Efficient computation of Fourier transforms on compact groups

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ABSTRACT. This paper generalizes the fast Fourier transform algorithm to the computation of Fourier transforms on compact Lie groups. The basic technique uses factorization of elements in the group and Gel'fand bases to simplify the computations, and may be extended to treat computation of finitely supported distributions on the group. Similar transforms may be calculated on homogeneous spaces; we show that special function properties of spherical functions lead to more efficient algorithms. These results may all be viewed as generalizations of the fast Fourier transform algorithms on the circle, and of the more recent results about Fourier transforms on finite groups. The techniques introduced make sense for finite groups, and give a general approach to the calculation of Fourier transforms on any group.

1. INTRODUCTION

Let G be a compact Lie group, X a finite subset of G, let $\mathcal{R}$ be a finite set of finite dimensional representations of G, and let f be a complex valued function on X. In this paper we shall concern ourselves with the efficient computation of the finite sums

$$\hat{f}(\rho) = \hat{f}(\rho)_X = \sum_{x \in X} f(x)\rho(x) \tag{1}$$

for all representations ρ in $\mathcal{R}$. By this we mean explicit computation with respect to a given set of bases for the representation spaces, V_ρ.

We shall also look at several generalizations of (1). We shall replace the factor $\rho(x)$ by $\rho(D_x)$, where D_x is a distribution supported on $\{x\}$, and we shall also consider the sum

$$\hat{f}(\rho)_{\mathcal{D}} = \sum_{D \in \mathcal{D}} f(D)\rho(D) \tag{2}$$

where $\mathcal{D}$ is a set of distributions on G and f is a complex valued function on $\mathcal{D}$. We consider the analogous problem on homogeneous spaces, and finally comment on a version of these results for tensor fields on the sphere.

The special case when G in a finite group has been treated in several papers [1, 2, 3, 4, 6, 17, 18, 19] and books [3]. When G in S^1 or $\mathbf{Z}/n\mathbf{Z}$ we obtain the fast

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Fourier transform. D. Healy & J. Driscoll and Dan Rockmore consider S^2 [7, 8]. Our techniques combine all these methods.

To compute the sum (1) directly using any basis for V_ρ takes $|X|\, d_\rho^2$ operations[1], and d_ρ is the dimension of V_ρ. To compute these sums for all ρ in the set $\mathcal{R}$ requires $|X| \sum_{\rho\in\mathcal{R}} d_\rho^2$ operations. A common situation is when the map from the function, f, on X to the collection of transform $\{\hat{f}(\rho)\}_{\rho\in\mathcal{R}}$ is one to one, so the function, f, is determined by its transforms. In that case $|X| \leq \sum_{\rho\in\mathcal{R}} d_\rho^2$, so the direct method takes at least $|X|^2$ operations.

When G is S^1, a variant of the fast Fourier transform computes the sums (1) much more efficiently. If $\mathcal{R}$ is the set of irreducible representations of weights ranging from 0 to $|X|-1$, then one may calculate $\hat{f}(\rho)$ for all ρ in $\mathcal{R}$ in order $|X|(\log|X|)^2$ operations. When X is the set of $|X|$-th roots of unity, the fast Fourier transform algorithms take $|X|\log|X|$ operations. For a discussion of such algorithms see the papers [5, 8, 20]

The computation of the sums (1) depends on the set of representations, $\mathcal{R}$, the set, X, and the bases chosen for the representation spaces. To apply our techniques we must restrict both the sets X and the bases. We make crucial use of Gel'fand bases, also called subgroup adapted bases, which behave well under restriction. The set X is assumed to be a subset of a group product of sets, each of which lies in or commutes with appropriate subgroups.

Our results are of two main types. First, general theorems concerning the complexity if Fourier transform algorithms, and second, complexity results for specific groups. To summarize the latter we introduce a notion of the size of an irreducible representation. Choose any norm on the dual of the Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$where $\mathfrak{g}$ is the Lie algebra of G, and for any positive integer, b, let $\mathcal{R}_b$ be the set of irreducible unitary representations of G that have highest weight with norm at most b. A naive approach top computing (1) at all representations in $\mathcal{R}_b$ takes order $b^{2\dim G}$ operations. We define subsets of the classical groups, $X = X_b$, for which the calculation of (1) at all representations in $\mathcal{R}_b$ may be performed in Gel'fand basis in order $b^{\dim G+\gamma_n}$, where γ_n is given by

K_n	$SO(n)$	$U(n)$	$SU(n)$	$Sp(n)$
$\gamma(K_n)$	$\lfloor\frac{n}{2}\rfloor$	$n-1$	n	$3n$

For a complete statement, see theorem 5.1.

For rank one homogeneous spaces of classical groups, the situation is even better; the analogous calculation (see section 7) may be performed in order $b^{\dim G/K}(\log b)^2$ operations. These results all require the use of adapted bases. We also extend our results to consider sums over more general distributions, as in (2), and finally we give a brief note on the expansion of tensor fields on the two-sphere in spherical harmonics.

The ultimate aim of these algorithms is to give a method for computing Fourier transforms of functions on the whole group, G. To do this one should relate the Fourier transform of a function on the group to sums of the form (2), which are Fourier transforms of finitely supported distributions. More precisely, if h is a C^n function on G and D is a distribution of order n supported on X, then the Fourier transform of the product distribution $h.D$ may be written as a sum of at most 2^n

[1]One operation counts for one complex multiplication together with one complex addition

sums of the form (2) and hence may be computed by the techniques of this paper, given values of derivatives of h as initial data.[2] In the thesis [14] and the paper [15] it is shown that under suitable conditions on h and D, the transform of $h.D$ approximates, or is even equal to, the Fourier transform of h. We must be sure that the sets, X, used in this paper are general enough for us to construct distributions, D, with the correct properties and supported on X. The construction of D, and the conditions on D and h for a good approximation are treated in [14] and [15]. It is shown there that the sets X we use in this paper are indeed general enough to treat the Fourier transforms of functions on the whole group (see lemma 4.3 of the current paper for a statement of such a result).

One can think of the techniques we shall use in a hierarchy, each technique improving on the results of the previous. Most fundamental is the use of Gel'fand bases or, equivalently, subgroup adapted sets of matrix representations. We introduce these in section 2 and use Schur's lemma to obtain results about the form of representation matrices. Next, in section 3, we use factorizations of X into coset representatives to get results analogous to those of Diaconis and Rockmore [6]. This gives us our first improvements over naive methods, and we also obtain a result on products of groups. In section 4, we refine the factorizations, adding more factors, and use commutativity between these factors and subgroups of G. Schur's lemma then simplifies the forms of the matrices involved. This idea is a new ingredient in the theory of fast Fourier transforms on noncommutative groups and yields new results in both the Lie group and finite group cases.

The techniques described so far work by replacing matrix additions and multiplications in (1) with still more matrix multiplications, but ensuring that the new matrix multiplications may be performed efficiently. To obtain better algorithms we work on the level of matrix coefficients. Viewing all the matrix equations obtained by previous techniques as sets of scalar equations gives additional flexibility in the ordering of factors; a multiple matrix product may be computed using matrix products in only two ways, right to left of left to right, where as an indexed sum of products may be computed in an arbitrary order. In section 5 we only develop this idea enough to prove our final results for the classical groups, but these methods also have a nice combinational formulation in terms of injections of diagrams into Bratteli diagrams [9] that enables one to write down explicit expressions for the complexities of the algorithms. A treatment of these combinatorial methods is given, in the finite group case, in [18].

Homogeneous spaces are treated in general in section 6, using similar techniques for computing sums of products. Treating the computations on the scalar level gives the possibility of using special function properties of the matrix coefficients: For the groups $SO(3)$, $SU(2)$, and the homogeneous spaces $SO(n)/SO(n-1)$, $SU(n)/SU(n-1)$, $Sp(n)/Sp(n-1)$ we may express the matrix coefficients in terms of classical orthogonal polynomials [12, 14]. In section 7 we use the algorithm of Driscoll and Healy, for computing expansions in such sequences, to improve our results on homogeneous spaces.

Section 8 treats the original problem of computing the sums (2). We reformulate our algorithms in the language of distributions and convolutions which allows us to treat the more general case by the same techniques. In section 9 we briefly indicate

[2] The sums are of the form (1), with $\rho(x)$ replaced by $\rho(D_x)$.

how our results yield a method for efficiently computing the expansion of tensor fields on S^2 in tensor harmonics. (see [10]).

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2. Gel'fand bases and adapted matrix representations

A set of bases for a set of finite dimensional representations of G is a set of Gel'fand bases relative to G, or is G-adapted, if a basis for each representation in the set is a union of bases for orthogonal irreducible representations making up that representation, and furthermore the matrix representations corresponding to occurrences of equivalent irreducible representations in any representations of the set are equal.

If H is a closed subgroup of G then a set of bases for representation in $\mathcal{R}$ is H-adapted, or a set of Gel'fand bases relative to H, if the set of bases is H-adapted when considered as bases for the set of restrictions of representations in $\mathcal{R}$ to H. Likewise, given a chain of subgroups of G, one may define a set of bases for $\mathcal{R}$ to be adapted to the chain, or a set of Gel'fand bases relative to the chain, if it is adapted to each subgroup in the chain.

Although we will need Gel'fand bases we will usually use an equivalent formulation in terms of matrix representations.

Definition 2.1. Assume G is a compact Lie group and $\mathcal{R}$ is a set of finite dimensional matrix representations of G.

(i) $\mathcal{R}$ is said to be G-adapted if there is a set, $\mathcal{R}_G$, of inequivalent irreducible matrix representations of G such that each representation in $\mathcal{R}$ is a matrix direct sum of representations in $\mathcal{R}_G$. The set $\mathcal{R}_G$ is uniquely determined by $\mathcal{R}$.

(ii) Assume H is a closed subgroup of G. Then $\mathcal{R}$ is said to be H-adapted if the set of restrictions of representations in $\mathcal{R}$ is H-adapted, i.e. $\mathcal{R}|H$ is H-adapted, where $\mathcal{R}|H = \{\rho|H : \rho \in \mathcal{R}\}$

(iii) Let $\mathcal{R}_H = (\mathcal{R}|H)_H$.

So a set of matrix representations is H-adapted if its set of restrictions to H may be constructed from a set of inequivalent irreducible matrix representations of H by taking matrix direct sums. The connection between sets of Gel'fand bases and adapted sets of matrix representations is obvious.

Lemma 2.1. (i) *The standard bases for an adapted set of matrix representations is adapted.*

(ii) *The sets of matrix representations corresponding to an adapted set of bases is adapted.*

The two approaches are completely equivalent; definition 2.1 and lemma 2.1 could be used as a definition of Gel'fand bases.

There is a natural way to index Gel'fand bases, which leads to a third definition of them. Suppose $\mathcal{R}$ is a set of representations of G, and $K_n \geq \ldots \geq K_0$ is a chain of closed subgroups of G. Let $\mathcal{R}_{K_i}$ be a set of inequivalent irreducible representations of K_i whose direct sums include representations equivalent to restrictions of representations of $\mathcal{R}$ to K_i. We construct a finite category with initial object, $\mathbf{C}$, and including all the representations in the $\mathcal{R}_{K_i}$ and in $\mathcal{R}$ as objects. The maps of this category are generated by maps from $\mathbf{C}$ to representations in $\mathcal{R}_{K_0}$ corresponding to bases of these representations, by maps injecting representations in $\mathcal{R}_{K_i}$ into ones in $\mathcal{R}_{K_{i+1}}$ according to chosen direct sum decompositions of representations in $\mathcal{R}_{K_{i+1}}$ restricted to K_i, and by maps injecting representations in $\mathcal{R}_{K_n}$ into representations in $\mathcal{R}$ according to restrictions of representations in $\mathcal{R}$ to K_n. Elements of the Gel'fand bases are then indexed by maps from $\mathbf{C}$ to representations in $\mathcal{R}$ in this category; each map indexing the vector which is the image of $1 \in \mathbf{C}$ under that map.

The main advantage of using adapted sets of representations is that they allow us to relate the form of the representation matrices, $\rho(a)$, to the properties of the group element, a. It is immediate from the definition that if a is an element of a subgroup of G and ρ is a matrix representation adapted to that subgroup, then $\rho(a)$ will have block diagonal form according to the decomposition of the restriction of ρ into irreducibles. If instead a commutes with all elements of a subgroup of G then $\rho(a)$ will have a block scalar form, provided ρ is adapted to the subgroup. This is the content of Schur's lemma.

Lemma 2.2 (Schur's Lemma). *Assume G is a compact Lie group and K is a closed subgroup of G. Assume ρ is a K-adapted representation of G such that $\rho|K = (\mathbf{C}^{m_1} \otimes \eta) \oplus \cdots \oplus (\mathbf{C}^{m_r} \otimes \eta_r)$, where $\eta_1, \ldots, \eta_r$ are inequivalent irreducible matrix representations of K and the action on $\mathbf{C}^{M_i}$ is trivial so η_i occurs in $\rho|K$ with multiplicity m_i. Then if a is in the centralizer of $K, \rho(a)$ is of the form*

$$(GL_{m_1}(\mathbf{C}) \otimes I_{d_{\eta_1}}) \oplus \cdots \oplus (GL_{m_r}(\mathbf{C}) \otimes I_{d_{\eta_r}}) \tag{3}$$

Where $I_{d_{\eta_i}}$ is the $d_{\eta_i} \times d_{\eta_i}$ identity matrix and d_{η_i} is the dimension of η_i. The tensor products are tensor products of matrices.

