i'\}$ be the image of the dual basis of $M'^{\vee}(1)$ under j_{et} (defined up to sign, see III 1g).

On the other hand, we have the relative De Rham cohomology sheaf $H^1_{DR}(\underline{A}/S^*)$ which admits a canonical locally free extension to S (where the Gauss–Manin connection acquires a logarithmic singularity with nilpotent residue); in fact this extension is free because S is local, and we denote by $\{\omega_j\}$ a basis of global sections. We are aiming to give some analytic recipe to compute the Fontaine–Messing "vanishing periods" $\frac{1}{t_p} < \mu_i', \omega_j(s) >$ of the fiber $\underline{A}_1(s)$, $s \in \overset{*}{\Delta}{}_v^n$, see III 1f.

b) Let us express the composed morphism

$$H^1_{DR}(\underline{A}/S^*)^{can} \longrightarrow H^1_{DR}(\hat{\underline{A}}/\hat{S}) \longrightarrow H^1_{DR}(\hat{T})_{\hat{S}} \simeq M' \otimes \mathcal{O}_S$$

(roof = formal completion) in terms of the bases ω_j, $d\mu_i/1 + \mu_i$. We get a (2g,r)–matrix (ω_{ij}) with entries in $\mathcal{R} = \tilde{E}[[x_1, \dots, x_n]]$.

Lemma 3. For any $s \in \overset{*}{\Delta}{}_v^n$, one has the relation $\omega_{ij}(s) = \pm \frac{1}{t_p} < \mu_i', \omega_j(s) >$. Moreover ω_{ij} is a bounded solution of the Gauss–Manin partial differential equations on Δ^n.

Proof: the first assertion is easily proved by considering Raynaud's incarnation $[M(s) \longrightarrow \underline{G}(s)]$ of the rigid 1–motive associated to $\underline{A}_1(s)$, together with the trivial computation of Fontaine–Messing periods of the split torus $T = \underline{T}(s) : < \mu_i', d\mu_j/1+\mu_j > = \pm \delta_{ij} t_p$. The second assertion follows from the horizontality of the map $H^1_{DR}(\underline{A}/S^*)^{can} \longrightarrow H^1_{DR}(\hat{A}/\hat{S})$ w.r.t. the Gauss–Manin connections ∇, and the fact that M' is formed of horizontal sections of $H^1_{DR}(\hat{T})_{\hat{S}}$ (see also [vM]).

c) Let $\tilde{\omega}$ denote a uniformizing parameter of R. We modify slightly the setting of 2. d) by assuming that f_1 extends to a semi–abelian scheme $\tilde{f} : \underline{A}^{\sim}{}_{\mathrm{Spec}\, R \cap E} \longrightarrow \tilde{S}$, proper outside the divisor $\tilde{\omega} x_1 \dots x_n = 0$, and with constant split toral part on this divisor. Again, the ω_{ij}'s converge on Δ_v^n, and for every point $s \in S_1^*(E) \cap \Delta_v^{*n}$, the v–adic evaluation of ω_{ij} at s may be interpreted as in lemma 3 (if furthermore E is a number field, the ω_{ij}'s are in fact globally bounded series). We next look for complex interpretation.

d) Let $\iota : E \hookrightarrow \mathbb{C}$ be a complex embedding. We now assume that $s \in S_1^*(E)$ satisfies the following property: $\tilde{S}(\mathbb{C})$ should contain the polydisk of radius $|x_i(s)|$ (to insure the convergence of the analytic solutions of Gauss–Manin in this polydisk).

By specializing to s, construction 2c provides an embedding: $\iota_B : M^v \hookrightarrow H^1_B(A_s \otimes_\iota \mathbb{C}, \mathbb{Z})$, where $A_s := \underline{A}_1(s)$. Dually, we also have an embedding:

$$j_B : M'^v(1) = 2i\pi M'^v \hookrightarrow H_{1B}(A_{s,\mathbb{C}}, \mathbb{Z}) .$$

In addition to the orientation of $\mathbb{C}_p$, we choose an orientation of $\mathbb{C}$; this eliminates all ambiguities of signs, and allows to identify $j_B(\mu_j^v(1))$ with μ'_j.

Proposition 4. The following diagram is commutative:

$$\begin{array}{ccccc}
 & & M^v & & \\
 & \swarrow^{\iota_{et}} & & \searrow^{\iota_B} & \\
H^1_{et}(A_{s,\bar{K}}, \mathbb{Q}_p) & & & & H^1_B(A_{s,\mathbb{C}}, \mathbb{Z}) \\
\downarrow{\scriptstyle F.M.^{-1}} & & & & \downarrow{\scriptstyle \mathcal{F}^1_\iota} \\
H^1_{DR}(A_s) \otimes_E B_{DR} & & & & H^1_{DR}(A_s) \otimes_E \mathbb{C} \\
 & \nwarrow^{\text{v–adic evaluation}} & H^1_{DR}(\underline{A}/S^*)^\nabla & \nearrow_{\text{complex evaluation}} &
\end{array}$$

In particular (by duality), the complex evaluation of ω_{ij} at s gives the "usual" period $\frac{1}{2i\pi} < \mu'_i, \omega_j(s) >$.

Proof: let us draw a middle vertical arrow $\begin{matrix} M^{\vee} \\ \downarrow \\ H^1_{DR}(\underline{A}/S^*)^{\nabla} \end{matrix}$, defined by the obvious embedding $M^{\vee} = \Gamma\underline{M}^{\vee} \hookrightarrow \Gamma H^1_{DR}[\underline{M} \xrightarrow{\psi} \underline{G}]_{/S^*} = \Gamma H^1_{DR}(\underline{A}/S^*)$ (or equivalently, when $n = 1$, by the analog of ι_{DR} in the rigid analytic category over the discretely valued field $E((x)))$.

Then the commutativity on the L.H.S. is essentially the content of prop. 3; the commutativity on the R.H.S. follows immediately from the definition of ι_B (details are left to the reader).

This proposition suggests the following open question: assume that E is a number field, and denote by $\overline{E}$ its algebraic closure of E inside $\mathbb{C}_p$. Does there exist $\gamma : \overline{E} \hookrightarrow \mathbb{C}$ above ι such that the following diagrams commute?

$$\begin{array}{ccc} M^{\vee} & \xrightarrow{\iota_B} & H^1_B(A_{\mathbb{C}}, \mathbb{Z}) \\ {\scriptstyle \iota_{et}}\downarrow & & \downarrow \\ H^1_{et}(A_{\overline{E}}, \mathbb{Q}_p) & \xrightarrow{\gamma^*} & H^1_{et}(A_{\mathbb{C}}, \mathbb{Q}_p) \end{array} \qquad \begin{array}{ccc} M'^{\vee}(1) & \xrightarrow{j_B} & H_{1B}(A_{\mathbb{C}}, \mathbb{Z}) \\ \downarrow {\scriptstyle j_{et}} & & \downarrow \\ T_p(A_{\overline{E}}) & \xleftarrow{\gamma^*_{\sim}} & T_p(A_{\mathbb{C}}) \end{array}$$

(We leave it as an exercise to answer positively, when A_E is an elliptic curve, with help of $[S_2]$).

