On simple ideal hyperbolic Coxeter polytopes

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Introduction

Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space and let P be a simple polytope in $\mathbb{H}^n$. P is called an *ideal polytope* if all vertices of P belong to the boundary of $\mathbb{H}^n$. P is called a *Coxeter polytope* if all dihedral angles of P are submultiples of π.

There is no complete classification of hyperbolic Coxeter polytopes. In [6] Vinberg proved that there are no compact hyperbolic Coxeter polytopes in $\mathbb{H}^n$ when $n \geq 30$. Prokhorov [5] and Khovanskij [3] proved that there are no Coxeter polytopes of finite volume in $\mathbb{H}^n$ for $n \geq 996$. Examples of bounded Coxeter polytopes are known only for $n \leq 8$, and examples of finite volume non-compact Coxeter polytopes are known only for $n \leq 19$ [8] and $n = 21$ [1].

In this paper, we prove that no simple ideal Coxeter polytope exists in $\mathbb{H}^n$ when $n > 8$.

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1 Preliminaries

1.1 Coxeter diagrams

It is convenient to describe Coxeter polytopes in terms of *Coxeter diagrams*.

A *Coxeter diagram* is one-dimensional simplicial complex with weighted edges, where weights are either of the type $\cos\frac{\pi}{m}$ for some integer $m \geq 3$ or positive real numbers no less than one. We can draw edges of Coxeter diagram by the following way:

if the weight equals $\cos\frac{\pi}{m}$ then the nodes are joined by either $(m-2)$-fold edge or simple edge labeled by m;

if the weight equals one then the nodes are joined by a bold edge;

if the weight is greater than one then the nodes are joined by a dotted edge labeled by its weight.

A *subdiagram* of Coxeter diagram is a subcomplex that can be obtained by deleting several nodes and all edges that are incident to these nodes.

Let Σ be a diagram with d nodes u_1,...,u_d. Define a symmetrical $d \times d$ matrix $Gr(\Sigma)$ by the following way: $g_{ii} = 1$; if two nodes u_i and u_j are adjacent then g_{ij} equals negative weight of the edge u_iu_j; if two nodes u_i and u_j are not adjacent then g_{ij} equals zero.

A *Coxeter diagram* $\Sigma(P)$ *of Coxeter polytope* P is a Coxeter diagram whose matrix $Gr(\Sigma)$ coincides with Gram matrix of P. In other words, nodes of Coxeter diagram correspond to facets of P. Two nodes are joined by either $(m-2)$-fold edge or m-labeled edge if the corresponding dihedral angle equals $\frac{\pi}{m}$. If the corresponding facets are parallel the nodes are joined by a bold edge, and if they diverge then the nodes are joined by a dotted edge.

By signature and rank of diagram Σ we mean the signature and the rank of the matrix $Gr(\Sigma)$.

A Coxeter diagram Σ is called *elliptic* if the matrix $Gr(\Sigma)$ is positively defined. A connected Coxeter diagram Σ is called *parabolic* if the matrix $Gr(\Sigma)$ is degenerated, and any subdiagram of Σ is elliptic. Elliptic and connected parabolic diagrams are exactly Coxeter diagrams of spherical and Euclidean Coxeter simplices respectively. They were classified by Coxeter [2]. The complete list of elliptic and connected parabolic diagrams is represented in Table 1.

A non-connected diagram is called *parabolic* if it is a disjoint union of connected parabolic diagrams. A diagram is called *indefinite* if it contains at least one connected component that is neither elliptic nor parabolic.

Let f be a k-dimensional face of P (by abuse of notation we write f *is a* k*-face of* P). If P is a simple n-dimensional polytope then α is an intersection of exactly $n-k$ facets. Let $f_1,\dots,f_{n-k}$ be the facets containing f and let $v_1,\dots,v_{n-k}$ be the corresponding nodes of $\Sigma(P)$. Let Σ_f be a subdiagram of $\Sigma(P)$ with nodes $v_1,\dots,v_{n-k}$. We say that Σ_f is the *diagram of the face* f.

The following properties of $\Sigma(P)$ and Σ_f are proved in [7].