If ρ is adapted with respect to a subgroup chain and a lies in a subgroup of the chain as well as commuting with other subgroups in the chain then $\rho(a)$ will be block diagonal according to the restriction to one subgroup, and after a permutation of rows and columns it will be block scalar corresponding to the subgroups that a commutes with. Block diagonal and block scalar matrices have many zero entries; this is the main source of the efficiency in our algorithms.

In order to describe the effect of the structure of $\rho(a)$ on the complexity of matrix multiplications we introduce some notation. Assume G is a compact Lie group and $H \geq K$ are closed subgroups of G. For any representation, ρ, of G let $\mathcal{M}_K^H(\rho)$ denote the maximum multiplicity of any irreducible representations of K in the restriction to K of an irreducible representation of H occurring in $\rho|H$. For any set of representations, $\mathcal{R}$, we let $\mathcal{M}_K^H(\mathcal{R})$ denote the maximum of $\mathcal{M}_K^H(\rho)$ as ρ varies over $\mathcal{R}$.

Lemma 2.3. *Assume $G \geq H \geq K$ are compact Lie groups, ρ is a representation of G adapted to the chain and a lies in H but commutes with K. Then we may calculate the matrix product $\rho(a).F$ where F is any $d_\rho \times d_\rho$ matrix in no more than $\mathcal{M}_K^G(\rho)d_\rho^2$ scalar multiplications and $(\mathcal{M}_K^G(\rho) - 1)d_\rho^2$ additions.*

Proof. Use Schur's Lemma to count the nonzero entries of $\rho(a)$. □

3. Reduction to Subgroups

We shall now adapt a method for computing Fourier transforms on finite groups, due to Diaconis and Rockmore [6], to the compact Lie group setting. This method relates the computation of the sums (1) to the computation of similar sums defined on a subgroup, under the assumption that the set over which we are summing factors nicely.

To keep track of the number of operations required by our methods we define:

Definition 3.1. Assume G is a compact Lie group, X is a finite subset of G and $\mathcal{R}$ is a set of finite dimensional matrix representations of G. Let $T_X(\mathcal{R})$ denote the minimum number of operations required to compute the sums (1) for all ρ in $\mathcal{R}$ for an arbitrary complex function, f, on X.

Lemma 3.1. *Assume X is any subset of the compact Lie group, G and $\mathcal{R}$ is a set of G-adapted matrix representations, then $T_X(\mathcal{R}) = T_X(\mathcal{R}_G)$.*

Assume G is a compact Lie group, K is a closed subgroup of G, Y is a subset of G, Z is a subset of K, and $X = Y.Z$. Then provided no two points of Y lie in the same coset of K, we have

$$\hat{f}(\rho) = \sum_{y \in Y} \rho(y) \sum_{z \in Z} f_y(z)\rho(z) \tag{4}$$

$$= \sum_{y \in Y} \rho(y) \hat{f}_y(\rho|K)_Z \tag{5}$$

where $f_y(z) = f(y.z)$. If some points of Y do lie in the same coset of K then it is possible that $x = y.z = y'.z'$ for more than one pair (y, z) in $Y \times Z$. In this case it is simple to redefine $f_y(z)$ so that (4) still holds, by choosing one pair (y, z) for which $x = y.z$ for each x, defining $f_y(z) = f(y.z)$ for that pair and setting $f_{y'}(z')$ to be zero for all other pairs (y', z') with $x = y'.z'$. A similar trick allows us to define $f_y(z)$ so that (4) holds when X is contained in $Y.Z$, but not necessarily equal to $Y.Z$.

Formula (4) suggests a method of computing $\hat{f}(\rho)$. First compute all the transforms $\hat{f}_y(\rho|K)_Z$, then multiply by the matrixes $\rho(y)$ and sum on Y. When ρ belongs to a K-adapted set of matrix representations, $\mathcal{R}$, the computation of the transforms at the restricted representations $\rho|K$ for ρ in $\mathcal{R}$ is equivalent to computing transforms at the set of representations $\mathcal{R}_K$. A simple count of the number of operations in this algorithm proves lemma 3.2, which is stated after the following definition:

Definition 3.2. Assume $\mathcal{R}$ is a set of matrix representations of G, Y is a subset of G and K is a closed subgroup of G. Let $M(\mathcal{R}, Y, K)$ denote the number of operations needed to compute $\sum_{y \in Y} \rho(y)F(y, \rho)$ for each ρ in $\mathcal{R}$, where each $F(y, \rho)$ is a $d_\rho \times d_\rho$ matrix in $\text{span}_{\mathbb{C}}\, \rho(K) = (\text{End}\, V_\rho)_K$.

Lemma 3.2. *Assume $X \subseteq Y.Z.$, where Z is a subset of K, then*

$$T_X(\mathcal{R}) \leq |Y| T_Z(\mathcal{R}|K) + M(\mathcal{R}, Y, K)$$

Theorem 3.3. *Assume G is a compact Lie group, K is a closed subgroup of G, $X \subseteq Y.Z$. Where $Z \subseteq K$ and $\mathcal{R}$ is a K-adapted set of matrix representations of G, then*

$$T_X(\mathcal{R}) \le |Y| T_Z(\mathcal{R}_K) + M(\mathcal{R}, Y, K)$$

There are two keys to effective use of this theorem. The first is to apply it to a chain of subgroups, and the second is to lower the bound on $M(\mathcal{R}, Y, K)$ by using special properties of the matrices $\rho(y)$. Before investigating these ideas let's see what a straight forward application of theorem 3.3 yields.

Corollary 3.4. *Assume G is a compact connected Lie group, K a closed connected subgroup, Y_b a subset of G with $|Y_b|$ of order $b^{\dim G - \dim K}$ and Z is a subset of K with $|Z_b|$ of order $b^{\dim K}$. Assume $X_b \subseteq Y_b.Z_b$. Choose a norm on the dual of the Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$, and let $\mathcal{R}_b$ be a K- adapted set of irreducible representations of G all of whose highest reights have norm no greater than b. Then $T_X(\mathcal{R}_b)$ has order $b^{\dim G + \gamma}$ where γ is the maximum of $\dim K$ and $\dim G - \frac{1}{2}\dim K - \frac{1}{2}\operatorname{rank} K$.*

Proof. Assume ρ is an irreducible representation of G and ρ' is an irreducible representation of K. then d_ρ is a polynomial of degree $\frac{1}{2}(\dim K - \operatorname{rank} G)$ and $d_{\rho'}$ is a polynomial of degree $\frac{1}{2}(\dim K - \operatorname{rank} K)$, both considered as functions of the highest weight of the representations. We may pick a Cartan subalgebra of $\mathfrak{k}^{\mathbf{C}}$ which embeds as a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$, and choose norms on the duals of both so that the highest weights of any representation in $(\mathcal{R}_b)_K$ has norm no greater than b. The maximum size of a block in the block matrix $\hat{f}_y(\rho|K)_Z$ is $\max\{d_{\rho'} : \rho' \in \{\rho\}\}_K$ which is clearly bounded by a term of order $b^{\frac{1}{2}(\dim K - \operatorname{rank} K)}$ when ρ is in $\mathcal{R}_b$. Therefore

$$\begin{aligned} T_X(\mathcal{R}_b) &\le |Y| T_Z((\mathcal{R}_b)_K) + M(\mathcal{R}_b, Y, K) \\ &\le |Y||Z| \sum_{\rho' \in (\mathcal{R}_b)_K} d_{\rho'}^2 + |Y| \left(\sum_{\rho \in \mathcal{R}_b} d_\rho^2 \right) \max\{d_{\rho'} : \rho' \in (\mathcal{R}_b)_K\} \end{aligned}$$

the first term has order $b^{\dim G + \dim K}$ and the second term has order $b^{2\dim G - \frac{1}{2}\dim K - \frac{1}{2}\operatorname{rank} K}$ □

Corollary 3.5. *Assume G is a connected simple compact Lie group, then T under the assumptions of corollary 3.4, $T_X(\mathcal{R}_b)$ has order $b^{\dim G + \gamma_G}$ where γ_G is asymptotic to $\frac{2}{3}\dim G$ as the dimension tends to infinity.*

Proof. We need only consider groups with Lie algebras of type A, B, E, or D. If G has rank r then choose K to be a subgroup of G from the same series but having rank the closest integer to $r\sqrt{\frac{2}{3}}$. □

Chains of Subgroups. We can apply theorem 3.3 recursively to a chain of subgroups.

Theorem 3.6. *Assume G is a Lie group and $G = K_n \geq \cdots \geq K_0 = 1$ is a chain of closed subgroups of G. Assume $\mathcal{R}$ is a set of representations adapted to this chain, and $X \subseteq Z_n.Z_{n-1}.\ldots.Z$, where $Z_i \subseteq K_i$, then*

$$T_X(\mathcal{R}) \leq |Z_n \ldots Z_2| T_{Z_1}(\mathcal{R}_{K_1}) + \sum_{i=2}^{n} |Z_n \ldots Z_{i+1}| M(\mathcal{R}_{K_i}, Z_i, K_{i-1})$$

Corollary 3.7. *Assume $G = K_n$ is one of the classical groups $SO(n)$, $SU(n)$, $U(n)$ or $Sp(n)$,[3] that $X_b \subseteq Z_n^b \ldots Z_1^b$ where $Z_i^b \subseteq K_i$, and $|Z_i^b|$ is of order $b^{\dim K_i - \dim K_{i-1}}$. Let $\mathcal{R}_b$ be a set of representations of K_n adapted to the chain $K_n \geq \cdots \geq K_0$ such that the norms of its highest weights are at most b. Then $T_{X_b}(\mathcal{R}_b)$ is of order $b^{\dim K_n + \gamma_n}$ where*

$$\gamma_n = \dim K_n - \frac{1}{2} \dim K_{n-1} - \frac{1}{2} \operatorname{rank} K_{n-1}$$

For any of these series of groups γ_n is asymptotic to $\frac{1}{2} \dim K_n$ as n tends to infinity.

Proof. Following the same line of argument as the proof of corollary 3.4, we find that $T_{X_b}(\mathcal{R}_b)$ is bounded by a sum of n terms with orders $b^{\dim K_n + \gamma_i}$ for $1 \leq i \leq n$. By explicit calculation we see that γ_i is always an increasing function of i and therefore $T_{X_b}(\mathcal{R}_b)$ is of order $b^{\dim K_n + \gamma_n}$. To verify that γ_n is asymptotic to $\frac{1}{2} \dim K_n$ we refer to table 3.1. □

K_n	$\dim K_n$	Number of positive roots	γ_n
$SU(n+1)$	$n^2 + 2n$	$\frac{n(n+1)}{2}$	$\frac{1}{2}n^2 + \frac{3}{2}n + 1$
$SO(2n+1)$	$2n^2 + n$	n^2	$n^2 + 2n$
$Sp(n)$	$2n^2 + n$	n^2	$n^2 + 2n$
$SO(2n)$	$2n^2 - n$	$n^2 - n$	$n^2 + n - 1$
$SO(n)$	$\frac{n(n-1)}{2}$	$\frac{1}{2} \left[\frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor \right]$	$\frac{1}{2} \cdot \frac{n(n-1)}{2} + \frac{1}{2} \lfloor \frac{n}{2} \rfloor$

Table 3.1: Fourier tranforms using coset decomposition.

Products. A special case of theorem 3.3 occurs when $G = H \times K$ is a direct product. In that case the irreducible representations of G are tensor products of irreducible representations of H and K and a matrix product of a matrix representation of H with one of K is both H and K adapted after a relabelling of rows and columns.

Theorem 3.8. *Assume $\mathcal{R}_1, \mathcal{R}_2$ are sets of matrix representations of H_1 and H_2 respectively and $\mathcal{R}_1 \otimes \mathcal{R}_2$ is the set of their tensor products. Assume $G = H_1 \times H_2$ and $X \subseteq Z_1.Z_2$ where $Z_1 \subseteq H_2$. Then*

$$T_X(\mathcal{R}_1 \otimes \mathcal{R}_2) \leq |Z_2| T_{Z_1}(\mathcal{R}_1) + |Z_1| T_{Z_2}(\mathcal{R}_2)$$

[3] We define $Sp(n)$ to be set set of $n \times n$ quaternionic matrices that are unitary with respect to a quaternionic inner product, i.e. $A^* A = I$.