4. The monodromy filtration.

a) In $[G_2]$, Grothendieck constructs and studies thoroughly a 3–step filtration on $T_p(A_{\overline{K}})$, the "monodromy filtration" (here, we turn back to the setting of section 1)). By duality, we get a filtration W_{et} on H^1_{et}; it turns out that this filtration is the natural weight filtration on the H^1_{et} of Raynaud's incarnation of the associated rigid 1–motive, loc. cit. § 14.

b) According to the semi–stable philosophy (motivated by higher dimensional motives), it should be natural to handle the monodromy business on the De Rham realization. The monodromy filtration $W_{-1} = 0$, $W_0 \simeq \underline{M}^{\vee}(K) \otimes_{\mathbb{Z}} K$, $W_2 = H^1_{DR}$, $Gr^W_2 \simeq \underline{M}'(K) \otimes_{\mathbb{Z}} K$, is the canonical filtration associated with the nilpotent operator of level 2 defined by:

$$
\begin{array}{llcl}
N: & H^1_{DR}(A) & \dashrightarrow & H^1_{DR}(A) \\
& \| & & \| \\
& H^1_{DR}[\underline{M} \to G] & & H^1_{DR}[\underline{M} \to G] \\
& \downarrow & & \uparrow \\
& H^1_{DR}[0 \to T] & & H^1_{DR}[\underline{M} \to 0] \\
& \| & & \| \\
& \underline{M}'(K) \otimes_{\mathbb{Z}} K & \longrightarrow & \underline{M}^{v}(K) \otimes_{\mathbb{Z}} K
\end{array}
$$

where the arrow at the bottom $\underline{M}'(K) \longrightarrow \underline{M}^v(K)$ is the map induced by opposite of Grothendieck's monodromy pairing: $v(q) : M \otimes M' \xrightarrow{q} \mathbb{G}_m\big|_{K^{nr}} \xrightarrow{v} \mathbb{Z}$ (v = valuation), ibid (we change the sign because we work on H^1_{DR}, not on the covariant H_{1DR}). Assume moreover that $M = \underline{M}(K)$. Then the cokernel of the map $M' \longrightarrow M^v$ inducing μ is canonically isomorphic to the group of connected components of the special fiber of the Néron model A, see [CF] III 8.1. The weight filtrations W and W_{et} are related via F.–M. :

Lemma 4 (for $M = \underline{M}(K)$) : $Gr_0^W \oplus Gr_2^W(1) \xrightarrow{\sim} (Gr_0^{W_{et}} \oplus Gr_2^{W_{et}}) \otimes_{\mathbb{Q}_p} K$.

In case A is a Jacobian variety, there is moreover a Picard–Lefschetz formula (loc. cit. § 12), where ${}^{\ell}et(M^v)$ appears once again as the module of covanishing cycles.

c) Like the Raynaud extension, the operator N admits a complex analog (which is well–known). In the situation 2 c), let $D_j = \Delta^{j-1} \times \{0\} \times \Delta^{n-j} \subset \Delta^n$ be the j^{th} divisor "at infinity". For any $s \in \Delta^{*n}$, there is a monodromy action "around D_j" : $M_j^{\infty} \in GL(H_1(A_s,\mathbb{Z}))$, which is unipotent of level 2. Set $N_j^{\infty} := \frac{1}{2i\pi} \log^t (M_j^{\infty})^{-1} \in \operatorname{End} H^1(A_s,\mathbb{C})$. These nilpotent operators are constant on Δ^{*n}, and can be computed on the limit fiber by: $N_j^{\infty} = -\operatorname{Res}_{D_j}\nabla$ (the opposite of the residue at D_j of the Gauss–Manin connection).

Under the identification $H^1(A_s,\mathbb{Q}) \simeq H^1_B[\underline{M}(s) \longrightarrow \underline{G}(s)] \otimes \mathbb{Q}$, the "monodromy" filtration on the L.H.S. associated with N_j^{∞} is just the standard <u>weight filtration</u> on the R.H.S. [D1].

d) One can mimic the construction a) over any complete discretely valued ring instead of K, e.g. over $\mathcal{R} = \tilde{E}[[x]]$, $I = (x)$, in the situation 3 d), with $n = 1$; We denote by $N^{for} \in \operatorname{End} H^1_{DR}\left[\underline{A}/\mathcal{R}\left[\frac{1}{x}\right]\right]$ the nilpotent endomorphism obtained this way.

Next, we wish to compare N , N^{for} and N^{∞} .

Let us consider a double embedding $\tilde{E} \begin{smallmatrix} \nearrow R \\ \searrow \mathbb{C} \end{smallmatrix}$ and let $s \in S_1(E)$. Assume that $|x(s)|_v < 1$ and that $S_1(\mathbb{C})$ contains the disk of radius $|x(s)|_2$.

At last, set $A_s = \underline{A}_{(s)}$.

Proposition 5. In this situation, the complex evaluation of N^{for} at s is $N^{\infty} \in \text{End } H^1_{DR}(A_s \otimes \mathbb{C}) \simeq \text{End } H^1(A_{s,\mathbb{C}}, \mathbb{C})$; the v–adic evaluation of N^{for} at s is $v(x(s))N \in \text{End } H^1_{DR}(A_s \otimes K)$.

Proof: the complex fact is well–known. The v–adic assertion relies on the equality $v(q_{ij}(s)) = (\text{val}_x q_{ij}) \cdot v(x(s))$, which follows immediately from lemma 2.

[Remarks: d_1) if we only assume that $\underline{A}^{\sim} \longrightarrow \tilde{S}_1$ is proper outside $\tilde{\omega} x = 0$ (instead of $x = 0$) , the monodromy filtrations corresponding to $N^{for}_{(s)}$ and $N = N_s$ still coincide at the limit.

d_2) A quite general definition of N is given in [CF] III 10.]

IV. <u>Frobenius and the p–adic Betti lattice</u>.

1. <u>Semi–stable Frobenius</u>.

We take up again the situation I 3b), and explain a construction of the Frobenius semi–linear endomorphism φ_β (due to Raynaud [R_2]).

a) Let β denote a branch of the logarithm on K^x. This amounts to the choice of some uniformizing parameter of R, say $\tilde{\omega}$, characterized (up to a root of unity) by the fact that $\beta : K^x \simeq \tilde{\omega}^{\mathbb{Z}} \times (R/\tilde{\omega}R)^x \times (1+\tilde{\omega}R) \longrightarrow K$ factorizes through $1+\tilde{\omega}R$.

b) Let A be an Abelian variety over K with semi–stable reduction, and let $[\underline{M} \xrightarrow{\psi} G]$ the Raynaud realization of the associated rigid 1–motive (G sits in an extension $0 \longrightarrow T \longrightarrow G \longrightarrow B \longrightarrow 0$, and ψ is described by $q : \underline{M} \times \underline{M}' \longrightarrow \mathbb{G}_m$).

Let us factorize $q = \tilde{\omega}^{v(q)} \cdot q^0$, so that $q^0 : \underline{M} \times \underline{M}' \longrightarrow \mathbb{G}_m$ extends over R. This amounts to a factorization $\psi = \chi_{\tilde{\omega}} \cdot \psi^0$, where $\chi_{\tilde{\omega}} : \underline{M} \longrightarrow T = \underline{\mathrm{Hom}}(M',\mathbb{G}_m)$ is induced by $\tilde{\omega}^{v(q)}$ and $\psi^0 : \underline{M} \longrightarrow G$ extends over R (we use the same notation ψ^0 for this extension). Because T is a torus, the universal $(\underline{M},\chi_{\tilde{\omega}})$–equivariant vectorial extension of T splits canonically, which yields a canonical isomorphism of (Hodge) filtered K–vector spaces:

$$\Delta_\beta : H^1_{DR}[\underline{M} \xrightarrow{\psi^0} G]_{/R} \otimes_R K \xrightarrow{\sim} H^1_{DR}[\underline{M} \xrightarrow{\psi} G] = H^1_{DR}(A).$$

For two uniformizing parameters $\tilde{\omega}_1$, $\tilde{\omega}_2$, the map Δ_{β_1}, Δ_{β_2} are related by:

(i) $\Delta_{\beta_2}\Delta_{\beta_1}^{-1} = \exp(-\log \tilde{\omega}_2/\tilde{\omega}_1 \cdot N)$, where N is the operator defined in the previous section.

