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REFINED SCATTERING DIAGRAMS AND THETA FUNCTIONS FROM ASYMPTOTIC ANALYSIS OF MAURER–CARTAN EQUATIONS

NAICHUNG CONAN LEUNG, ZIMING NIKOLAS MA, AND MATTHEW B. YOUNG

Abstract. We further develop the asymptotic analytic approach to the study of scattering diagrams. We do so by analysing the asymptotic behavior of Maurer–Cartan elements of a differential graded Lie algebra constructed from a (not-necessarily tropical) monoid-graded Lie algebra. In this framework, we give alternative differential geometric proofs of the consistent completion of scattering diagrams, originally proved by Kontsevich–Soibelman, Gross–Siebert and Bridgeland. We also give a geometric interpretation of theta functions and their wall-crossing. In the tropical setting, we interpret Maurer–Cartan elements, and therefore consistent scattering diagrams, in terms of the refined counting of tropical disks.

Introduction

Motivation. The notion of a scattering diagram was introduced by Kontsevich–Soibelman [15] and Gross–Siebert [13] in their studies of the reconstruction problem in Strominger–Yau–Zaslow mirror symmetry [23]. In this setting, scattering diagrams encode and control the combinatorial data required to consistently glue together local pieces of the mirror manifolds. Since their introduction, scattering diagrams have found important applications to integrable systems [16], cluster algebras [11], enumerative geometry [12] and combinatorics [22], amongst other areas. A different perspective on scattering diagrams was developed in [6], motivated by Fukaya's approach to the reconstruction problem [10]. In this paper, we further develop the asymptotic analytic approach of [6] to give a differential geometric approach to refined and Hall algebraic scattering diagrams.

The most basic form of scattering diagrams is based on the Lie algebra of Poisson vector fields on a torus. For many applications, it is necessary to study refined, or quantum, variants of scattering diagrams, in which case the torus Lie algebra is replaced by the so-called quantum torus Lie algebra or, more generally, an abstract monoid-graded Lie algebra satisfying a tropical condition [16, 11, 18]. For example, a number of conjectures in the theory of quantum cluster algebras were proved using scattering diagrams in [11]. Refined scattering diagrams were related to the refined tropical curve counting of Block–Göttsche [1] by Filippini–Stoppa [9] and Mandel [18]. This refined curve counts are also related to the refined enumeration of real plane curves by the work of Mikhalkin [21], and to higher genus Gromov–Witten invariants by the work of Bousseau [2]. These connections could be anticipated from the role of scattering diagrams in the reconstruction problem.

A further generalization of scattering diagrams was introduced by Bridgeland [4] under the name $\mathfrak{g}$-complex. Here $\mathfrak{g}$ is a (non-necessary tropical) monoid-graded Lie algebra. This additional flexibility allows one to define, for example, scattering diagrams based on the commutator Lie algebra of the motivic Hall algebra of a three dimensional Calabi–Yau category. Bridgeland showed that each quiver with potential (Q, W) defines a consistent $\mathfrak{g}$-complex with values in the motivic Hall–Lie algebra, the wall-crossing automorphisms of the $\mathfrak{g}$-complex encoding the motivic Donaldson–Thomas invariants of (Q, W). The (refined) cluster scattering diagram of (Q, W) is then obtained by applying a Hall algebra integration map to this $\mathfrak{g}$-complex. Using these ideas, Bridgeland was able to connect scattering diagrams to the geometry of the space of stability conditions on the triangulated category associated to (Q, W).

In [10], Fukaya suggested that instanton corrections to the B-side complex structure which arise near the large volume limit could be described in terms of certain Maurer–Cartan elements of the Kodaira–Spencer Lie algebra. In the context of scattering diagrams, these ideas were made precise and put into practice in [6], where it was shown that the asymptotic behavior of solutions to the Maurer–Cartan equation of a certain differential graded algebra admits an alternative interpretation in terms of consistent (classical) scattering diagrams. Moreover, the passage from an initial scattering diagram to its consistent completion, a procedure which exists due to works of Kontsevich–Soibelman [15, 16] and Gross–Siebert [13], can be understood in terms of the perturbative construction of Maurer–Cartan elements. These ideas were further pursued in [7] in the setting of toric mirror symmetry to study the deformation theory of the Landau–Ginzburg mirror of a toric surface X and the counting of tropical disks in X.

Main results. In this paper we further develop the asymptotic approach to scattering diagrams by treating both the refined, or more generally, tropical and Hall algebraic (or non-tropical) cases. To describe our results, we introduce some notation. Let M be a lattice of rank r and let $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\sigma \subset M_{\mathbb{R}}$ be a strictly convex cone and set $M_\sigma^+ = (M \cap \sigma) \setminus \{0\}$. Let $\mathfrak{h}$ be a M_σ^+-graded Lie algebra, which for the moment we assume to be tropical.

Following [6, 7], we consider differential forms on $M_{\mathbb{R}}$ which depend on a parameter $\hbar \in \mathbb{R}_{>0}$. We will be interested in the dg algebra $\mathcal{W}_*^0$ of such differential forms which approach a bump form along a closed tropical polyhedral subset $P \subset M_{\mathbb{R}}$ as $\hbar \to 0$; see Figure 1. The subspace $\mathcal{W}_*^{-1} \subset \mathcal{W}_*^0$ of differential forms which satisfy $\lim_{\hbar \to 0} \alpha = 0$ is a dg ideal. Then

$$\mathcal{H}^* := \bigoplus_{m \in M_\sigma^+} \left(\mathcal{W}_*^0 / \mathcal{W}_*^{-1}\right) \otimes_{\mathbb{C}} \mathfrak{h}_m$$

inherits the structure of a tropical dg Lie algebra. Our goal is to describe Maurer–Cartan elements of $\mathcal{H}^*$.

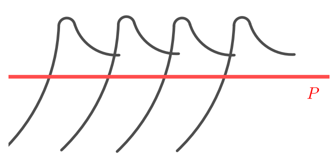

FIGURE 1. Bump form concentrating along P.

Our first result relates Maurer–Cartan elements of $\mathcal{H}^*$ to the counting of tropical disks in $M_{\mathbb{R}}$. Let $\mathcal{D}_{in}$ be an initial scattering diagram. To each wall $\mathbf{w}$ of $\mathcal{D}_{in}$, whose support is a hyperplane $P_{\mathbf{w}}$ and whose wall-crossing factor is $\log(\Theta_{\mathbf{w}})$, we associate the term

$$\Pi^{\mathbf{w}} := -\delta_{P_{\mathbf{w}}} \log(\Theta_{\mathbf{w}}) \in \mathcal{H}^*.$$

Here $\delta_{P_{\mathbf{w}}}$ is a 1-form which concentrates along $P_{\mathbf{w}}$ as $\hbar \to 0$. We take $\Pi = \sum_{\mathbf{w} \in \mathcal{D}_{in}} \Pi^{\mathbf{w}}$ as input data to solve the Maurer–Cartan equation. Our first main result, whose proof uses a modification of a method of Kuranishi [17], describes the Maurer–Cartan element $\varPhi$ constructed perturbatively from Π.

Theorem (See Theorems 3.9 and 3.14)**.** *The Maurer–Cartan element $\varPhi$ can be written as a sum over tropical disks L in $(M_{\mathbb{R}}, \mathcal{D}_{in})$:*

$$\varPhi = \sum_L \frac{1}{|\mathrm{Aut}(L)|} \alpha_L g_L.$$

Here α_L is a 1-form concentrated along $P_L \subset M_{\mathbb{R}}$, the locus traced out by the stop of L as it varies in its moduli, and g_L is the Block–Göttsche-type multiplicity of L. Moreover, when $\dim_{\mathbb{R}}(P_L) = r-1$, there exists a polyhedral decomposition $\mathcal{P}_L$ of P_L such that, for each maximal cell σ of $\mathcal{P}_L$, there exists a constant $c_{L,\sigma}$ such that $\lim_{\hbar\to 0}\int_\varrho \alpha_L = -c_{L,\sigma}$ for any affine line ϱ intersecting positively with σ in its relative interior.

Furthermore, if we generically perturb $\mathcal{D}_{in}$, then $c_{L,\sigma} = 1$, so that $\lim_{\hbar\to 0}\int_\varrho \alpha_L$ is a count of tropical disks.

In Section 3.2 we associate to the Maurer–Cartan element Φ a scattering diagram $\mathcal{D}(\Phi)$ whose walls are labeled by the maximal cells σ of the polyhedral decompositions $\mathcal{P}_L$ and whose wall-crossing automorphisms are $\exp(\frac{c_{L,\sigma}}{|\mathrm{Aut}(L)|} g_L)$. The diagram $\mathcal{D}(\Phi)$ extends $\mathcal{D}_{in}$ and, by the main result of [6], is in fact a consistent scattering diagram; see Proposition 3.16. In this way, we obtain an enumerative interpretation of the consistent completion of $\mathcal{D}_{in}$.

Next, we turn to the study of theta functions. In this setting, we further assume that we are given a monoid P satisfying $M \cap \sigma \subset P \subset M$ and a graded action of $\mathfrak{h}$ on a P-graded algebra A. The dg Lie algebra $\mathcal{H}^*$ then acts naturally on the dg algebra

$$\mathcal{A}^* := \bigoplus_{m\in P} \left(\mathcal{W}^0_*/\mathcal{W}^{-1}_*\right) \otimes_{\mathbb{C}} A_m. \tag{0.1}$$

Given a Maurer–Cartan element $\Phi \in \mathcal{H}^*$, it is natural to study the space of flat sections $\mathrm{Ker}(d_\Phi)$ of the deformed differential $d_\Phi = d + [\Phi, \cdot]$. The algebra structure on $\mathcal{A}^*$ induces an algebra structure on $\mathrm{Ker}(\Phi)$. The following result describes the wall-crossing behavior of the $\hbar \to 0$ limits of flat sections.

Theorem (See Theorem 3.17)**.** *Let $s \in \mathrm{Ker}(d_\Phi)$ and $Q, Q' \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathcal{D}(\Phi))$. Then, for any path $\gamma \subset M_{\mathbb{R}} \setminus \mathrm{Joints}(\mathcal{D})$ from Q to Q', we have $\lim_{\hbar\to 0} s_{Q'} = \Theta_{\gamma,\mathcal{D}}(\lim_{\hbar\to 0} s_Q)$, where $\Theta_{\gamma,\mathcal{D}(\Phi)}$ is the associated wall-crossing factor and $s_{Q'}$ and s_Q are the restrictions of s to Q and Q', respectively.*

In Lemma 3.18 we associate to each $m \in P$ a flat section $\theta_m \in \mathrm{Ker}(d_\Phi)$. This section is determined by the requirement that it agrees with a certain distinguished element $z^m \in A_m$ on some neighborhood V of the ray $\mathbb{R}_{\geq c} \cdot m \subset M_{\mathbb{R}}$ for some sufficiently large c; see Figure 2. The next result relates the section θ_m to the theta function $\vartheta_{m,Q}$ defined by counting broken lines, that is, piecewise linear maps $\gamma : (-\infty, 0] \to M_{\mathbb{R}}$ which bend only at the walls of $\mathcal{D}(\Phi)$. Each broken line determines a weight $a_\gamma \in A$.

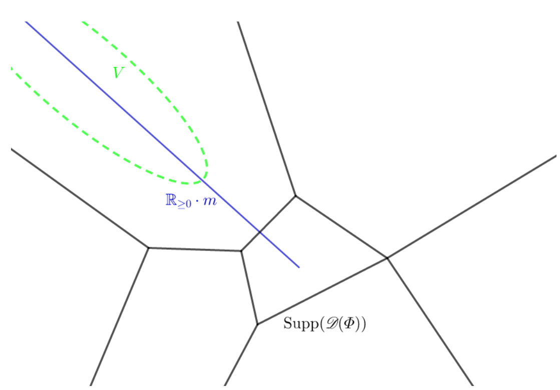

FIGURE 2. Definition of θ_m.

Theorem (See Theorem 3.19)**.** *For each $Q \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathcal{D}(\Phi))$, the equality*

$$\lim_{\hbar \to 0} \theta_m(Q) = \vartheta_{m,Q} := \sum_{\substack{\text{broken lines } \gamma \\ \text{ending at } (m,Q)}} a_\gamma$$

holds, where $\theta_m(Q)$ is the value of θ_m at Q.

Finally, in Section 3.4 we develop the above asymptotic methods in the setting of non-tropical Lie algebras. As already mentioned, the main case of interest is that of the Hall algebra scattering diagram [4]. We impose a mild commutativity condition on the wall-crossing automorphisms of the walls of $\mathcal{D}_{in}$ which, for example, is satisfied in the Hall algebra setting when Q has no edge loops. We obtain corresponding generalizations each of the above theorems; see Theorems 3.29 and 3.32. In particular, we show that the Hall algebra theta functions defined by Bridgeland [4] admit an alternative interpretation as of flat sections of the deformed differential d_Φ associated to a Maurer–Cartan element.

Remark. After this paper was completed, the paper [8] appeared, in which algebraic methods are used to address problems similar to those in this paper.

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1. Scattering diagrams and theta functions

We collect background material related to scattering diagrams and theta functions.

Fix a lattice M of rank r with dual lattice $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Write $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ for the canonical pairing. Let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

1.1. Tropical Lie algebras and scattering diagrams.

Following [11, 18], we recall the definition of scattering diagrams. Compared to [18], the roles of M and N are reversed.

1.1.1. *Tropical Lie algebras.* Fix a strictly convex polyhedral cone $\sigma \subset M_{\mathbb{R}}$. Let $M_\sigma = \sigma \cap M$ and $M_\sigma^+ = M_\sigma \setminus \{0\}$. For each $k \in \mathbb{Z}_{>0}$, set $kM_\sigma^+ = \{m_1 + \cdots + m_k \mid m_i \in M_\sigma^+\}$.

Let $\mathfrak{h} = \bigoplus_{m \in M_\sigma^+} \mathfrak{h}_m$ be a M_σ^+-graded Lie algebra over $\mathbb{C}$. For each $k \in \mathbb{Z}_{>0}$, set $\mathfrak{h}^{\geq k} = \bigoplus_{m \in kM_\sigma^+} \mathfrak{h}_m$. Then $\mathfrak{h}^{<k} := \mathfrak{h}/\mathfrak{h}^{\geq k}$ is a nilpotent Lie algebra. Associated to the pro-nilpotent Lie algebra $\hat{\mathfrak{h}} := \varprojlim_k \mathfrak{h}^{<k}$ is the exponential group $\hat{G} := \exp(\hat{\mathfrak{h}})$. Similarly, for each $m \in M_\sigma^+$, set $\mathfrak{h}_m^{\parallel} = \bigoplus_{k \geq 1} \mathfrak{h}_{km}$ and $\hat{\mathfrak{h}}_m^{\parallel} = \prod_{k \in \mathbb{Z}_{>0}} \mathfrak{h}_{km} \subset \hat{\mathfrak{h}}$ with associated exponential group $\hat{G}_m^{\parallel}$.

To define theta functions, we require a second (not necessarily strictly) convex polyhedral cone $\mathcal{C} \subsetneq M_{\mathbb{R}}$ which contains σ. Let $P = \mathcal{C} \cap M$ be the corresponding monoid. Suppose that $\mathfrak{h}$ acts on a P-graded $\mathbb{C}$-algebra $A = \bigoplus_{m \in P} A_m$ by derivations so that $\mathfrak{h}_m \cdot A_{m'} \subset A_{m+m'}$. Then $A^{\geq k} := \bigoplus_{m \in kM_\sigma^+ + P} A_m$ is a graded ideal of A. Set $A^{<k} = A/A^{\geq k}$ and $\hat{A} = \varprojlim_k A^{<k}$. There is an induced action of $\hat{\mathfrak{h}}$, and hence of $\hat{G}$, on the algebra $\hat{A}$.

More generally, given a sublattice $L \subset M$, let $\mathfrak{h}_L = \bigoplus_{m \in L \cap M_\sigma^+} \mathfrak{h}_m$ and $A_L = \bigoplus_{m \in L \cap P} A_m$ with associated completions $\hat{\mathfrak{h}}_L$ and $\hat{A}_L$.

Let $K \subset M$ be a saturated sublattice which satisfies the following conditions:

(1) $\mathfrak{h}_K$ is a central Lie subalgebra of $\mathfrak{h}$.
(2) The induced $\mathfrak{h}_K$-action on A is trivial.

(3) The induced $\mathfrak{h}$-action on A_K is trivial.

Denote by $\pi_K : M \to \overline{M} := M/K$ the canonical projection and by $\overline{N} := \overline{M}^\vee \hookrightarrow N$ the embedding of $\overline{M}^\vee$ into N as the orthogonal $K^\perp$.

The following assumption will be used in Section 1.2.

Assumption 1.1 ([18]). *(1) The monoid P satisfies $\overline{M} = \pi_K(P)$.*

(2) There is a fan structure on $\overline{M}_\mathbb{R}$ and a piecewise linear section $\varphi : \overline{M}_\mathbb{R} \to M_\mathbb{R}$ of π_K which satisfies $\varphi(0) = 0$ and $P = \varphi(\overline{M}) + (K \cap P)$.

(3) We are given elements $z^{\varphi(\mathsf{m})} \in A_{\varphi(\mathsf{m})}$, $\mathsf{m} \in \overline{M}$, which satisfy
   - *(a) $z^{\varphi(0)} = 1$,*
   - *(b) for any $a \in \hat{A}_K$ setminus$\{0\}$ and $\mathsf{m} \in \overline{M}$, we have $az^{\varphi(\mathsf{m})} \neq 0$, and*
   - *(c) for any $\mathsf{m} \in \overline{M}$, we have $A_{\varphi(\mathsf{m})+P\cap K} = z^{\varphi(\mathsf{m})}A_K$.*

Definition 1.2. *The Lie algebra $\mathfrak{h}$ is called* tropical *if, for each pair $(m, n) \in M_\sigma^+ \times \overline{N}$ satisfying $\langle m, n\rangle = 0$, it is equipped with a subspace $\mathfrak{h}_{m,n} \subset \mathfrak{h}_m$. These subspaces are required to satisfy*

(1) $\mathfrak{h}_{m,0} = \{0\}$ and $\mathfrak{h}_{m,kn} = \mathfrak{h}_{m,n}$ for each $k \neq 0$,
(2) $[\mathfrak{h}_{m_1,n_1}, \mathfrak{h}_{m_2,n_2}] \subset \mathfrak{h}_{m_1+m_2,n}$, where $n = \langle m_2, n_1\rangle n_2 - \langle m_1, n_2\rangle n_1$, and
(3) $\mathfrak{h}_{m_1,n} \cdot A_{m_2} = \{0\}$ if $\langle m_2, n\rangle = 0$.

Examples of tropical and non-tropical $\mathfrak{h}$ can be found in [18, Example 2.1]. See also Section 3.4.1. Until mentioned otherwise, we will assume that $\mathfrak{h}$ is tropical.

Observe that for each $(m, n) \in M_\sigma^+ \times \overline{N}$ with $\langle m, n\rangle = 0$, the direct sum $\mathfrak{h}^{\|}_{m,n} := \bigoplus_{k\in\mathbb{Z}_{>0}} \mathfrak{h}_{km,n}$ is an abelian Lie subalgebra of $\mathfrak{h}^{\|}_m$. Denote by $\hat{\mathfrak{h}}^{\|}_{m,n}$ the completion of $\mathfrak{h}^{\|}_{m,n}$.

Finally, given a commutative unital $\mathbb{C}$-algebra R, there are R-linear versions of the above definitions. For example, $\mathfrak{h}_R := \mathfrak{h} \otimes_\mathbb{C} R$ is a Lie algebra over R which acts on $A \otimes_\mathbb{C} R$ by the R-linear extension of the rule $t_1h \cdot t_2a = t_1t_2(h \cdot a)$. The completion $\hat{\mathfrak{h}}\hat{\otimes}_\mathbb{C}R$ acts on $\hat{A}\hat{\otimes}_\mathbb{C}R$. The corresponding exponential group is G_R with completion $\hat{G}_R$. Similarly, there are abelian Lie subalgebras $\hat{\mathfrak{h}}^{\|}_{m,n,R} \subset \hat{\mathfrak{h}}^{\|}_{m,R}$ and, given a saturated sublattice $L \subset M$, we can form $\mathfrak{h}_{L,R}$, $A_{L,R}$ and so on.

1.1.2. *Scattering diagrams.* We continue to follow [11, 18]. Fix a commutative unital $\mathbb{C}$-algebra R. Recall that r is the rank of M.