- [cor. of Th. 2.1] the signature of $Gr(\Sigma(P))$ equals $(n,1)$;
- [cor. of Th. 3.1] if a k-face f is not an ideal vertex of P (i.e. f is not a point at the boundary of $\mathbb{H}^n$), then Σ_f is an elliptic diagram of rank $n-k$;
- [cor. of Th. 3.2] if f is an ideal vertex of P then Σ_f is a parabolic diagram of rank $n-1$; if f is a simple ideal vertex of P then Σ_f is connected;
- [cor. of Th. 3.1 and Th. 3.2] any elliptic subdiagram of $\Sigma(P)$ corresponds to a face of P; any parabolic subdiagram of $\Sigma(P)$ is a subdiagram of the diagram of a unique ideal vertex of P.

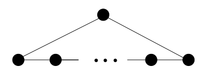
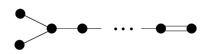
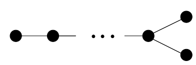
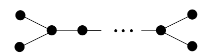
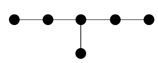
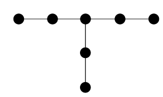
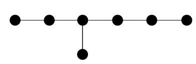
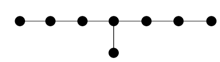
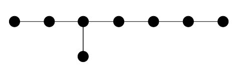
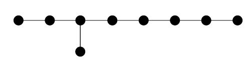
As a corollary, for simple ideal Coxeter polytope $P \subset \mathbb{H}^n$ we obtain:

(I) Any two non-intersecting indefinite subdiagrams of $\Sigma(P)$ are joined in $\Sigma(P)$.

(II) Any elliptic subdiagram of $\Sigma(P)$ contains less than n nodes;

(III) Any parabolic subdiagram of $\Sigma(P)$ is connected and contains exactly n nodes;

Table 1: Connected elliptic and parabolic Coxeter diagrams are listed in left and right columns respectively.

$\mathbf{A_n}$ $(n \geq 1)$		$\widetilde{\mathbf{A}}_\mathbf{1}$ $\widetilde{\mathbf{A}}_\mathbf{n}$ $(n \geq 2)$	 
$\mathbf{B_n} = \mathbf{C_n}$ $(n \geq 2)$		$\widetilde{\mathbf{B}}_\mathbf{n}$ $(n \geq 3)$ $\widetilde{\mathbf{C}}_\mathbf{n}$ $(n \geq 2)$	
$\mathbf{D_n}$ $(n \geq 4)$		$\widetilde{\mathbf{D}}_\mathbf{n}$ $(n \geq 4)$	
$\mathbf{G_2}$		$\widetilde{\mathbf{G}}_\mathbf{2}$	
$\mathbf{F_4}$		$\widetilde{\mathbf{F}}_\mathbf{4}$	
$\mathbf{E_6}$		$\widetilde{\mathbf{E}}_\mathbf{6}$	
$\mathbf{E_7}$		$\widetilde{\mathbf{E}}_\mathbf{7}$	
$\mathbf{E_8}$		$\widetilde{\mathbf{E}}_\mathbf{8}$	
$\mathbf{H_3}$			
$\mathbf{H_4}$			

Note, that a connected parabolic diagram with more than 3 nodes contains neither bold nor k-fold edges for $k > 2$. Hence, a Coxeter diagram of simple ideal Coxeter polytope in $\mathbb{H}^n$, $n > 3$, contains only simple edges, 2-fold edges and dotted edges.

Notation

Let F be a k-face of P and let $f_1, \dots, f_{n-k}$ be the facets of P containing F. Let $v_1, \dots, v_{n-k}$ be the corresponding nodes of $\Sigma(P)$.

• We denote by Σ_F the subdiagram of $\Sigma(P)$ spanned by $v_1, \dots, v_{n-k}$.

• We also write $\Sigma_F = <v_1, \dots, v_{n-k}>$ and $\Sigma_F = <v_1, \Theta>$, where $\Theta = <v_2, \dots, v_{n-k}>$. We denote by $\Sigma \backslash \{v_1, ..., v_m\}$ the subdiagram of Σ spanned by all nodes of Σ different from $v_1, ..., v_m$.

• For elliptic and parabolic diagrams we use standard notation (see Table 1). For example, we write $\Sigma_F = \widetilde{A}_n$.