[Note the similarity with the definition of the canonical extension in the theory of regular connections, and also with [CF] III 9].

c) Let BT denote the Barsotti–Tate group attached to the reduction mod. $\tilde{\omega}$ of $[\underline{M} \xrightarrow{\psi^0} G]_{/R}$ (1), and let H^1_{crys/K^0} denote the K^0–space obtained by inverting p in its

---

(1) Remember that the Barsotti–Tate group attached to $[\underline{M} \xrightarrow{\psi^0} G]_{/R}$ is given by the image of ψ^0 under the connecting homomorphism $\mathrm{Hom}(\underline{M}, {}_{p^n}G) \longrightarrow \mathrm{Ext}(\underline{M}, {}_{p^n}G)$ associated with the exact sequence $0 \longrightarrow {}_{p^n}G \longrightarrow G \xrightarrow{p^n} G \longrightarrow 0$.

first crystalline cohomology group with coefficients in R^0. Up to isogeny, BT splits into the sum of two Barsotti–Tate groups: the constant one $\underline{M}(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p$, and $\underset{p^n}{\overleftarrow{\lim}}\ {}_{p^n}G \otimes_R R/\tilde{\omega}R$. [It follows that H^1_{crys/K^0} does not depend on $\tilde{\omega}$; in fact, it depends only on $A_R \otimes R/\tilde{\omega}^2R$, which determines $G \underset{R}{\otimes} R/\tilde{\omega}^2R$.]

The K^0–structure H^1_0 mentioned in I 3 b_1) is just the image of H^1_{crys/K^0} under Δ_β inside $H^1_{DR}(A)$; the element u_β is $u_\beta := -\mathrm{LOG}\ \tilde{\omega}$ (defined up to translation by $\mathbb{Z}_p(1) \subset B^+_{cris}$).

By transport of structure, the σ–semi–linear Frobenius on H^1_{crys}/K^0 provides the σ–semi–linear endomorphism $\varphi = \varphi_\beta$ on H^1_0 (σ = Frobenius on K^0). Using (i), one gets the following relation:

(ii) $$\varphi_{\beta_2} \circ \varphi_{\beta_1}^{-1} = \exp(-\tfrac{1}{p}\log(\tilde{\omega}_2/\tilde{\omega}_1)^{p-\sigma} \cdot N).$$

From the functoriality of Raynaud extensions G and of the rigid analytic isomorphisms $G^{rig}/M = A^{rig}$, it follows that the semi–stable structure (H^1_0, φ, N) is functorial in A.

e) That construction of Raynaud may be extended to the relative situation III 2, i.e. over $\mathcal{R} = \tilde{\omega}$ – adically complete noetherian normal R^0–algebra.

Let $U \subset \mathrm{Spec}\ \mathcal{R}$ be as in loc. cit., and let us choose a lifting $\sigma \in \mathrm{End}\ U$ of the char. p Frobenius. By analogy with step c), we can construct, locally for the "loose" topology on U, a horizontal morphism $\phi_\beta(\sigma) : \sigma^* H^1_{DR}(\underline{A}/U) \longrightarrow H^1_{DR}(\underline{A}/U)$; furthermore, this morphism "stabilizes" $\underline{M}^{\vee}_U$, and it can be globally defined there. [This is the "stability of vanishing cycles" mentioned in [Dw]; indeed, when say $\mathcal{R} = R\widehat{[\tilde{\omega}x]}$, $\sigma : x \longmapsto x^p$, ϕ_β is nothing but the analytic Dwork–Frobenius mapping].

If A is the fiber $\underline{A}(s)$ of $\underline{A}$ at some point $s \in U$ fixed under σ, we recover $\phi_\beta(\sigma) = \varphi_\beta$.

2. <u>Construction of $H^1_\beta(\underline{A})$</u>.

From now onwards, we shall assume that A <u>has multiplicative reduction</u>.

a) With our previous notations, we then obtain the following consequences:

a_1): $G = T$, $r = g$,

a_2): the Hodge filtration splits canonically:

$$H^1_{DR} = (\underline{M}^v(K) \otimes_{\mathbb{Z}} K) \oplus F^1$$

a_3): the monodromy filtration consists of only two steps:

$$Gr^{W_{et}} H^1_{et} \simeq (M^v \otimes_{\mathbb{Z}} \mathbb{Q}_p) \oplus (M' \otimes_{\mathbb{Z}} \mathbb{Q}_p(-1)) \ \ (\text{via } \iota_{et} \text{ and } j_{et}) ,$$

$$Gr^W H^1_{DR} \simeq (M^v(K) \otimes_{\mathbb{Z}} K) \oplus (\underline{M}'(K) \otimes_{\mathbb{Z}} K) ,$$

(these isomorphisms being compatible via F.M., by prop. 3 and its dual) $\underline{M}^v(K) \otimes_{\mathbb{Z}} K = \mathrm{Ker}\, N$, and F^1 projects onto $\underline{M}'(K) \otimes_{\mathbb{Z}} K$ (this isomorphic projection being given by $F^1 = \mathrm{Colie}\ A^{rig} \simeq \mathrm{Colie}\ T^{rig} \xrightarrow{\sim} \underline{M}'(K) \otimes_{\mathbb{Z}} K)$.

a_4): the Fontaine–Messing isomorphism F.M. is described in I 4 c).

b) The splitting of BT (up to isogeny) reflects on H^1_0 , and translates into an isomorphism:

$$\Sigma_\beta : Gr^W H^1_0 \xrightarrow{\sim} H^1_0$$

(φ acts trivially on $Gr_0 = \underline{M}^v(K) \otimes_{\mathbb{Z}} K^0$, and by multiplication by p on the image of $Gr_1 = \underline{M}'(K) \otimes_{\mathbb{Z}} K^0)$.

Let us now choose an orientation of $\mathbb{C}_p$ (see III 1f): $\mathbb{Z}(-1) := X^*(\mathbb{G}_m) \hookrightarrow \mathbb{Z}t_p^{-1} \subset B_{DR}$,and let us consider the etale lattice $\underline{\Lambda} := \underline{M}^v \oplus \underline{M}'(-1)$, and let $\Lambda := \underline{\Lambda}(\overline{K}) = \underline{\Lambda}(\overline{R}) = \underline{\Lambda}(K^{nr})$, where K^{nr} denotes the maximal subfield of $\overline{K}$ non ramified over K .

Using Σ_β and the orientation, we can embed Λ into $H^1_{DR} \otimes_K K^{nr}\left[\frac{1}{t_p}\right] \subset H^1_{DR} \otimes_K B_{DR}$, and we call p–adic Betti lattice its image, which we denote by H^1_β [This is the dual of the lattice L_β mentioned in the introduction. The introduction of t_p , the "p–adic $2i\pi$" , is motivated by the fact that the complex Betti lattice (in the setting III 4c) is stable under $2i\pi N_\infty$, not N_∞] .

We thus get a tautological isomorphism:

$$\mathcal{P}_\beta : H^1_{DR} \underset{K}{\otimes} K^{nr}[t_p] \xrightarrow{\sim} H^1_\beta \underset{\mathbb{Z}}{\otimes} K^{nr}[t_p]$$

where in fact K^{nr} could be replaced by some finite extension of K, or else by K itself if T is split.