Definition 1.3. *A* wall $\mathbf{w}$ *(over R) in $M_\mathbb{R}$ is a tuple (m, n, P, Θ) consisting of*

- *a primitive element $m \in M_\sigma^+$ and an element $n \in \overline{N} \setminus \{0\}$ which satisfy $\langle m, n\rangle = 0$,*
- *an $(r-1)$-dimensional closed convex rational polyhedral subset P of $m_0 + n^\perp \subset M_\mathbb{R}$ for some $m_0 \in M_\mathbb{R}$, called the* support *of $\mathbf{w}$, and*
- *an element $\Theta \in \hat{G}_{m,n,R} := \exp(\hat{\mathfrak{h}}^{\|}_{m,n,R})$, called the* wall-crossing automorphism *of $\mathbf{w}$.*

A wall $\mathbf{w} = (m, n, P, \Theta)$ is called *incoming* (resp. *outgoing*) if $P + tm \subset P$ for all $t \in \mathbb{R}_{>0}$ (resp. $t \in \mathbb{R}_{\le 0}$). The vector $-m$ is called the *direction* of $\mathbf{w}$.

Definition 1.4. *A* scattering diagram $\mathcal{D}$ *over R is a countable set of walls $\{(m_i, n_i, P_i, \Theta_i)\}_i$ such that, for each $k \in \mathbb{Z}_{>0}$, the image of $\log(\Theta_i)$ in $\mathfrak{h}^{<k} \otimes_\mathbb{C} R$ is zero for all but finitely many i.*

Let $k \in \mathbb{Z}_{>0}$. Using the canonical projection $\hat{\mathfrak{h}}_R \to \mathfrak{h}^{<k} \otimes_\mathbb{C} R$, a scattering diagram $\mathcal{D}$ induces a finite scattering diagram $\mathcal{D}^{<k}$ with wall-crossing automorphisms in $\exp(\mathfrak{h}^{<k} \otimes_\mathbb{C} R)$.

The *support* and *singular set* of a scattering diagram $\mathcal{D}$ are

$$\mathrm{Supp}(\mathcal{D}) := \bigcup_{\mathbf{w}\in\mathcal{D}} P_{\mathbf{w}}, \qquad \mathrm{Joints}(\mathcal{D}) := \bigcup_{\mathbf{w}\in\mathcal{D}} \partial P_{\mathbf{w}} \cup \bigcup_{\substack{\mathbf{w}_1,\mathbf{w}_2\in\mathcal{D}\\ \dim(\mathbf{w}_1\cap\mathbf{w}_2)=r-2}} P_{\mathbf{w}_1}\cap P_{\mathbf{w}_2},$$

respectively.

1.1.3. *Path ordered products.* An embedded path $\gamma : [0,1] \to N_{\mathbb{R}} \setminus \mathrm{Joints}(\mathcal{D})$ is said to *intersect* $\mathcal{D}$ *generically* if $\gamma(0), \gamma(1) \notin \mathrm{Supp}(\mathcal{D})$, $\mathrm{Im}(\gamma) \cap \mathrm{Joints}(\mathcal{D}) = \emptyset$ and γ intersects all walls of $\mathcal{D}$ transversally. Given such a path γ, the *path ordered product* is $\Theta_{\gamma,\mathcal{D}} := \varprojlim_k \Theta^{<k}_{\gamma,\mathcal{D}}$, where $\Theta^{<k}_{\gamma,\mathcal{D}} := \prod^{\gamma}_{\mathbf{w}\in\mathcal{D}^{<k}} \Theta_{\mathbf{w}} \in \exp(\mathfrak{h}^{<k} \otimes_{\mathbb{C}} R)$ is defined in [12, §1.3].

Definition 1.5. *(1) A scattering diagram $\mathcal{D}$ is called* consistent *if $\Theta_{\gamma,\mathcal{D}} = \mathrm{Id}$ for any embedded loop γ intersecting $\mathcal{D}$ generically.*

(2) Two scattering diagrams $\mathcal{D}_1$, $\mathcal{D}_2$ are called equivalent *if $\Theta_{\gamma,\mathcal{D}_1} = \Theta_{\gamma,\mathcal{D}_2}$ for any embedded path γ intersecting both $\mathcal{D}_1$ and $\mathcal{D}_2$ generically.*

The following result is fundamental in the theory of scattering diagrams.

Theorem 1.6 ([15, 13])**.** *Let $\mathcal{D}_{in}$ be a scattering diagram consisting of finitely many walls supported on full affine hyperplanes. Then there exists a scattering diagram $\mathcal{S}(\mathcal{D}_{in})$ which is consistent and obtained from $\mathcal{D}_{in}$ by adding only outgoing walls. Moreover, such a scattering diagram $\mathcal{S}(\mathcal{D}_{in})$ is unique up to equivalence.*

Using asymptotic analytic techniques, an independent proof of the existence part of Theorem 1.6 will be given in Proposition 3.16.

1.2. **Broken lines and theta functions.** We follow [18] to define broken lines. Fix a consistent scattering diagram $\mathcal{D}$ over R.

Definition 1.7. *A* broken line γ with end $(\mathsf{m}, Q) \in \overline{M}\setminus\{0\}\times M_{\mathbb{R}}\setminus\mathrm{Supp}(\mathcal{D})$ *is the data of a partition $-\infty < t_0 \leq t_1 \leq \cdots \leq t_l = 0$, a piecewise linear map $\gamma : (-\infty, 0] \to M_{\mathbb{R}} \setminus \mathrm{Joints}(\mathcal{D})$ and elements $a_i \in A_{m_i} \otimes_{\mathbb{C}} R$, $i = 0, \ldots, l$, with $m_i \neq 0$. This data is required to satisfy the following conditions:*

(1) $a_0 = z^{\varphi(\mathsf{m})}$.
(2) $\gamma(0) = Q$.
(3) $\{t_0, \ldots, t_{l-1}\} \subseteq \gamma^{-1}(\mathrm{Supp}(\mathcal{D}))$.
(4) $\gamma'|_{(t_{i-1},t_i)} \equiv -m_i$ for $i = 0, \ldots, l$, where $t_{-1} := -\infty$, and all bends $m_{i+1} - m_i$ *are non-zero.*
(5) For each $i = 0, \ldots, l-1$, set $\Theta_i := \prod_{\substack{\mathbf{w}\in\mathcal{D}\\ \gamma(t_i)\in P_{\mathbf{w}}}} \Theta_{\mathbf{w}}^{\mathrm{sgn}(m_i, n_{\mathbf{w}})} \in \hat{G}_R$. Then a_{i+1} is a homogeneous summand of $\Theta_i \cdot a_i$.

In the notation of Definition 1.7, we will write a_γ for a_l.

Definition 1.8. *Given $(\mathsf{m}, Q) \in \overline{M}\setminus\{0\}\times M_{\mathbb{R}}\setminus\mathrm{Supp}(\mathcal{D})$, define $\vartheta_{\mathsf{m},Q} = \sum_{\mathrm{End}(\gamma)=(\mathsf{m},Q)} a_\gamma \in \hat{A}\hat{\otimes}_{\mathbb{C}}R$, the sum being over all broken lines with end (m, Q). Define also $\vartheta_{0,Q} = 1$.*

In the present setting, well-definedness of theta functions was proved in [18]. Observe that $\vartheta_{\mathsf{m},Q} \in z^{\varphi(\mathsf{m})} + \hat{A}_{\varphi(\mathsf{m})+M^+_\sigma}$, where $\hat{A}_{\varphi(\mathsf{m})+M^+_\sigma}$ is the completion of $A_{\varphi(\mathsf{m})+M^+_\sigma} \otimes_{\mathbb{C}} R$.

Proposition 1.9 ([5, 18])**.** *Under Assumption 1.1, the following statements hold.*

(1) For each $Q \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathcal{D})$, the set $\{\vartheta_{\mathsf{m},Q}\}_{\mathsf{m}\in\overline{M}}$ is linearly independent over $\hat{A}_K\hat{\otimes}_{\mathbb{C}}R$ and, for each $k \in \mathbb{Z}_{>0}$, additively generates $A^{<k} \otimes_{\mathbb{C}} R$ over $A^{<k}_K \otimes_{\mathbb{C}} R$.
(2) Let $\mathcal{D} = \mathcal{S}(\mathcal{D}_{in})$ and let $\rho : [0,1] \to M_{\mathbb{R}} \setminus \mathrm{Joints}(\mathcal{D})$ be a path with generic endpoints which do not lie in $\mathrm{Supp}(\mathcal{D})$*. Then the equality $\vartheta_{\mathsf{m},\rho(1)} = \Theta_{\rho,\mathcal{D}}(\vartheta_{\mathsf{m},\rho(0)})$ holds for all $\mathsf{m} \in \overline{M}$.*

1.3. **Tropical disk counting.** We recall some definitions of [18], modified so as to incorporate the work of [6]. Fix a scattering diagram $\mathcal{D}_{in} = \{\mathbf{w}_i = (m_i, n_i, P_i, \Theta_i)\}_{i\in I}$ and let $g_i = \log(\Theta_i)$. Write

$$g_i = \sum_{j\geq 1} g_{ji} \in \big(\prod_{j\geq 1} \mathfrak{h}_{jm_i}\big) \cap \hat{\mathfrak{h}}^{\parallel}_{m_i,n_i}, \qquad g_{ji} \in \mathfrak{h}_{jm_i}. \tag{1.1}$$

For each $l \geq 0$, define commutative rings $R = \mathbb{C}[\{t_i \mid i \in I\}]$, and $R_l = \mathbb{C}[\{t_i \mid i \in I\}]/\langle t_i^{l+1} \mid i \in I\rangle$, as in [12, 18]. There is a ring homomorphism

$$R_l \to \tilde{R}_l := \frac{\mathbb{C}\left[\{u_{ij} \mid i \in I,\ 1 \leq j \leq l\}\right]}{\langle u_{ij}^2 \mid i \in I,\ 1 \leq j \leq l\rangle}, \qquad t_i \mapsto \sum_{j=1}^{l} u_{ij}. \tag{1.2}$$

Definition 1.10. *A perturbation $\tilde{\mathcal{D}}_{in,l}$ of $\mathcal{D}_{in}$ over $\tilde{R}_l$ is a scattering diagram over $\tilde{R}_l$ consisting of a wall $\mathbf{w}_{iJ} = (m_i, n_i, P_{iJ}, \Theta_{iJ})$ for each $i \in I$ and $J \subset \{1, \ldots, l\}$ with $\#J \geq 1$ such that*

(1) each P_{iJ} is a translate of $n_i^{\perp}$ and $P_{iJ} \neq P_{i'J'}$ unless $i = i'$ and $J = J'$, and
(2) the equality $\log(\Theta_{iJ}) = (\#J)! g_{(\#J)i} \prod_{s\in J} u_{is}$ holds.

We follow [6, 9, 12, 20] and introduce the notion of tropical disks in $\mathcal{D}_{in}$ or $\tilde{\mathcal{D}}_{in,l}$.

Definition 1.11. *A* (directed) k-tree T *is the data of finite sets of vertices $\bar{T}^{[0]}$ and edges $\bar{T}^{[1]}$, a decomposition $\bar{T}^{[0]} = T_{in}^{[0]} \sqcup T^{[0]} \sqcup \{v_o\}$ into incoming, internal and outgoing vertices and boundary maps $\partial_{in}, \partial_{out} : \bar{T}^{[1]} \to \bar{T}^{[0]}$. This data is required to satisfy the following conditions:*

(1) The set $T_{in}^{[0]}$ has cardinality k.
(2) Each vertex $v \in T_{in}^{[0]}$ is univalent and satisfies $\#\partial_{out}^{-1}(v) = 0$ and $\#\partial_{in}^{-1}(v) = 1$.
(3) Each vertex $v \in T^{[0]}$ is trivalent and satisfies $\#\partial_{out}^{-1}(v) = 2$ and $\#\partial_{in}^{-1}(v) = 1$.
(4) We have $\#\partial_{out}^{-1}(v_o) = 1$ and $\#\partial_{in}^{-1}(v_o) = 0$.
(5) The topological realization $|\bar{T}| := \left(\coprod_{e\in\bar{T}^{[1]}}[0,1]\right) / \sim$, where $\sim$ is the equivalence relation which identifies boundary points of edges if their images in $T^{[0]}$ agree, is connected and simply connected.

Two k-trees are isomorphic if there are bijections between their sets of vertices and edges which preserve the respective decompositions and boundary maps. Set $T_{\infty}^{[0]} = T_{in}^{[0]} \sqcup \{v_0\}$ and $T^{[1]} = \bar{T}^{[1]} \setminus \partial_{in}^{-1}(T_{in}^{[0]})$. The edge $e_o := \partial_{out}^{-1}(v_o)$ is called the *outgoing edge*. The *root vertex* v_r is the unique vertex satisfying $e_o = \partial_{in}^{-1}(v_r)$.

Definition 1.12. *(1) A* labeled k-tree *is a k-tree L with a labeling of each edge $e \in \partial_{in}^{-1}(L_{in}^{[0]})$ by a wall $\mathbf{w}_{i_e} = (m_{i_e}, n_{i_e}, P_{i_e}, \Theta_{i_e})$ in $\mathcal{D}_{in}$ and an element $m_e \in M_\sigma^+$ such that $m_e = k_e m_{i_e}$ for some $k_e \in \mathbb{Z}_{>0}$.*

(2) A weighted k-tree *is a k-tree Γ with a weighting of each edge $e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]})$ by a wall $\mathbf{w}_{i_eJ_e} = (m_{i_e}, n_{i_e}, P_{i_eJ_e}, \Theta_{i_eJ_e})$ in $\tilde{\mathcal{D}}_{in,l}$ and a pair $(m_e, u^{\vec{J}_e})$, where $u^{\vec{J}_e} := \prod_{i\in I}\prod_{j\in J_{e,i}} u_{ij} \in \tilde{R}_l$ such that $m_e = (\#J_e)m_{i_e}$ and $\vec{J}_e$ is an I-tuple of finite subsets of $\{1, \ldots, l\}$ such that $J_{e,i_e} = J_e$ and $J_{e,j} = \emptyset$ for $j \in I \setminus \{i_e\}$. Moreover, the weights of incoming edges are required to be pairwise distinct.*

Two labeled (resp. weighted) k-trees are isomorphic if they are isomorphic as k-trees by a label (resp. weight) preserving isomorphism. The set of isomorphism classes of labeled (resp. weighted) k-trees will be denoted by $\mathtt{LT}_k$ (resp. $\mathtt{WT}_k$).

Let L be a labeled k-tree. Inductively define a labeling of all edges of L by requiring that for a vertex $v \in L^{[0]}$ with incoming edges e_1, e_2 (so that $\partial_{out}^{-1}(v) = \{e_1, e_2\}$) and outgoing edge e_3, the equality $m_{e_3} = m_{e_1} + m_{e_2}$ holds. A similar procedure applies to a weighted k-tree, where we additionally require $u^{\vec{J}_{e_3}} = u^{\vec{J}_{e_1}} u^{\vec{J}_{e_2}}$.

Definition 1.13. *A* labeled ribbon k-tree $\mathcal{L}$ *is a labeled k-tree together with a ribbon structure, that is, a cyclic ordering of $\partial_{in}^{-1}(v) \sqcup \partial_{out}^{-1}(v)$ for each internal vertex $v \in \mathcal{L}^{[0]}$.*

Two labeled ribbon k-trees are isomorphic if they are isomorphic as k-trees by an isomorphism which preserves both the ribbon structure and the labels. The set of isomorphism classes of labeled ribbon k-trees will be denoted by $\mathtt{LR}_k$. Similarly, $\mathtt{WRT}_k$ is the set of isomorphism classes of weighted ribbon k-trees. Note that the topological realization of a (labeled) ribbon k-tree $\mathcal{L}$ admits a canonical embedding into the unit disc D so that $\mathcal{L}_{\infty}^{[0]} \subset \partial D$.

Given a labeled k-tree L (resp. weighted k-tree Γ), we will write m_L (resp. $(m_\Gamma, u^{\vec{J}_\Gamma})$) for the label (resp. weight) of the edge e_o. We use analogous notation in the ribbon cases.

Definition 1.14 ([18])**.** *Given a labeled k-tree L (resp. weighted k-tree Γ), associate to each $e \in \bar{L}^{[1]}$ (resp. $e \in \bar{\Gamma}^{[1]}$) a pair $\pm(n_e, g_e)$, defined up to sign,*[1] *with $n_e \in \overline{N}$ and $g_e \in \mathfrak{h}_{m_e,n_e}$ (resp. $g_e \in \mathfrak{h}_{m_e,n_e,\tilde{R}_l}$), inductively along the direction of the tree as follows:*

(1) Associated to each $e \in \partial_{in}^{-1}(L_{in}^{[0]})$ (resp. $e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]})$) is a unique initial wall $\mathbf{w}_{i_e} = (m_{i_e}, n_{i_e}, P_{i_e}, \Theta_{i_e})$ (resp. $\mathbf{w}_{i_e J_e} = (m_{i_e}, n_{i_e}, P_{i_e J_e}, \Theta_{i_e J_e})$). Set $n_e = n_{i_e}$ and $g_e = g_{k_e i_e}$ (resp. $g_e = g_{(\# J_{e,i_e}) i_e}$), where g_{ji} is given by equation (1.1).

(2) At a trivalent vertex $v \in L^{[0]}$ (resp. $v \in \Gamma^{[0]}$) with incoming edges e_1, e_2 and outgoing edge e_3, set $n_{e_3} = \langle m_{e_2}, n_{e_1} \rangle n_{e_2} - \langle m_{e_1}, n_{e_2} \rangle n_{e_1}$ and $g_{e_3} = [g_{e_1}, g_{e_2}]$.

For a labeled (resp. weighted) ribbon tree $\mathcal{L}$ (resp. $\mathcal{T}$), the label (n_e, g_e) of $e \in \bar{\mathcal{L}}^{[1]}$ (resp. $e \in \bar{\mathcal{T}}^{[1]}$) can be defined without the sign ambiguity by requiring $\{e_1, e_2, e_3\}$ to be clockwise oriented.

Write (n_L, g_L) (resp. (n_Γ, g_Γ)) for the pair associated to e_o. Note that if $v \in \Gamma^{[0]}$ has incoming edges e_1, e_2 and outgoing edge e_3 and $m_{e_1}, m_{e_2} \in M_{\mathbb{R}}$ are linearly dependent, then $n_{e_3} = 0$ and hence $g_L = 0$ (resp. $g_\Gamma = 0$), as follows from the vanishing $\mathfrak{h}_{m,0} = \{0\}$.

Given a weighted k-tree Γ and $\vec{s} := (s_e)_{e \in \Gamma^{[1]}} \in (\mathbb{R}_{<0})^{|\Gamma^{[1]}|}$, the associated *realization* of Γ is $|\Gamma_{\vec{s}}| := \Big(\big(\bigsqcup_{e \in \partial_{out}^{-1}(\Gamma_{in}^{[0]})} (\mathbb{R}_{\leq 0})_e \big) \sqcup \big(\bigsqcup_{e \in \Gamma^{[1]}} [s_e, 0] \big) \Big) / \sim$. Here $(\mathbb{R}_{\leq 0})_e$ is a copy of $\mathbb{R}_{\leq 0}$ and $\sim$ is the equivalence relation which identifies boundary points of edges if their images in $\Gamma^{[0]}$ agree.

Definition 1.15. *A* tropical disk *in $(M_{\mathbb{R}}, \mathcal{D}_{in})$ (resp. $(M_{\mathbb{R}}, \tilde{\mathcal{D}}_{in,l})$) consists of*

- *a labeled k-tree L (resp. weighted k-tree Γ), with labeling of $e \in \partial_{in}^{-1}(L_{in}^{[0]})$ by a wall $\mathbf{w}_{i_e} = (m_{i_e}, n_{i_e}, P_{i_e}, \Theta_{i_e})$ and $m_e \in M_\sigma^+$ (resp. labeling of $e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]})$ by a wall $\mathbf{w}_{i_e J_{e,i_e}} = (m_{i_e}, n_{i_e}, P_{i_e J_{e,i_e}}, \Theta_{i_e J_{e,i_e}})$ and $(m_e, u^{\vec{J}_e})$),*
- *a set of parameters $\vec{s} = (s_e)_{e \in L^{[1]}} \in (\mathbb{R}_{<0})^{|L^{[1]}|}$ (resp. $\vec{s} = (s_e)_{e \in \Gamma^{[1]}} \in (\mathbb{R}_{<0})^{|\Gamma^{[1]}|}$), and*
- *a proper map $\varsigma : |L_{\vec{s}}| \to M_{\mathbb{R}}$ (resp. $\varsigma : |\Gamma_{\vec{s}}| \to M_{\mathbb{R}}$)*

such that the following conditions are satisfied:

(1) For each $e \in \partial_{in}^{-1}(L_{in}^{[0]})$ (resp. $e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]})$), we have $\varsigma|_{(\mathbb{R}_{\leq 0})_e}(0) \in P_{i_e}$ (resp. $\varsigma|_{(\mathbb{R}_{\leq 0})_e}(0) \in P_{i_e J_{e,i_e}}$) and $\varsigma|_{(\mathbb{R}_{\leq 0})_e}(s) = \varsigma|_{(\mathbb{R}_{\leq 0})_e}(0) + s(-m_e)$ for all $s \in \mathbb{R}_{\leq 0}$.