Proof. Let $\{v_1^{i_1}\}$ be a basis for V_{ρ_1} and $\{V_2^{i_2}\}$ be a basis for V_{ρ_2} where ρ_k is a representation in $\mathcal{R}_k$. Then

$$\left[\hat{f}(\rho_1 \otimes \rho_2)_{Z_1.Z_2}\right]_{v_1^{i_1}\otimes v_1^{j_1},v_2^{i_2}\otimes v_2^{j_2}} = \sum_{z_1\in Z_1,z_2\in Z_2} f(z_1,z_2)\langle\rho_1(z_1)v_1^{i_1},v_1^{j_1}\rangle\langle\rho_2(z_2)v_2^{i_2},v_2^{j_2}\rangle$$
$$= \sum_{z_2\in Z_2}\langle\rho_2(z_2)v_2^{i_2},v_2^{j_2}\rangle\left[\sum_{z_1\in Z_1} f(z_1,z_2)\langle\rho_1(z_1)v_1^{i_1},v_1^{j_1}\rangle\right]$$

For fixed Z_2 the inner sum is a transform on H_1 whereas for fixed ρ_1 the outer sum is a transform as H_2 □

Theorem 3.8 allows us to restrict our consideration to transforms on simple compact Lie groups, when necessary.

4. Sums of Products

Theorems 3.3 and 3.6 examine the effect of factoring elements of the set X, on which we perform our transform, into products where elements came from subgroups of G. In the notation of theorem 3.3, we shall now see how factoring the elements of Y improves the bound on $M(\mathcal{R}, Y, K)$.

The basic idea is very simple. We start with a set $Y \subseteq G$, a representation ρ, of G, which is adapted to a chain of closed subgroups $G = K_n \geq \cdots \geq K_0 = 1$, and a $d_\rho \times d_\rho$ matrix valued function, F, on Y. We wish to compute the sum

$$\sum_{y\in Y}\rho(y)F(y) \tag{6}$$

efficiently. By lemma 2.3 we know that if y lies in or commutes with some subgroups of the chain, then the special structure of the matrix $\rho(y)$ allows us to perform the matrix multiplication $\rho(y).F(y)$ efficiently. If $y = a_1 \dots a_n$ and each matrix $\rho(a_i)$ has a special structure, then we can use the equation $\rho(y) = \rho(a_1)\dots\rho(a_n)$ to calculate $\rho(y).F(y)$ efficiently.

To quantify this we make the following definition. Assume a is an element of the compact Lie group G, and

$$G = K_n \geq \cdots \geq K_0 = 1$$

is a chain of subgroups. Let $c^+(a)$ be the minimum nonnegative integer such that a lies in $K_{c^+(a)}$ and let $c^-(a)$ be the maximum number such that a commutes with all elements of $K_{c^-(a)}$. Now define $\mathcal{M}(a,\rho) = \mathcal{M}_{K_{c^-(a)}}^{K_{c^+(a)}}(\rho)$ and $\mathcal{M}(a,\mathcal{R}) = \mathcal{M}_{K_{c^-(a)}}^{K_{c^+(a)}}(\mathcal{R})$ for any representation, ρ, of G, or any set of representations, $\mathcal{R}$.

Lemma 4.1. *Assume $\tilde{Y}$ is a set of words in G. Whose corresponding set of elements of G is precisely Y. Then the sum* (6) *may be computed in no more than*

$$\left(\sum_a \mathcal{M}(a,\rho)\right) d_\rho^2$$

where a ranges over all occurrences of all symbols in words of $\tilde{Y}$.

Corollary 4.2. *Assume G is a compact Lie group, $G = K_n \geq \cdots \geq K_0 = 1$ is a chain of closed subgroups of G, K is a closed subgroup of G and $\mathcal{R}$ is a set of matrix representations of G adapted to the subgroup chain. Assume $Y \subseteq G$ and $\tilde{Y}$ is a set of words in G whose corresponding set of group elements is precisely Y. Then*

$$M(\mathcal{R}, Y, K) \leq \left(\sum_a \mathcal{M}(a, \mathcal{R})\right)\left(\sum_{\rho\in\mathcal{R}} d_\rho^2\right) \tag{7}$$

where a ranges over all occurrences of all symbols in words of $\tilde{Y}$.

Proof. Lemma 4.1 and corollary 4.2 both result from a simple count of the number and complexity of the matrix products in the sum

$$\sum_{a_1 \ldots a_m \in \tilde{Y}} \rho(a_1)\cdots\rho(a_m)F(a_1 \ldots a_m)$$

No special technique for simplifying this sum is used. □

The Main Example: Classical Groups. For each of the classical groups we will now define sets of elements, X_b, and sets of representations, $\mathcal{R}_b$, which we shall use for the rest of the paper. We shall also choose subgroup chains for these groups and use the results of this section to bound $T_{X_b}(\mathcal{R}_b)$ for these choices, assuming adapted bases for the representations.

The sets X_b^n defined in the examples below are all products of sets of size $O(b)$ which are contained in one parameter subgroups of the compact group. For the purposes of our complexity results, the number of points chosen in these factor sets could be different from the examples, provided it is always $O(b)$. The sets $\mathcal{R}_b$ are chosen according to particular choices of norms on the duals of the Cartan subalgebras. The choices for these sets can be varied provided there is always a norm for which $\mathcal{R}_b$ is contained in the set of irreducible representations whose highest weight has norm no greater than b. It is easy to see, by scaling the variable b, that our results imply the results for these more general sets X_b^n and $\mathcal{R}_b^n$.

Let $U_{n-1} \times U_1$ be the group of block diagonal unitary matrices with one block of size $n-1$ followed by a block of size 1, and let $S(U_{n-1} \times U_1)$ be the subgroup of $U_{n-1} \times U_1$ of matrices with determinant 1. Similarly, $S_p(n-1) \times S_p(1)$ and $S_p(n-1) \times U(1)$ are identified with subgroups of block diagonal quaternionic matrices in $Sp(n)$. The sets, X_b, we shall construct, come from parametrizations of these groups analogous to generalized Euler angles.

Define the following one parameter subgroups: let

$$r_n(\theta) = \left(\begin{array}{c|cc|c} I_{n-2} & & & \\ \hline & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ \hline & & & I \end{array}\right)$$

where the rotation block appears in columns $n-1$ and n, and let $t_n(\theta)$ be the diagonal matrix with ones on the diagonal except for $e^{i\theta}$ in the n^{th} diagonal position. Define $q_n(\theta) = t_1(-\theta)\cdots t_n(-\theta)t_{n+1}(n\theta)$, and let $u_n(\theta)$ be the diagonal matrix with ones except for $e^{j\theta}$ in the n^{th} diagonal position, where j is a unit quaternion.

The Special Orthogonal Groups. In this case we shall always use the subgroup chain

$$SO(n) \geq SO(n-1) \geq \cdots \geq SO(2) \geq 1 \tag{8}$$

Assume b is a positive integer. For any $n \geq 2$ we let A_2^n be a set of $2b+1$ distinct points in $r_2([0,2\pi))$ and for $2 < k \leq n$ let A_k^n be a set of $b+1$ distinct points in $r_k([0,\pi])$. We now define the sets $X_b^n \subseteq SO(n)$ as follows:

$$\begin{aligned} X_b^2 &= A_2^2 \\ X_b^n &= A_2^n.\ldots.A_n^n.X_b^{n-1} \end{aligned}$$

To define $\mathcal{R}_b$, we note that relative to a standard basis for the dual of the Cartan subalgebra the highest weight $m_{1,n}, \ldots, m_{r,n}$, where $r = \lfloor \frac{n}{2} \rfloor$ is the rank of $SO(n)$. These integers satisfy the relation

$$m_{1,n} \geq \cdots \geq |m_{r,n}|$$

and when n is odd $m_{r,n} \geq 0$. We define $\mathcal{R}_b^n$ (up to isomorphism) to be the set of irreducible representations with highest weight $m_{1,n}, \ldots, m_{r,n}$, such that $|m_{1,n}| \leq b$. The 'betweenness' relations for the restriction of representations from $SO(n)$ to $SO(n-1)$, see [23], show that the set of representations occurring in the restrictions of $\mathcal{R}_b^n$ to $SO(n-1)$ is precisely $\mathcal{R}_b^{n-1}$. Therefore we may choose $\mathcal{R}_n^n$ to be an adapted set of matrix representations for the chain, and have $(\mathcal{R}_b^n)_{SO(n-1)} = \mathcal{R}_b^{n-1}$.

The Unitary Groups. We use the subgroup chain

$$U(n) \geq U_{n-1} \times U_1 \geq U(n-1) \geq \cdots \geq U(i) \geq 1 \tag{9}$$

For $n \geq 2$ and $2 \leq k \leq n$, let A_k^n be a set of $b+1$ distinct points in $r_k\left(\left[0, \frac{\pi}{2}\right]\right)$. For $n \geq 1$ and $1 \leq k \leq n$ let B_k^n be a set of $2b+1$ distinct points in $t_k([0,2\pi))$. Define

$$\begin{aligned} X_b^1 &= B_1^1 \\ X_b^{n,1} &= B_{n+1}^{n+1}.X_b^n \subseteq U_n \times U_{n-1} \\ X_b^n &= (B_1^n \cdots B_{n-1}^n A_2^n).X_b^{n-1,1} \subseteq U(n) \end{aligned}$$

Irreducible λ representations of $U(n)$ have highest weights, given in a standard basis by an n tuple of integers $m_{1,n}, \ldots, m_{n,n}$ such that,

$$m_{1,n} \geq \cdots \geq m_{n,n} \tag{10}$$

Irreducible representations of $U_n \times U_1$ are therefore indexed by an n tuple $m_{1,n}$, $\ldots$, $m_{n,n}$ satisfying (10), together with an integer $m_{n+1,n}$ with indexes a representation of $U(1)$. We define $\mathcal{R}_b^n$ to be the set of irreducible representations of $U(n)$ for which $b \geq m_{1,n} \geq m_{n,n} \geq -b$, and let $\mathcal{R}_b^{n,1}$ be the set of representations of $U_n \times U_1$ for which $b \geq m_{1,n} \geq m_{n,n} \geq -b$ and $|m_{n+1,n}| \leq b$. The betweeness relations for restricting from $U(n)$ to $U(n-1)$ (as in [23]) show we may choose matrix representations so that $(\mathcal{R}_b^n)_{U_{n-1}\times U_1} = \mathcal{R}_b^{n-1,1}$, $(\mathcal{R}_b^{n-1,1})_{U(n-1)} \subseteq \mathcal{R}_b^{n-1}$, and $(\mathcal{R}_b^n)_{U(n-1)} = \mathcal{R}_b^{n-1}$.

The special unitary group. We use the subgroup chain

$$SU(n) \geq S(U_{n-1} \times U_1) \geq \cdots \geq S(U_1 \times U_1) \geq 1 \tag{11}$$

For ≥ 2 and $2 \leq k \leq n$, let A_k^n be a set of $\left[\frac{b}{2}\right] + 1$ distinct points in $r_k\left(\left[0, \frac{\pi}{2}\right]\right)$. For $n \geq 1$ and $1 \leq k \leq n$, let B_k^n be a set of $2kb + 1$ distinct points in $q_k\left(\left[0, \frac{2\pi}{k}\right)\right)$. Define

$$\begin{aligned} X_b^{1,1} &= B_1^1 \\ X_b^n &= B_1^n \ldots B_{n-1}^n A_2^n \ldots A_n^n . X_b^{n-1,1} \subseteq SU(n) \\ X_b^{n,1} &= B_n^n X_b^n \subseteq S(U_{n-1} \times U_1) \end{aligned}$$

Irreducible representations of $SU(n)$ have highest weights indexed by $(n-1)$-tuples of integers $m_{1,n}, \ldots, m_{n-1,n}$, such that

$$m_{1,n} \geq \cdots \geq m_{n-1,n} \geq 0 \tag{12}$$

Irreducible representations of $S(U_n \times U_1)$ are indexed by an $(n-1)$-tuple satisfying (12) together with an integer, $m_{n,n}$, indexing a representation of the subgroup $q_n(\mathcal{R})$. We let $\mathcal{R}_b^n$ be the set of irreducible representations of $SU(n)$ for which $m_{1,n} \leq b$, and let $\mathcal{R}_b^{n,1}$ be the set of irreducible representations of $S(U_n \times U_1)$ for which $m_{1,n} \leq b$ and $|m_{n,n}| \leq nb$. It is easy to see [14] that we may choose these sets to be sets of matrix representations satisfying $(\mathcal{R}_b^n)_{S(U_{n-1} \times U_1)} \subseteq \mathcal{R}_b^{n-1,1}, (\mathcal{R}_b^{n,1})_{SU(n)} = \mathcal{R}_b^n$, and $(\mathcal{R}_b^n)_{SU(n-1)} = \mathcal{R}_b^{n-1}$.