From formulae (i) (ii), it follows:

(iii) $$H^1_{\beta_2} = \exp(-\log \tilde{\omega}_2/\tilde{\omega}_1 \cdot N) \cdot H^1_{\beta_1} .$$

From the very construction of H^1_β and the formula $\varphi t_p = pt_p$, we get:

<u>Lemma 5</u>: The lattice H^1_β spans the $\mathbb{Q}_p$–space of φ_β–invariants in $H^1_{DR} \underset{K}{\otimes} K^{nr}[t_p]$.

<u>Remark</u>: the image of $\mathcal{P}_\beta^{-1} H^1_\beta$ under F.M. does <u>not</u> lie in $H^1_{et}(A,\mathbb{Q}_p)$; compare with lemma 4.

c) Let us now describe the complex analog of $\Sigma_\beta : \Lambda \longrightarrow H^1_\beta$. So let $A_{\mathbb{C}}$ be a complex Abelian variety in Jacobi form $T_{\mathbb{C}}/M$ (the quotient being alternatively described by $q : M \otimes M' \longrightarrow \mathbb{C}^\times$, where $M' = X^*(T_{\mathbb{C}})$). Let us orient $\mathbb{C}$, and choose a branch β_∞ of the complex logarithm, and compose with $q : M \otimes M' \xrightarrow{\beta_\infty \circ q} \mathbb{C}$. We get an embedding $M \hookrightarrow M'^{\vee} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathrm{Lie}\, T_{\mathbb{C}} \simeq H_{1B}(A_{\mathbb{C}},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ which factorizes through $H_{1B}(A_{\mathbb{C}},\mathbb{Z})$. This in turn provides an isomorphism $\Sigma_{\beta_\infty} : \Lambda = M^\vee \oplus M'(-1) = M^\vee \oplus \frac{1}{2i\pi} M' \xrightarrow{\sim} H^1_B(A_{\mathbb{C}},\mathbb{Z})$ (the injectivity is a consequence of the Riemann condition $\mathrm{Re}\, \beta_\infty(q) < 0$).

[d) One can imitate the construction of the p–adic lattice in the case of an Abelian variety B with ordinary good reduction over $K = K_0$. Over $\widehat{K^{nr}}$ indeed, the Barsotti–Tate group $B(p) = \varinjlim {}_{p^n}B$ becomes isomorphic to the B.–T. group associated to a 1–motive $[M \xrightarrow{\psi} T]$, where ψ is given by the Serre–Tate parameters [K]. However, in contrast to the multiplicative reduction case, the lattice $\simeq M^\vee \oplus M'(-1)$ obtained in this way is <u>not</u> functorial, as is easily seen from the case of complex multiplication $(\psi = 1)$.

e) The construction of Frobenius generalizes easily to the case of 1–motives. This allows to construct p–adic Betti lattices for 1–motives whose Abelian part has multiplicative reduction. We shall not pursue this generalization any further here.]

3. Computation of periods.

a) We shall compute the matrix of the restriction of $\mathcal{P}_\beta$ to $F^1H^1_{DR}$ w.r.t. the bases $\{d\mu_j/1+\mu_j\}_{j=1}^g$ in F^1, $\{\mu''_i = \Sigma_\beta(\mu_i(-1))$, $m^\vee_i\}_{i=1}^g$ in H^1_β, assuming that T splits over K. In other words, we compute half of the (β)–p–adic period matrix.

Proposition 6. Let $q_{ij} = q(m_i,\mu_j)$, as in I 4 c). The following identity holds in $H^1_{DR}(A_K)\otimes_K K[t_p]$:

$$d\mu_j/1+\mu_j = t_p\mu''_j + \sum_{i=1}^{g} \beta(q_{ij})m^\vee_i .$$

b) Proof: it relies on a deformation argument. First of all, one may replace M by a sublattice of finite index, such that $q \equiv \tilde{\omega}^{v(q)}q^0$ with $q^0 \equiv 1 \bmod \tilde{\omega}$ (in this situation BT splits actually, not only up to isogeny). Let us consider the analytic deformation $\left[M \xrightarrow{\underline{\Psi} = X_{\tilde{\omega}}\cdot\underline{\Psi}^0} T\right]$ of $\left[M \xrightarrow{\Psi = X_{\tilde{\omega}}\cdot\Psi^0} T\right]$ over $\mathcal{R} = R[[\xi_{ij}-\delta_{ij}]]^g_{i,j=1}$ δ_{ij} = Kronecker symbol, $\underline{\Psi}^0$ being given by the matrix ξ_{ij} (so that $[M \xrightarrow{\psi} T]$ arise as the fiber at $\xi_{ij} = q^0_{ij}$). For the fiber at $\xi_{ij} = \delta_{ij}$: $[M \xrightarrow{X_{\tilde{\omega}}} T]$, the $F^1H^1_{DR}$ coincides with $\Sigma_\beta(Gr^W_1H^1_{DR})$; more precisely $d\mu_j/1+\mu_j = t_p\mu''_j$, at $\xi_{ij} = \delta_{ij}$. By definition of the Kodaira–Spencer mapping K.S. (see e.g. [CF] III. 9), one deduces that

$$d\mu_j/1+\mu_j = t_p\mu''_j + \left(\int_{\xi_{ij}=\delta_{ij}}^{q^0_{ij}} K.S.\right)m^\vee_i \text{ , at } \xi_{ij} = q^0_{ij} .$$

But in our bases, K.S. is expressed by the matrix $d\xi_{ij}/\xi_{ij}$ (see [Ka], or [CF] ibid, where there is a minus sign because of a slightly different convention). One concludes by noticing that $\log q^0_{ij} = \beta(q_{ij})$.

c) One could also argue as follows, using F.M.: it follows from 2 a 3) that $d\mu_j/1+\mu_j$ may be expressed in the form $t_p\mu''_j + \Sigma\beta_{ij}m^\vee_i$, $\beta_{ij}\in K$; furthermore, these coefficients β_{ij} are uniquely determined by the property that $d\mu_j/1+\mu_j - \Sigma\beta_{ij}m^\vee_i$ lies in $H^1_0 \otimes_{K^0} B_{ss}$ and is multiplied by p under φ_p. Let us show that $\beta_{ij} = \beta(q_{ij})$ satisfies this property: by I 4 c), we have

$$d\mu_j/1+\mu_j = t_p FH^{-1}(\tilde{\mu}_j) + \Sigma\, LOG(q_{ij}) m_i^v \text{ , so that}$$

$$d\mu_j/1+\mu_j - \Sigma\, \beta_{ij} m_i^v = \Sigma(LOG(q_{ij}) - \beta(q_{ij})) m_i^v + t_p FM^{-1}(\tilde{\mu}_j) \,.$$

Because $\tilde{\mu}_i \in H^1_{et}$, $t_p FM^{-1}(\tilde{\mu}_j) \in (H^1_0 \otimes B_{ss})^{\varphi=p}$, and we conclude by the following:

Lemma 6: let $c \in K^x$. Then "the" element $LOG\, c - \beta c$ of B_{ss} is multiplied by p under the Frobenius φ_β .