(2) For each $e \in L^{[1]}$ (resp. $e \in \Gamma^{[1]}$), we have $\varsigma|_{[s_e,0]}(s) = \varsigma|_{[s_e,0]}(0) + s(-m_e)$.

[1] As the signs of n_{e_3} and g_{e_3} depend on the cyclic ordering e_1, e_2, e_3 in the same way, only $\pm(n_{e_3}, g_{e_3})$ is defined.

The point $\varsigma(v_o) := \varsigma|_{[s_{e_0},0]}(0) \in M_{\mathbb{R}}$ is called the *stop* of the tropical disk ς. Given a tropical disk ς, denote by $\pm(n_\varsigma, g_\varsigma)$ the pair $\pm(n_L, g_L)$ (resp. $\pm(n_\Gamma, g_\Gamma)$) associated to the underlying labeled (resp. weighted) tree.

One can also define a tropical disk in $M_{\mathbb{R}}$ of type L without specifying a scattering diagram by relaxing condition (1) in Definition 1.15 to

$$\varsigma|_{(\mathbb{R}_{\leq 0})_e}(s) = \varsigma|_{(\mathbb{R}_{\leq 0})_e}(0) + s(-m_e) \text{ for all } s \in \mathbb{R}_{\leq 0}$$

and allowing each s_e to take on the value 0. Tropical disks in $M_{\mathbb{R}}$ then form a moduli space $\mathfrak{M}_L(M_{\mathbb{R}})$. Under the identification $\mathfrak{M}_L(M_{\mathbb{R}}) \cong \mathbb{R}_{\leq 0}^{|L^{[1]}|} \times M_{\mathbb{R}}$, the evaluation map $ev_o : \mathfrak{M}_L(M_{\mathbb{R}}) \to M_{\mathbb{R}}$, obtained by taking the stop of tropical disks, is the projection to $M_{\mathbb{R}}$. Similar comments and notation apply to tropical disks in $M_{\mathbb{R}}$ of type Γ.

We denote by $\mathfrak{M}_L(M_{\mathbb{R}}, \mathscr{D}_{in})$ (resp. $\mathfrak{M}_\Gamma(M_{\mathbb{R}}, \tilde{\mathscr{D}}_{in,l})$) the set of all tropical disks in $(M_{\mathbb{R}}, \mathscr{D}_{in})$ (resp. $(M_{\mathbb{R}}, \tilde{\mathscr{D}}_{in,l})$) when $n_L \neq 0$ (resp. $n_\Gamma \neq 0$ and $u^{\vec{J}_\Gamma} \neq 0$) with underlying labeled k-tree L (resp. weighted k-tree Γ). By definition, we have $\mathfrak{M}_L(M_{\mathbb{R}}, \mathscr{D}_{in}) \subset \mathfrak{M}_L(M_{\mathbb{R}})$. We use $\overline{\mathfrak{M}}_L(M_{\mathbb{R}}, \mathscr{D}_{in})$ to denote the closure inside $\mathfrak{M}_L(M_{\mathbb{R}})$. We also define an affine subspace by $P_L = (ev_o)(\overline{\mathfrak{M}}_L(M_{\mathbb{R}}, \mathscr{D}_{in})) \subset M_{\mathbb{R}}$. Similar comments and notation apply in the perturbed setting.

When M has rank two, P_Γ is a line when $k = 1$ and is a ray when $k > 1$. The latter case is illustrated Figure 3.

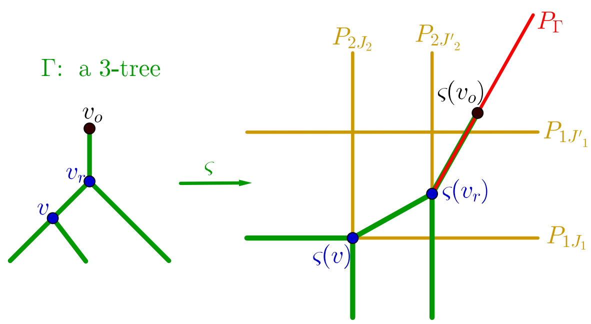

FIGURE 3. P_Γ from moduli of tropical disks.

Lemma 1.16. *If P_Γ is non-empty, then it is orthogonal to n_Γ.*

Proof. We proceed by induction on the cardinality of $\Gamma^{[0]}$. In the initial case, $\Gamma^{[0]} = \emptyset$, the only tree is that with a unique edge and the statement is trivial.

For the induction step, suppose that $v_r \in \Gamma^{[0]}$ is adjacent to the outgoing edge e_0 and incoming edges e_1, e_2. Split Γ at v_r, thereby obtaining trees Γ_1 and Γ_2 with outgoing edges e_1 and e_2 and k_1 and k_2 incoming edges, respectively. We have

$$\left(\overline{\mathfrak{M}}_{\Gamma_1}(M_{\mathbb{R}}, \tilde{\mathscr{D}}_{in,l})_{ev_o} \times_{ev_o} \overline{\mathfrak{M}}_{\Gamma_2}(M_{\mathbb{R}}, \tilde{\mathscr{D}}_{in,l})\right) \times \mathbb{R}_{\geq 0} \cdot (-m_\Gamma) \cong \overline{\mathfrak{M}}_\Gamma(M_{\mathbb{R}}, \tilde{\mathscr{D}}_{in,l}),$$

implying that $P_\Gamma = (P_{\Gamma_1} \cap P_{\Gamma_2}) + \mathbb{R}_{\geq 0} \cdot (-m_\Gamma)$. By the induction hypothesis, n_{Γ_i} is orthogonal to P_{Γ_i}, $i = 1, 2$, and hence n_Γ is orthogonal to $P_{\Gamma_1} \cap P_{\Gamma_2}$. A direct computation using the definition of n_Γ shows that $\langle m_\Gamma, n_\Gamma \rangle = 0$. The lemma follows. □

Definition 1.17. *A scattering diagram $\tilde{\mathscr{D}}_{in,l}$ is called* generic *if for any two weighted trees Γ_1, Γ_2 such that $u^{\vec{J}_{\Gamma_1}} \cdot u^{\vec{J}_{\Gamma_2}} \neq 0$ and P_{Γ_1} intersects P_{Γ_2} transversally,[2] the intersection $P_{\Gamma_1} \cap P_{\Gamma_2} \subset M_{\mathbb{R}}$ has codimension two and is contained in the boundary of neither P_{Γ_1} nor P_{Γ_2}*

[2]Meaning the unique affine subspaces containing P_{Γ_1} and P_{Γ_2} intersect transversally.

The next result, which was proved by various authors in increading levels of generality, relates consistent scattering diagrams to the counting of tropical disks.

Theorem 1.18 ([12, 9, 18]). *Let $\tilde{\mathfrak{D}}_{in,l}$ be a generic initial scattering diagram. There is a bijective correspondence between walls $\mathbf{w} \in \mathcal{S}(\tilde{\mathfrak{D}}_{in,l})$ and weighted trees Γ with $\mathfrak{M}_\Gamma(M_\mathbb{R}, \tilde{\mathfrak{D}}_{in,l}) \neq \emptyset$ under which a wall $\mathbf{w} = (m, n, P, \Theta)$ corresponds to the weighted tree Γ with $(n_\Gamma, P_\Gamma) = (n, P)$ and $\log(\Theta) = \left(\prod_{e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]})} (\#J_{e,i_e})!\right) g_\Gamma u^{\vec{J}_\Gamma}$.*

2. Pertubative solution of the Maurer–Cartan equation

We introduce a differential graded (dg) Lie algebra whose Maurer–Cartan equation governs the scattering process from $\mathfrak{D}_{in}$ to $\mathcal{S}(\mathfrak{D}_{in})$, or its generic perturbation.

2.1. **Differential forms with asymptotic support.**

We begin by recalling some background material from [6, §4.2.3] and [7, §3.2].

Let U be a convex open subset of $M_\mathbb{R}$, or more generally, of an integral affine manifold , as in [7, §3.2]. Introduce the notation $\Omega^k_\hbar(U) = \Gamma(U \times \mathbb{R}_{>0}, \bigwedge^k T^\vee U)$, where the coordinate of $\mathbb{R}_{>0}$ is $\hbar$. Let $\mathcal{W}_k^{-\infty}(U) \subset \Omega^k_\hbar(U)$ be the set of k-forms α such that, for each $q \in U$, there exists a neighborhood $q \in V \subset U$ and constants $D_{j,V}$, c_V such that $\|\nabla^j \alpha\|_{L^\infty(V)} \leq D_{j,V} e^{-c_V/\hbar}$ for all $j \geq 0$. Similarly, let $\mathcal{W}_k^{\infty}(U) \subset \Omega^k_\hbar(U)$ be the set of k-forms α such that, for each $q \in U$, there exists a neighborhood $q \in V \subset U$ and constants $D_{j,V}$ and $N_{j,V} \in \mathbb{Z}_{>0}$ such that $\|\nabla^j \alpha\|_{L^\infty(V)} \leq D_{j,V} \hbar^{-N_{j,V}}$ for all $j \geq 0$. The assignment $U \mapsto \mathcal{W}_k^{-\infty}(U)$ (resp. $U \mapsto \mathcal{W}_k^{\infty}(U)$) defines a sheaf $\mathcal{W}_k^{-\infty}$ (resp. $\mathcal{W}_k^{\infty}$) on $M_\mathbb{R}$. Note that $\mathcal{W}_k^{-\infty}$ and $\mathcal{W}_k^{\infty}$ are closed under the wedge product, $\nabla_{\frac{\partial}{\partial x}}$ and the de Rham differential d. Since $\mathcal{W}_k^{-\infty}$ is a dg ideal of $\mathcal{W}_k^{\infty}$, the quotient $\mathcal{W}_*^{\infty}/\mathcal{W}_*^{-\infty}$ is a sheaf of dg algebras when equipped with the de Rham differential.

By a *tropical polyhedral subset* of U we mean a connected convex subset which is defined by finitely many affine equations or inequalities over $\mathbb{Q}$.

Definition 2.1. *A k-form $\alpha \in \mathcal{W}_k^{\infty}(U)$ is said to have* asymptotic support on a closed codimension k tropical polyhedral subset $P \subset U$ with weight s, *denoted $\alpha \in \mathcal{W}_P^s(U)$, if the following conditions are satisfied:*

(1) For any $p \in U \setminus P$, there is a neighborhood $p \in V \subset U \setminus P$ such that $\alpha|_V \in \mathcal{W}_k^{-\infty}(V)$.
(2) There exists a neighborhood $W_P \subset U$ of P such that
$$\alpha = h(x,\hbar)\nu_P + \eta$$
on W_P, where $\nu_P \in \bigwedge^k N_\mathbb{R}$ is the unique affine k-form which is normal to P, $h(x,\hbar) \in C^\infty(W_P \times \mathbb{R}_{>0})$ and $\eta \in \mathcal{W}_k^{-\infty}(W_P)$.
(3) For any $p \in P$, there exists a convex neighborhood $p \in V \subset U$ equipped with an affine coordinate system $x = (x_1, \ldots, x_n)$ such that $x' := (x_1, \ldots, x_k)$ parametrizes codimension k affine linear subspaces of V parallel to P, with $x' = 0$ corresponding to the subspace containing P. With the foliation $\{(P_{V,x'})\}_{x' \in N_V}$, where $P_{V,x'} = \{(x_1, \ldots, x_n) \in V \mid (x_1, \ldots, x_k) = x'\}$ and N_V is the normal bundle of V, we require that, for all $j \in \mathbb{Z}_{\geq 0}$ and multi-indices $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{Z}^k_{\geq 0}$, the estimate

$$\int_{x'} (x')^\beta \left(\sup_{P_{V,x'}} |\nabla^j (\iota_{\nu_P^\vee} \alpha)| \right) \nu_P \leq D_{j,V,\beta} \hbar^{-\frac{j+s-|\beta|-k}{2}} \tag{2.1}$$

holds for some constant $D_{j,V,\beta}$ and $s \in \mathbb{Z}$, where $|\beta| = \sum_l \beta_l$ and $\nu_P^\vee = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_k}$.

Observe that $\nabla_{\frac{\partial}{\partial x_l}} \mathcal{W}_P^s(U) \subset \mathcal{W}_P^{s+1}(U)$ and $(x')^\beta \mathcal{W}_P^s(U) \subset \mathcal{W}_P^{s-|\beta|}(U)$. It follows that

$$(x')^\beta \nabla_{\frac{\partial}{\partial x_{l_1}}} \cdots \nabla_{\frac{\partial}{\partial x_{l_j}}} \mathcal{W}_P^s(U) \subset \mathcal{W}_P^{s+j-|\beta|}(U). \tag{2.2}$$

The weight s defines a filtration of $\mathcal{W}_k^\infty$ (we will drop the U dependence from the notation whenever it is clear from the context):[3]

$$\mathcal{W}_k^{-\infty} \subset \cdots \subset \mathcal{W}_P^{-1} \subset \mathcal{W}_P^0 \subset \mathcal{W}_P^1 \subset \cdots \subset \mathcal{W}_k^\infty \subset \Omega_\hbar^k(U). \tag{2.3}$$

This filtration, which keeps track of the polynomial order of $\hbar$ for k-forms with asymptotic support on P, provides a convenient tool to prove and express results in asymptotic analysis.

Definition 2.2. *A differential k-form α is in $\tilde{\mathcal{W}}_k^s(U)$ if there exist polyhedral subsets $P_1, \ldots, P_l \subset U$ of codimension k such that $\alpha \in \sum_{j=1}^l \mathcal{W}_{P_j}^s(U)$. If, moreover, $d\alpha \in \tilde{\mathcal{W}}_{k+1}^{s+1}(U)$, then we say α is in $\mathcal{W}_k^s(U)$. For every $s \in \mathbb{Z}$, let $\mathcal{W}_*^s(U) = \bigoplus_k \mathcal{W}_k^{s+k}(U)$.*

We say that closed tropical polyhedral subsets $P_1, P_2 \subset U$ of codimension k_1, k_2 *intersect transversally* if the affine subspaces of codimension k_1 and k_2 which contain P_1 and P_2, respectively, intersect transversally. This definition applies also when $\partial P_i \neq \emptyset$.

Lemma 2.3 ([7, Lemma 3.11])**.**
(1) Let $P_1, P_2, P \subset U$ be closed tropical polyhedral subsets of codimension k_1, k_2 and $k_1 + k_2$, respectively, such that P contains $P_1 \cap P_2$ and is normal to $\nu_{P_1} \wedge \nu_{P_2}$. Then $\mathcal{W}_{P_1}^s(U) \wedge \mathcal{W}_{P_2}^r(U) \subset \mathcal{W}_P^{r+s}(U)$ if P_1 and P_2 intersect transversally and $\mathcal{W}_{P_1}^s(U) \wedge \mathcal{W}_{P_2}^r(U) \subset \mathcal{W}_{k_1+k_2}^{-\infty}(U)$ otherwise.
(2) We have $\mathcal{W}_{k_1}^{s_1}(U) \wedge \mathcal{W}_{k_2}^{s_2}(U) \subset \mathcal{W}_{k_1+k_2}^{s_1+s_2}(U)$. In particular, $\mathcal{W}_*^0(U) \subset \mathcal{W}_*^\infty(U)$ is a dg subalgebra and $\mathcal{W}_*^{-1}(U) \subset \mathcal{W}_*^0(U)$ is a dg ideal.

2.1.1. *Homotopy operators.* Let $P \subset U$ be a closed tropical polyhedral subset. In the remainder of this section, we study the behavior of $\mathcal{W}_P^s(U)$ under the application of a homotopy-type operator I. To do so, fix a reference tropical hyperplane $R \subset U$ which divides U into $U \setminus R = U_+ \cup U_-$. Fix also an affine vector field v (meaning $\nabla v = 0$) which is not tangent to R and points into U_+.

By shrinking U if necessary, we can assume that, for any $p \in U$, the unique flow line of v in U passing through p intersects R at a unique point, say $x \in R$. The time t flow along v then defines a diffeomorphism $\tau : W \to U,\ (t,x) \mapsto \tau(t,x)$, where $W \subset \mathbb{R} \times R$ is the maximal domain of definition of τ. For each $x \in R$, set $\tau_x(t) = \tau(t,x)$. Let $P_\pm = P \cap \overline{U}_\pm$ and define

$$I(P)_+ = (P_+ + \mathbb{R}_{\geq 0} \cdot v) \cap U, \quad I(P)_- = (P_- + \mathbb{R}_{\leq 0} \cdot v) \cap U. \tag{2.4}$$

Define an integral operator I by

$$I(\alpha)(t,x) = \int_0^t \iota_{\frac{\partial}{\partial s}} (\tau^*(\alpha))(s,x) ds, \qquad \alpha \in \mathcal{W}_P^s(U). \tag{2.5}$$

Despite the notation, I depends on the choice of the tropical hyperplane R and the vector field v.

Lemma 2.4 (*cf.* [7, Lemmas 3.12, 3.15])**.** *Let $\alpha \in \mathcal{W}_P^s(U)$. We have $I(\alpha) \in \mathcal{W}_{k-1}^{-\infty}(U)$ if v is tangent to P and $I(\alpha) \in \mathcal{W}_{I(P)_+}^{s-1}(U) + \mathcal{W}_{I(P)_-}^{s-1}(U)$ otherwise. Moveover, if $\alpha \in \tilde{\mathcal{W}}_k^s(U)$ (resp. $\alpha \in \mathcal{W}_k^s(U)$), then $I(\alpha) \in \tilde{\mathcal{W}}_{k-1}^{s-1}(U)$ (resp. $I(\alpha) \in \mathcal{W}_{k-1}^{s-1}(U)$).*

Using the affine coordinates determined by τ, we define a tropical hypersurface $\mathbf{i} : R \to U$, $x \mapsto (0,x)$, and an affine projection $\mathbf{p} : U \to R$, $(t,x) \mapsto x$.

[3]Note that k is equal to the codimension of $P \subset U$.

Lemma 2.5 (*cf.* [7, Lemmas 3.13, 3.14]). *(1) Let $k = \operatorname{codim}_{\mathbb{R}}(P \subset U)$. For $\alpha \in \mathcal{W}_P^s(U)$, we have $\mathbf{i}^*\alpha \in \mathcal{W}_Q^s(R)$ if P intersects R transversally, where $Q \subset R$ is any codimension k polyhedral subset which contains $P \cap R$ and is normal to $\mathbf{i}^*\nu_P$, and $\mathbf{i}^*\alpha \in \mathcal{W}_k^{-\infty}(R)$ otherwise. Moreover, the pullback along $\mathbf{i}$ is a map $\mathbf{i}^* : \mathcal{W}_k^s(U) \to \mathcal{W}_k^s(R)$.*

(2) For $\alpha \in \mathcal{W}_P^s(R)$, we have $\mathbf{p}^*\alpha \in \mathcal{W}^s_{\mathbf{p}^{-1}(P)}(U)$. Moreover, the pullback along $\mathbf{p}$ is a map $\mathbf{p}^* : \mathcal{W}_k^s(R) \to \mathcal{W}_k^s(U)$.

Finally, we extend the above construction to define an integral operator which retracts U to a chosen point q_0. Consider a chain of affine subspaces $\{q_0\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_r = U$ with $\dim_{\mathbb{R}}(U_j) = j$. Denote by $\mathbf{i}_j : U_j \to U_{j+1}$ and $\mathbf{p}_j : U_{j+1} \to U_j$ the inclusions and affine projections, respectively. Let v_j be a constant affine vector field on U_{j+1} which is tangent to the fiber of $\mathbf{p}_j$. Composition of the inclusion operators gives $\mathbf{i}_{i,j} : U_i \to U_j$, $i < j$, and similarly for the projection operators. Let $I_j : \mathcal{W}_k^s(U_{j+1}) \to \mathcal{W}_{k-1}^{s-1}(U_{j+1})$ be the integral operator defined using v_j, as above. For the purpose of solving the Maurer–Cartan equation, we will choose q_0 to be an irrational point in U_1. While $\{q_0\}$ is not a tropical polyhedral subset of U_1, the definition of $\mathbf{p}_{0,j}^*$ remains valid if it is treated as the inclusion of constant functions. The operator I_0 defines a map $\mathcal{W}_1^s(U_1) \to \mathcal{W}_0^{s-1}(U_1)$, even if q_0 is irrational, as each $\alpha \in \mathcal{W}_1^s(U_1)$ can be written as a finite sum $\sum_l \alpha_l$ with $\alpha_l \in \tilde{\mathcal{W}}_{P_l}^s(U_1)$ for some rational points P_l of U_1 which, in particular, are distinct from q_0. It follows that $I_0(P_l)$ is still a tropical subspace of U_1.