The Symplectic Groups. We use the subgroup chain

$$Sp(n) \geq Sp(n-1) \times Sp(1) \geq Sp(n-1) \times U(1) \geq Sp(n-1) \geq \cdots \geq 1 \tag{13}$$

For $n \geq 2$ and $2 \leq k \leq n$, let A_k^n be a set of $b+1$ distinct points in $r_k\left(\left[0, \frac{\pi}{2}\right]\right)$. For $n \geq 1$ and $1 \leq k \leq n$, let $B_k^{n,1}$ and $B_k^{n,2}$ be sets of $2b+1$ points in $t_k([0, 2\pi))$ and let C_k^n be a set of $\left[\frac{b}{2}\right] + 1$ points in $u_k\left(\left[0, \frac{\pi}{2}\right]\right)$. Define

$$\begin{aligned} X_b^{0,1} &= B_1^{1,1} \\ X_b^{n,2} &= B_{n+1}^{n+1,2} C_{n+1}^{n+1} X_b^{n,1} \subseteq Sp(n) \times Sp(1) \\ X_b^n &= (B_1^{n,2} C_1^n B_1^{n,1} A_2^n) \ldots (B_{n-1}^{n,2} C_{n-1}^n B_{n-1}^{n,1} A_n^n) X_b^{n-1,2} \subseteq Sp(n) \\ X_b^{n,1} &= B_{n+1}^{n+1,1} X_b^n \subseteq Sp(n) \times U(1) \end{aligned}$$

Irreducible representations of $Sp(n)$ are indexed by n-tuples of integers $m_{1,n}, \ldots, m_{n,n}$ such that

$$m_{1,n} \geq \cdots \geq m_{n,n} \geq 0$$

Irreducible representations of $Sp(n) \times Sp(1)$ and $Sp(n) \times U(1)$ indexed by an irreducible representation of $Sp(n)$ together an irreducible representation of $Sp(1)$ or $U(1)$ respectively, which we index by integers m_n and p_n respectively; clearly $m_{1,n} \leq b$. Also let $\mathcal{R}_b^{n,1}$ and $\mathcal{R}_b^{n,2}$ be the sets of irreducible representations of $Sp(n) \times U(1)$ and $Sp(n) \times Sp(1)$ for which $m_{1,n} \leq b$ and $|p_n| \leq b$ or $m_n \leq b$ respectively. We may choose the irresucible representations with these highest eights to be matrix representations such that $(\mathcal{R}_b^n)_{Sp(n-1) \times Sp(i)} \subseteq \mathcal{R}_b, (\mathcal{R}_b^{n,2})_{Sp(n)} \subseteq \mathcal{R}_b^{n,1}$, $(\mathcal{R}_b^{n,1})_{Sp(n)} = \mathcal{R}_b^n$ and $(\mathcal{R}_b^n)_{Sp(n-1)} = \mathcal{R}_b^{n-1}$.

Results for the Classical Groups. The rationale for our particular choice of the sets $\mathcal{R}_b$ and X_b is the following lemma, which shows that function on the set X_b may be used to approximate continuous functions on the group. This result is not required anywhere in this paper, but it does show that the conditions on sets of points and representations required for approximating continuous functions are compatible with these required to use the algorithms we have developed. A proof is given in [14] and [15] for the $SO(n)$, $SU(n)$ and $Sp(n)$ cases. The $U(n)$ case is similar.

Lemma 4.3 ([14, 15]). *Assume X_b^n and $\mathcal{R}_b^n$ are as in one of the examples above. Then for any integer, b, there is a complex function, $f : X_b^n \to \mathbf{C}$ such that $\hat{f}(\rho) = 0$. When $\rho \in \mathcal{R}_b^n$ and $\rho \neq 1_G$, and $\hat{f}(1_G) = 1$, where 1_G is the trivial representation.*

Before proving complexity results, we require a lemma bounding the multiplicity of restrictions from subgroups appearing in the examples.

Lemma 4.4. *Assume $G > K$ are two groups appearing in one of the chains (8), (9), (11) or (13), and the number of steps down the chain from G to K is at least 1 in the (8) case, 2 in the (9), (11) cases or 3 in the (8) case. Let $\mathcal{R}_b$ be the set of representations of G defined in the example. Then*

$$\mathcal{M}_K^G(\mathcal{R}_b) \leq 0(b^{\sigma(G,K)}) \tag{14}$$

where $\sigma(G,K) = N_G - N_K - \operatorname{rank} K$, and N_G, N_K are the numbers of positive roots of G and K respectively.

Proof. We first note that if $G > H > K$ are groups from one of the chains and (14) holds for $G|H$ and $G|K$, then

$$\begin{aligned} \mathcal{M}_K^G(\mathcal{R}_b) &\leq \mathcal{M}_H^G(\mathcal{R}_b)\mathcal{M}_K^H((\mathcal{R}_b)_H)|(\mathcal{R}_b)_H| \\ &\leq O(b^{\sigma(G,H)+\sigma(H,K)+\operatorname{rank} H}) \end{aligned}$$

so the result also holds for $G|K$. Similar reasoning shows that if (14) holds for $G|H \times K$ and also for $H|H'$, then it holds for $G|H' \times K$. It clearly holds for $G|1$. In addition it is easy to see that if (14) holds for $H|H'$ and $K|K'$, then it must also hold for $H \times K|H' \times K'$. Taking this all into consideration we see that the only cases we need verify are $SO(n)|SO(n-1), U(n)|U(n-1), SU(n)|S(U_{n-1} \times U_1), Sp(n)|Sp(n-1) \times Sp(1)$, and $Sp(1)|U(1)$. These cases follow from the results in [23]. □

Theorem 4.5. *Let K_n denote one of the series of groups, $SO(n)$, $U(n)$, $SU(n)$, or $Sp(n)$. Then using the sets X_b^n, $\mathcal{R}_b^n$ defined in the main example, we have*

$$T_{X_b^n}(\mathcal{R}_b^n) \leq O\left(b^{\dim K_n + (\dim K_n - \dim K_{n-1}) + \sigma(K_n, K_{n-2})}\right)$$

Proof. We prove this for $SO(n)$; the proofs for the other series of groups are similar. First note that $|X_b^n| = |A_2^2| \dots |A_n^n| \, |X_b^{n-1}|$; though we do not need this property, it does simplify the proof. By theorem 3.6, we have

$$T_{X_b^n}(\mathcal{R}_b^n) \leq \frac{|X_b^n|}{|X_b^2|} T_{X_b^2}(\mathcal{R}_b^2) + \sum_{i=3}^{n} \frac{|X_b^n|}{|X_b^i|} M(\mathcal{R}_b^i, A_2^i \dots A_i^i, K_{i-1})$$

but by corollary 4.2 and lemma 14,

$$\begin{aligned}
&\frac{1}{|X_b^i|} M(\mathcal{R}_b^i, A_2^i \dots A_i^i, K_{i-1}) \\
&\leq \frac{1}{|X_b^i|} \left(\sum_{\rho \in \mathcal{R}_b^i} d_\rho^2 \right) (i-1) \left|A_2^i\right| \dots \left|A_i^i\right| \max_{2 \leq j \leq i} \mathcal{M}_{K_{i-2}}^{K_i}(\mathcal{R}_b^i) \\
&\leq O\left(b^{\dim K_i - \dim K_{i-1} + \sigma(K_i, K_{i-2})}\right)
\end{aligned}$$

Now, $\dim K_i - \dim K_{i-1} + \sigma(K_i, K_{i-2}) = i + \lfloor \frac{i-1}{2} \rfloor - 1$ is an increasing function of i, and $\frac{1}{|X_b^2|} T_{X_b^2}(\mathcal{R}_b^2) \leq O(b)$. Therefore

$$T_{X_b^n} \leq |X_b^n| \, O\left(b^{\dim K_i - \dim K_{i-1} + \sigma(K_i, K_{i-2})}\right)$$

□

Theorem 4.5 does not make full use of the subgroup chains in the examples—it only uses the subgroup chain $K_n \geq K_{n-1} \geq \dots \geq 1$. Making complete use of the chains, (8), (9), (11) or (13), gives slightly better results, which we record in the following theorem. The proof is almost identical to theorem 4.5 with the exception of using a finer subgroup chain. Of course, for $SO(n)$ we have already used the full subgroup chain.

Theorem 4.6. *Assume $n \geq 2$, then*

(i) *For $U(n)$ we have* $T_{X_b^n}(\mathcal{R}_b^n) \leq 0(b^{\dim U(n) + 3n - 3})$
(ii) *For $SU(n)$ we have* $T_{X_b^n}(\mathcal{R}_b^n) \leq 0(b^{\dim SU(n) + 3n - 2})$
(iii) *For $Sp(n)$ we have* $T_{X_b^n}(\mathcal{R}_b^n) \leq 0(b^{\dim Sp(n) + 6n - 6})$

5. Scalar Sums of Products

We shall now use an adapted basis to look at (1) as a scalar equation relating elements of the matrix $\hat{f}(\rho)$ to sums of products of matrix elements. As scalar multiplication is commutative, this gives us more flexibility in choosing the order in which we multiply the matrix elements.

Theorem 5.1. *Let K_n denote one of the series of groups, $SO(n)$, $U(n)$, $SU(n)$, or $Sp(n)$. Define the sets X_b^n, $\mathcal{R}_b^n$ as in the main example of section 4, and let L_n denote the subgroup immediately below K_n in the chain defined for that series. Then*

$$T_{X_b^n}(\mathcal{R}_b^n) \leq O\left(b^{\dim K_n + 1 + \operatorname{rank} K_{n-2} + \sigma(L_n, K_{n-2}) + \sigma(K_{n-1}, K_{n-2})}\right)$$

Before proving the theorem, let's make the statement more explicit. Let

$$\gamma(K_n) = 1 + \operatorname{rank} K_{n-2} + \sigma(L_n, K_{n-2}) + \sigma(K_{n-1}, K_{n-2}).$$

Then γ_{K_n} is given by the following table.

K_n	$SO(n)$	$U(n)$	$SU(n)$	$Sp(n)$
$\gamma(K_n)$	$\lfloor \frac{n}{2} \rfloor$	$n-1$	n	$3n$

In each of these cases $\gamma(K_n)$ is asymptotic to $\sqrt{\alpha \dim K_n}$, where α is $\frac{1}{2}$ for $SO(n)$, α is 1 for $U(n)$ and $SU(n)$, and $\frac{9}{2}$ for $Sp(n)$.

Proof of theorem 5.1. we shall prove the theorem in detail for $SO(n)$ and then indicate how the proof generalizes to the other chains of groups. we start by nothing that

$$(15)\qquad T_{X_b^n}(\mathcal{R}_b^n) \le |X_b^n| \left[T_{X_b^2}(\mathcal{R}_b^2) + \sum_{i=3}^{n} \frac{1}{|X_b^i|} M(\mathcal{R}_b^i, A_2^i, \ldots A_i^i, SO(i-1)) \right]$$

so we shall first concentrate on bounding $M(\mathcal{R}_b^n, A_2^n, \ldots A_n^n, SO(n-1))$, which is the number of operations required to compute the sum

$$\sum_{\substack{a_i \in A_i^n \\ 2 \le i \le n}} \rho(a_2) \ldots \rho(a_n) F(a_2 \ldots a_n, \rho)$$

in an adapted basis, where $F(a_2 \ldots a_n, \rho)$ is a $d_\rho \times d_\rho$ matrix in $(\text{End } V_\rho)_{SO(n-1)}$, and ρ is any representation in $\mathcal{R}_b^n$. An adapted basis for V_ρ is indexed by a chain of representations (see section 2)

$$\rho_n \longleftarrow \rho_{n-1} \longleftarrow \cdots\cdots\cdots \longleftarrow \rho_2 \longleftarrow 0$$

where $\rho_n = \rho$, ρ_i is in $\mathcal{R}_b^i$ and an arrow from ρ_i to ρ_{i+1} indicates that ρ_i occurs in the restriction of ρ_{i+1} to $SO(i)$. We shall indicate such chains of representations by Λ, Λ' etc. Using Schur's lemma it is easy to find the form of the matrices $\rho(a_i)$ in an adapted basis

$$[\rho(a_i)]_{\Lambda,\Lambda'} = \left(\prod_{j \ne i-1} \delta_{\rho_j, \rho'_j} \right) P^i_{\rho_i, \rho_{i-1}, \rho'_{i-1}, \rho_{i-2}}(a_i)$$

where $P^i_{\rho_i,\rho_{i-1},\rho'_{i-1},\rho_{i-2}}$ is a scalar function and by convention we always take $\rho_0 = 0$, $\rho_1 = 0$. Similarly,

$$[F(a_2 \ldots a_n; \rho)]_{\Lambda,\Lambda'} = \delta_{\rho_{n-1},\rho'_{n-1}} F_1(\rho'_{n-1}, \ldots, \rho'_2, \rho_{n-2}, \ldots, \rho_2; a_2, \ldots, a_n)$$

for some scalar function F_1. Therefore (5) becomes

(16)

$$\sum_{\substack{a_i \in A_i^n \\ 2 \le i \le n}} \sum_{\rho'_2, \ldots, \rho'_{n-2}} \left[\prod_{i=2}^{n} P^i_{\rho_i, \rho_{i-1}, \rho'_{i-1}, \rho'_{i-2}}(a_i) \right] F_1(\rho^2_{n-1}, \ldots, \rho^2_2, \rho'_{n-2}, \ldots, \rho'_2; a_2, \ldots, a_n)$$

where we have the convention that $\rho'_{n-1} = \rho^2_{n-1}$, $\rho_n = \rho$, and where the indices $\rho_i, \rho'_i, \rho^2_i$ satisfy the restriction relations of the following diagram