Proof: let us write $c = \tilde{\omega}^{v(c)} c^0$, so that $LOG\, c - \beta c = -v(c) u_\beta + LOG\, c^0 - \log c^0$. Now $LOG\, c^0 - \log c^0 = -\log \lim (\tilde{c}_n)^{p^n}$ in B^+_{cris} , where $\tilde{c}_n$ is any lifting of $c_n = (c^0)^{p^{-n}} \in \overline{R}$. Let $c'_n = (\ldots c_{n+1}, c_n) \in \varprojlim_{x \mapsto x^p} \overline{R}$, and let $\tilde{c}_n$ be the Teichmüller representative $[c'_n] \in W(\varprojlim \overline{R})$. We have $[c'_n]^\varphi = [c'^p_n] = [c'_{n-1}] = \tilde{c}_{n-1}$, whence $(\lim \tilde{c}_n p^n)^\varphi = (\lim \tilde{c}_n^{p^n})^p$. It remains only to take logarithms and remind that $\varphi_\beta u_\beta = p u_\beta$.

d) Let us examine the complex counterpart, as in 2 c). The lattice $M \oplus M'^v(1) = \Lambda^v \xrightarrow[\sim]{({}^t\Sigma_{\beta_\infty})^{-1}} H_{1B}(A_{\mathbb{C}}, \mathbb{Z})$ embeds into Lie $T_{\mathbb{C}}$; the subspace $F^0 H_{1DR}(A_{\mathbb{C}})$ of $H_1(A_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H_{1DR}(A_{\mathbb{C}})$ is just the kernel of the complexification of this embedding. It follows that the canonical lifting $\tilde{m}_i$ of m_i inside $F^0 H_{1DR}(A_{\mathbb{C}})$ is given by $\tilde{m}_i = m_i - \frac{1}{2i\pi} \Sigma \beta_\infty(q_{ij}) \mu'_i$ (we set $\mu'_i = ({}^t\Sigma_{\beta_\infty})^{-1}(\mu_i^v(1))$, and $\mu''_j = \Sigma_{\beta_\infty}(\mu_j(1)))$. By orthogonality $(F^1 H^1_{DR} = (F_0 H_{1DR})^\perp)$, we obtain:

Proposition 7: the following identity holds in $H^1_{DR}(A_{\mathbb{C}})$:

$$d\mu_j/1+\mu_j = 2i\pi\mu''_j + \Sigma\beta_\infty(q_{ij}) m_i^v \,.$$

[The compatibility (resp. analogy) between prop. 6 and formula (iii) resp. prop. 7., is a good test for having got the right signs. Although μ''_j is defined quite differently in the p–adic , resp. complex case, the exterior derivative of the coefficients of m_i^v's describes in both cases the Kodaira–Spencer mapping.]

4. Periods in the relative case, and Dwork's p–adic cycles.

a) Let us consider the relative situation as in 1. d with $r = g$; U being subject to be the complement of divisor with normal crossings $\tilde{\omega}x_1 \dots x_n = 0$. We set $\mathcal{R} = R[[x_1, \dots, x_n]]$, and we denote by $\mathcal{S}$ the K–algebra generated by $\mathcal{R}\left[\frac{1}{x_1 \cdots x_n}\right]$ and (β)–logarithms of non–zero elements of $\mathcal{R}\left[\frac{1}{x_1 \cdots x_n}\right]$. The construction of H^1_β can be transposed to this relative setting: We use "the" relative Frobenius $\phi_\beta(\sigma)$ to construct an embedding

$$\underline{\Lambda} \xrightarrow{\sim} \underline{H}^1_{\beta,\sigma} \subseteq H^1_{DR}(\underline{A}/\mathcal{S}[t_p]) ,$$

such that $\phi_\beta(\sigma)\Big|_{\mathrm{Im}\,\underline{\Lambda}} = \sigma_*$. Of course, when $\sigma s = s$, we recover $\underline{H}^1_{\beta,\sigma}(s) = H^1_\beta$.

Because $\phi_\beta(\sigma)$ is horizontal, so is $\underline{H}^1_{\beta,\sigma}$ (it is locally constant w.r.t. the loose topology), and we get:

Lemma 7: $H^1_{DR}(\underline{A}/\mathcal{S}[t_p])^\nabla = \underline{H}^1_{\beta,\sigma} \otimes_{\mathbb{Z}} K[t_p]$.

b) In order to interpret the lattice $\underline{H}^1_{\beta,\sigma}$ (for $n = 1$, $\phi : x \longmapsto x^p$) in terms of Dwork's p–adic cycles [Dw], one forgets about t_p (or better, one specializes t_p to $1 : K[t_p] \longrightarrow K$, $\underline{H}^1_{\beta,\sigma} \simeq \underline{M}^\vee \oplus \underline{M}'$) . Let us for instance take back the example III 2g (Legendre). For $K = \mathbb{Q}_p(\sqrt{-1})$ $(p \neq 2)$, we have $M = \underline{M}(K)$, with base m . Setting $v = uw$, the period of the differential of the first kind $\omega = \frac{du}{2v}$ for the covanishing cycle $m^\vee$ at $x = 0$ is given by the residue of $\frac{du}{2uw} = \frac{dw}{w^2+1}$ at one of the two points above $u = 0$ on the rational curve $w^2 = u - 1$; namely, this is $\frac{\sqrt{-1}}{2}$.

Let μ be the basis of $M' = \underline{M}'(K)$ lifted to H^1_β , such that $q = q(m,\mu)$ is given by the formula displayed in III 2. g. Then after specializing t_p to 1 , the matrix of $\mathcal{P}_\beta$ in terms of the bases ω , $\omega' = \nabla(x\frac{d}{dx})\omega$ is

$$\frac{\sqrt{-1}}{2}\begin{bmatrix} F & x\,\dot{F} \\ F.\log q & x(F\,\dot{\log}\,q) \\ = F\,\log x - \log 16 + \dots & = 1 + x\dot{F}\,\log x + \dots \end{bmatrix} \quad \text{(with determinant } (4x(x-1))^{-1}) .$$

Here "log" is standing for the branch β, and $\dot{F}$ for $\frac{d}{dx}F$.

In fact, Dwork prefers to get rid of the constants $\log 16$ and $\frac{\sqrt{-1}}{2}$, by changing the basis $\{\mu, m^v\}$ into $-2\sqrt{-1}\{\mu + (\log 16)m^v, m^v\}$. In this new basis, the entries of the period matrix lie in $\mathbb{Q}[[x]][[\log x]]$, and the matrix of $\phi_\beta(x \longmapsto x^p)$ becomes $(-1)^{\frac{p-1}{2}} \begin{bmatrix} p & 0 \\ \log 16^{1-p} & 1 \end{bmatrix}$ see [Dw] 8. 11.

c) In section 3, we computed periods of one–forms of the first kind. The "horizontality lemma" 7 then allows to obtain other periods by taking derivatives; still, we have to show that, in the multiplicative reduction case, any one–form of the second kind is the Gauss–Manin derivative of some one–form of the first kind. In other words:

Lemma 8. Let us consider a relative situation, as in III 2c or 2d. If $r = g$, then for any $k = 1, \dots, n$, the smallest $\mathcal{O}_S[\nabla(x_k \partial/\partial x^k)]$–submodule of $H^1_{DR}(\underline{A}/S^*)$ containing F^1 is $H^1_{DR}(\underline{A}/S^*)$.