With the above notation, define $I : \mathcal{W}_*^s(U) \to \mathcal{W}_{*-1}^{s-1}(U)$ by

$$I = \mathbf{p}_{1,r}^* I_0 \mathbf{i}_{1,r}^* + \cdots + \mathbf{p}_{r-1,r}^* I_{r-2} \mathbf{i}_{r-1,r}^* + I_{r-1}. \tag{2.6}$$

Write $\mathbf{i}^* := \mathbf{i}_{0,r}^*$ for evaluation at q_0 and $\mathbf{p}^* := \mathbf{p}_{0,r}^*$.

Proposition 2.6 (*cf.* [7, Lemma 3.16]). *The equality $dI + Id = \mathrm{Id} - \mathbf{p}^*\mathbf{i}^*$ holds.*

2.2. The tropical differential graded Lie algebra.

2.2.1. *Abstract tropical dg Lie algebras.* Consider a M_σ^+-graded tropical Lie algebra $\mathfrak{h}$ acting on P-graded algebra A, as in Section 1. Fix a convex open subset $U \subset M_{\mathbb{R}}$.

Definition 2.7. *The* tropical dg Lie algebra associated to $\mathfrak{h}$ *is*

$$\mathcal{H}^*(U) := \bigoplus_{m \in M_\sigma^+} \left(\mathcal{W}_*^0(U)/\mathcal{W}_*^{-1}(U)\right) \otimes_{\mathbb{C}} \mathfrak{h}_m,$$

with differential $d(\alpha h) = (d\alpha)h$ and Lie bracket $[\alpha h, \beta h'] = (\alpha \wedge \beta)[h, h']$, where $\alpha, \beta \in \mathcal{W}_*^0(U)/\mathcal{W}_*^{-1}(U)$ and $h, h' \in \mathfrak{h}$.

Denote by $\hat{\mathcal{H}}^*(U)$ the completion associated to the monoid ideals $kM_\sigma^+ \subset M_\sigma^+$, $k \in \mathbb{Z}_{>0}$. Given a commutative algebra R, set $\mathcal{H}_R^*(U) = \mathcal{H}^*(U) \otimes_{\mathbb{C}} R$ and $\hat{\mathcal{H}}_R^*(U) = \hat{\mathcal{H}}^*(U) \hat{\otimes}_{\mathbb{C}} R$. For later convenience, we also introduce the dg Lie algebra $\mathcal{G}^*(U) := \bigoplus_{m \in M_\sigma^+} \mathcal{W}_*^0(U) \otimes_{\mathbb{C}} \mathfrak{h}_m$ and its dg Lie ideal $\mathcal{I}^*(U) := \bigoplus_{m \in M_\sigma^+} \mathcal{W}_*^{-1}(U) \otimes_{\mathbb{C}} \mathfrak{h}_m$. Observe that $\mathcal{G}^*(U)/\mathcal{I}^*(U) \simeq \mathcal{H}^*(U)$. When $U = M_{\mathbb{R}}$, we will often omit U from the notation.

We will be interested in solving the Maurer–Cartan equation in $\hat{\mathcal{H}}^*$ or $\mathcal{H}_R^*$, which reads

$$d\varphi + \frac{1}{2}[\varphi, \varphi] = 0. \tag{2.7}$$

Definition 2.8. *The* tropical dg algebra associated to A *is $\mathcal{A}^*(U) := \bigoplus_{m \in P} \left(\mathcal{W}_*^0(U)/\mathcal{W}_*^{-1}(U)\right) \otimes_{\mathbb{C}} A_m$, with differential $d(\alpha f) = (d\alpha)f$ and product $(\alpha f) \wedge (\beta g) = (\alpha \wedge \beta)(fg)$, where $\alpha \in \mathcal{W}_*^0(U)/\mathcal{W}_*^{-1}(U)$ and $f \in A$.*

There is a natural left $\mathcal{H}^*(U)$-action on $\mathcal{A}^*(U)$ given by $(\alpha h) \cdot (\beta f) := (\alpha \wedge \beta)(h \cdot f)$. As for $\mathcal{H}^*(U)$, we can also define $\hat{\mathcal{A}}^*(U)$, $\mathcal{A}_R(U)$ and $\hat{\mathcal{A}}_R(U)$.

2.2.2. *Homotopy operator.* We will solve equation (2.7) using Kuranishi's method [17], in which a solution is written as a sum over trivalent trees. We take $U = M_{\mathbb{R}}$ for the remainder of this section.

Fix an affine metric g_0 on $M_{\mathbb{R}}$. For each $m \in M_\sigma^+$, fix a chain of affine subspaces $\{pt\} = U_0^m \subseteq U_1^m \subseteq \cdots \subseteq U_r^m = M_{\mathbb{R}}$. We assume that U_0^m is an irrational point of U_1^m. Denote by $\mathbf{p}_j^m$ the affine projection determined by the vector field v_j^m, with the convention that $v_1^m = -m$.

Given these choices, we obtain a homotopy operator $\mathsf{H}_m : \mathcal{W}_*^0 \to \mathcal{W}_{*-1}^0$ using equation (2.6) (denoted there by I). Let $\mathsf{P}_m : \mathcal{W}_*^0 \to \mathcal{W}_0^0(U_0^m)$ be the projection $\mathsf{P}_m(\alpha) := \alpha|_{U_0^m}$ and let $\iota_m : \mathcal{W}_0^0(U_0^m) \to \mathcal{W}_*^0$ be given by $\iota_m(\alpha) := \alpha$, the embedding of constant functions on $M_{\mathbb{R}}$. As in [6], these operators satisfy

$$d\mathsf{H}_m + \mathsf{H}_m d = \mathrm{Id} - \iota_m \mathsf{P}_m, \tag{2.8}$$

so that $\mathsf{H}_* = \bigoplus_m \mathsf{H}_m$ is a homotopy retracting $\mathcal{W}_*^0$ to its cohomology $H^*(\mathcal{W}_*^0, d) \simeq \mathcal{W}_0^0(U_0^m)$. Moreover, these operators descend to the quotient $\mathcal{W}_*^0/\mathcal{W}_*^{-1}$, thereby contracting its cohomology to $\mathbb{C} \cong \mathcal{W}_0^0(U_0^m)/\mathcal{W}_0^{-1}(U_0^m)$.

Definition 2.9. *(1) For each $m \in M_\sigma^+$, let $\mathcal{H}_m^* := \left(\mathcal{W}_*^0/\mathcal{W}_*^{-1}\right) \otimes_{\mathbb{C}} \mathfrak{h}_m$ and define the homotopy operator $\mathsf{H}_m : \mathcal{H}_m^{*+1} \to \mathcal{H}_m^*$ by $\mathsf{H}_m(\alpha h) = \mathsf{H}_m(\alpha) h$. Denote by $\mathsf{H} = \bigoplus_m \mathsf{H}_m$ the induced operator on $\mathcal{H}^*$.*
(2) Define operators $\mathsf{P} = \bigoplus_m \mathsf{P}_m$ and $\iota = \bigoplus_m \iota_m$ similarly.

Similar definitions apply to define operators on $\hat{\mathcal{H}}^*$, $\mathcal{H}_R^*$ and $\hat{\mathcal{H}}_R^*$.

2.3. Solving the Maurer–Cartan equation.

2.3.1. *Input of the Maurer–Cartan equation.* Consider an initial scattering diagram $\mathcal{D}_{in}$, or its perturbation $\tilde{\mathcal{D}}_{in,l}$. We are going to associate to each wall of the scattering diagram a term in $\hat{\mathcal{H}}^1$ or $\hat{\mathcal{H}}^1_{\tilde{R}_l}$ to serve as inputs to solve the Maurer–Cartan equation.

Consider first $\mathcal{D}_{in} = \{\mathbf{w}_i = (m_i, n_i, P_i, \Theta_i)\}_{i \in I}$, with $\log(\Theta_i) = \sum_j g_{ji}$ as in equation (1.1). Consider an affine function $\eta_i = \langle \cdot, n_i \rangle + c$ such that $P_i = \{x \in M_{\mathbb{R}} \mid \eta_i(x) = 0\}$ and set

$$\delta_{P_i} = \big(\frac{1}{\pi\hbar}\big)^{1/2} e^{-(\eta_i^2)/\hbar} d\eta_i. \tag{2.9}$$

Lemma 2.10 ([6, §4])**.** *We have $\delta_{P_i} \in \mathcal{W}^1_{P_i}(M_{\mathbb{R}})$.*

Set

$$\Pi^{(i)} = -\delta_{P_i} \log(\Theta_i) \in \hat{\mathcal{H}}^1 \tag{2.10}$$

and take $\Pi = \sum_{i \in I} \Pi^{(i)}$ as the input to solve the Maurer–Cartan equation.

If instead we begin with a perturbed diagram $\tilde{\mathcal{D}}_{in,l}$, then we have walls $\mathbf{w}_{iJ} = (m_{iJ}, n_{iJ}, P_{iJ}, \Theta_{iJ})$, leading to $\delta_{P_{iJ}} \in \mathcal{W}^1_{P_i iJ}(M_{\mathbb{R}})$ and $\tilde{\Pi} = \sum_{i,J} \tilde{\Pi}_J^{(i)} \in \mathcal{H}^1_{\tilde{R}_l}$.

2.3.2. *Summation over trees.* Motivated by Kuranishi's method [17] of solving the Maurer–Cartan equation of the Kodaira–Spencer dg Lie algebra, and its generalization to general dg Lie algebras (see, for example, [19]), instead of solving equation (2.7), we first look for solutions $\breve{\Phi} \in \hat{\mathcal{H}}^1$ of the equation

$$\breve{\Phi} = \Pi - \frac{1}{2}\mathsf{H}[\breve{\Phi}, \breve{\Phi}]. \tag{2.11}$$

In the perturbed setting, we look for solutions $\tilde{\breve{\Phi}} \in \mathcal{H}^1_{\tilde{R}_l}$ of the equation $\tilde{\breve{\Phi}} = \tilde{\Pi} - \frac{1}{2}\mathsf{H}[\tilde{\breve{\Phi}}, \tilde{\breve{\Phi}}]$.

Proposition 2.11. *If $\breve{\Phi}$ satisfies equation* (2.11)*, then $\breve{\Phi}$ satisfies equation* (2.7) *if and only if* $\mathsf{P}[\breve{\Phi},\breve{\Phi}] = 0$. *An analgous statement holds for $\breve{\tilde{\Phi}}$.*

The unique solution $\breve{\Phi}$ of equation (2.11) can be expressed as a sum over directed trees, as we now recall. An analogous statement holds for $\breve{\tilde{\Phi}}$. Further details can be found in [6, §5.1].

Definition 2.12. *Given a labeled (resp. weighted) ribbon k-tree $\mathcal{L}$ (resp. $\mathcal{T}$), number the incoming vertices by $v_1, \ldots, v_k$ according to their cyclic ordering and let $e_1, \ldots, e_k$ be their incoming edges. Define an operator $\mathfrak{l}_{k,\mathcal{L}} : (\hat{\mathcal{H}}^{*+1})^{\otimes k} \to \hat{\mathcal{H}}^{*+1}$ (resp. $\mathfrak{l}_{k,\mathcal{T}} : (\mathcal{H}^{*+1}_{\tilde{R}_l})^{\otimes k} \to \mathcal{H}^{*+1}_{\tilde{R}_l}$) so that its value on $\zeta_1, \ldots, \zeta_k \in \hat{\mathcal{H}}^{*+1}_{(\tilde{R}_l)}$ is given by*

1. *first extracting component of ζ_i in $\mathcal{H}^{*+1}_{m_{e_i}}$ (resp. $\mathcal{H}^{*+1}_{m_{e_i}} u^{J_{e_i}}$) and aligning it as the input at v_i,*
2. *then applying m_2 at each vertex in $\mathcal{L}^{[0]}$ (resp. $\mathcal{T}^{[0]}$), where $m_2 : \hat{\mathcal{H}}^{*+1} \otimes \hat{\mathcal{H}}^{*+1} \to \hat{\mathcal{H}}^{*+1}$ is the graded symmetric operator $m_2(\alpha, \beta) = (-1)^{\bar{\alpha}(\bar{\beta}+1)}[\alpha, \beta]$, where $\bar{\alpha}$ and $\bar{\beta}$ denote the degrees of α and β, and finally*
3. *applying the homotopy operator $-\mathsf{H}$ to each edge in $\mathcal{L}^{[1]}$ (resp. $\mathcal{T}^{[1]}$).*

Having defined $\mathfrak{l}_{k,\mathcal{L}}$ and $\mathfrak{l}_{k,\mathcal{T}}$, we can write

$$\breve{\Phi} = \sum_{k \geq 1} \frac{1}{2^{k-1}} \sum_{\mathcal{L} \in \mathtt{LR}_k} \mathfrak{l}_{k,\mathcal{L}}(\Pi, \ldots, \Pi), \qquad \breve{\tilde{\Phi}} = \sum_{k \geq 1} \frac{1}{2^{k-1}} \sum_{\mathcal{T} \in \mathtt{WRT}_k} \mathfrak{l}_{k,\mathcal{T}}(\tilde{\Pi}, \ldots, \tilde{\Pi}). \tag{2.12}$$

It is not hard to see that the sum defining $\breve{\Phi}$ converges in $\hat{\mathcal{H}}^*$. The sum defining $\breve{\tilde{\Phi}}$ is finite in $\mathcal{H}^*_{\tilde{R}_l}$ because the maximal ideal of $\tilde{R}_l$ is nilpotent.

3. Tropical counting and theta functions from Maurer–Cartan solutions

3.1. **Tropical counting from Maurer–Cartan solutions.** The goal of this section is to relate Maurer–Cartan solution in $\mathcal{H}^1_{\tilde{R}_l}$ with the counting of tropical disks in $(M_{\mathbb{R}}, \tilde{\mathfrak{D}}_{in,l})$. Similar results for M of rank two can be found in [7].

3.1.1. *A partial homotopy operator.* Recall the homotopy operator $\mathsf{H} = \bigoplus_{m \in M_\sigma^+} \mathsf{H}_m$ from Section 2.2.2. It will be useful to replace H_m with a partial homotopy operator

$$\mathbf{H}_m(\alpha)(x) := \int_{-\infty}^{0} (\iota_{\frac{\partial}{\partial s}} (\tau^m)^*(\alpha)(s, x)) ds, \tag{3.1}$$

where $\tau^m : \mathbb{R} \times M_{\mathbb{R}} \to M_{\mathbb{R}}$ is the flow with respect to the vector field $-m$. Given a labeled (resp. weighted) ribbon k-tree $\mathcal{L}$ (resp. $\mathcal{T}$), we denote by $\mathbf{L}_{k,\mathcal{L}}$ (resp. $\mathbf{L}_{k,\mathcal{T}}$) the operation obtained by replacing the operator H with $\mathbf{H}$ in Definition 2.12 (denoted there by $\mathfrak{l}$ instead of $\mathbf{L}$).

Geometrically defined? Or what is canonical?Here I wanna say that $\mathbf{H}$ is not depending on the extra choices, for each m we have an operator $\mathbf{H}_m$.

Remark 3.1. The reason for introducing the operators $\mathbf{H}$ and $\mathbf{L}_{k,\mathcal{L}}$ (resp. $\mathbf{L}_{k,\mathcal{T}}$) is that the operator H of Section 2.2.2 depends on the choice of a chain of affine subspaces $U^m_\bullet$ for each $m \in M_\sigma^+$. A drawback of the operator $\mathbf{H}$, whose definition is indepdendent of the choice $U^m_\bullet$, is that $\mathbf{H}_m(\alpha)$ is defined only when α is suitably behaved at infinity; see Lemma 3.5. Furthermore, additional arguments are required to verify that the formula given by summation over trees above using $\mathbf{L}_{k,\mathcal{L}}$ (resp. $\mathbf{L}_{k,\mathcal{T}}$) is well-defined and in fact solves the Maurer–Cartan equation; see Lemma 3.8.

An alternative way of proceeding is to instead make a careful choice of the chains $U^m_\bullet$ so as to directly relate the Maurer–Cartan solution $\breve{\Phi}$ defined using H with scattering diagrams; this approach is taken in [6, 7].

3.1.2. *Modified Maurer–Cartan solutions.* In this section we prove that $\mathbf{L}_{k,\mathcal{T}}(\tilde{\Pi},\dots,\tilde{\Pi})$ is well-defined; the proof for $\mathbf{L}_{k,\mathcal{L}}(\Pi,\dots,\Pi)$ is similar.

Given a weighted ribbon k-tree $\mathcal{T}$, denote by $\mathfrak{M}_{\mathcal{T}}(M_{\mathbb{R}}) \cong \mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|} \times M_{\mathbb{R}}$ the space of tropical disks in $M_{\mathbb{R}}$ for the underlying weighted tree.

Definition 3.2. *Given a directed path $\mathfrak{e} = (e_0, e_1, \dots, e_l)$ in $\mathcal{T}$, considered as a sequence of edges, define a map $\tau^{\mathfrak{e}} : \mathbb{R}_{\leq 0}^{|\mathcal{T}_{\mathfrak{e}}^{[1]}|} \times M_{\mathbb{R}} \to M_{\mathbb{R}}$ by $\tau^{\mathfrak{e}}(\vec{s}, x) = \tau_{s_0}^{e_0} \circ \tau_{s_1}^{e_1} \circ \cdots \circ \tau_{s_l}^{e_l}(x)$, where $\mathcal{T}_{\mathfrak{e}}^{[1]}$ is the subset $\{e_0, \dots, e_l\} \subset \mathcal{T}^{[1]}$. The map $\tau^{\mathfrak{e}}$ extends to a map $\hat{\tau}^{\mathfrak{e}} : \mathfrak{M}_{\mathcal{T}}(M_{\mathbb{R}}) \cong \mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|} \times M_{\mathbb{R}} \to \mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]} \setminus \mathcal{T}_{\mathfrak{e}}^{[1]}|} \times M_{\mathbb{R}}$ by taking the Cartesian product with $\mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]} \setminus \mathcal{T}_{\mathfrak{e}}^{[1]}|}$.*

Note that the previous definition does not use the ribbon structure of $\mathcal{T}$, so it applies also to a weighted k-tree Γ.

Recall that the differential form δ_{P_i} depends on an affine function η_i which vanishes on P_i. Let N_i be the space of leaves obtained by parallel translation of the hyperplane P_i, equipped with the natural coordinate function η_i. Recall that associated to each edge $e \in \mathcal{T}_{in}^{[1]}$ is a wall $\mathbf{w}_{i_e}$. Define an affine map

$$\vec{\tau} : \mathfrak{M}_{\mathcal{T}}(M_{\mathbb{R}}) \to \prod_{e \in \mathcal{T}_{in}^{[1]}} N_{i_e} \tag{3.2}$$

by requiring $\vec{\tau}^*(\eta_{i_e}) = \eta_{i_e}(\tau^{\mathfrak{e}}(\vec{s}, x))$. Write $\mathcal{I}_x$ for $\mathbb{R}_{\leq 0}^{\mathcal{T}^{[1]}} \times \{x\}$.

Definition 3.3. *Assign a differential form ν_e on $\mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|}$ to each $e \in \bar{\mathcal{T}}^{[1]}$ recursively as follows. Set $\nu_e = 1$ if $e \in \partial_{in}^{-1}(\mathcal{T}_{in}^{[0]})$. If v is an internal vertex with $\partial_{out}^{-1}(v) = \{e_1, e_2\}$ and $\partial_{in}^{-1}(v) = \{e_3\}$ such that $\{e_1, e_2, e_3\}$ is clockwise oriented, then set $\nu_{e_3} = (-1)^{|\nu_{e_2}|} \nu_{e_1} \wedge \nu_{e_2} \wedge ds_{e_3}$, where $|\nu_{e_2}|$ is the cohomological degree of ν_{e_2}.*

The form $\nu_{\mathcal{T}}$ attached to the outgoing edge $e_o \in \mathcal{T}^{[1]}$ is a volume form on $\mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|}$.

The following result can be proved in the same way as [6, Lemma 5.33].