• Let v and u be two nodes of $\Sigma(P)$. We write

$[v, u] = 0$ if u and v are disjoint in $\Sigma(P)$;

$[v, u] = 1$ if u and v are joined by a simple edge;

$[v, u] = 2$ if u and v are joined by a 2-fold edge;

$[v, u] = \infty$ if u and v are joined by a dotted edge.

1.2 Nikulin's estimate

Let P be an n-dimensional polytope. Denote by α_i, $i = 0, 1, \dots, n-1$, the number of i-faces of P. For a face f of P denote by α_i^f the number of i-faces of f (e.g. $\alpha_i = \alpha_i^P$). Denote by

$$\alpha_k^{(i)} = \frac{1}{\alpha_k} \sum_{dim f = k} \alpha_i^f$$

the average number of i-faces of a k-face of P.

Proposition 1 (Nikulin [4]). *For every simple convex bounded polytope P in $\mathbb{R}^n$ for $i < k \le [n/2]$ the following estimate holds:*

$$\alpha_k^{(i)} < \binom{n-i}{n-k} \frac{\binom{[n/2]}{i} + \binom{[(n+1)/2]}{i}}{\binom{[n/2]}{k} + \binom{[(n+1)/2]}{k}}.$$

Using this theorem for 2-faces ($i = 0$ and $k = 2$), Vinberg proved that no compact Coxeter polytope exists in $\mathbb{H}^n$, $n \ge 30$.

In [3], Khovanskij proved that Nikulin's estimate holds for edge-simple polytopes (a polytope is called *edge-simple* if any edge is the intersection of exactly $n-1$ facets). This was used by Prokhorov [5] when he proved that no Coxeter polytope of finite volume exists in $\mathbb{H}^n$ for $n \ge 996$.

In this paper, we study simple ideal hyperbolic Coxeter polytopes. Any hyperbolic Coxeter polytope of finite volume is edge-simple (see [3]). Thus, we can use Nikulin's estimate. We consider the combinatorics of Coxeter diagrams of simple ideal hyperbolic Coxeter polytopes and prove that such a polytope has no triangular 2-faces and that the number of quadrilateral 2-faces of such a polytope is relatively small. This falls into a contradiction with Nikulin's estimate in dimensions greater than 8.

2 Absence of triangular 2-faces and estimate for quadrilateral 2-faces.

Let P be a simple ideal Coxeter polytope in $\mathbb{H}^n$ and let V be a vertex of P. Since P is simple, the vertex V is contained in exactly n edges VV_i, $i = 1, \dots, n$. Denote by v_i the node of Σ_V such that $\Sigma_{VV_i} = \Sigma_V \backslash \{v_i\}$. Denote by u_i the node of $\Sigma(P)$ such that $\Sigma_{V_i} = <u_i, \Sigma_{VV_i}>$.

Now, starting from the diagram Σ_V, we want to describe all possible diagrams $<v_i, u_i, \Sigma_{VV_i}>$. For example, suppose that $\Sigma_V = \widetilde{A}_{n-1}$, $n \neq 3, 8, 9$. Then $\Sigma_{VV_i} = \Sigma_V \setminus v_i = A_{n-1}$. It is easy to see, that if $n \neq 3, 8, 9$ then $\widetilde{A}_{n-1}$ is the only parabolic diagram with n nodes containing a subdiagram A_{n-1}. Thus, $\Sigma_{V_i} = \widetilde{A}_{n-1}$. Note, that $[v_i, u_i] \neq 0$ and $[v_i, u_i] \neq 1$, otherwise $<v_i, u_i, \Sigma_{VV_i}>$ does not satisfy property (III). Hence, either $[v_i, u_i] = 2$ or $[v_i, u_i] = \infty$, and the subdiagram $<v_i, u_i, \Sigma_{VV_i}>$ is one of two diagrams shown in Figure 1.