Indeed, this amounts to the surjectivity of K.S., which follows from the invertibility of its residue at $x_k = 0$; this follows in turn from the fact that this residue $(F^1)_0^{can} \simeq \underline{M}'(\mathcal{K}) \otimes E \to (H^1_{DR}/F^1)_0^{can} \simeq \underline{M}^v(\mathcal{K}) \otimes E$ is induced by the non–degenerate pairing $\mathrm{val}(x_k) \circ q$. In the situation of III 3 a) b), we can now complete the analytic description of the period matrix: take a basis ω_j of the canonical extension of $H^1_{DR}(\underline{A}/S^*)$ in the form

$$\begin{cases} \omega_j \in F^1 \\ \omega_{j+g} = \nabla(x_k \partial/\partial x_k)\omega_j \end{cases} \quad j = 1, \dots g.$$

Lemma 9. The matrix of $\mathcal{P}_\beta$ w.r.t. the bases $\{\omega_j\}$, $\{\mu_i'', m_i\}$ has the form:

$$\begin{bmatrix} \pm t_p \omega_{ij}(s) & \pm t_p (x_k \partial/\partial x_k \omega_{ij})(s) \\ \omega_{ij}(s) \log q_{ij}(s) & (x_k \partial/\partial x_k (\omega_{ij} \log q_{ij}))(s) \end{bmatrix} \quad (\text{for } A = \underline{A}_1(s))$$

d) We are now in position to state the main result of this section IV, relating p–adic and complex Betti lattices.

Data: d_1): a field E, doubly embedded $E \nearrow \mathbb{C}$, $E \searrow K$; orientations of $\mathbb{C}$ and $\mathbb{C}_p$. A branch β (resp. β_∞) of the logarithm on K^x (resp. on $\mathbb{C}^x$); a uniformizing parameter $\tilde{\omega}$ such that $\beta(\tilde{\omega}) = 0$.

d_2): an affine curve S_1 over E ; a smooth point $0 \in S_1(E)$, and a local parameter x around 0 ; a regular model $\tilde{S}_1$ of S_1 over $E \cap R$.

d_3): a semi–abelian scheme $f : \underline{A} \longrightarrow \tilde{S}_1$, proper outside the divisor $\tilde{\omega}x = 0$, and given by a split torus on this divisor. To f , one attaches as before the constant sheaf of lattices $\Lambda = M^{\vee} \oplus M'(-1)$ (outside $x = 0$) , and the bilinear form $q : M \otimes M' \longrightarrow \mathbb{G}_m$ (outside $\tilde{\omega}x = 0$) . Taking bases of M , resp. M' , one may expand the entries of a matrix of q into Laurent series: $q_{ij} = \eta_{ij} x^{n_{ij}}$ + h.o.t., and consider the double homomorphism from the E–algebra $E_1 := E[\log \eta_{ij}, t] \begin{smallmatrix} \nearrow K[t_p] \\ \searrow \mathbb{C} \end{smallmatrix}$ induced by β , $t \longmapsto t_p$ (resp. β_∞ , $t \longmapsto 2i\pi$) .

d_4): a simply connected open neighborhood of 0 in $S_{\mathbb{C}}$, say $\mathcal{U}$; over $\mathcal{U} \backslash 0$, Λ is identified with the graded form (w.r.t. the local monodromy N_∞) of $R^1 f_*^{an} \mathbb{Z}$.

d_5): a point $s \in S_1(E)$ such that $s \in \mathcal{U}$ and $|x(s)|_v < 1$ (from this last condition, it follows that the fiber $\underline{A}(s)$ has multiplicative reduction mod $\tilde{\omega}$) .

Combining the previous lemma with propositions 4, 6, and 7, we obtain:

<u>Theorem 2</u>. The following diagram is commutative:

$$\begin{array}{ccc}
 & \Lambda & \\
\Sigma_\beta \swarrow & & \searrow \Sigma_{\beta_\infty} \\
H^1_\beta(\underline{A}(s)) \otimes_{\mathbb{Z}} K[t_p] & & H^1_B(\underline{A}(s) \otimes \mathbb{C}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \\
\nwarrow & & \nearrow \\
 & H^1_{DR}(\underline{A}/E_1[[x]][\log x])^\nabla &
\end{array}$$

Left arrow: p–adic evaluation at s (via $\log x \longmapsto \beta(x(s))$ and $E_1 \longrightarrow K[t_p]$)

Right arrow: complex evaluation at s (via $\log x \longmapsto \beta_\infty(x(s))$ and $E_1 \longrightarrow \mathbb{C}$)

(In the example III 2g, E_1 is just $\mathbb{Q}(\sqrt{-1})[t]$, and the parameter $x = \lambda$ should be replaced by $x = 16\lambda$) .

V. p–Adic lattice and Hodge classes.

1. Rationality of Hodge classes.

a) Let A_E be an Abelian variety over a number field E. Let v be a finite place of E where A_E has multiplicative reduction, and let $K = E_v$ denote the completion.

Conjecture 6. Let $\xi \in (\mathrm{End}\, H^1_{DR}(A_E))^{\otimes n}$ be some Hodge class [(1)]. Then for every branch β of the logarithm on K^x, the image of ξ under $\mathcal{P}_\beta$ lies in the rational subspace $(\mathrm{End}\, H^1_{\beta,\mathbb{Q}})^{\otimes n}$, where $H^1_{\beta,\mathbb{Q}} := H^1_\beta \otimes_{\mathbb{Z}} \mathbb{Q}$. (For instance, this holds if $n = 1$ just by functoriality of H^1_β).

b) Let $\iota : E \hookrightarrow \mathbb{C}$ and let Sh be the connected Shimura variety associated to the Hodge structure $H^1_B(A_E \otimes_\iota \mathbb{C}, \mathbb{Z})$ and to some (odd prime–to–p) N–level–structure; Sh descends to an algebraic variety over some finite extension E' of E, and $A_{E'}$ is the fiber of an Abelian scheme $\underline{A} \longrightarrow \mathrm{Sh}$ at some point $s \in \mathrm{Sh}(E')$. In terms of Siegel's modular schemes $A_{g,N}$ [CF] IV, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh} & \hookrightarrow & A_{g,N} \otimes_{\mathbb{Z}\left[\frac{1}{N},\zeta_N\right]} E' \\ \cap & & \cap \\ \overline{\mathrm{Sh}} & \hookrightarrow & \overline{A}_{g,N} \otimes_{\mathbb{Z}\left[\frac{1}{N},\zeta_N\right]} E' \end{array},$$

where the superscript $-$ denotes suitable projective toroidal compactifications, see [H].

In fact $\underline{A} \longrightarrow \mathrm{Sh}$ extends to a semi–abelian scheme over a normal projective model $\widetilde{\mathrm{Sh}}$ of $\overline{\mathrm{Sh}}$ over $\mathcal{O}_{E'}$ (namely $\widetilde{\mathrm{Sh}}$ = normalization of the schematic adherence of $\overline{\mathrm{Sh}}$ in $\overline{A}_{g,N} \otimes_{\mathbb{Z}\left[\frac{1}{N},\zeta_N\right]} \mathcal{O}_{E'}$).

---

(1) Some authors prefer to look at Hodge classes in the more general twisted tensor spaces $(H^1_B)^{\otimes m_1} \otimes (H^{1\vee}_B)^{\otimes m_2}(m_3)$. However such spaces contain Hodge classes only if $m_1 + m_2$ is even (in fact if $m_1 - m_2 = 2m_3$), and any polarization then provides an isomorphism of rational Hodge structures $H^{1\ \otimes m_1}_B \otimes (H^{1\vee}_B)^{m_2}(m_3) \simeq (\mathrm{End}\, H^1_B)^{\otimes \frac{m_1+m_2}{2}}$. In particular, these extra Hodge classes do not change the Hodge group.

We consider the following condition:

(*) There exists a zero–dimensional cusp in $\overline{\mathrm{Sh}}$, say 0 , such that 0 and s have the same reduction mod. the maximal ideal of R' . In fancy terms, this means that any Abelian variety with multiplicative reduction in characteristic p should also degenerate multiplicatively (in characteristic 0) inside the family "of Hodge type" that it defines [M].