Lemma 3.4. *We have $\vec{\tau}^*(d\eta_{i_{e_1}} \wedge \cdots \wedge d\eta_{i_{e_k}}) = c\nu_{\mathcal{T}} \wedge n_{\mathcal{T}} + \varepsilon$ for some $c > 0$, where $n_{\mathcal{T}} \in N$ is a 1-form on $M_{\mathbb{R}}$, $\nu_{\mathcal{T}}^{\vee}$ is the top polyvector field on $\mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|}$ dual to $\nu_{\mathcal{T}}$ and $\iota_{\nu_{\mathcal{T}}^{\vee}} \varepsilon = 0$. In particular, $\vec{\tau}|_{\mathcal{I}_x}$ is an affine isomorphism onto its codimension one image $C(\vec{\tau}, x) \subset \prod_{e \in \mathcal{T}_{in}^{[1]}} N_{i_e}$ when $n_{\mathcal{T}} \neq 0$.*

The well-definedness of $\mathbf{L}_{k,\mathcal{T}}(\tilde{\Pi},\dots,\tilde{\Pi})$ depends on the convergence of the integral in the following lemma. Write α_j in place of $\delta_{P_{i_{e_j}} J_{e_j}}$ and, for each $L > 0$, set $\mathcal{I}_{x,L} = [-L, 0]^{\mathcal{T}^{[1]}} \times \{x\}$.

Lemma 3.5. *(1) The integral*

$$\alpha_{\mathcal{T}}(x) := -\int_{\mathcal{I}_x} (\tau^{\mathfrak{e}_1})^*(\alpha_1) \wedge \cdots \wedge (\tau^{\mathfrak{e}_k})^*(\alpha_k)$$

is well-defined. Moreover, $\alpha_{\mathcal{T}} = 0$ if $n_{\mathcal{T}} = 0$ and $\alpha_{\mathcal{T}} \in \mathcal{W}_1^{-\infty}(M_{\mathbb{R}})$ if $P_{\mathcal{T}} = \emptyset$, where $P_{\mathcal{T}}$ is defined as in Section 1.3 by forgetting the ribbon structure of $\mathcal{T}$.

(2) The integral $\alpha_{\mathcal{T},L}(x) := -\int_{\mathcal{I}_{x,L}} (\tau^{\mathfrak{e}_1})^*(\alpha_1) \wedge \cdots \wedge (\tau^{\mathfrak{e}_k})^*(\alpha_k)$ uniformly converges to $\alpha_{\mathcal{T}}(x)$ for x in any pre-compact open subset $K \subset M_{\mathbb{R}}$. Furthermore, $(\alpha_{\mathcal{T}} - \alpha_{\mathcal{T},L})|_K \in \mathcal{W}_1^{-\infty}(K)$ for sufficiently large L.

(3) If $P_{\mathcal{T}} \neq \emptyset$, so that $\dim_{\mathbb{R}}(P_{\mathcal{T}}) = r-1$, and $\varrho : (a,b) \to M_{\mathbb{R}}$ is an embedded affine line intersecting $P_{\mathcal{T}}$ positively[4] *and transversally in its relative interior* $\mathrm{Int}_{re}(P_{\mathcal{T}})$, *then* $\lim_{\hbar \to 0} \int_{\varrho} \alpha_{\mathcal{T}} = -1$.

Proof. Explicitly, the integral $\alpha_{\mathcal{T}}(x)$ under consideration is

$$\Big(\frac{1}{\pi\hbar}\Big)^{k/2} \int_{\mathcal{I}_x} \vec{\tau}^* \Big(\prod_{j=1}^{k} e^{-(\eta_{i_{e_j}}^2)/\hbar} d\eta_{i_{e_j}}\Big) = \Big(\frac{1}{\pi\hbar}\Big)^{k/2} \int_{\mathcal{I}_x} e^{-(\sum_{j=1}^{k} (\tau^{\mathfrak{e}_j})^* \eta_{i_{e_j}}^2)/\hbar} \vec{\tau}^*(d\eta_{i_{e_1}} \cdots d\eta_{i_{e_k}}).$$

By Lemma 3.4, the only case that we need consider is when $n_{\mathcal{T}} \neq 0$, in which case $\vec{\tau}|_{\mathcal{I}_x}$ is an affine isomorphism onto its image $C(\vec{\tau}, x)$, a codimension one closed affine subspace. The well-definedness of the integral is due to the fact that $\int_{C(\vec{\tau},x)} e^{-(\sum_{j=1}^{k} (\tau^{\mathfrak{e}_j})^* \eta_{i_{e_j}}^2)/\hbar} \mu_{C(\vec{\tau},x)} < \infty$ for any affine linear volume form $\mu_{C(\vec{\tau},x)}$ on $C(\vec{\tau}, x)$. When $P_{\mathcal{T}} = \emptyset$, we have $0 \notin C(\vec{\tau}, x)$ for any $x \in M_{\mathbb{R}}$, which implies $\alpha_{\mathcal{T}} \in \mathcal{W}_1^{-\infty}(M_{\mathbb{R}})$.

Notice that

$$\bigcap_{e \in \mathcal{T}_{in}^{[1]}} \{(\tau^{\mathfrak{e}})^* \eta_{i_e} = 0\} = \overline{\mathfrak{M}}_{\mathcal{T}}(M_{\mathbb{R}}, \tilde{\mathfrak{D}}_{in,l}) \subset \mathfrak{M}_{\mathcal{T}}(M_{\mathbb{R}}).$$

Furthermore, for each pre-compact subset $K \subset M_{\mathbb{R}}$ and any $b > 0$, there exists an L_b such that $(\mathcal{I}_x \setminus \mathcal{I}_{x,L_b}) \cap \bigcap_{e \in \mathcal{T}_{in}^{[1]}} \{|(\tau^{\mathfrak{e}})^* \eta_{i_e}| \leq b\} = \emptyset$ for all $x \in K$, as follows from the fact that $\vec{\tau}|_{\mathcal{I}_x}$ is an affine isomorphism onto its image. This implies that $\alpha_{\mathcal{T},L}$ converges uniformly to $\alpha_{\mathcal{T}}$ on K and that $(\alpha_{\mathcal{T}} - \alpha_{\mathcal{T},L})|_K \in \mathcal{W}_1^{-\infty}(K)$ for sufficiently large L.

Suppose now that $P_{\mathcal{T}}$ and ϱ are as in the final statement of the lemma. Consider the affine subspace $\mathcal{I}_{\varrho} := \bigcup_{t \in (a,b)} \mathcal{I}_{\varrho(t)} \subset \mathfrak{M}_{\mathcal{T}}(M_{\mathbb{R}})$. We have

$$\int_{\varrho} \alpha_{\mathcal{T}} = -\int_{\mathcal{I}_{\varrho}} (\tau^{\mathfrak{e}_1})^*(\alpha_1) \wedge \cdots \wedge (\tau^{\mathfrak{e}_k})^*(\alpha_k) = -\int_{\vec{\tau}(\mathcal{I}_{\varrho})} \alpha_1 \wedge \cdots \wedge \alpha_k,$$

where $\vec{\tau}(\mathcal{I}_{\varrho}) \subset \prod_{e \in \mathcal{T}_{in}^{[1]}} N_{i_e}$. Here we apply Lemma 3.4 above to conclude that $\vec{\tau}$ is an affine isomorphism onto its image $\vec{\tau}(\mathcal{I}_{\varrho})$ when $P_{\mathcal{T}} \neq \emptyset$. Since ϱ intersects $P_{\mathcal{T}}$ in $\mathrm{Int}_{re}(P_{\mathcal{T}})$ and $\tilde{D}_{in,l}$ is generic, we have $0 \in \mathrm{Int}(\vec{\tau}(\mathcal{I}_{\varrho}))$. Together with the explicit form of $\alpha_1 \wedge \cdots \wedge \alpha_k$, we then obtain $\lim_{\hbar \to 0} \int_{\varrho} \alpha_{\mathcal{T}} = -1$. ☐

3.1.3. *Relation with tropical counting.* The following result is a modification of [6, §5].

Lemma 3.6. *For each $\mathcal{T} \in \mathtt{WRT}_k$, we have* $\mathbf{L}_{k,\mathcal{T}}(\tilde{\Pi}, \ldots, \tilde{\Pi}) = \Big(\prod_{e \in \partial_{in}^{-1}(\mathcal{T}_{in}^{[0]})} (\#J_{e,i_e})!\Big) \alpha_{\mathcal{T}} g_{\mathcal{T}} u^{\vec{J}_{\mathcal{T}}}$.

Proof. We proceed by induction on the cardinality of $\mathcal{T}^{[0]}$. In the initial case, $\mathcal{T}^{[0]} = \emptyset$, the only tree is that with a unique edge and there is nothing to prove.

For the induction step, the root vertex $v_r \in \mathcal{T}^{[0]}$ is adjacent to the outgoing edge e_0 and the two incoming edges, say e_1 and e_2. Assume that $\{e_1, e_2, e_o\}$ are in clockwise orientation. Split $\mathcal{T}$ at v_r, thereby obtaining trees $\mathcal{T}_1$ and $\mathcal{T}_2$ with outgoing edges e_1 and e_2 and k_1 and k_2 incoming edges, respectively. By the induction hypothesis, we can write $\mathbf{L}_{k_i,\mathcal{T}_i}(\tilde{\Pi}, \ldots, \tilde{\Pi}) = \Big(\prod_{e \in \partial_{in}^{-1}((\mathcal{T}_i)_{in}^{[0]})} (\#J_{e,i_e})!\Big) \alpha_{\mathcal{T}_i} g_{\mathcal{T}_i} u^{\vec{J}_{\mathcal{T}_i}}$, $i = 1, 2$. We therefore have

$$\mathbf{L}_{k,\mathcal{T}}(\tilde{\Pi}, \ldots, \tilde{\Pi}) = -\Big(\prod_{e \in \partial_{in}^{-1}(\mathcal{T}_{in}^{[0]})} (\#J_{e,i_e})!\Big) \mathbf{H}(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})[g_{\mathcal{T}_1}, g_{\mathcal{T}_2}] u^{\vec{J}_{\mathcal{T}_1}} u^{\vec{J}_{\mathcal{T}_2}}.$$

[4] Intersecting positively means $\langle \varrho', n_{\mathcal{T}} \rangle > 0$.

By definition, $g_{\mathcal{T}} = [g_{\mathcal{T}_1}, g_{\mathcal{T}_2}]$ and $u^{\vec{J}_{\mathcal{T}}} = u^{\vec{J}_{\mathcal{T}_1}} u^{\vec{J}_{\mathcal{T}_2}}$. Finally, the proof of [6, Lemma 5.31] shows that

$$\mathbf{H}(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) = \int_{\mathcal{I}_x} (\tau^{\mathfrak{e}_1})^*(\alpha_1) \cdots (\tau^{\mathfrak{e}_k})^*(\alpha_k) = -\alpha_{\mathcal{T}}(x).$$

Note that the well-definedness of $\mathbf{H}(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})$ is guaranteed by Lemma 3.5. □

Lemma 3.7. *For each $\mathcal{T} \in \mathtt{WRT}_k$, we have $\alpha_{\mathcal{T}} \in \mathcal{W}^1_{P_{\mathcal{T}}}(M_{\mathbb{R}}) \cap \mathcal{W}^1_1(M_{\mathbb{R}})$ if $P_{\mathcal{T}} \neq \emptyset$.*

Proof. We proceed by induction on the cardinality of $\mathcal{T}^{[0]}$. The initial case, $\mathcal{T}^{[0]} = \emptyset$, holds by Lemma 2.10.

For the induction step, split $\mathcal{T}$ at the root vertex $v_r \in \mathcal{T}^{[0]}$ to obtain trees $\mathcal{T}_1$ and $\mathcal{T}_2$, as in the proof Lemma 3.6. We can assume that each $P_{\mathcal{T}_i}$ is non-empty and that $P_{\mathcal{T}_1}$ and $P_{\mathcal{T}_2}$ intersect transversally and generically, as in Definition 1.17. Then $Q = P_{\mathcal{T}_1} \cap P_{\mathcal{T}_2}$ is a codimension two affine subspace of $M_{\mathbb{R}}$. The induction hypothesis implies $\alpha_{\mathcal{T}_i} \in \mathcal{W}^1_{P_{\mathcal{T}_i}}(M_{\mathbb{R}}) \cap \mathcal{W}^1_1(M_{\mathbb{R}})$ and Lemma 2.3 gives $\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2} \in \mathcal{W}^2_Q(M_{\mathbb{R}}) \cap \mathcal{W}^2_2(M_{\mathbb{R}})$. Arguing as in the proof of Lemma 3.6, we find $\alpha_{\mathcal{T}} = -\mathbf{H}_{m_{e_o}}(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})$, which is nonzero only if $n_{\mathcal{T}} \neq 0$. Note that if $n_{\mathcal{T}} \neq 0$, then $-m_{\mathcal{T}} = -m_{e_o}$ is not tangent to Q.

We would like to apply Lemma 2.4 to conclude our result. However, the operator $\mathbf{H}_{m_{e_o}}$ is slightly different from that appearing in Lemma 2.4. So, we must slightly modify the proof.

To simplify notation, write $m = m_{e_o}$ for the remainder of the proof. Since m is not tangent to Q, we can assume that the chain of affine subspaces $\{pt\} = U^m_0 \subseteq U^m_1 \subseteq \cdots \subseteq U^m_r = M_{\mathbb{R}}$ used to define H_m is such that U^m_{r-1} separates $M_{\mathbb{R}}$ into $M^-_{\mathbb{R}}$ and $M^+_{\mathbb{R}}$, $-m$ points into $M^+_{\mathbb{R}}$ and $Q \subset M^+_{\mathbb{R}}$. With this choice, we obtain a homotopy operator H_m as in Section 2.2.2 which, by Lemma 2.4, satisfies $\mathsf{H}_m(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) \in \mathcal{W}^1_{P_{\mathcal{T}}}(M_{\mathbb{R}}) \cap \mathcal{W}^1_1(M_{\mathbb{R}})$. Since $U^m_{r-1} \cap Q = \emptyset$, we find that the difference $\mathbf{H}_{m,L}(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) - \mathsf{H}_m(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})$ lies in $\mathcal{W}^{-\infty}_1(M_{\mathbb{R}})$. It follows that $\mathbf{H}_{m,L}(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})$ satisfies the desired property. □

Lemma 3.8. *The element $\tilde{\Phi} := \sum_{k \geq 1} \frac{1}{2^{k-1}} \sum_{\mathcal{T} \in \mathtt{WRT}_k} \mathbf{L}_{k,\mathcal{T}}(\tilde{\Pi}, \ldots, \tilde{\Pi})$ is well-defined in $\mathcal{G}^* \otimes_{\mathbb{C}} \tilde{R}_l$. Furthermore, it solves equation (2.7).*

Proof. Well-definedness of $\tilde{\Phi}$ follows from Lemmas 3.5 and 3.6. The same reasoning as in Section 2.3.2 then shows that $\tilde{\Phi} = \tilde{\Pi} - \frac{1}{2}\mathbf{H}[\tilde{\Phi}, \tilde{\Phi}]$.

We will use Proposition 2.11 to show that $\tilde{\Phi}$ solves equation (2.7). Fix a pre-compact open subset $K \subset M_{\mathbb{R}}$ and consider the restriction of equation (2.7) to K. By Lemma 3.5, by choosing L sufficiently large we can ensure that the truncation $\tilde{\Phi}_L := \sum_{k \geq 1} \frac{1}{2^{k-1}} \sum_{\substack{\mathcal{T} \in \mathtt{WRT}_k \\ u^{\vec{J}_{\mathcal{T}}} \neq 0}} \alpha_{\mathcal{T},L} g_{\mathcal{T}} u^{\vec{J}_{\mathcal{T}}}$ satisfies $\alpha_{\mathcal{T}} - \alpha_{\mathcal{T},L} \in \mathcal{W}^{-\infty}_1(K)$. Indeed, this is possible because there are only finitely many terms with $u^{\vec{J}_{\mathcal{T}}} \neq 0$ in the expression for $\tilde{\Phi}_L$, as the maximal ideal of $\tilde{R}_l$ is nilpotent. Notice that $\tilde{\Phi}_L$ satisfies $\tilde{\Phi}_L = \tilde{\Pi} - \frac{1}{2}\mathbf{H}_L[\tilde{\Phi}_L, \tilde{\Phi}_L]$, where $\mathbf{H}_L := \bigoplus_{m \in M^+_\sigma} \mathbf{H}_{L,m}$ and $\mathbf{H}_{L,m}(\alpha)(x) := \int_{-L}^0 (\iota_{\frac{\partial}{\partial s}} (\tau^m)^*(\alpha)(s,x))ds$.

Similar to Proposition 2.11, it suffices to show that $\mathbf{P}_L[\tilde{\Phi}_L, \tilde{\Phi}_L] = 0$ on K, where $\mathbf{P}_L := \bigoplus_{m \in M^+_\sigma} \mathbf{P}_{L,m}$ and $\mathbf{P}_{L,m}(\beta) := (\tau^m_{-L})^*\beta$. Since $\tilde{\Phi}_L$ is a sum over trees, we consider weighted trees $\mathcal{T}_i \in \mathtt{WRT}_{k_i}$, $i = 1, 2$, with $u^{\vec{J}_{\mathcal{T}_i}} \neq 0$ and the associated terms $\alpha_{\mathcal{T}_i,L} g_{\mathcal{T}_i}$, where $g_{\mathcal{T}_i} \in \mathfrak{h}_{m_{\mathcal{T}_i}, n_{\mathcal{T}_i}, \tilde{R}_l}$. We join the trees $\mathcal{T}_1$ and $\mathcal{T}_2$ to give $\mathcal{T}$. It suffices to assume that $n_{\mathcal{T}} \neq 0$, since $[g_{\mathcal{T}_1}, g_{\mathcal{T}_2}] \in \mathfrak{h}_{m_{\mathcal{T}}, 0, \tilde{R}_l} = \{0\}$ when $n_{\mathcal{T}} = 0$. If $n_{\mathcal{T}} \neq 0$, then $m_{\mathcal{T}}$ is not tangent to $P_{\mathcal{T}_1} \cap P_{\mathcal{T}_2}$. We may therefore choose L sufficiently large so that $\tau^{m_{\mathcal{T}}}_{-L}(K) \cap P_{\mathcal{T}_1} \cap P_{\mathcal{T}_2} = \emptyset$. As a result, we must have $(\tau^{m_{\mathcal{T}}}_{-L})^*(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) = 0$ in $\mathcal{H}^2_{\tilde{R}_l}(K)$. □

Let Γ be a weighted k-tree with $P_\Gamma \neq \emptyset$. Since the monomial weights $u^{\vec{J}_e}$ at incoming edges $e \in \Gamma^{[1]}_{in}$ are distinct, there are, up to isomorphism, exactly 2^{k-1} ribbon structures on Γ. Note that

$\mathbf{L}_{k,\mathcal{T}}(\tilde{\Pi},\dots,\tilde{\Pi})$ does not depend[5] on the ribbon structure of $\mathcal{T}$, since $\tilde{\Pi} \in \mathcal{H}^1_{\tilde{R}_l}$ and $\tilde{\Pi}$ commutes with odd elements of $\mathcal{H}^1_{\tilde{R}_l}$. It follows that $\tilde{\Phi} = \sum_{k\geq 1}\sum_{\Gamma\in \mathtt{WT}_k} \mathbf{L}_{k,\mathcal{T}}(\tilde{\Pi},\dots,\tilde{\Pi})$, where $\mathcal{T}$ is any ribbon tree whose underlying tree $\underline{\mathcal{T}}$ is Γ. Combining Lemmas 3.6, 3.7 and 3.8, we conclude the following theorem.

Theorem 3.9. *The Maurer–Cartan solution $\tilde{\Phi} \in \mathcal{H}^1_{\tilde{R}_l}$ of Lemma 3.6 can be expressed as the following sum over trees:*

$$\tilde{\Phi} = \sum_{k\geq 1} \sum_{\substack{\Gamma \in \mathtt{WT}_k \\ \mathfrak{M}_\Gamma(M_{\mathbb{R}},\tilde{\mathfrak{D}}_{in,l})\neq\emptyset}} \alpha_\Gamma \log(\Theta_\Gamma).$$

Here $\mathfrak{M}_\Gamma(M_{\mathbb{R}},\tilde{\mathfrak{D}}_{in,l})\neq\emptyset$ indicates the existence of a tropical disk in $(M_{\mathbb{R}},\tilde{\mathfrak{D}}_{in,l})$ of combinatorial type Γ, the wall-crossing factor Θ_Γ is given by

$$\log(\Theta_\Gamma) = \Big(\prod_{e\in\partial_{in}^{-1}(\Gamma_{in}^{[0]})} (\#J_{e,i_e})!\Big) g_\Gamma u^{\vec{J}_\Gamma}$$

and α_Γ is a 1-form with asymptotic support on P_Γ which satisfies $\lim_{\hbar\to 0}\int_\varrho \alpha_\Gamma = -1$ for any affine line ϱ intersecting positively with P_Γ.