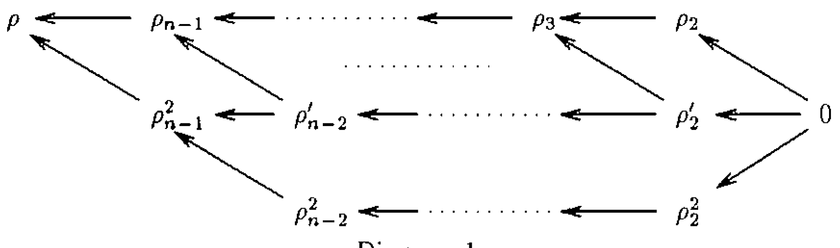


Diagram 1.

we shall calculate the sum (16) in $n-1$ steps as follows. We calculate functions F_i for $2 \le i \le n$ where F_i is given in terms of F_{i-1} by

$$(17)\quad F_i(\rho^2_{n-1}, \dots, \rho^2_2; \rho'_{n-2}, \dots \rho'_{i-1}; \rho_i, \dots, \rho_2; a_{i+1}, \dots, a_n) = \sum_{a_i \in A^n_i} \sum_{\rho'_{i-2}} P^i_{\rho_i, \rho_{i-1}, \rho'_{i-1}, \rho'_{i-2}}(a_i) F_{i-1}(\rho^2_{n-1}, \dots, a_n)$$

The representations appearing as indices in the expression (17) satisfy the relations

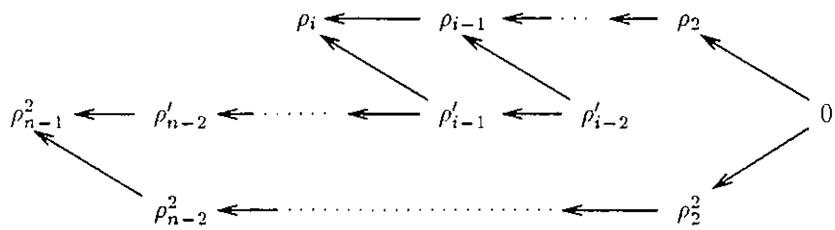


Diagram 2.

when $2 \le i \le n-1$. It is easy to see that F_n is the sum we wish to compute.

The number of operations we take to calculate the inner sum of (17) for any given values of $a_i, \dots, a_n$ and all allowed values of the representations $\rho_j, \rho'_j, \rho^2_j \in \mathcal{R}^j_b$ is the number of ways of filling in Diagram 2 with representations $\rho_j, \rho'_j, \rho^2_j \in \mathcal{R}^j_b$ subject to the restriction relations represented by the arrows in that diagram. TO count the number of ways of filling in this diagram, we first choose representations ρ^2_{n-1}, ρ_i, ρ_{i-1}, ρ'_{i-1}, ρ'_{i-2} subject to the relations

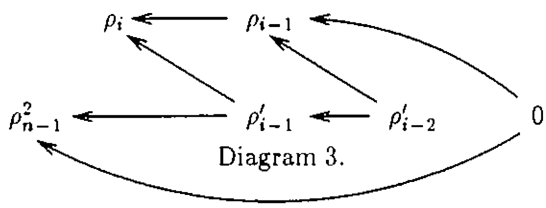


Diagram 3.

The arrows of diagram 3 are in one to one correspondence with chains of arrows in diagram 2[4]. Given a choice of ρ^2_{n-1}, ρ_i, ρ_{i-1}, ρ'_{i-1}, ρ'_{i-2}, the ways of filling in these chains of representations are independent and the number of ways of filling in the chain corresponding to an edge from β to α in diagram 3 is the multiplicity $M_\alpha(\beta)$. Hence the number of operations we take to compute the inner sum given $a_i, \dots, a_n$ for allowed index values is

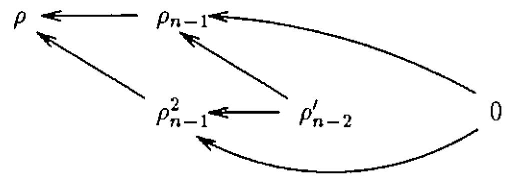
$$(18)\quad \sum_{\substack{\rho^2_{n-1}, \rho_i, \rho_{i-1} \\ \rho'_{i-1}, \rho'_{i-2}}} \prod_{\substack{\text{arrows from } \alpha \\ \text{to } \beta \text{ in diagram 3}}} M_\alpha(\beta)$$

[4]The chains in diagram 2 that we consider are precisely the maximal chains of arrow such that each intermediate point in the chain has exactly one arrow entering it and one arrow leaving it

But $M_{\rho_i}(\rho'_{i-1})$, $M_{\rho'_{i-1}}(\rho'_{i-2})$, $M_{\rho_{i-1}}(\rho'_{i-2})$ and $M_{\rho_i}(\rho_{i-1})$ are each 1 whenever the relations of diagram 3 are satisfied. So the number of operations required to calculate F_i given F_{i-1} is bounded by

$$\begin{aligned} |A_i^n|\ldots|A_n^n| & \sum_{\substack{\rho^2_{n-1},\rho_i,\rho_{i-1}\\ \rho'_{i-1},\rho'_{i-2}}} d_{\rho^2_{n-1}} d_{\rho_{i-1}} M_{\rho^2_{n-1}}(\rho'_{i-1}) \\ \leq\ & O(b^{n-i+1} b^{\text{rank}\,SO(n-1)+\text{rank}\,SO(i)+2\,\text{rank}\,SO(i-1)+\text{rank}\,SO(i-2)} \times \\ & \times b^{\sigma(SO(n-1),SO(i-1))+N_{SO(n-1)}+N_{SO(i-1)}}) \\ =\ & O(b^{\dim SO(n)+\text{rank}\,SO(i)}) \end{aligned}$$

When $i = n$ diagram 3 becomes



and the number of operations required to calculate F_n given F_{n-1} is

$$\begin{aligned} |A_n^n| \sum_{\substack{\rho,\rho_{n-1}\\ \rho^2_{n-1},\rho'_{n-2}}} & d_{\rho^2_{n-1}} d_{\rho_{n-1}} M_{\rho^2_{n-1}}(\rho'_{1-1}) \\ & \leq O(b^{n-i+1} b^{\dim SO(n)+1+\text{rank}\,SO(n-2)+2\sigma(SO(n-1),SO(n-2))}) \\ & = O(b^{\dim SO(n)+\text{rank}\,SO(n)}) \end{aligned}$$

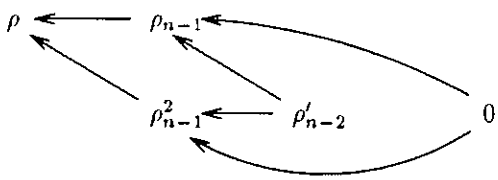
From this we wee that

$$M(\mathcal{R}_b^n, A_2^n \ldots A_n^n, SO(n-1)) \leq \sum_{i=2}^{n} O(b^{\dim SO(n)+\text{rank}\,SO(i)}).$$

Substituting this bound into (15) gives

$$T_{X_b^n}(\mathcal{R}_b^n) \leq \sum_{i=2}^{n} O(b^{\dim SO(n)+\text{rank}\,SO(i)}) = O(b^{\dim SO(n)+\text{rank}\,SO(n)})$$

This proves the theorem for $K_n = SO(n)$. The method of the theorem generalizes to the other chains of groups transparently (even when the restrictions in the chain are not multiplicity 1). The complexity is bounded by a sum of terms associated with different diagrams generalizing diagram 3. The dominant term comes from a diagram of the form



where each ρ is a representation in $\mathcal{R}^n_b$, ρ_{n-1} is in $\mathcal{R}^{n-1}_n$, ρ'_{n-2} is in $\mathcal{R}^{n-2}_b$, and ρ^2_{n-1} is a representation in $(\mathcal{R}^n_b)_{L_{n-1}}$. Using (18 this dominant term is easily bounded by

$$O\left(b^{\dim K_n+1+\operatorname{rank} K_{n-2}+\sigma(L_n,K_{n-2})+\sigma(K_{n-1},K_{n-2})}\right)$$

□

A method for obtaining the functions $P^i_{\rho_i,\rho_{i-1},\rho'_{i-1},\rho_{i-2}}$ is given in [13], and in the case of unitary groups explicit expressions are given ([13] p. 395).

6. Homogeneous spaces

We shall now consider a problem analogous to the computation of the sum (1) or of the introduction, but on a homogeneous space.

Assume φ is a distribution on the compact Lie group, G, and ρ is a finite dimensional representation of G. Then define the Fourier transform of φ at ρ to be $\hat{\varphi}(\rho)$, where

$$\langle v^*, \hat{\varphi}(\rho)v\rangle = \langle \varphi, (x \mapsto \langle v^*, \rho(x)v\rangle)\rangle$$

for any v in V_ρ, v^* in V^*_ρ, and where $\langle\ ,\ \rangle$ denotes either the dual pairing between V_ρ and V^*_ρ or between functions and distributions. This transform is also denoted $\rho(\varphi)$. This definition is simply related to the sum (1), for if X is a finite subset of G, f is a complex function on X, and we define a distribution $f_\delta = \sum_{x\in X} f(x)\delta_x$, then using the notation of (1), $\hat{f} = \hat{f}_\delta$.

Assume now that K is a closed subgroup of G. Then we may project functions on G to functions on G/K by integrating along the fibers of the canonical projection $G \longrightarrow G/K$. The dual of this projection defines an injection, $i_{G/K}$ from distributions on G/K to distributions on G. If $\bar{Y}$ is a finite subset of G/K and g is a complex function on $\bar{Y}$, then define $g_\delta = \sum_{\bar{y}\in\bar{Y}} g(\bar{y})\delta_{\bar{y}}$, and

$$\hat{g} = \hat{g}_{\bar{Y}} = i_{G/K}(\hat{g}_\delta)$$

When $\mathcal{R}$ is a set of matrix representations of G, we let $T_{\bar{Y}}(\mathcal{R})$ be the number of operations required to compute $\hat{g}(\rho)$ for all ρ in $\mathcal{R}$, given any complex function, g on $\bar{Y}$.

Now we shall relate $\hat{g}$ to a sum with a form we have already met. For any representation, ρ, we define V^K_ρ to be the set of K-invariant vectors in V_ρ, and let $P^K_\rho = \int_K \rho(k)dk$ be the canonical projection from V_ρ onto V^K_ρ. we say that ρ is class 1 with respect to K if V^K_ρ is nonzero, and for any set of representations, $\mathcal{R}$, we let $\mathcal{R}^K$ denote the set of representations in $\mathcal{R}$ that are class 1. If f is a complex function on a finite subset $Y \subset G$, then we define

$$\hat{f}^K(\rho) = \hat{f}.P^K_\rho = \sum_{y\in Y} \rho(y)\left[f(y)P^K_\rho\right] \tag{19}$$

When $\mathcal{R}$ is a set of matrix representations, we define $T_Y^K(\mathcal{R})$ to be the number of operations required to compute $\hat{f}^K(\rho)$ for all ρ in $\mathcal{R}$, given any complex function, f, on Y. The following properties are trivial.

Lemma 6.1. (i) *$\hat{f}^K(\rho)$ lies in $\mathrm{Hom}(V_\rho^K; V_\rho)$.*

(ii) *If ρ is not class 1 with respect to K, then $\hat{f}^K(\rho) = 0$.*

(iii) *Assume $\bar{Y}$ is a finite subset of G/K, g is a complex function on $\bar{Y}$, Y is a set of coset representatives for the cosets in $\bar{Y}$, and f is the complex function on Y defined by $f(y) = g(yK)$ for any y on Y. Then*

$$\hat{g}(\rho) = \hat{f}^K(\rho)$$

Corollary 6.2. (i) *Assume $\bar{Y}$ is a finite subset of G/K and Y is a set of coset representatives for $\bar{Y}$. Then $T_{\bar{Y}}(\mathcal{R}) = T_Y^K(\mathcal{R})$*

(ii) $T_Y^K(\mathcal{R}) = T_Y^K(\mathcal{R}^K)$.

The sum (19) used to define $\hat{f}^K(\rho)$ is of the same form as (6) considered in section 4—simply substitute $f(y)P_\rho^K$ for $F(y)$. It is easy to see that the techniques of section 4 do not give an efficient way of computing the sum (19), but under the right conditions on the set, Y, the techniques of section 5 do give good results.