Remark: condition (*) should follow from Gerritzen classification [Ge] of endomorphism rings of rigid analytic tori (which is the same in equal or unequal characteristics), in the special case of Shimura families of p_{EL}–type [Sh] (i.e. characterized by endomorphisms).

Theorem 3. Conjecture 6 follows from (*).

Proof: by definition of the Shimura variety, and by the theory of absolute Hodge classes [D_2], $\xi = \underline{\xi}(s)$ is the fiber at s of a global horizontal section $\underline{\xi} \in \Gamma(\mathrm{End}\, H^1_{DR}(\underline{A}/\mathrm{Sh})^{\otimes n})^{\nabla}$.

Let S_1 be an algebraic curve on $\overline{\mathrm{Sh}}$, joining 0 and s , and smooth at 0 ; let x be a local parameter around 0 , with $|x(s)|_v < 1$. Then because 0 is a 0–dimensional cusp, $\underline{A}$ degenerates multiplicatively at 0 and we are in the situation where theorem 2 applies.

The β–periods of $\underline{\xi}$ admit an expansion in the form $\sum_{\ell=0}^{n} \alpha_\ell \log^\ell x$, with $\alpha_0 \in E'[[x]]$, $\alpha_\ell \in E'_1[[x]]$, whose complex evaluation (w.r.t $\iota : E' \hookrightarrow \mathbb{C}$) gives the corresponding complex period of $\underline{\xi}$, according to theorem 2. Since $\underline{\xi}$ is a global horizontal section and a Hodge class at s , the complex periods are rational constants: $\alpha_\ell = 0$ for $\ell > 1$, and $\alpha_0 \in \mathbb{Q}$. Thus the β–periods of $\xi = \underline{\xi}(s)$ are rational numbers.

Remark: it follows (inconditionally) from theorem 1 and Fontaine' semi–stable theorem that the image of ξ under $\mathcal{P}_\beta$ lies in $(\mathrm{End}\, H^1_\beta)^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

2. p–Adic Hodge classes.

Let E' be some finitely generated extension of E . We define a p–adic Hodge class on $A_{E'}$ to be any element ξ of $F^0(\mathrm{end}\, H^1_{DR}(A_{E'})^{\otimes n})$ such that for every E–embedding of E' into any finite extension K' of K , and for every branch β of the logarithm on $K'^{\times}$, the image of ξ under $\mathcal{P}_\beta$ lies in the rational subspace $(\mathrm{End}\, H^1_{\beta,\mathbb{Q}})^{\otimes n}$. Conjecture 6 predicts that any Hodge class is a p–adic Hodge class, and conjecture 2 would identify the two notions.

Proposition 8: if E is algebraically closed in E', then any p–adic Hodge class ξ comes from $(\mathrm{End}\, H^1_{DR}(A_E))^{\otimes n}$, and is sent into $[(\mathrm{End}\, H^1_{et})^{\otimes n}]^{\mathcal{G}}$ by F.M.

Proof: the first assertion follows Deligne's proof in the complex case $[D_2]$. To prove the second one, we remark that $\xi \in F^0[(\mathrm{End}\, H^1_0)^{\otimes n}]^{\varphi=1}$; moreover, by changing β continuously, the lattice H^1_β is moved by $\exp(-\log u.N)$, $u \in R^\times$. Since ξ has to remain rational w.r.t. all these lattices, we deduce that $N\xi = 0$, and we conclude by Fontaine semi–stable theorem.

Remark: it is essential to take all E–embedding $E' \hookrightarrow K$ into account; for instance, $m^v \in F^0 H^1_{DR}(A_{E'})$ for $E' = K$, and $m^v \in H^1_\beta$, $FM(m^v) \in (H^1_{et})^{\mathcal{G}}$, but it is highly probable that m^v is not defined over $\overline{E} \cap K$.

3. A p–adic period conjecture.

For any E–algebra E', the E'–linear bijections $H^1_{DR}(A_E) \otimes_E E' \xrightarrow{\sim} (H^1_{\beta,\mathbb{Q}}) \otimes_{\mathbb{Q}} E'$ which preserve p–adic Hodge classes form the set of E'–valued points of an irreducible E–torsor P_β under the "p–adic Hodge group" of A_E (which is by definition the algebraic subgroup of $GL\, H^1_{DR}(A_E)$ which fixes the p–adic Hodge classes; conjecture 2 would identify this group with the Hodge group). One has a canonical $K[t_p]$–valued point of P_β given by $\mathcal{P}_\beta$. A variant of conjecture 1 may be stated as follows:

Conjecture 1′: for sufficiently general β, $\mathcal{P}_\beta$ is a (Weil) generic point of P_β.

The next section will offer two partial positive answers.

4. Period relations of bounded degree.

a) We denote by $E[\mathcal{P}_{\beta_v}]_{\leq \delta}$ the quotient of the polynomial ring in $4g^2$ indeterminates over E by the ideal generated by relations of degree $\leq \delta$ among (β_v)–p–adic periods $(v\,|\,p)$. Hence for sufficiently large δ, there is a natural embedding $\mathrm{Spec}\, E[\mathcal{P}_{\beta_v}]_{\leq \delta} \subset P_{\beta_v}$. The same construction works simultaneously at several places of multiplicative reduction: $E[(\mathcal{P}_{\beta_v})_{v \in V}]_{\leq \delta} \subset \prod_v E_v[t_p]$, and we have projections $\mathrm{Spec}\, E[(\mathcal{P}_{\beta_v})_{v\in V}]_{\leq \delta} \longrightarrow P_{\beta_v}$.

b) Assume that A_E is the fiber at $s \in S_1(E)$ of a semi–abelian scheme $\underline{A} \longrightarrow S_1$ over an affine curve $S_1/\mathrm{Spec}\, E$, proper outside some smooth point $0 \in S_1(E)$, and degenerating to a split torus at this point. Let x be a local parameter around 0, and let $\delta >> 0$.

We lay down an extra <u>normalization hypothesis</u>:

(**) the entries of the q–matrix expand $q_{ij} = \eta_{ij} x^{n_{ij}} + \dots$ where η_{ij} are <u>roots of unity</u> (this is the case in example III 2g), if we set $x = 16\lambda$ and $E = \mathbb{Q}(\sqrt{-1})$.

In these circumstances, we have the following two results:

<u>Theorem 4</u>. Assume that $|x(s)|_v$ is sufficiently small – w.r. to δ – so that in particular $A_E = \underline{A}(s)$ has multiplicative reduction at v. Let us choose $\beta = \beta_v$ such that $\beta(x(s)) = 0$.

Then $\mathrm{Spec}\, E[\mathcal{P}_\beta]_{\leq \delta} = P_\beta$, and moreover any p–adic Hodge class on A_E is a Hodge class.