Remark 3.10. Theorem 3.9 gives a bijective correspondence between tropical disks and summands of the Maurer–Cartan solution. Together with Proposition 3.16 below, which relates Maurer–Cartan solutions with consistent scattering diagrams. This provides an alternative realization of the enumerative interpretation of Theorem 1.18.

3.2. **Non-perturbed initial scattering diagram.** In this section we study the relationship between Maurer–Cartan elements and unperturbed scattering diagrams. We are motivated by the fact that it may not always be possible (or natural) to perturb the incoming diagram. This is the case, for example, for the Hall algebra scattering diagrams. With appropriate modifications, we find that most of the results of Sections 3.1.2 and 3.1.3 remain true without perturbation.

Given an initial diagram $\mathfrak{D}_{in}$, consider $\mathbf{L}_{k,\mathcal{L}}(\Pi,\dots,\Pi)$ as in Section 3.1. The main difference is that, when $P_{\mathcal{L}}\neq\emptyset$, we have $\dim_{\mathbb{R}}(P_{\mathcal{L}}) = r-1$ in the perturbed case, whereas we only have $0 \leq \dim_{\mathbb{R}}(P_{\mathcal{L}}) \leq r-1$ in the unperturbed case.

To begin, note that the first two parts of Lemma 3.5 remains true in the context of labeled ribbon k-trees $\mathcal{L}$. However, $\lim_{\hbar\to 0}\int_\varrho \alpha_{\mathcal{L}}$ need not equal -1, even when $\dim_{\mathbb{R}}(P_{\mathcal{L}}) = r-1$. Indeed, we only have $0\in\vec{\tau}(\mathcal{I}_\varrho)$, as opposed to $0 \in \mathrm{Int}(\vec{\tau}(\mathcal{I}_\varrho))$, so the relevant part of the proof of Lemma 3.5 does not apply. The replacement of the second half of Lemma 3.5 will be given in Lemma 3.13.

Lemma 3.11. *For each $\mathcal{L}\in\mathtt{LR}_k$, we have $\mathbf{L}_{k,\mathcal{L}}(\Pi,\dots,\Pi) = \alpha_{\mathcal{L}} g_{\mathcal{L}}$, where $g_{\mathcal{L}}$ is as in Definition 1.14.*

Proof. This can be proved in the same way as Lemma 3.6. □

The next result gives the required modification of Lemma 3.7.

Lemma 3.12. *Let $\mathcal{L}\in\mathtt{LR}_k$ and let $P_{\mathcal{L}}\subset P$ be the codimension one hyperplane normal to $n_{\mathcal{L}}$.*

(1) We have $\alpha_{\mathcal{L}} \in \mathcal{W}^1_{P_{\mathcal{L}}}(M_{\mathbb{R}})\cap\mathcal{W}^1_1(M_{\mathbb{R}})$ if $\dim_{\mathbb{R}}(P_{\mathcal{L}}) = r-1$ and $\alpha_{\mathcal{L}}\in\mathcal{W}^1_P(M_{\mathbb{R}})\cap\mathcal{W}^1_1(M_{\mathbb{R}})$ otherwise. In either case, $\alpha_{\mathcal{L}}|_{M_{\mathbb{R}}\setminus P_{\mathcal{L}}}\in\mathcal{W}^{-\infty}_1(M_{\mathbb{R}}\setminus P_{\mathcal{L}})$.

(2) If $\dim_{\mathbb{R}}(P_{\mathcal{L}}) = r-1$, then there exists a polyhedral decomposition $\mathcal{P}_{\mathcal{L}}$ of $P_{\mathcal{L}}$ such that $d(\alpha_{\mathcal{L}})|_{M_{\mathbb{R}}\setminus|\mathcal{P}^{[r-2]}_{\mathcal{L}}|}\in\mathcal{W}^{-\infty}_2(M_{\mathbb{R}}\setminus|\mathcal{P}^{[r-2]}_{\mathcal{L}}|)$, where $\mathcal{P}^{[l]}$ denotes the set of l-dimensional strata and $|\mathcal{P}^{[l]}|$ is the underlying set of $\mathcal{P}^{[l]}$.

[5] This can also be deduced from the proof of Lemma 3.6 by observing that the dependence of $\alpha_{\mathcal{T}}$ and $g_{\mathcal{T}}$ on the ribbon structure of $\mathcal{T}$ cancels out in the formula for $\mathbf{L}_{k,\mathcal{T}}(\tilde{\Pi},\dots,\tilde{\Pi})$.

Proof. We proceed by induction on the cardinality of $\mathcal{L}^{[0]}$. The initial case, $\mathcal{L}^{[0]} = \emptyset$, holds by Lemma 2.10.

For the induction step, split $\mathcal{L}$ at the root vertex $v_r \in \mathcal{L}^{[0]}$ to obtain $\mathcal{L}_1$ and $\mathcal{L}_2$, as in the proof Lemma 3.6. We can assume that $n_{\mathcal{L}} \neq 0$, as otherwise $\alpha_{\mathcal{L}} = 0$ by Lemma 3.5. By the induction hypothesis, we have $\alpha_{\mathcal{L}_i} \in \mathcal{W}^1_{P_i}(M_{\mathbb{R}}) \cap \mathcal{W}^1_1(M_{\mathbb{R}})$, with $P_i = n^{\perp}_{\mathcal{L}_i}$ containing $P_{\mathcal{L}_i}$, $i = 1, 2$. Since $n_{\mathcal{L}} \neq 0$, P_1 and P_2 intersect transversally. Applying Lemma 2.3 then gives $\alpha_{\mathcal{L}_1} \wedge \alpha_{\mathcal{L}_2} \in \mathcal{W}^2_Q(M_{\mathbb{R}}) \cap \mathcal{W}^2_2(M_{\mathbb{R}})$, where $Q = P_1 \cap P_2$. We have $\alpha_{\mathcal{T}} = -\mathbf{H}_{m_{e_o}}(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})$, as in Lemma 3.6. Similar to the proof of Lemma 3.7, since $Q - \mathbb{R}_{\geq 0} m_{\mathcal{L}} \subset P$, we can apply Lemma 2.4 to conclude that $\alpha_{\mathcal{L}} \in \mathcal{W}^1_P(M_{\mathbb{R}}) \cap \mathcal{W}^1_1(M_{\mathbb{R}})$. Using the induction hypothesis and the relation $P_{\mathcal{L}} = (P_{\mathcal{L}_1} \cap P_{\mathcal{L}_2}) - \mathbb{R}_{\geq 0} m_{\mathcal{L}}$, we have $(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})|_{M_{\mathbb{R}} \setminus (P_{\mathcal{L}_1} \cap P_{\mathcal{L}_2})} \in \mathcal{W}^{-\infty}_2(M_{\mathbb{R}} \setminus (P_{\mathcal{L}_1} \cap P_{\mathcal{L}_2}))$, which gives $\alpha_{\mathcal{L}}|_{M_{\mathbb{R}} \setminus P_{\mathcal{L}}} \in \mathcal{W}^{-\infty}_1(M_{\mathbb{R}} \setminus P_{\mathcal{L}})$.

Consider now $d\alpha_{\mathcal{L}}$. Since $\alpha_{\mathcal{L}} \in \mathcal{W}^1_1(M_{\mathbb{R}})$, we can write $d\alpha_{\mathcal{L}} = \sum_j \beta_j$, where $\beta_j \in \mathcal{W}^2_{Q_j}(M_{\mathbb{R}})$ for some codimension two polyhedral subsets $Q_j \subset M_{\mathbb{R}}$. In particular, $d\alpha_{\mathcal{L}}|_{M_{\mathbb{R}} \setminus \bigcup_j Q_j} \in \mathcal{W}^{-\infty}(M_{\mathbb{R}} \setminus \bigcup_j Q_j)$ and $d\alpha_{\mathcal{L}}|_{M_{\mathbb{R}} \setminus P_{\mathcal{L}}} \in \mathcal{W}^{-\infty}(M_{\mathbb{R}} \setminus P_{\mathcal{L}})$. Letting $\mathcal{P}_{\mathcal{L}}$ be a polyhedral decomposition of $P_{\mathcal{L}}$ such that $|\mathcal{P}^{[r-2]}_{\mathcal{L}}|$ contains $P_{\mathcal{L}} \cap \bigcup_j Q_j$, we obtain the desired result. $\square$

Lemma 3.13. *Let $\mathcal{P}_{\mathcal{L}}$ be a polyhedral decomposition of $P_{\mathcal{L}}$ which satisfies the second part of Lemma 3.12 and let $\sigma \in \mathcal{P}^{[r-1]}_{\mathcal{L}}$. Then there exists a constant $c_{\mathcal{L},\sigma} > 0$ such that $\lim_{\hbar \to 0} \int_{\varrho} \alpha_{\mathcal{L}} = -c_{\mathcal{L},\sigma}$ for any embedded affine line ϱ which intersects positively and transversally with σ in $\mathrm{Int}_{re}(\sigma)$.*

Proof. For any such path ϱ, we have $\lim_{\hbar \to 0} \int_{\varrho} \alpha_{\mathcal{L}} = -\int_{\vec{\tau}(\mathcal{I}_{\varrho})} \alpha_1 \wedge \cdots \wedge \alpha_k$; see the proof of Lemma 3.5. Although $0 \in \vec{\tau}(\mathcal{I}_{\varrho})$ instead of $0 \in \mathrm{Int}(\vec{\tau}(\mathcal{I}_{\varrho}))$, we still have $\int_{\vec{\tau}(\mathcal{I}_{\varrho})} \alpha_1 \wedge \cdots \wedge \alpha_k = -c$ for some constant $c > 0$. It remains to argue that c is independent of ϱ.

Let ϱ_1 and ϱ_2 be two paths of the above type. Join the end points of ϱ_1 and ϱ_2 by paths γ_0 and γ_1 which do not intersect in $P_{\mathcal{L}}$ to form a cycle C. Then $\lim_{\hbar \to 0} \int_{\gamma_i} \alpha_{\mathcal{L}} = 0$ and $\lim_{\hbar \to 0} \int_C \alpha_{\mathcal{L}} = \lim_{\hbar \to 0} \int_D d\alpha_{\mathcal{L}} = 0$ for some 2-chain D with $D \cap |\mathcal{P}^{[r-2]}_{\mathcal{L}}| = \emptyset$. It follows that $\lim_{\hbar \to 0} \int_{\varrho_1} \alpha_{\mathcal{L}} = \lim_{\hbar \to 0} \int_{\varrho_2} \alpha_{\mathcal{L}}$. $\square$

We claim that $\varPhi := \sum_{k \geq 1} \frac{1}{2^{k-1}} \sum_{\mathcal{L} \in \mathtt{LR}_k} \mathbf{L}_{k,\mathcal{L}}(\Pi, \ldots, \Pi)$ defines an element of $\hat{\mathcal{G}}^*$ which satisfies equation (2.7) in $\hat{\mathcal{H}}^*$, as in Lemma 3.8. Indeed, if we consider this claim in $\mathcal{G}^{<k,*} := \mathcal{W}^0_* \otimes_{\mathbb{C}} \mathfrak{h}^{<k}$ and $\mathcal{H}^{<k,*} := \left(\mathcal{W}^0_* / \mathcal{W}^{-1}_*\right) \otimes_{\mathbb{C}} \mathfrak{h}^{<k}$, we will have a finite number of terms and the proof of Lemma 3.8 applies. The claim in $\hat{\mathcal{G}}^*$ and $\hat{\mathcal{H}}^*$ then follows by taking limits.

Let L be a labeled k-tree. Since $\mathbf{L}_{k,\mathcal{L}}(\Pi, \ldots, \Pi)$ does not depend on the ribbon structure of $\mathcal{L}$, we can make sense of the sum $\mathbf{L}_{k,L}(\Pi, \ldots, \Pi)$. Since the labeling of the incoming edges $e \in L^{[1]}_{in}$ need not be distinct, we have

$$\frac{1}{|\mathrm{Aut}(L)|} \mathbf{L}_{k,L}(\Pi, \ldots, \Pi) = \sum_{\underline{\mathcal{L}} = L} \frac{1}{2^{k-1}} \mathbf{L}_{k,\mathcal{L}}(\Pi, \ldots, \Pi) \tag{3.3}$$

and hence $\varPhi = \sum_{k \geq 1} \sum_{L \in \mathtt{LT}_k} \frac{1}{|\mathrm{Aut}(L)|} \mathbf{L}_{k,L}(\Pi, \ldots, \Pi)$. Combining the above arguments yields the following modification of Theorem 3.9.

Theorem 3.14. *The Maurer–Cartan solution $\varPhi \in \hat{\mathcal{H}}^*$ can be expressed as a sum over trees,*

$$\varPhi = \sum_{k \geq 1} \sum_{\substack{L \in \mathtt{LT}_k \\ \mathfrak{M}_L(M_{\mathbb{R}}, \mathcal{D}_{in}) \neq \emptyset}} \frac{1}{|\mathrm{Aut}(L)|} \alpha_L g_L, \tag{3.4}$$

with $\alpha_L \in \mathcal{W}^1_P(M_{\mathbb{R}}) \cap \mathcal{W}^1_1(M_{\mathbb{R}})$ for the codimension one affine subspace $P_L \subset P$ normal to n_L.

Furthermore, when $\dim_{\mathbb{R}}(P_L) = r-1$, *there exists a polyhedral decomposition* $\mathcal{P}_L$ *of* P_L *such that, for each* $\sigma \in \mathcal{P}_L^{[r-1]}$, *there is a constant* $c_{L,\sigma}$ *such that* $\lim_{\hbar\to 0}\int_{\varrho}\alpha_L = -c_{L,\sigma}$ *for any affine line* ϱ *intersecting positively.*[6] *with* σ *in* $\mathrm{Int}_{re}(\sigma)$

Definition 3.15. *Let* Φ *be as in Theorem 3.14. Define a scattering diagram* $\mathcal{D}(\Phi)$ *as follows. For each* $L \in \mathtt{LT}_k$ *with* $\dim_{\mathbb{R}}(P_L) = r-1$, *let* $\sigma \in \mathcal{P}_L^{[r-1]}$ *be a maximal cell with associated constant* $c_{L,\sigma}$ *(see Lemma 3.13). Define a wall* $\mathbf{w}_{L,\sigma} = (m_L, n_L, P_{L,\sigma}, \Theta_{L,\sigma})$ *as follows:*

(1) m_L *and* n_L *are defined as in the case of weighted k-trees (see Definitions 1.12 and 1.14).*
(2) $P_{L,\sigma} = \sigma$ *and* $\Theta_{L,\sigma} = \exp(\frac{c_{L,\sigma}}{|\mathrm{Aut}(L)|}g_L)$.

We claim that $\mathcal{D}(\Phi)$ is equivalent to $\mathcal{S}(\mathcal{D}_{in})$. We will use the main result of [6] to conclude that $\mathcal{D}(\Phi)$ is a consistent extension of $\mathcal{D}_{in}$. Since this result cannot be applied directly to our current situation, we supply the required modifications. Firstly, we have $\mathcal{D}(\Phi)^{<k} = \mathcal{D}(\Phi^{<k})$, where $\Phi^{<k}$ is the image Φ in $\mathcal{H}^{<k,*}$ and $\mathcal{D}(\Phi)^{<k}$ is the diagram obtained by replacing the wall-crossing automorphisms with their images under $\hat{\mathfrak{h}} \to \mathfrak{h}^{<k}$. To prove consistency of $\mathcal{D}(\Phi)$, it suffices to prove consistency of $\mathcal{D}(\Phi^{<k})$ for each $k \in \mathbb{Z}_{>0}$. For the latter, consider a polyhedral decomposition $\mathcal{J}(\mathcal{D}(\Phi^{<k}))$ of $\mathrm{Joints}(\mathcal{D}(\Phi^{<k}))$ such that, for each $\mathfrak{j} \in \mathcal{J}(\mathcal{D}(\Phi^{<k}))^{[r-2]}$, the intersection $P_L \cap \mathfrak{j}$ is a facet of $\mathfrak{j}$ for all labeled trees L with $g_L \neq 0 \in \mathfrak{h}^{<k}$. It then suffices to prove consistency at each joint $\mathfrak{j}$.

Fix a joint $\mathfrak{j}$ and let U be a convex neighborhood of $\mathrm{Int}_{re}(\mathfrak{j})$ such that $(U \setminus \mathfrak{j}) \cap P_L \neq \emptyset$ only if $\dim_{\mathbb{R}}(P_L) = r-1$. There is a decomposition $\Phi|_U = \sum_{(L,\sigma)\in\mathbb{W}} \Phi^{(L,\sigma)} + \mathcal{E}$. Here $\mathbb{W}$ is the set of pairs (L,σ) for which $\dim_{\mathbb{R}}(P_L) = r-1$ and $\sigma \in \mathcal{P}_L$ and $\sigma \cap \mathrm{Int}_{re}(\mathfrak{j}) \neq \emptyset$. Restricted to $U\setminus\mathfrak{j}$, the summand $\Phi^{(L,\sigma)}$ is equal to $\frac{1}{|\mathrm{Aut}(L)|}\alpha_L g_L$. The final term $\mathcal{E} = \sum_{\substack{P_L\cap U\neq\emptyset \\ \dim_{\mathbb{R}}(P_L)<r-1}} \frac{1}{|\mathrm{Aut}(L)|}\alpha_L g_L$ satisfies $\mathcal{E}|_{U\setminus\mathfrak{j}} = 0$. Indeed, this follows from our assumptions on $\mathcal{J}(\mathcal{D}(\Phi^{<k}))$ and the fact that $(U\setminus\mathfrak{j})\cap P_L = \emptyset$ for those P_L satisfying $\dim(P_L) < r-1$.

Since the sum $\sum_{(L,\sigma)\in\mathbb{W}} \Phi^{(L,\sigma)}$ satisfies Assumptions I and II of [6, Introduction], the following result can be proved using the methods of [6].

Proposition 3.16. *The scattering diagram* $\mathcal{D}(\Phi)$ *is consistent.*

It now follows from Theorem 1.6 that the scattering diagrams $\mathcal{D}(\Phi)$ and $\mathcal{S}(\mathcal{D}_{in})$ are equivalent. For the perturbed case, we have $\mathcal{D}(\tilde{\Phi})$ equivalent to $\mathcal{S}(\tilde{\mathcal{D}}_{in,l})$.

3.3. **Theta functions as flat sections.** Let $\Phi \in \hat{\mathcal{H}}^1$ be a Maurer–Cartan element. Then $d_\Phi = d + [\Phi, -]$ is a differential which acts on the graded algebra $\hat{\mathcal{A}}^*$. The space of flat sections of d_Φ is

$$\mathrm{Ker}(d_\Phi) = \{s \in \hat{\mathcal{A}}^0 \mid d_\Phi(s) = 0\}. \tag{3.5}$$

The product on $\hat{\mathcal{A}}^*$ induces a product $*$ on $\mathrm{Ker}(d_\Phi)$. In the same way, $\mathrm{Ker}^{<k}(d_\Phi)$ inherits a product from $\mathcal{A}^{<k,*}$. The goal of this section is to relate the algebra $\mathrm{Ker}^{<k}(d_\Phi)$ with the theta functions introduced in Section 1.2.

3.3.1. *Wall-crossing of flat sections.* In this section we will prove a wall-crossing formula for flat sections $\mathrm{Ker}^{<k}(d_\Phi)$ using arguments similar to those of [7, Introduction].

Consider a polyhedral decomposition $\mathcal{P}^{<k}$ of $\mathrm{Supp}(\mathcal{D}(\Phi^{<k}))$ with the property that, for every $0 \leq l \leq r-1$ and $\sigma \in \mathcal{P}^{<k,[l]}$, we have $\sigma \subset P_{\mathbf{w}}$ for some wall $\mathbf{w} \in \mathcal{D}(\Phi^{<k})$ and $P_L \cap \sigma$ is a facet of σ for every P_L with $g_L \neq 0 \in \mathfrak{h}^{<k}$. Fix a maximal cell $\sigma \in \mathcal{P}^{<k,[r-1]}$. Let $U \subset M_{\mathbb{R}} \setminus |\mathcal{P}^{<k,[r-2]}|$ be a

[6] Here positivity depends on n_L, which is defined up to sign. However, this sign ambiguity cancels with that of g_L, as mentioned in Definition 1.14.

contractible open subset intersecting $\mathrm{Int}_{re}(\sigma)$ and which is separated by $\mathrm{Int}_{re}(\sigma)$ into two connected components, U_+ and U_-. Associated to σ is the wall-crossing automorphism

$$\Theta_\sigma = \prod_{\substack{\mathbf{w}\in\mathcal{D}(\Phi^{<k}) \\ P_\mathbf{w}\cap U\cap\sigma\neq\emptyset}} \Theta_\mathbf{w}^{\mathrm{sgn}(n_\mathbf{w},v)}, \tag{3.6}$$

where $v \neq 0$ points into U_+. Results from [6, §4] imply that there is a unique gauge φ which solves the equation

$$e^{\mathrm{ad}_\varphi} d e^{-\mathrm{ad}_\varphi} = d_\Phi \tag{3.7}$$

and satisfies $\varphi|_{U_-} = 0$. Moreover, this gauge is necessarily given by

$$\varphi = \begin{cases} \log(\Theta_\sigma) & \text{on } U_+, \\ 0 & \text{on } U_-. \end{cases} \tag{3.8}$$

In words, Φ behaves like a delta function supported on σ and φ behaves like a step function which jumps across σ. See Figure 4.