For each of the classical groups considered in section 4 we define sets Y_b^n in terms of the main examples.

$$SO(n)Y_b^n = A_2^n \dots A_n^n \tag{20}$$

$$U(n)\ Y_b^n = B_1^n \dots B_{n-1}^n A_2^n \dots A_n^n \tag{21}$$

$$SU(n)Y_b^n = B_1^n \dots B_{n-1}^n A_2^n \dots A_n^n \tag{22}$$

$$Sp(n)\ Y_b^n = (B_1^{n,2} C_1^n B_1^{n,1} A_2^n) \dots (B_{n-1}^{n,2} C_{n-1}^n B_{n-1}^{n,1} A_n^n) \tag{23}$$

where the sets A_i^n, B_i^n, C_i^n. $B_i^{n,1}$, $B_i^{n,2}$ are defined for each of the series of groups according to the definitions in section 4. For these subsets of the classical groups we may use the argument of the proof of theorem 5.1 to bound $T_{Y_b^n}^{L_n}(\mathcal{R}_b^n)$.

Theorem 6.3.

$$\begin{array}{lll}
\text{(i)} & T_{Y_b^n}^{SO(n-1)}(\mathcal{R}_b^n) & \leq\ O(b^{\dim[SO(n)/SO(n-1)]+1}) \\
\text{(ii)} & T_{Y_b^n}^{U_{n-1}\times U_1}(\mathcal{R}_b^n) & \leq\ O(b^{\dim[U(n)/(U_{n-1}\times U_1)]+1}) \\
\text{(iii)} & T_{Y_b^n}^{S(U_{n-1}\times U_1)}(\mathcal{R}_b^n) & \leq\ O(b^{\dim[SU(n)/S(U_{n-1}\times U_1)]+2}) \\
\text{(iv)} & T_{Y_b^n}^{Sp(n-1)\times Sp(1)}(\mathcal{R}_b^n) & \leq\ O(b^{\dim[Sp(n)/(Sp(n-1)\times Sp(1))]+3})
\end{array}$$

Remark. The results stated for $SU(n)$ and $Sp(n)$ are clearly not as good as those for $SO(n)$ and $U(n)$. This is an indication that the indexing for adapted bases of class 1 representations implicit in the proof is not completely appropriate. In the case of $SU(n)/S(U_{n-1} \times U_1)$ we shall give explicit formulae (see section 7) for the associated spherical functions that allow us to improve this bound. Such formulae also exist for $Sp(n)/(Sp(n-1) \times Sp(1))$, though we do not treat this case here.

Proof of theorem 6.3. As previously mentioned, the proof is simply an adaptation of the proof of theorem 5.1. We shall present it in the case of $SO(n)$ and then briefly indicate how to generalize to the other classical groups.

Using the same notation for indexing adapted bases as in 5.1, we see that

$$\left[P_\rho^{SO(n-1)}\right]_{\Lambda,\Lambda'} = \delta_{\rho_{n-1},0}\delta_{\rho'_{n-1},0}$$

and hence that

$$\left[\hat{f}^{SO(n-1)}(\rho)\right]_{\Lambda,\Lambda^2} = \delta_{\rho^2_{n-1},0} \sum_{\substack{a_i \in A_i^n \\ 2\le i\le n}} \left[\prod_{i=2}^{n} P^i_{\rho_i,\rho_{i-1},0,0}(a_i)\right] f(a_2\dots a_n) \tag{24}$$

Only the case $\rho^2_{n-1} = 0$ is important; in this case the sum (24) is exactly the same as (16) with $\rho^2_{n-1}, \dots, \rho^2_2, \rho'_{n-1}, \dots, \rho'_2$ all set to the trivial representation, 0, and $F_1(0,\dots,0,a_2\dots a_n)$ equal to $f(a_2\dots a_n)$. Thus we may calculate this sum in $n-1$ steps by defining

(25)

$$F_i(\rho_i,\dots,\rho_2;a_{i+1}\dots a_n) = \sum_{a_i\in A_i^n}\prod_{i=2}^{n} P^i_{\rho_i,\rho_{i-1},0,0}(a_i)F_{i-1}(\rho_{i-1},\dots,\rho_2;a_i\dots a_n)$$

Diagram 3, which gives the restriction relations between representations occurring in the sum becomes

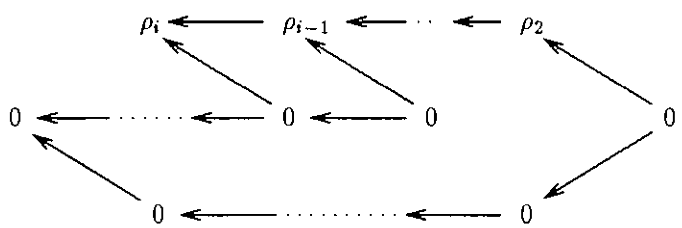


Diagram 4.

So ρ_i and ρ_{i-1} are class 1 relative to $SO(i-1)$, $SO(i-2)$ respectively. The number of operations required to obtain F_i from F_{i-1} is bounded by

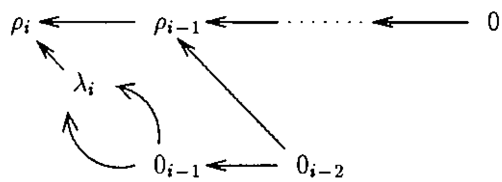
$$\begin{aligned} & |A_i^n|\dots|A_n^n| \sum_{\rho_i\in(\mathcal{R}_b^i)^{SO(i-1)}} d_{\rho_i} \\ \le\ & O(b^{n-i+1}b^{\dim SO(i)/SO(i-1)}) \\ =\ & O(b^n) = O(b^{\dim SO(n)/SO(n-1)+1}) \end{aligned}$$

Therefore

$$T_{Y_b^n}^{SO(n-1)}(\mathcal{R}_b^n) \le \sum_{k=2}^{n} O(b^{\dim SO(n)/SO(n-1)+1}) = O(b^{\dim SO(n)/SO(n-1)+1})$$

This proves the result for $SO(n)/SO(n-1)$. For the other classical groups the proof is similar. For $U(n)/(U_{n-1}\times U_1)$ and $SU(n)/S(U_{n-1}\times U_1)$ the most important diagrams to consider have the same form as diagram 4. For $Sp(n)$ one should sum over sets in the order: $B_1^{n,1}$, $B_2^{n,1}$, C_1^n; $B_2^{n,1}$, $B_2^{n,2}$, C_1^n, A_2^n, $B_3^{n,1}$, ...,

A^n_{n-1}. The diagrams corresponding to the dominant terms in the complexity sums have the form



where λ_i is a representation of $Sp(i-1) \times Sp(1)$ and 0_j denotes the trivial representation of $Sp(j)$. □

7. Fast Transforms on Spheres

The proof of theorem 6.3 works by factoring the associated spherical functions of a homogeneous space and then calculating the sum for $\hat{f}^K(\rho)$ one factor at a time. For the spaces $SO(n)/SO(n-1)$, $U(n)/U(n-1)$, $SU(n)/SU(n-1)$ and $Sp(n)/Sp(n-1)$, the associated spherical functions may be written, in a polyspherical coordinate system, as products of complex exponentials and Jacobi polynomials. In terms of the proof of theorem 6.3, we will try to use explicit expressions for the functions $P^i_{\rho_i,\rho_{i-1},0,0}$ to compute the sum (25) efficiently.

We shall develop fast transforms for the associated spherical functions of the spaces $SO(n)/SO(n-1)$ and $SU(n)/SU(n-1)$. The key to performing fast transforms on these homogeneous spaces is the use of efficient algorithms to project functions of one variable onto spaces of polynomials. These have been developed by Driscoll and Healy.

The techniques of Driscoll and Healy rely on the three term recurrence relation satisfied by a orthogonal polynomial sequence.

Theorem 7.1 (Driscoll, Healy, Rockmore). *Assume $\{\Phi_l : 0 \leq l < n\}$ is a set of complex functions defined at the points $z_0, \dots, z_{n-1} \in \mathbf{C}$ and satisfying a three term recurrence relation, at these points, of the form $\Phi_{l+1} = (a_l x + b_l)\Phi_l + c_l\Phi_{l-1}$, for $1 \leq l < n-1$, and that $f : \{0, \dots, n-1\} \longrightarrow \mathbf{C}$. Then the following computations can be completed in $O(n(\log n)^2)$ operations given a precomputed data structure of size $O(b(\log b)^2)$.*

(a) *Calculation of the sums $\sum_{j=0}^{n-1} f(j)\Phi_l(z_j)$ for $0 \leq l < n$,*
(b) *Calculation of the sums $\sum_{j=0}^{n-1} f(j)\overline{\Phi_l(z_j)}$ for $0 \leq l < n$.*

For the proof of this result see [7] and [8], or [16].

Example: The Fast Transform on S^{n-1}. The Gel'fand basis vectors for the class one representation, $\Delta_{(m,0,\dots)}$ of $SO(n)$ relative to $SO(n-1)$ are determined by the integers $m_n = m_{1,n}, \dots, m_2 = m_{1,2}$ which satisfy the constraints

$$m = m_n \geq \dots \geq m_3 \geq |m_2|$$

Let us denote the corresponding Gel'fand basis vector by $\mathbf{M}$, and the $SO(n-1)$-invariant basis vector for which $m_n = m, m_{n-1} = \dots = m_2 = 0$, by $\mathbf{0}_m$. The matrix coefficients that appear in the Plancherel formula for S^{n-1} can be written, with respect to this basis, in the following form

$$\langle \Delta_{(m,0,\ldots)}(r_2(\theta_2)\cdot\ldots\cdot r_n(\theta_n)g)0_m, \mathbf{M}\rangle = A^n_{\mathbf{M}}\left[\prod_{j=3}^{n} C^{\frac{j-2}{2}+|m_{j-1}|}_{m_j-|m_{j-1}|}(\cos\theta_j)\sin^{|m_{j-1}|}(\theta_j)\right] e^{im_2\theta_2}$$

where g can be any element of $SO(n-1)$, and C^p_k is a Gegenbauer polynomial.

$$A^n_{\mathbf{M}} = \sqrt{\frac{m_n!\Gamma(n-1)}{(2m_n+n-2)}} \times \sqrt{\frac{1}{\Gamma(\frac{n}{2})}\prod_{j=3}^{n}\frac{2^{2|m_{j-1}|+j-4}(m_j-|m_{j-1}|)!(j+2m_j-2)(\Gamma(\frac{j-2}{2}+|m_{j-1}|))^2}{\sqrt{\pi}\Gamma(m_j+|m_{j-1}|+j-2)}}$$

is a normalization constant. A proof of this formula may be found in [21].

Given such an explicit expression for the spherical functions, it is now straightforward to improve the algorithm of section 6 for a Fourier transform on S^{n-1}.

Theorem 7.2. *Assume that $\theta_{0,2},\ldots,\theta_{2b,2}$ are points in $[0,2\pi)$, and that for $2<k\leq n$, $\theta_{0,k},\ldots,\theta_{b,k}$ are points in $[0,\pi]$. Then for any complex function, f, defined on $\{0,\ldots,2b\}\times\{0,\ldots,b\}^{n-2}$ the sums*

$$\sum_{j_2=0}^{2b}\sum_{j_3,\ldots,j_n=0}^{b} f(j_2,\ldots,j_n)\left[\prod_{k=3}^{n} C^{\frac{k-2}{2}+|m_{k-1}|}_{m_k-|m_{k-1}|}(\cos\theta_{j_k,k})\sin^{|m_{k-1}|}(\theta_{j_k,k})\right] e^{im_2\theta_{j_2,2}}$$

can be computed for all $m_n,\ldots,m_2$ with $b\geq m_j-|m_{j-1}|\geq 0$ for all j, in a total of $O(b^{n-1}(\log b)^2)$ operations, given a precomputed data structure of size $O(b^2(\log b)^2)$.

Proof. For $2\leq l\leq n$, let us define the partial transform,

$$f^{m_2,\ldots m_l}(j_{l+1},\ldots,j_n) = \sum_{j_2=0}^{2b}\sum_{j_3,\ldots,j_l=0}^{b} f(j_2,\ldots,j_l)\times \left[\prod_{k=3}^{l} C^{\frac{k-2}{2}+|m_{k-1}|}_{m_k-|m_{k-1}|}(\cos\theta_{j_k,k})\sin^{|m_{k-1}|}(\theta_{j_k,k})\right] e^{im_2\theta_{j_2,2}}$$

Then f^{m_2} can be obtained from f, for all $|m_2|\leq b$, by a normal fast Fourier transform on the j_2 variable, in $b^{n-2}O(b\log b)$ operations. The Gegenbauer polynomials satisfy the recurrence relations

$$C^\lambda_{m+1}(x) = \frac{2(\lambda+m)}{m+1}xC^\lambda_m - \frac{2\lambda+m-1}{m+1}C^\lambda_{m-1}$$

so by theorem7.1, we can compute $f^{m_2,\ldots m_{k+1}}$, for all $m_{k+1},\ldots,m_2$ with $b\geq m_j-|m_{j-1}|\geq 0$, starting with the data $f^{m_2,\ldots,m_k}.\sin^{|m_k|}(\theta_{j_k,k})$, by a one variable polynomial transform, in $O(b^{n-1}(\log b)^2)$ operations. After $n-2$ such steps, we have computed $f^{m_2,\ldots,m_n}$ in the desired number of operations. □

Corollary 7.3. $T^{SO(n-1)}_{Y^n_b}(\mathcal{R}^n_b)\leq O(b^{\dim S^{n-1}}(\log b)^2) = O(b^{n-1}(\log b)^2)$

Example: The Fast Transform on $SU(n)/SU(n-1)$. We shall identify $SU(n)/SU(n-1)$ with the $(2n-1)$-sphere, S^{2n-1}. The Gel'fand basis vectors for class one representations of $SU(n)$ relative to $SU(n-1)$ are determined by integers $m_n, \ldots, m_2, \nu_n, \ldots, \nu_2, \lambda_1$, which satisfy

$$m_n \geq \ldots \geq m_2 \geq 0; \quad \nu_n \geq \ldots \geq \nu_2 \geq 0$$

$$m_2 + \nu_2 \geq |\lambda_1| \text{ and } m_2 + \nu_2 + \lambda_1 \text{ even.}$$

These are related to the $m_{i,j}$ of the section 4 dealing with $SU(n)$, by

$$m_{1,k} = m_k + \nu_k,\ m_{2,k} = \nu_k, \ldots,\ m_{k-1,k} = \nu_k$$
$$m_{k,k} = k(m_{k+1} - \nu_{k+1}) - (k+1)(m_k - \nu_k)$$

for $2 \leq k < n$, and $m_{1,1} = \lambda_1$.