<u>Theorem 5</u>. Assume that $\underline{A} \longrightarrow S_1$ extends to a semi–abelian scheme over some regular model of S_1 over $\mathcal{O}_E$, proper outside the divisor $\nu x = 0$, $\nu \in \mathbb{N}$. Let $V(s)$ denote the finite set of finite places v of E where $|x(s)|_v < |\nu|_v$ (so that $\underline{A}(s)$ has multiplicative reduction at $v \in V$). Let us choose β_v such that $\beta_v(x(s)) = 0$, $v \in V(s)$, and let $\varepsilon > 0$. If for every $\iota : E \hookrightarrow \mathbb{C}$, $|x(s)|_\iota \geq \varepsilon$, then the projections $\mathrm{Spec}\, E[(\mathcal{P}_{\beta_v})_{v \in V}]_{\leq \delta} \longrightarrow P_{\beta_v}$ are surjective, except possibly if s belong to a certain finite exceptional set (depending on δ, ε).

c) In fact, the proof shows a little bit more: one can replace P_{β_v} in the statements by the specialization at s of the S_1–torsor formed of isomorphisms $H^1_{DR}(\underline{A}/S_1^*) \otimes ? \longrightarrow \underline{H}^1_\beta \otimes ?$ preserving global horizontal classes; this makes sense because any such class is automatically a $\mathcal{O}_{S_1^*}$–linear combination of <u>relative</u> Hodge classes, in virtue of:

<u>Proposition 9</u> (Mustafin). On an Abelian scheme $\underline{A} \longrightarrow S_1^*$ degenerating to a torus at $0 \in S_1 \backslash S_1^*$, any element ξ of $\Gamma(\mathrm{End}\, H^1_{DR}(\underline{A}/S_1^*)^{\otimes n})^\nabla$ is a linear combination of relative Hodge cycles.

See e.g. [A] IX 3.2. The argument given in the course of proving theorem 3 then shows that ξ is also a linear combination of relative p–adic Hodge cycles.

d) We thus have to show that any relation (resp. "global relation" for theorem 5) of degree $\leq \delta$ with coefficients in E between (β)–periods of $\underline{A}(s)$ is the specialization at s of some relation of degree $\leq \delta$ with coefficients in $E[x]$ between the relative β_v–periods (which belong to $E[t_p, \log x][[x]]$ in virtue of (**) and lemma 9).

Because t_p is transcendental over E_v, and $\beta_v(\eta_{ij} x^{n_{ij}}(s)) = 0$, it suffices to replace in this statement β_v–periods by the v–adic evaluations of the G–functions ω_{ij}, ω'_{ij}, $\omega_{ij}\log q^1_{ij}$, $(\omega_{ij}\log q^1_{ij})'$, where $q^1_{ij} = \frac{1}{\eta_{ij}} q_{ij} x^{-n_{ij}} = 1 + \ldots$

This can be now deduced from standard results in G–function theory [A] VII thm. 4.3, resp. 5.2. See also, ibid IX for more details about the proof of a (complex) analogous statement.

References

[A] André Y.; G–functions and Geometry, Aspects of Mathematics E13, Vieweg 1988.

[Be] Berkovich V.G.; Spectral theory and analytic geometry over non–archimedean fields II. Preprint IHES 1987.

[B] Berthelot P.; Cohomologie rigide, to appear in the "Astérisque" series.

[Bo] Borovoi, M.; On the action of the Galois group on rational cohomology classes of type (p.p) of Abelian varieties, Mat. Sbornik 94 (1974) n°4, 649–652 (Russian).

[BL] Bosch S., Lütkebohmert W.; stable reduction and Uniformization of Abelian varieties II, Inv. Math. 78, 257–297 (1984).

[Bl 1] Bosch S., Lütkebohmert W.; Degenerating Abelian Varieties, preprint Münster 1988.

[CF] Chai C.L., Faltings G.; Semiabelian degeneration and compactification of Siegel modular spaces. Forthcoming book (1990).

[C] Christol G.; Globally bounded solutions of differential equations, to appear in Proc. of the Tokyo conference on analytic number theory.

[Cn] Coleman R.; The universal vectorial bi–extension, preprint Berkeley 1989.

[Co] Colmez P.; Périodes des variétés abéliennes de type CM, preprint Bonn 1987.

[D_1] Deligne P.; Théorie de Hodge III, IHES Publ. Math. N° 44 (1974), 5–78.

[D_2] Deligne P.; Hodge cycles on Abelian varieties, in Lecture Notes in Math. 900 Springer–Verlag 1982 (notes by J. Milne).

[Dw] Dwork B.; p–adic cycles, IHES Publ. Math. N° 37 (1969), 27–116.

[Fa] Faltings G.; preprint Princeton on the "De Rham conjecture" 1989.

[F_1] Fontaine J.M.; Sur certains types de représentations p–adiques du groupe de Galois d'un corps local, construction d'un anneau de Barsotti–Tate, Ann. of Math. 115, (1982), 529–577.

[F_2] Fontaine J.M.; contribution to the proc. of IHES seminar 88 on "p–adic periods".

[F.M.] Fontaine J.M.; Messing W.; p–adic periods and p–adic etale cohomology, Contemporary mathematics, Vol. 67 AMS (1985), 179–207.

[Ge] Gerritzen L.; On multiplication algebras of Riemann matrices, Math. Ann. 194 (1971), 109–122.

[G_1] Grothendieck A.; On the De Rham cohomology of algebraic varieties, IHES Publ. Math. 29 (1966) (especially footnote 10).

[G_2] Grothendieck A.; Modèles de Néron et monodromie, exp. IX, SGA 7, in Lecture Notes in Math. 288 Springer–Verlag 1970.

[G_3] Grothendieck A.; Groupes de Barsotti–Tate et cristaux de Dieudonné, Séminaire de Mathématiques Supérieures, n° 45, Presses de l'Université de Montréal 1974.

[H] Harris J.; Fonctorial properties of toroidial compactifications of locally symmetric varieties. Proc. London Math. Soc. (3) 59 (1989), 1–22.

[Ka] Katz N.; Serre–Tate local moduli, in Surfaces Algébriques, exposé 5 bis, Springer LNM 868 (1981), 138–202.

[K] Kiehl R.; Die De Rham Kohomologie algebraischer Mannigfaltigkeiten über einem bewerteten Körper. IHES Publ. Math. 33 (1967), 5–20.

[lS] Le Stum B.; Cohomologie rigide et variétés abéliennes, thèse Univ. Rennes 1985.

[M] Mumford D.; Families of Abelian varieties, Algebraic groups and Discontinuous subgroup, Proc. Symp. Pure Math. vol. 9, AMS, Providence R.I. (1966), 347–351.

[Mu] Mustafin G.A.; Families of Algebraic varieties and invariant cycles, Math. USSR Izvestiya Vol 27 (1986) n° 2, 251–278.

[O] Ogus A.; Contribution to this conference.

[R_1] Raynaud M.; Variétés abéliennes et géométrie rigide. Actes du congrès international de Nice 1970, t.1, 473–477.

[R_2] Raynaud M.; Contribution to the proc. of IHES seminar 88 on "p–adic periods".

[S_1] Serre J.P.; Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, Inv. Math., 15, 1972, 259–331.

[S_2] Serre J.P.; Abelian ℓ–adic representations and elliptic curves. Benjamin, New–York, 1968.

[SGA_4] Artin M., Grothendieck A., Verdier J.–L.; Théorie des topos et cohomologie étale des schémas, t. 3, LNM 269 Springer Verlag (1972).

[SR] Shimura G.; On analytic families of polarized abelian varieties and automorphic functions, Annals of Math. 78 (1963), 149.

[St] Steenbrink J.; Limits of Hodge structures, Inv. Math. 31 (1975/76), 229–257.

[T] Tate J.; p–divisible groups. Proc. of a Conf. on local fields. Springer Verlag (1967).

[U] Ullrich P.; Rigid analytic covering maps. Proc. of the Conf. on p–adic analysis, Hengelhoef (1986), 159–171.

[vM] Van der Marel B.; thesis Rijksuniversiteit Groningen 1987.

[W] Wintenberger J.P.; p–adic Hodge theory for families of abelian varieties, preprint 1989.