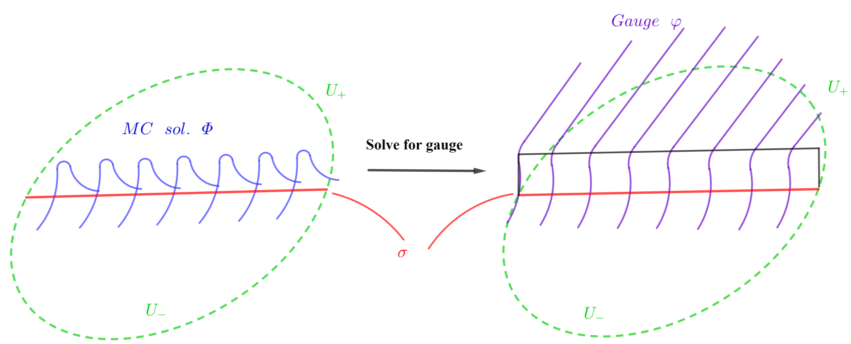

FIGURE 4. The gauge φ as step function.

Let $s \in \mathrm{Ker}^{<k}(\Phi)$. Since $\Phi^{<k}|_{U_\pm} = 0 \in \mathcal{H}^{<k,*}(U_\pm)$, we have $d(s|_{U_\pm}) = 0$. We can therefore treat $s|_{U_\pm}$ as a constant section over $U_\pm$, which we henceforth denote by $s_\pm \in A^{<k}$.

Using equation (3.7), the condition $d_\Phi(s) = 0$ is seen to be equivalent to the condition that the function $e^{-\mathrm{ad}_\varphi}(s)$, which is defined on U, is d-flat. On the other hand, equation (3.8) gives

$$e^{-\mathrm{ad}_\varphi}(s) = \begin{cases} \Theta_\sigma^{-1}(s_+) & \text{on } U_+, \\ s_- & \text{on } U_-. \end{cases}$$

We therefore conclude $\Theta_\sigma(s_-) = s_+$. By applying this argument to a path γ crossing finitely many walls generically in $\mathcal{D}(\Phi^{<k})$, we obtain the following wall-crossing formula.

Theorem 3.17. *Let $s \in \mathrm{Ker}(d_\Phi)$ and $Q, Q' \in M_\mathbb{R}\backslash\mathrm{Supp}(\mathcal{D}(\Phi))$. Then*

$$s_{Q'} = \Theta_{\gamma,\mathcal{D}}(s_Q) \tag{3.9}$$

for any path $\gamma \subset M_\mathbb{R} \setminus \mathrm{Joints}(\mathcal{D})$ joining Q to Q', where $s_{Q'}$ and s_Q are restrictions of s to sufficiently small neighborhoods containing Q and Q', respectively, and are treated as constant A^0-valued sections.

3.3.2. *Theta functions as elements of* $\mathrm{Ker}(d_\Phi)$. In this section we define, for each $\mathsf{m} \in \overline{M} \setminus \{0\}$, an element $\theta_\mathsf{m} \in \mathrm{Ker}(d_\Phi)$ and relate it to the theta function $\vartheta_{m,Q}$ of Section 1.2. Since $\mathrm{Ker}(d_\Phi) = \varprojlim_k \mathrm{Ker}(d_\Phi^{<k})$, we will define θ_m as an inverse limit $\varprojlim_k \theta_\mathsf{m}^{<k}$, where $\theta_\mathsf{m}^{<k}$ is defined by the following lemma.

Lemma 3.18. *For sufficiently large $c > 0$, there exists a unique section $\theta_\mathsf{m}^{<k} \in \mathrm{Ker}^{<k}(d_\Phi)$ such that $\theta_\mathsf{m}^{<k}|_U = z^{\varphi(\mathsf{m})}$ in a neighborhood U of $\mathbb{R}_{\geq c} \cdot \varphi(\mathsf{m})$.*

Proof. Equation (3.4), and the resulting equation for $\Phi^{<k}$ obtained by passing to $\mathcal{H}^{<k,*}$, exhibits $\Phi^{<k}$ as a finite sum. For each L with $g_L \neq 0 \in \mathfrak{h}^{<k}$, consider the polyhedral subset $P_L \subset M_\mathbb{R}$. Because the set of such P_L is finite, there exists a constant $c > 0$ such that either $P_L \cap (\mathbb{R}_{\geq c} \cdot \varphi(\mathsf{m})) = \emptyset$ or $\mathbb{R}_{\geq c} \cdot \varphi(\mathsf{m}) \subset P_L$. It follows that there exists a neighborhood U of $\mathbb{R}_{\geq c} \cdot \varphi(\mathsf{m})$ such that $P_L \cap U = \emptyset$ for those P_L which do not contain $\mathbb{R}_{\geq c} \cdot \varphi(\mathsf{m})$. Therefore, we have

$$\Phi^{<k}|_U = \sum_{k \geq 1} \sum_{\substack{L \in \mathsf{LT}_k \\ P_L \cap U \neq \emptyset}} \frac{1}{|\mathrm{Aut}(L)|} \alpha_L g_L$$

with $g_L \in \mathfrak{h}_{m_L, n_L}$. The construction gives $\mathbb{R}_{\geq c} \cdot \varphi(\mathsf{m}) \subset P_L \subset n_L^\perp$ whenever $P_L \cap U \neq \emptyset$, and therefore $g_L \cdot z^{\varphi(\mathsf{m})} = 0$ using Definition 1.2. This gives $\left(\Phi^{<k} \cdot z^{\varphi(\mathsf{m})}\right)|_U = 0$ and therefore $z^{\varphi(\mathsf{m})}|_U \in \mathrm{Ker}^{<k}(d_\Phi|_U)$.

Since $H^1(\mathcal{H}^{<k,*}, d) = 0$, the deformation of d by any $\Phi^{<k}$ defined on $M_\mathbb{R}$ is gauge equivalent to d by some gauge $\varphi \in \mathcal{H}^{<k,0}$, that is, $e^{\mathrm{ad}_\varphi} d e^{-\mathrm{ad}_\varphi} = d_{\Phi^{<k}}$. We may choose φ so that $\varphi(m_0) = 0$ at some point $m_0 \in U$. For the existence part of the lemma, we may simply take $\theta_\mathsf{m}^{<k} = e^{-\mathrm{ad}_\varphi}(z^{\varphi(\mathsf{m})})$. As for uniqueness, note that if $s \in \mathrm{Ker}^{<k}(\Phi)$ is such that $s|_U = 0$, then $e^{-\mathrm{ad}_\varphi}(s)$ is constant section which restricts to zero on U. It follows that $e^{-\mathrm{ad}_\varphi}(s)$, and hence s, is zero. □

Theorem 3.19. *For $Q \in M_\mathbb{R} \setminus \bigcup_L P_L$, we have $\vartheta_{\mathsf{m},Q} = \theta_\mathsf{m}(Q)$, the value of the section θ_m at Q.*

Proof. Let $k \in \mathbb{Z}_{>0}$ and consider the equation $\vartheta_{\mathsf{m},Q} = \theta_\mathsf{m}(Q)$ in $\mathcal{A}^{<k}$. From Theorem 3.17 and Lemma 3.18 we learn that $\theta_\mathsf{m}^{<k}$ satisfies the wall-crossing formula of Theorem 3.17 and that $\theta_\mathsf{m}^{<k}$ and $z^{\varphi(\mathsf{m})}$ agree in a sufficiently small neighborhood U of $\mathbb{R}_{\geq c} \cdot \varphi(\mathsf{m})$. From Proposition 1.9, we learn that $\vartheta_{\mathsf{m},Q}^{<k}$ satisfies the same wall-crossing formula as $\theta_\mathsf{m}^{<k}$. It remains to check that $\vartheta_{\mathsf{m},Q}^{<k} = z^{\varphi(\mathsf{m})}$ for $Q \in U \setminus \mathrm{Supp}(\mathcal{D}(\Phi^{<k}))$, which follows from the definition of $\vartheta_{\mathsf{m},Q}^{<k}$. □

The next proposition, which is parallel to Proposition 1.9, relates the set $\{\theta_\mathsf{m}\}_{\mathsf{m} \in \overline{M}}$ to the ring $\mathrm{Ker}(d_\Phi)$. Note that the natural inclusion $\hat{A}_K \hookrightarrow \mathrm{Ker}(d_\Phi)$ gives $\mathrm{Ker}(d_\Phi)$ the structure of a $\hat{A}_K$-module.

Proposition 3.20. *The set $\{\theta_\mathsf{m}\}_{\mathsf{m} \in \overline{M}}$ is linearly independent over $\hat{A}_K$ and, for each $k \in \mathbb{Z}_{>0}$, additively generates $\mathrm{Ker}^{<k}(d_\Phi)$ over $A_K^{<k}$.*

Proof. Consider a finite $\hat{A}_K$-linear relation $\sum_\mathsf{m} a_\mathsf{m} \theta_\mathsf{m} = 0$. By Theorem 3.19, we have $\theta_\mathsf{m}(Q) = \vartheta_{\mathsf{m},Q}$ for each $Q \in M_\mathbb{R} \setminus \bigcup_L P_L$. It follows that $\sum_\mathsf{m} a_\mathsf{m} \vartheta_{\mathsf{m},Q} = 0$, whence each a_m vanishes by Proposition 1.9. This establishes the claimed linear independence.

For the generating property, fix $k \in \mathbb{Z}_{>0}$ and consider $s \in \mathrm{Ker}^{<k}(d_\Phi)$. Let $\mathfrak{C}$ be a chamber of $M_\mathbb{R} \setminus \bigcup_{0 \neq g_L \in \mathfrak{h}^{<k}} P_L$. Then $s|_\mathfrak{C}$ is a constant section which can be treated as element of $A^{<k}$, henceforth denoted by $s_\mathfrak{C}$. Similar to the proof of Lemma 3.18, we have $H^1(\mathcal{H}^{<k,*}, d) = 0$ and therefore $e^{\mathrm{ad}_\varphi} d e^{-\mathrm{ad}_\varphi} = d_{\Phi^{<k}}$ for some gauge φ which satisfies $\varphi|_\mathfrak{C} = 0$. It follows that $s = e^{-\mathrm{ad}_\varphi}(s_\mathfrak{C})$, where $s_\mathfrak{C}$ is treated as a constant section via $A^{<k} \hookrightarrow \mathcal{A}^{<k,0}$. By Proposition 1.9, if $Q \in \mathfrak{C}$, then $s_\mathfrak{C}$ is contained in the subalgebra generated by $\{\theta_{\mathsf{m},Q}^{<k}\}_{\mathsf{m} \in \overline{M}}$ over $A_K^{<k}$. We also have $\theta_\mathsf{m}^{<k} = e^{-\mathrm{ad}_\varphi}(\vartheta_{\mathsf{m},Q}^{<k})$. We therefore conclude that s is contained in the subalgebra generated by $\{\theta_\mathsf{m}^{<k}\}_{\mathsf{m} \in \overline{M}}$ over $A_K^{<k}$. □

3.4. **Hall algebra scattering diagrams.** An interesting class of scattering diagrams which are not covered by the tropical case are the Hall algebra scattering diagrams, introduced in [4]. In this section we investigate the analogues of the results of Sections 3.1-3.3 in the non-tropical case.

3.4.1. *Motivic Hall algebras.* We recall the definition of Joyce's motivic Hall algebra. While the Hall algebra scattering diagrams are the main example of non-tropical scattering diagrams, we will not use anything technical about Hall algebras in the paper. We will therefore be brief. The reader is referred to [14, 3] for details. See also [4, §§4, 5].

Let Q be a quiver[7] with finite sets of nodes Q_0 and arrows Q_1. Let $N^{\oplus} = \mathbb{Z}_{\geq 0}Q_0$ be the monoid of dimensino vectors. For each $d \in N^{\oplus}$, denote by $R_d = \prod_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}})$ the affine variety of complex representations of Q of dimension vector d. The reductive group $\mathsf{GL}_d = \prod_{i \in Q_0} \mathsf{GL}_{d_i}(\mathbb{C})$ acts on R_d by change of basis. The quotient stack R_d/GL_d, denoted by $\mathcal{M}_d$, is the moduli stack of representations of dimension vector d. Set $\mathcal{M} = \bigsqcup_{d \in N^{\oplus}} \mathcal{M}_d$.

Similarly, given $d_1, d_2 \in N^{\oplus}$, let $\mathcal{M}_{d_1,d_2}$ be the moduli stack which parameterizes short exact sequences $0 \to U_1 \to U_2 \to U_3 \to 0$ of representations in which U_1 and U_3 have dimension vector d_1 and d_2, respectively. There is a canonical correspondence

$$\mathcal{M}_{d_1} \times \mathcal{M}_{d_2} \xleftarrow{\pi_1 \times \pi_3} \mathcal{M}_{d_1,d_2} \xrightarrow{\pi_2} \mathcal{M}_{d_1+d_2}, \tag{3.10}$$

a short exact sequence being sent by $\pi_1 \times \pi_3$ to its first and third terms and by π_2 to its second term. The map $\pi_1 \times \pi_3$ is of finite type while π_2 is proper and representable.

Let $\mathcal{H}_Q := K_0(\mathsf{St}/\mathcal{M}) = \bigoplus_{d \in N^{\oplus}} K_0(\mathsf{St}/\mathcal{M}_d)$, the Grothendieck ring of finite type stacks with affine stabilizers over $\mathcal{M}$. Push-pull along the correspondence (3.10) induces a product on $\mathcal{H}_Q$, making it into a $N^{\oplus}$-graded associative algebra, called the motivic Hall algebra of Q. The augmentation ideal $\mathfrak{g}_Q \subset \mathcal{H}_Q$, with its commutator bracket, is then a N^{+}-graded Lie algebra, the motivic Hall–Lie algebra.

In the setting of scattering diagrams, it is convenient to use a specialization of $\mathcal{H}_Q$. Write St in place of $\mathsf{St}/_{\mathrm{Spec}(\mathbb{C})}$. Cartesian product with $\mathrm{Spec}(\mathbb{C})$ makes $\mathcal{H}_Q$ into a $K_0(\mathsf{St})$-module. Let $\Upsilon : K_0(\mathsf{St}) \to \mathbb{C}(t)$ be the unique ring homomorphism which sends the class of a smooth projective variety to the Poincaré polynomial of its singular cohomology with complex coefficients. Using this, $\mathcal{H}_Q^{\Upsilon} := K_0(\mathsf{St}/\mathcal{M}) \otimes_{K_0(\mathsf{St})} \mathbb{C}(t)$ becomes a $\mathbb{C}(t)$-algebra, called the Hall algebra of stack functions [14]. The augmentation ideal $\mathfrak{g}_{\mathcal{H}}^{\Upsilon}$ is again a N^{+}-graded Lie algebra. The Hall algebra scattering diagram of [4] takes values in the (non-tropical) Lie algebra $\mathfrak{g}_{\mathcal{H}}^{\Upsilon}$.

3.4.2. *Non-tropical Maurer–Cartan solutions.* We begin by describing an abstract setting in which scattering diagrams can be defined without the tropical assumption. We largely follow [4], introducing modifications where required. Let $\mathfrak{h}$ be a M_{σ}^{+}-graded Lie algebra; we do not assume the tropical condition. We will use the notation $\hat{\mathfrak{h}}$, $\mathfrak{h}_L$ and $\hat{\mathfrak{h}}_L$ from Section 1.1.1.

In this section we will consider scattering diagrams in $N_{\mathbb{R}}$ instead of $M_{\mathbb{R}}$. The relevant modification of Definition 1.3 is as follows.

Definition 3.21. *A* wall $\mathbf{w}$ *in* $N_{\mathbb{R}}$ *is a pair* (P, Θ) *consisting of a codimension one closed convex rational polyhedral subset* $P \subset N_{\mathbb{R}}$ *and an element* $\Theta \in \hat{G}_{P^{\perp}} := \exp(\hat{\mathfrak{h}}_{P^{\perp}})$*, where* $P^{\perp}$ *consists of those* $m \in M$ *which are perpendicular to any* $n \in N_{\mathbb{R}}$ *which is tangent to* P*.*

We also require a modified definition of scattering diagrams.

Definition 3.22. *A* scattering diagram $\mathcal{D}$ *consists of data* $\{\mathcal{D}^{<k}\}_{k \in \mathbb{Z}_{>0}}$*, where* $\mathcal{D}^{<k} = \{(P_{\alpha}, \Theta_{\alpha})\}_{\alpha}$ *is a finite collection of walls with* $\dim_{\mathbb{R}}(P_{\mathbf{w}_1} \cap P_{\mathbf{w}_2}) < r - 1$ *for any two distinct walls* $\mathbf{w}_1$, $\mathbf{w}_2$*. The*

[7]For simplicity, we restrict attention to the case of trivial potential.

diagrams $\mathcal{D}^{<k+1}$ (mod $\mathfrak{h}^{\geq k}$) *and* $\mathcal{D}^{<k}$ *are required to be equal up to refinement by taking polyhedral decompositions of the polyhedral subsets of* $\mathcal{D}^{<k}$ *and by adding walls with trivial wall-crossing automorphisms.*

Remark 3.23. Suppose that $\mathfrak{h}$ is in fact tropical. Then, given a scattering diagram $\mathcal{D}$ in the sense of Definition 1.4, we can subdivide the walls and recombine overlapping walls in such a way that Definition 3.22 is satisfied.

Definition 3.22 agrees with the notion of a $\mathfrak{h}$-complex in [4, Section 2] if we further require that each P_α is rational polyhedral cone. We will henceforth restrict attention to this case. Following [4], fix an ordered basis $(f_1, \ldots, f_r)$ of M, thereby identifying M with $\mathbb{Z}^r$. We take $\sigma = \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0} \cdot f_r$ to be the standard cone and consider an initial scattering diagram $\mathcal{D}_{in} = \{\mathbf{w}_i = (P_i, \Theta_i)\}_{1 \leq i \leq r}$ with $P_i = f_i^\perp \subset N_\mathbb{R}$. Write $g_i := \log(\Theta_i) = \sum_{j>1} g_{ji}$ with $g_{ji} \in \mathfrak{h}_{jf_i}$. We assume that $[g_{j_1 i}, g_{j_2 i}] = 0$ for each initial wall.

Example 3.24. *Let Q be a quiver without edge loops. For any vertex $i \in Q_0$ and integer $k \geq 0$, the stack $\mathcal{M}_{ki}$ is isomorphic to the classifying stack $B\mathsf{GL}_k(\mathbb{C})$. Let $\mathcal{M}_{\langle i \rangle} = \bigsqcup_{k \geq 0} \mathcal{M}_{ki}$. Then the element $\Theta_i = [\mathcal{M}_{\langle i \rangle} \to \mathcal{M}]$ of the dimension-completed motivic Hall algebra satisfies the above assumptions.*

Using the affine structure on $N_\mathbb{R}$, we can again define the dg Lie algebras $\mathcal{H}^*$, $\hat{\mathcal{H}}^*$ and $\mathcal{H}^{<k,*}$. The discussions in Section 2.2 continue to hold without the tropical assumption on $\mathfrak{h}$. Let $\Pi = \sum_{i=1}^r \Pi^{(i)}$ with $\Pi^{(i)}$ as in equation (2.10). To define an operator $\mathbf{H}_m$ via equation (3.1), we must first choose a suitable direction $v^m \in N_\mathbb{Q} \setminus \{0\}$ along which to define the flow τ^m. For that purpose, choose a line segment $\lambda : [0, 1] \to N_\mathbb{R}$ from $(-1, \ldots, -1)$ to the dual cone $N_\sigma := M_\sigma^\vee$ with slope $(a_1, \ldots, a_r) \in \mathbb{R}^r$. We assume that $0 < a_r < \cdots < a_1$ and that $\{a_1, \ldots, a_r\}$ are algebraically independent over $\mathbb{Q}$.