Even though we are using Gel'fand bases coming from a chain of subgroups for $SU(n)$, it is convenient for us to parametrize S^{2n-1} using the action of $U(n)$, and the isomorphism $S^{2n-1} = U(n)/U(n-1)$. This parametrizatation is merely a notational convenience, and is simply related to one coming from the action of $SU(n)$, see [14] for an explanation.

In order to write down the associated spherical functions for this homogeneous space, let us denote the Gel'fand basis vector corresponding to the integers $m_n, \ldots m_2, \nu_n, \ldots, \nu_2, \lambda_1$, by $\mathbf{M}$ and the $SU(n-1)$-invariant basis vector by $\mathbf{0}_n$. Let T be the standard maximal torus of $U(n)$, let $\hat{T}$ be the dual group of T, and let $\tilde{A}_j : \hat{T} \longrightarrow \mathbf{Z}$ be the j-th coordinate function with respect to this basis. Let $T_{\mathbf{M}\mathbf{0}_n}$ be the associated spherical function corresponding to the $\mathbf{M}, \mathbf{0}_n$ matrix element. Then,

$$T_{\mathbf{M}\mathbf{0}_n}(t.r_2(\theta_2)\ldots r_n(\theta_n)\begin{pmatrix} \vdots \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} n-2+\min\{m_n, \nu_n\} \\ n-2 \end{pmatrix} A^n_{\mathbf{0}_n} A^n_{\mathbf{M}} \times \varphi_{\mathbf{M}}(t)$$
$$\times \prod_{j=2}^{n} (\cos\theta_j)^{A_j} (\sin\theta_j)^{B_j} P_{k_j}^{j-2+B_j, A_j}(\cos 2\theta_j)$$

for $t \in T$, where $\varphi_{\mathbf{M}}$ is the element of $\hat{T}$ defined by

$$\begin{aligned} \tilde{A}_j(\varphi_{\mathbf{M}}) &= (m_j - \nu_j) - (m_{j-1} - \nu_{j-1}) \text{ for } j \geq 3 \\ \tilde{A}_2(\varphi_{\mathbf{M}}) &= \frac{1}{2}(m_2 - \nu_2 + \lambda_1) \\ \tilde{A}_1(\varphi_{\mathbf{M}}) &= \frac{1}{2}(m_2 - \nu_2 - \lambda_1) \end{aligned}$$

and

$$k_j = \min\{m_j - m_{j-1}, \nu_j - \nu_{j-1}\},\ \text{for } j \geq 3, \qquad k_2 = \frac{1}{2}(\lambda_1 + m_2 + \nu_2)$$
$$A_j = \left|\tilde{A}_j(\varphi_{\mathbf{M}})\right|,\ \text{for } j \geq 3, \qquad A_2 = -\tilde{A}_2$$
$$B_j = m_{j-1} + \nu_{j-1}, \qquad B_2 = \tilde{A}_1(\varphi_{\mathbf{M}})$$
$$= 2k_{j-1} + A_{j-1} + B_{j-1},\ \text{for } j \geq 3$$

The constant A^n_{M} can be written as

$$A^n_{\mathrm{M}} = \sqrt{\frac{1}{2^{n-1}(n-1)!}\prod_{j=2}^{n}\left[(B_{j+1}+j-1)\frac{\begin{pmatrix} B_{j+1}-k_j+j-2 \\ A_j \end{pmatrix}}{\begin{pmatrix} A_j+k_j \\ A_j \end{pmatrix}}\right]}$$

These formulae are proven in the thesis [14] (see also [12] for the unitary group case). Note that if $b \geq m_n, \nu_n$, then $\left|\tilde{A}_j(\varphi_{\mathrm{M}})\right| \leq b$ and $0 \leq k_j \leq b$, for $1 \leq j \leq n$, but also that $k_2 \geq \max\{-A_2, -B_2, -(A_2+B_2), 0\}$. Define a subset of $\hat{T}$, by $\Gamma_b = \{\phi \in \hat{T} : \left|\tilde{A}_j(\phi)\right| \leq b, \text{ for } 1 \leq j \leq n\}$.

Theorem 7.4. *Assume $x_0, \ldots, x_{K-1}$ are points on T, such that for any function $h : \{0, \ldots, K-1\} \longrightarrow \mathbf{C}$ we can compute $\sum_j h(j)\phi(x_j)$ for all characters, ϕ, of T in Γ_b in a total time $O(b^n(\log b)^2)$. Assume that for $2 \leq k \leq n$, we have chosen points $\theta_{0,k}, \ldots, \theta_{\lfloor \frac{b}{2} \rfloor, k}$ in $[0, \frac{\pi}{2}]$. Then for any complex function, f, defined on $\{0, \ldots, K-1\} \times \{0, \ldots, \lfloor \frac{b}{2} \rfloor\}^{n-1}$, the sums*

$$\sum_{j_1=0}^{K-1} \sum_{j_2,\ldots,j_n=0}^{[\frac{b}{2}]} f(j_1, \ldots, j_n)\varphi_{\mathrm{M}}(x_{j_1}) \prod_{i=2}^{n} (\cos\theta_i)^{A_i}(\sin\theta_{j_i,i})^{B_i} P_{k_i}^{i-2+B_i, A_i}(\cos 2\theta_{j_i,i})$$

can be computed for all $m_n \geq \ldots \geq m_2 \geq 0$, $\nu_n \geq \ldots \geq \nu_2 \geq 0$, $m_2 + \nu_2 \geq |\lambda_1|$, and $m_2 + \nu_2 + \lambda_1$ even, in a total of $O(b^{2n-1}(\log b)^2)$ operations.

Proof. We proceed in a similar way to the S^{n-1} case. However, we shall use the transform variables $\phi = \varphi_{\mathrm{M}}$, and $k_2, \ldots, k_n$. It is easy to write the variables m_i, ν_i and λ_1 in terms of ϕ and the k_i, and the calculation of one set of variable from the other can be performed in a fixed number of operations depending only on n. To perform the transform, first do an abelian Fourier transform with respect to the x_j transforming to the variable, ϕ. By hypothesis, this can be done for fixed $k_2, \ldots, k_n$, in $O(b^n(\log b)^2)$ operations. Do this for all necessary values of $k_2, \ldots, k_n$. Now transform to the variables $k_2, \ldots, k_n$ in that order, making use of the recurrence relations for Jacobi polynomials at each step. The only difficult step is the transform to k_2, the problem being that A_2 and B_2 could be negative. But, we may restrict k_2 so that $k_2 \geq \max\{-A_2, -B_2, -(A_2+B_2), 0\}$, and for such values of k_2, the Jacobi polynomials are well defined and have a nontrivial recurrence relation. □

The following corollary and theorem are now immediate.

Corollary 7.5. $T^{SU(n-1)}_{Y^n_b, B^n_n}(\mathcal{R}^n_b) \leq O(b^{\dim S^{2n-1}}(\log b)^2) = O(b^{2n-1}(\log b)^2)$

Theorem 7.6. $T^{S(U_{n-1}\times U_1)}_{Y^n_b}(\mathcal{R}^n_b) \leq O(b^{\dim \mathbf{CP}^{n-1}}(\log b)^2) = O(b^{2n-2}(\log b)^2)$

A result analogous to theorem 7.4 for the homogeneous space $U(n)/U(n-1)$ has an almost identical proof, as the spherical functions for that case differ in only one factor. A similar argument also works for the space $Sp(n)/Sp(n-1)$ using the expression for the spherical functions given on p. 400 [12].

8. Fourier Transforms of Distributions

Assume $\mathcal{D}$ is a finite set of distributions on G, and $f : \mathcal{D} \longrightarrow \mathbf{C}$. We now consider the computation of

$$\hat{f}(\rho)_{\mathcal{D}} = \sum_{D \in \mathcal{D}} f(D)\rho(D) \tag{26}$$

for all representations, ρ, in $\mathcal{R}$, where $\mathcal{R}$ is a finite set of finite dimensional representations of G.

The arguments of sections 3, 4, 5, and 6 generalize to this new situation; one simply replaces group elements, x, by distributions, D, and replaces multiplication of groups elements by convolution of distributions. We say that a distribution, D, commutes with a group element, k, if $\delta_k * D = D * \delta_k$, and we say D commutes with a subgroup if it commutes with all elements of that subgroup.

Lemma 8.1. *Assume D is a distribution on G, K is a closed subgroup of G, and ρ is a finite dimensional representation of G.*

(i) *If D is in $C^\infty(K)'$[5], then $\rho(D)$ is in $\mathrm{span}_{\mathbf{C}}(\rho(K)$.*
(ii) *If D commutes with K, then $\rho(D)$ commutes with each element of $\rho(K)$.*

Therefore, when K is a closed subgroup of G, the appropriate replacements for the two conditions, 'x is a member of K' and 'x commutes with all elements of K', are 'D is an element of $C^\infty(K)'$ ' and 'D commutes with K'.

We now generalize the definitions of $T_X(\mathcal{R})$ and $M(\mathcal{R}, Y, K)$ so that the statement of theorems in sections 3-6 remain unchanged when we replace group elements by distributions. For a set of distributions, $\mathcal{D}$, and a set of matrix representations, $\mathcal{R}$, of G, we let $T_{\mathcal{D}}(\mathcal{R})$ be the minimum number of operations needed to compute $\hat{f}(\rho)_{\mathcal{D}}$ for all ρ in $\mathcal{R}$, at any given function $f : \mathcal{R} \longrightarrow \mathbf{C}$. If K is a closed subgroup of G, we let $T^K_{\mathcal{D}}(\mathcal{R}) = T^K_{\mathcal{D}*\{c_K\}}(\mathcal{R})$, where c_K is the characteristic distribution of K. We also let $M(\mathcal{R}, \mathcal{D}, K)$ denote the number of operations needed to compute

$$\sum_{D \in \mathcal{D}} \rho(D).F(D, \rho)$$

at each ρ in $\mathcal{R}$, where for each ρ in $\mathcal{R}$ and D in $\mathcal{D}$, $F(D, \rho)$ is a $d_\rho \times d_\rho$ matrix in $\mathrm{span}_{\mathbf{C}}\, \rho(K)$. The generalization of theorem 3.3 becomes

Theorem 8.2. *Assume G is a compact Lie group, K is a closed subgroup of G, $\mathcal{D} \subset \mathcal{A} * \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are sets of distributions on G and $\mathcal{B} \subset C^\infty(K)'$. Assume that $\mathcal{R}$ is a K-adapted set of matrix representations of G. Then*

$$T_{\mathcal{D}}(\mathcal{R}) \leq |\mathcal{A}|\, T_{\mathcal{B}}(\mathcal{R}_K) + M(\mathcal{R}, \mathcal{A}, K)$$

[5] $C^\infty(K)'$ is the space of all distributions on K, and may be embedded in $C^\infty(G)'$ using the mapping on distributions induced from the embedding of K in G

Distributions on the classical groups. In order to complete the translation of the results on classical groups into this new context, we need to redefine the sets A_k^n, B_k^n, C_k^n and $B_k^{n,i}$ of the main example, section 4. We now choose all these sets to be sets of distributions, each with $O(b)$ elements, ans satisfying the properties which follow. If we denote the series of classical groups under consideration K_n, then we require that all elements of A_i^n are distributions on K_i which commute with K_{i-2}. We also make the following assumptions in each case.

- $U(n)$ The elements of B_k^n should be distributions on $U_{k-1} \times U_1$ that commute with U_{k-1}.
- $SU(n)$ The elements of B_k^n should be distributions on $S(U_k \times U_1)$ that commute with $SU(k-1)$.
- $Sp(n)$ The elements of $B_k^{n,1}$ and $B_k^{n,2}$ should be distributions on $Sp(k-1) \times U_1$ that commute with $Sp(k-1)$. The elements of C_k^n should be distributions on $Sp(k-1) \times Sp(1)$ that commute with $Sp(k-1)$.

Define the sets X_b^n, $X_b^{n,i}$, and Y_b^n as before, but replacing multiplication of group elements by convolution of distributions. Under these assumptions the following result holds.

Theorem 8.3. *With the new definitions above, the statements of theorem 4.6, theorem 5.1 and theorem 6.3 still hold.*

In a further generalization, all these theorems can be translated into statements about countable chains of multi-matrix algebras.

Special functions and distributions. To generalize the results of section 7 we need to be more careful. The following theorems treat the case of finitely supported distributions on the spheres S^{n-1} or S^{2n-1}.

Theorem 8.4. *Let D be a linear differential operator on $C^\infty((0,2\pi) \times (0,\pi)^{n-2})$, let $\theta_{0,2}, \dots, \theta_{2b,2}$ be points in $(0,2\pi)$, and for $2 < k \leq n$, let $\theta_{0,k}, \dots, \theta_{b,k}$ be points in $(0,\pi)$. Then for any complex function, f, defined on $\{0,\dots,2b\} \times \{0,\dots,b\}^{n-2}$ the sums*

$$\sum_{j_2=0}^{2b} \sum_{j_3,\dots,j_n=0}^{b} f(j_2,\dots,j_n) \times$$
$$\times D\left[\prod_{k=3}^{n} C_{m_k-|m_{k-1}|}^{\frac{k-2}{2}+|m_{k-1}|}(\cos\theta_k)\sin^{|m_{k-1}|}(\theta_k)\right] e^{im_2\theta_2}\Big|_{(\theta_{j_2,2},\dots,\theta_{j_n,n})}$$

can be computed for all $m_n,\dots,m_2$ with $b \geq m_j - |m_{j-1}| \geq 0$ for all j, in a total of $O(b^{n-1}(\log b)^2)$ operations, given a precomputed data structure of size $O(b^2(\log b)^2)$. In addition, these bounds can be chosen to depend only on b, n, and the order of D.

Proof. For clarity let's look at the case where $n=3$ and the order of D is 1. In this case, D has the form $D = a_1(\theta_2,\theta_3) + a_2(\theta_2,\theta_3)\frac{\partial}{\partial\theta_2} + a_3(\theta_2,\theta_3)\frac{\partial}{\partial\theta_3}$, and so the sum we need to compute becomes

$$(27)\quad \sum_{j_2,j_3} f(j_2,j_3)\big[a_1(\theta_{j_2,2},\theta_{j_3,3})A_0(\theta_{j_3,3}) + a_3(\theta_{j_2,2},\theta_{j_3,3})A_1(\theta_{j_3,3}) + a_3(\theta_{j_2,2},\theta_{j_3,3})A_2(\theta_{j_3,3}) + a_2(\theta_{j_2,2},\theta_{j_3,3})im_2A_3(\theta_{j_3,3})\big]e^{im_2\theta_2}$$

where

$$\begin{aligned} A_1(\theta) &= -(2\,|m_2|+1)(\sin^{|m_2|+1}\theta)C^{\frac{3}{2}+|m_2|}_{m_3-|m_2|-1}(\cos\theta) \\ A_2(\theta) &= |m_2|\,(\sin^{|m_2|-1}\theta)(\cos\theta)C^{\frac{1}{2}+|m_2|}_{m_3-|m_2|}(\cos\theta) \\ A_3(\theta) &= (\sin^{|m_2|}\theta)C^{\frac{1}{2}+|m|}_{m_3-|m_2|}(\cos\theta) \end{aligned}$$

The sum (27) splits into 4 sums corresponding to the 4 terms within the large brackets. By the same arguments as used for theorem 7.2 we see that each of these four sums may be computed in $O(b^2(\log b)^2)$ operations. In the general case, D has the form

$$D = \sum_\alpha a_\alpha(\theta_2,\dots,\theta_n)\frac{\partial^{|\alpha|}}{(\partial\theta)_\alpha}$$

where α runs over all multi-indices of order at most the order of D. Hence, if l denotes the order of D, then using Leibnitz's rule the sum we need to compute may be broken up into less than $(l+1)^n2^l$ sums, each of which may be computed in $O(b^{n-1}(\log b)^2)$ operations.[6] □

Corollary 8.5. *Let Y_b^n be the subset of $SO(n)$ defined by (20), section 6, and assume $\mathcal{Y}_b^n$ is a set consisting of a distribution supported on p for each point, p, of Y_b^n. Then*

$$T^{SO(n-1)}_{\mathcal{Y}_b^n}(\mathcal{R}_b^n) \le O(b^{\dim S^{n-1}}(\log b)^2) = O(b^{n-1}(\log b)^2)$$

In addition these bounds can be chosen to depend only on b, n, and the maximum order of the distributions.

Theorem 8.6. *Assume D is a linear differential operator on $C^\infty(T\times(0,\pi)^{n-1})$ and $x_0,\dots,x_{K-1}$ are points on T such that for any function $h:\{0,\dots,K-1\}\longrightarrow \mathbf{C}$ we can compute $\sum_j h(j)\phi(x_j)$ for all characters, ϕ, of T in Γ_b in a total time $O(b^n(\log b)^2)$. Assume that for $2\le k\le n$, we have chosen points $\theta_{0,k},\dots,\theta_{\lfloor\frac{k}{2}\rfloor,k}$ in $(0,\frac{\pi}{2})$. Then for any complex function, f, defined on $\{0,\dots,K-1\}\times\{0,\dots,\lfloor\frac{b}{2}\rfloor\}^{n-1}$, the sums*

$$\sum_{j_1=0}^{K-1}\sum_{j_2,\dots,j_n=0}^{\lfloor\frac{k}{2}\rfloor} f(j_1,\dots,j_n) \times D\left[\varphi_{\mathrm{M}}(x_{j_1})\prod_{i=2}^{n}(\cos\theta_i)^{A_i}(\sin\theta_i)^{B_i}P_{k_i}^{i-2+B_i,A_i}(\cos 2\theta_i)\right]\Bigg|_{(x_{j_1},\theta_{j_2,2},\dots\theta_{j_n,n})}$$

---

[6] For the purposes of calculating transforms of finitely supported distributions, as corollary 8.5, it suffices to consider the case where $D = a(\theta_2,\dots,\theta_n)\frac{\partial^{|\alpha|}}{(\partial\theta)_\alpha}$. In this case the sum splits into at most 2^l other sums.

can be computed for all $m_n \geq \ldots \geq m_2 \geq 0$, $\nu_n \geq \ldots \geq \nu_2 \geq 0$, $m_2 + \nu_2 \geq |\lambda_1|$, *and* $m_2 + \nu_2 + \lambda_1$ *even, in a total of* $O(b^{2n-1}(\log b)^2)$ *operations. This bound can be chosen to depend only on* b, n, *and the order of* D.

Proof. The proof here follows the same ideas as that for the S^n transform. One simply evaluates the derivatives using Leibnitz's rule to get a collection of sums, each of which may be calculated in $O(b^{2n-1}(\log b)^2)$ operations. This is possible because $(P_k^{\alpha,\beta})' = \frac{1}{2}(k+\alpha+\beta+1)P_{k-1}^{\alpha+1,\beta+1}$. □

Corollary 8.7. *Let* $Y_b^n.B_n^n$ *be the subset of* $SU(n)$ *used in corollary 7.5, and assume* $\mathcal{Y}_b^n$ *is a set consisting of a distribution supported on* p *for each point,* p, *of* $Y_b^n.B_n^n$. *Then*

$$T_{\mathcal{Y}_b^n}^{SU(n-1)}(\mathcal{R}_b^n) \leq O(b^{\dim S^{2n-1}}(\log b)^2) = O(b^{2n-1}(\log b)^2)$$

In addition these bounds can be chosen to depend only on b, n, *and the maximum order of the distributions.*

9. Fast Transforms on Homogeneous Vector Bundles over S^2

Monopole Harmonics. Let τ_n be the representation of $SO(2)$ of weight n, and let E_n be the subbundle of $(TS^{2\mathbf{C}})^{\otimes|n|}$ isomorphic to $SO(3)\times_{\tau_n}\mathbf{C}$.

Define

$$\begin{aligned}
\omega_{+1} &= \frac{1}{\sqrt{2}}(d\theta - i\sin\theta d\varphi) \\
\omega_{-1} &= \frac{1}{\sqrt{2}}(d\theta + i\sin\theta d\varphi) \\
\omega_n &= \left\{\begin{array}{ll} \omega_{+1}^{\otimes n} & \text{if } n>0 \\ 1 & \text{if } n=0. \\ \omega_{-1}^{\otimes(-n)} & \text{if } n<0. \end{array}\right]
\end{aligned}$$

Then is is clear that ω_n is a section of E_n, defined everywhere except at the poles, and that $\bar{\omega}_n = \omega_{-n}$. For any C^∞ function, f, on S^2 define

$$\begin{aligned}
D_{+1}(f\omega_n) &= \frac{1}{2}\left[(B - n\cot\theta)f\right]\omega_{n+1} \\
D_{-1}(f\omega_n) &= \frac{1}{2}\left[(\bar{B} + n\cot\theta)f\right]\omega_{n-1}
\end{aligned}$$

where $B = \frac{\partial}{\partial\theta} + i\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$, and $\bar{B} = \frac{\partial}{\partial\theta} - i\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$. Calculating using B, it is straightforward to see that

$$D_{-1} = \bar{D}_{+1} = -D_{+1}^*$$

$$\begin{aligned}
D_{+1}^*D_{+1} &= \frac{1}{2}\left(\Delta - n(n+1)\right) \\
D_{+1}D_{+1}^* &= \frac{1}{2}\left(\Delta - (n-1)n\right)
\end{aligned}$$

on $\Gamma^\infty(E_n)$, where $\bar{D} = (D^*)^t$, $(\ )^*$ is the formal pre-Hilbert space adjoint, and $(\ )^t$ is the usual vector space adjoint. Δ is the -1 times the Casimir operator for the regular representation in E_n. Set D_k to be D_{+1}^k when $k \geq 0$, and D_{-1}^{-k} when

$k \leq 0$. D_k is $|k|$-th order and generates the space of invariant differential operators from $\Gamma^\infty(E_n)$ to $\Gamma^\infty(E_{n+k})$ as a bimodule over the algebras of invariant differential operators on $\Gamma^\infty(E_n)$ and $\Gamma^\infty(E_{n+k})$[7].

We use these invariant differential operators to construct a basis for the space of square integrable sections $L^2(E_n)$, from the spherical harmonic basis for functions on the sphere. If $l \geq |m|$ are integers, then define

$$Y_{lm} = A_{lm} C_{l-|m|}^{\frac{1}{2}+|m|}(\cos\theta)(\sin\theta)^{|m|} e^{im\varphi}$$

where

$$A_{lm} = \frac{1}{2^{|m|+1}} \sqrt{l! \begin{pmatrix} 2|m| \\ |m| \end{pmatrix} \Big/ \begin{pmatrix} l+|m| \\ 2|m| \end{pmatrix}}$$

which differs form the usual normalization by a factor of i^m. Using the formula for $D_{+1}^* D_{+1}$ we see that

$$\langle D_n Y_{lm}, D_n Y_{lm} \rangle = \frac{1}{2^{|n|}} \prod_{k=0}^{|n|-1} (l(l+1) - k(k+1)) = C_l^n$$

and hence that for fixed n, the set of all sections, $Y_{lm}^n = \frac{1}{\sqrt{C_l^n}} D_n Y_{lm}$ where $l \geq |n|$ and $l \geq |m|$, forms a complete orthonormal set in $L^2(E_n)$. These sections are called the monopole harmonics.

Assume $\theta_0, \ldots, \theta_{b-1}$ are points in $(0, \pi)$, $\varphi_0, \ldots, \varphi_{2b-2}$ are in $[0, 2\pi)$, and for each j, k we have a vector, $v_{jk} = a_{jk}.\omega_n(\theta_j, \varphi_k)$ in the fibre, $E_n|_{(\theta_j,\varphi_k)}$ of E_n over the point of S^2 with coordinates (θ_j, φ_k). Then

$$(28) \qquad \sum_{jk} \langle Y_{lm}^{-n}(\theta_j, \varphi_k), v_{jk} \rangle = \frac{1}{2^{|n|}\sqrt{C_l^n}} \left[\sum_{jk} a_{jk} \left[(B_{-n} Y_{lm})(\theta_j, \varphi_j) \right] \right]$$

where $B_n = (B - (n-1)\cot\theta) \ldots (B - \cot\theta)B$, for $n \geq 0$, and $B_{-n} = \bar{B}_n$. It is clear from theorem 8.4 of the previous section that this sum can be computed for all l, m with $b > l \geq |n|$ and $l \geq |m|$ in a total time $O(b^2(\log b)^2)$.

An alternative approach to computing the sum (28) is to write the section Y_{lm}^n directly as a linear combination of sections of the form $(\sin\theta)^{|n|} Y_{p,m}.\omega_n$. This approach to the problem is explored in [10].

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[7]These algebras are both of the form C[Δ], where Δ is the Laplacian in the corresponding space.

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