We introduce some notation. Let $m \in M_\sigma^+$. Consider the polyhedral decomposition $\mathfrak{P}_m$ of the hyperplane $m^\perp$ induced by the finite hyperplane arrangement whose hyperplanes are of the form $m_1^\perp \cap m^\perp$, where $m_1 \nparallel m \in M_\sigma^+$ and $m_1 + m_2 = m$ for some $m_2 \in M_\sigma^+$. By construction, $\lambda \cap m^\perp$ is contained in $\mathrm{Int}_{re}(-\sqsupset)$ for some maximal cone $\sqsupset \in \mathfrak{P}_m^{[r-1]}$. If $m = kf_i$ for some $k > 0$ and $1 \leq i \leq r$, then we set $v^m = -kf_i^\vee$. For any other m, we take $v^m \in \mathrm{Int}_{re}(\sqsupset)$ to be a rational point.

With the above notation, we obtain operations[8] $\mathbf{L}_{k,\mathcal{L}}(\Pi, \ldots, \Pi)$ as in Section 3.1.1, and hence also $\mathbf{L}_{k,L}(\Pi, \ldots, \Pi)$ by equation (3.3). We modify Definition 1.15 to talk about tropical disks in $(N_\mathbb{R}, \mathcal{D}_{in})$ of type L, which are a proper maps $\varsigma : |L_{\vec{s}}| \to N_\mathbb{R}$ (instead of $M_\mathbb{R}$) whose slope at an edge $e \in \bar{L}^{[1]}$ is v^m (instead of $-m$). The moduli space $\mathfrak{M}_L(N_\mathbb{R}, \mathcal{D}_{in})$ is defined accordingly and $P_L := ev_o(\overline{\mathfrak{M}}_L(N_\mathbb{R}, \mathcal{D}_{in}))$ is now a subset of $m_L^\perp$. Lemma 3.4 holds after replacing $n_\mathcal{L}$ with $m_\mathcal{L}$, with the caveat that we can now only conclude $c \neq 0$, instead of $c > 0$.

With $\alpha_\mathcal{L}$ defined as in Lemma 3.5, the first parts of Lemmas 3.5 and 3.12 hold by the same argument (after replacing $n_\mathcal{L}$ with $m_\mathcal{L}$ and $M_\mathbb{R}$ with $N_\mathbb{R}$). Lemma 3.13 is again valid, except that the sign of $c_{\mathcal{L},\sigma} \neq 0$ cannot be determined. By Lemma 3.11, we have $\mathbf{L}_{k,\mathcal{L}}(\Pi, \ldots, \Pi) = \alpha_\mathcal{L} g_\mathcal{L}$ and, by equation (3.3), we can write $\mathbf{L}_{k,L}(\Pi, \ldots, \Pi) = \alpha_L g_L$.

Lemma 3.25. *The sum*

$$\varPhi := \sum_{k \geq 1} \sum_{\substack{\mathcal{L} \in \mathtt{LR}_k \\ \mathfrak{M}_\mathcal{L}(N_\mathbb{R}, \mathcal{D}_{in}) \neq \emptyset}} \frac{1}{2^{k-1}} \alpha_\mathcal{L} g_\mathcal{L} = \sum_{k \geq 1} \sum_{\substack{L \in \mathtt{LT}_k \\ \mathfrak{M}_L(N_\mathbb{R}, \mathcal{D}_{in}) \neq \emptyset}} \frac{1}{|\mathrm{Aut}(L)|} \alpha_L g_L$$

is a Maurer–Cartan element in $\hat{\mathcal{H}}^*$.

Proof. The equality in the statement for the theorem holds by the same reasoning as in the tropical case. So we focus on proving that $\varPhi$ is a Maurer–Cartan element.

[8] Labeled (ribbon) k-trees are defined as in Definitions 1.12 and 1.13 using the walls of Definition 3.21.

Fix $k \in \mathbb{Z}_{>0}$ and work over $\mathcal{H}^{<k,*}$. Let $K \subset N_\mathbb{R}$ be a compact subset. As in the proof of Lemma 3.8, we must show that, for a sufficiently large real number $L > 0$, we have $\mathbf{P}_L[\Phi_L, \Phi_L] = 0$ on K for the cut-off element $\Phi_L := \Pi - \mathbf{H}_L[\Phi_L, \Phi_L]$, where $\mathbf{P}_{L,m}(\beta) = (\tau^m_{-L})^*(\beta)$. We consider labeled ribbon trees $\mathcal{L}_1, \mathcal{L}_2$ with associated terms $\alpha_{\mathcal{L}_i,L} g_{\mathcal{L}_i}$, where $\alpha_{\mathcal{L}_i,L} \in \mathcal{W}^1_{P_i}(N_\mathbb{R}) \cap \mathcal{W}^1_1(N_\mathbb{R})$ and $P_i = m^\perp_{\mathcal{L}_i}$. We further assume that $m_{\mathcal{L}_1} \nparallel m_{\mathcal{L}_2}$, as otherwise $\alpha_{\mathcal{L}_1,L} \wedge \alpha_{\mathcal{L}_2,L} \in \mathcal{W}^{-\infty}(N_\mathbb{R})$ and hence $[\alpha_{\mathcal{L}_1,L} g_{\mathcal{L}_1}, \alpha_{\mathcal{L}_2,L} g_{\mathcal{L}_2}] = 0 \in \mathcal{H}^{<k,*}$. Joining the trees $\mathcal{L}_i$ to obtain $\mathcal{L}$, the transversal intersection $P_{\mathcal{L}_1} \cap P_{\mathcal{L}_2} \subset P_\mathcal{L} = m^\perp_\mathcal{L}$ is contained in the $(r-2)$-dimensional strata of the polyhedral decomposition $\mathfrak{P}_{m_\mathcal{L}}$. By our choice of $v^{m_\mathcal{L}}$, the flow $\tau^{m_\mathcal{L}}$ is not tangent to $P_{\mathcal{L}_1} \cap P_{\mathcal{L}_2}$. As in proof of Lemma 3.8, we can therefore choose L sufficiently large so that $(\tau^{m_\mathcal{L}}_{-L})^*(\alpha_{\mathcal{L}_1} \wedge \alpha_{\mathcal{L}_2}) = 0$ on K. □

Notice that, by our construction of v^m, the line segment λ is disjoint from each $P_\mathcal{L}$.

Given Lemma 3.25, the conclusion of Theorem 3.14 holds after replacing $M_\mathbb{R}$ with $N_\mathbb{R}$.

Remark 3.26. Suppose that we are given an integral (for simplicity) skew-symmetric bilinear form ω on M and a map $p : M \to N$ which satisfies $\langle m', p(m)\rangle = \omega(m, m')$. Let $K = \mathrm{Ker}(p)$. This is a common setting in the theory of scattering diagrams; see [11]. Suppose, in addition, that $\omega(e_i, e_j) < 0$ whenever $i < j$, as in [4, Theorem 1.5], then we can decompose $m = (m_1, \dots, m_r) \in M^+_\sigma$ as $m = m^{\leq i} + m^{>i}$ where $m^{\leq i} = (m_1, \dots, m_i, 0, \dots, 0)$ and $m^{>i} = (0, \dots, 0, m_{i+1}, \dots, m_r)$. If both $m^{\leq i}$ and $m^{>i}$ are non-zero, then $\omega(m^{\leq i}, m^{>i}) < 0$ and hence $\langle m^{>i}, p(m)\rangle < 0$.

We can choose $v^m := -p(m)$ and prove a modification of Lemma 3.25 as follows. To begin, we show inductively that $P_\mathcal{L} \subset \{x \in N_\mathbb{R} \mid \langle m^{>i}_\mathcal{L}, x\rangle \geq 0\}$ for each $i = 1, \dots, r-1$ and all $\mathcal{L}$. The initial case is trivial. For the induction step, spilt $\mathcal{L}$ into $\mathcal{L}_1, \mathcal{L}_2$. Then we have $P_{\mathcal{L}_1} \cap P_{\mathcal{L}_2} \subset \{x \in N_\mathbb{R} \mid \langle m^{>i}_\mathcal{L}, x\rangle \geq 0\}$. From the relation $\langle m^{>i}_\mathcal{L}, -p(m)\rangle \geq 0$ we obtain the result. To conclude the proof of Lemma 3.25, we notice that by taking $0 \neq m^{>i}_\mathcal{L} \neq m_\mathcal{L}$ we obtain a hyperplane $(m^{>i}_\mathcal{L})^\perp \cap m^\perp_\mathcal{L}$ separating $P_\mathcal{L}$ and $p(m_\mathcal{L})$ in $m^\perp_\mathcal{L}$. From this we deduce that, for any compact subset $K \subset N_\mathbb{R}$, we can choose L sufficiently large so that $\tau^{m_\mathcal{L}}_{-L}(K) \cap P_\mathcal{L} = \emptyset$.

3.4.3. *Consistent scattering diagrams from non-tropical Maurer–Cartan solutions.* We establish the relation between Maurer–Cartan solutions $\Phi \in \hat{\mathcal{H}}^*$ and consistent scattering diagrams. By construction, $\Phi = \varprojlim_k \Phi^{<k}$, where $\Phi^{<k} = \sum_P \alpha_P g_P$ is a finite sum indexed by polyhedral subsets P of $N_\mathbb{R}$. From the discussion in Section 3.4.2, we have $g_P \in \mathfrak{h}^{<k}$ and $\alpha_P \in \mathcal{W}^1_{\tilde{P}}(N_\mathbb{R}) \cap \mathcal{W}^1_1(N_\mathbb{R})$, where $\tilde{P} \subset N_\mathbb{R}$ is a codimension one polyhedral subset containing P and $\alpha_P|_{N_\mathbb{R}\setminus P} \in \mathcal{W}^{-\infty}(N_\mathbb{R} \setminus P)$. Similarly to Section 3.3.1, consider a polyhedral decomposition $\mathcal{P}^{<k}$ of $\bigcup_{0\neq g_P \in \mathfrak{h}^{<k}} P$ such that, for every $0 \leq l \leq r-1$ and $\sigma \in \mathcal{P}^{<k,[l]}$, we have $\sigma \subset P$ for some $\dim_\mathbb{R}(P) = r-1$ and $P \cap \sigma$ is a facet of σ for every P with $0 \neq g_P \in \mathfrak{h}^{<k}$.

Let U be a convex open set such that $U \cap \tau = \emptyset$ whenever $\tau \neq \sigma \in \mathcal{P}^{<k}$ and $U \setminus \sigma = U_+ \cup U_-$ is a decomposition into connected components. Since U is contractible, $H^1(\mathcal{H}^{<k,*}(U), d) = 0$, whence $\Phi^{<k}|_U$ is gauge equivalent to 0, that is, $e^{\mathrm{ad}_\varphi} d e^{-\mathrm{ad}_\varphi} = d_{\Phi^{<k}}$ on U, where φ satisfies $\varphi|_{U_-} = 0$. We will use a homotopy operator $\hat{\mathsf{I}}$ acting on $\mathcal{H}^{<k,*}$ to solve for φ. Assume that we are given a chain of affine subspaces $U_\bullet$ of U, as in Section 2.1.1, such that v_1 is transversal to σ and $U_{1,+}$, the half space over which v_1 points inwards, contains $U_+ \cup \sigma$. See Figure 5. Such a choice yields a homotopy operator $I : \mathcal{W}^0_*(U) \to \mathcal{W}^0_{*-1}(U)$, as in equation (2.6). By Lemma 2.4, the operator I descends to $\mathcal{W}^0_*(U)/\mathcal{W}^{-1}_*(U)$. As in Definition 2.9, we then obtain a homotopy operator $\hat{\mathsf{I}}$, defined using (the m-independent) I in place of H_m, and operators $\hat{\mathsf{P}}$ and $\hat{\iota}$ on $\hat{\mathcal{H}}^*(U)$.

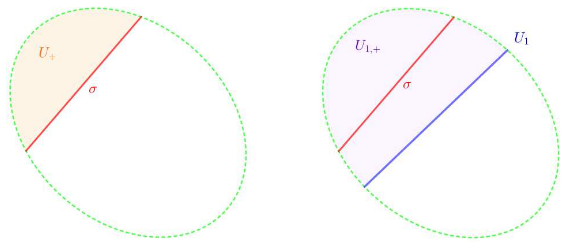

FIGURE 5. $U_{1,+}$ and U_+

Arguments of [6, §4] show that the unique gauge satisfying $\hat{\mathsf{P}}(\varphi) = 0$ is given by $\varphi = \varprojlim_k \varphi^{<k}$, where $\varphi^{<k} \in \mathcal{H}^{<k,0}$ is constructed inductively by

$$\varphi^{<(k+1)} = -\hat{\mathsf{I}} \left(\varPhi + \sum_{l \geq 0} \frac{\mathrm{ad}^l_{\varphi^{<k}}}{(l+1)!} d\varphi^{<k} \right). \tag{3.11}$$

Using Lemmas 2.3 and 2.4, we inductively obtain

$$\varphi^{<k} \in \Big(\frac{\mathcal{W}^0_{\overline{U}_+}(U) \cap \mathcal{W}^0_0(U) + \mathcal{W}^{-1}_0(U)}{\mathcal{W}^{-1}_0(U)} \Big) \otimes_{\mathbb{C}} \mathfrak{h}^{<k}_{\sigma^\perp} \tag{3.12}$$

$$\frac{\mathrm{ad}^l_{\varphi^{<s}}}{(l+1)!} d\varphi^{<s} \in \Big(\frac{\mathcal{W}^1_\sigma(U) \cap \mathcal{W}^1_1(U) + \mathcal{W}^{-1}_0(U)}{\mathcal{W}^{-1}_0(U)} \Big) \otimes_{\mathbb{C}} \mathfrak{h}^{<k}_{\sigma^\perp} \tag{3.13}$$

for all $s \leq k$ and $l \geq 0$. Here $\sigma^\perp$ is the subspace perpendicular to the tangents of σ.

Setting $l = 0$ in equation (3.13) gives $(d\varphi^{<k})|_{U_+} = 0$, which suggests that $\lim_{\hbar \to 0} \varphi^{<k}|_{U_+}$ is a constant which can be treated as an element in $\mathfrak{h}^{<k}$; denote this element by $\log(\Theta^{<k}_\sigma)$. Note that $\log(\Theta^{<k}_\sigma)$ is independent of U, as follows from the uniqueness of φ on the common intersection of two such open sets.

Remark 3.27. When $\mathfrak{h}$ is tropical, we have $\varPhi|_U = \sum_{k \geq 1} \sum_{\substack{L \in \mathtt{LT}_k \\ P_L \cap U \neq \emptyset}} \frac{1}{|\mathrm{Aut}(L)|} \alpha_L g_L$ with $g_L \in \mathfrak{h}_{m_L, n_L}$. This forces $[g_{L_1}, g_{L_2}]$ to vanish whenever $P_{L_i} \cap U \neq \emptyset$, $i = 1, 2$, because $\dim_{\mathbb{R}}(P_{L_i}) = r - 1$ and $P_{L_i} \cap U = \sigma \cap U$ by our choice of polyhedral decomposition $\mathcal{P}^{<k}$. The normals n_{L_1} and n_{L_2} to P_{L_1} and P_{L_2} are parallel while the vectors m_{L_i} are tangent to $P_{L_i} \cap U = \sigma \cap U$, $i = 1, 2$. This gives $\langle m_{L_j}, n_{L_i} \rangle = 0$ for $i, j = 1, 2$. By an induction argument, this implies $\frac{\mathrm{ad}^l_{\varphi^{<s}}}{(l+1)!} d\varphi^{<s} = 0 \in \mathcal{H}^*(U)$ for all s, l. If $\mathfrak{h}$ is not tropical, then $\frac{\mathrm{ad}^l_{\varphi^{<s}}}{(l+1)!} d\varphi^{<s}$ need not vanish and so contributes to the recursive construction of $\varphi^{<k}$.

Definition 3.28. *Let $\varPhi$ be as in Theorem 3.14. For each $k \in \mathbb{Z}_{>0}$, let $\mathcal{D}(\varPhi^{<k})$ be the scattering diagram with walls $\mathbf{w}_\sigma = (P_\sigma = \sigma, \Theta^{<k}_\sigma)$ indexed by the maximal cells $\sigma \in \mathcal{P}^{<k,[r-1]}$.*

Denote by $\mathcal{D}(\varPhi)$ the scattering diagram determined by $\{\mathcal{D}(\varPhi^{<k})\}_{k \in \mathbb{Z}_{>0}}$.

Theorem 3.29.
(1) The scattering diagram $\mathcal{D}(\varPhi)$ is consistent and the path ordered product $\Theta_{\lambda, \mathcal{D}(\varPhi)}$ is equal to the product $\Theta_r \cdots \Theta_1$ of the wall-crossing factors associated to initial walls.
(2) The scattering diagram $\mathcal{D}(\varPhi)$ is equivalent to the scattering diagram (or $\mathfrak{h}$-complex) $\mathcal{D}(g)$ constructed in [4, Lemma 3.2], *where $g = \Theta_r \cdots \Theta_1$.*

Proof. The proof of Proposition 3.16 carries over with only minor changes to show that $\mathcal{D}(\varPhi)$ is consistent. The discussion after the proof of Lemma 3.25 shows that the path λ does not intersect any walls of $\mathcal{D}(\varPhi)$ which are supported on $P_{\mathcal{L}}$. By construction, λ crosses the initial walls $\mathbf{w}_1, \ldots, \mathsf{w}_r$ consecutively. The assumption $[g_{j_1 i}, g_{j_2 i}] = 0$ ensures that the wall-crossing factor Θ_i, as in Definition 3.28, agrees with the wall-crossing automorphism of the wall $\mathbf{w}_i$ determined by the gauge φ constructed above; see also Remark 3.27. The path ordered product is therefore as stated. The equivalence between $\mathcal{D}(\varPhi)$ and $\mathcal{D}(g)$ is achieved by using [4, Proposition 3.3] to show $\mathcal{D}(\varPhi^{<k})$ and $\mathcal{D}(g^{<k})$ are equivalent for all order $k \in \mathbb{Z}_{>0}$. □

Example 3.30. *If the quiver Q is acyclic or, more generally, the quiver with potential (Q, W) is genteel in the sense of* [4, §11.5] *(and we modify the motivic Hall algebra so as to include the potential), then the consistent completion $\mathcal{D}(\varPhi)$ of the initial scattering diagram $\mathcal{D}_{in}$, with $\Theta_i = [\mathcal{M}_{\langle i \rangle} \to \mathcal{M}]$, is the Hall algebra scattering diagram of* [4]. *In the non-genteel case, additional walls must be added to $\mathcal{D}_{in}$ so as to recover the Hall algebra scattering diagram.*

3.4.4. *Non-tropical theta functions.* To talk about theta functions, we consider a $M_\sigma \oplus N$-graded algebra $B = \bigoplus_{(m,n) \in M_\sigma \times N} B_{m,n}$ together with a M_σ-graded action of $\mathfrak{h}$ by derivations so that $\mathfrak{h}_m \cdot B_{0,n} = 0$ whenever $\langle m, n \rangle = 0$. We further assume that, for each $n \neq 0$, there is a distinguished element $z^n \in B_{0,n}$ which we use to identify $B_{0,n}$ with $\mathbb{C} \cdot z^n$. We again define a dg algebra $\mathcal{B}^*(U)$ as in Definition 2.8, although it need not be graded commutative. Nevertheless, the dg Lie algebra $\mathcal{H}^*(U)$ acts on $\mathcal{B}^*(U)$, so we can still talk about theta functions as elements in $\mathrm{Ker}(d_\Phi)$.

Proposition 3.31. *For sufficiently large $c > 0$, there exists a unique section $\theta_n^{<k} \in \mathrm{Ker}^{<k}(d_\varPhi)$ such that $\theta_n^{<k}|_U = z^n$ in a neighborhood U of $\mathbb{R}_{\geq c} \cdot n$.*

Proof. The condition $\mathfrak{h}_m \cdot B_{0,n} = 0$ whenever $\langle m, n \rangle = 0$ ensures that, for each fixed k, there is a neighborhood U of $\mathbb{R}_{\geq c} \cdot n$ for some sufficiently large c such that $z^n \in \mathrm{Ker}^{<k}(d_\varPhi|_U)$. The remainder of the proof is as in that of Lemma 3.18. □

We can now state our final result.

Theorem 3.32. *Let $n \in N \setminus \{0\}$ and define θ_n by Proposition 3.31.*

(1) For any path $\gamma \subset N_{\mathbb{R}} \setminus \mathrm{Joints}(\mathcal{D}(\varPhi))$ from Q to Q', the wall-crossing formula
$$\theta_n(Q') = \Theta_{\gamma, \mathcal{D}(\varPhi)}(\theta_n(Q))$$
holds.

(2) The Hall algebra theta function $\vartheta_{n,Q}$, as defined in [4, §10.5], *is related to θ_n by the formula $\theta_n(Q) = \vartheta_{n,Q}$.*

Proof. The statements can be proved in the same way as Theorem 3.17 and 3.19. □

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