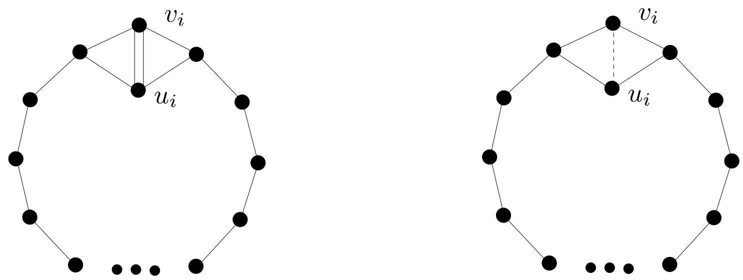


Figure 1: Two possibilities for $<v_i, u_i, \Sigma_{VV_i}>$, if $\Sigma_V = \widetilde{A}_{n-1}$, $n \neq 3, 8, 9$.

Similarly, one can list all possible diagrams $<u_i, v_i, \Sigma_{VV_i}>$ for any other type of Σ_V. Recall that Σ_V is one of the diagrams shown in the right column of Table 1. A case-by-case check using properties (I)–(III) shows the following:

Lemma 1. *Suppose that $n > 5$. In the notation above $[v_i, u_i] \neq 0$. If $[v_i, u_i] = 1$ then, up to interchange of v_i and u_i, the diagram $<v_i, u_i, \Sigma_{VV_i}>$ coincides with one of the diagrams shown in Figure 2.*

Figure 2: Two possibilities for $<v_i, u_i, \Sigma_{VV_i}>$ when $[v_i, u_i] = 1$.

A node v of a diagram Σ is called a *leaf* of Σ if Σ contains exactly one node joined with v.

Lemma 2. *Assume that* $n > 3$*. Then for the diagram* $< v_i, u_i, \Sigma_{VV_i} >$ *the following property holds: if* v_i *is a leaf of* Σ_V *and* u_i *is not a leaf of* Σ_{V_i} *then* $\Sigma_V = \widetilde{E}_k$*,* $\Sigma_{V_i} = \widetilde{A}_k$*, where* $k = 7$ *or* 8*. In this case* $<v_i, u_i, \Sigma_{VV_i}>$ *is one of the diagrams shown in Figure 3.*

Proof. Consider the subdiagram Σ_{VV_i}. Since $\Sigma_V = <\Sigma_{VV_i}, v_i>$ and v_i is a leaf of Σ_V, Σ_{VV_i} is connected. Since u_i is not a leaf of $\Sigma_{V_i} = <\Sigma_{VV_i}, u_i>$, there are at least two edges joining u_i with Σ_{VV_i}. Hence, Σ_{V_i} contains a cycle. Combined with (III), this implies that $\Sigma_{V_i} = \widetilde{A}_k$. Hence, $\Sigma_{VV_i} = A_k$. The only parabolic diagrams with $k+1$ nodes containing a subdiagram A_k are $\widetilde{A}_k$, $\widetilde{G}_2$, $\widetilde{E}_7$ and $\widetilde{E}_8$. Since $n > 3$ and Σ_V has at least one leaf v_i, $\Sigma_V = \widetilde{E}_7$ or $\widetilde{E}_8$.

We are left to show that $[v_i, u_i] = 2$ or $[v_i, u_i] = \infty$. This follows from Lemma 1.

□

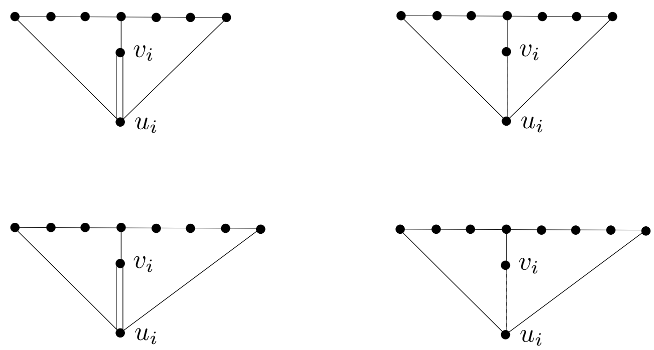


Figure 3: Possibilities for $<v_i, u_i, \Sigma_{VV_i}>$ when v_i is a leaf of Σ_V and u_i is not a leaf of Σ_{V_i}

Lemma 3. *Let* P *be a simple ideal Coxeter polytope in* $\mathbb{H}^n$*,* $n > 5$*. Then* P *has no triangular 2-faces.*

Proof. Suppose that UVW is a triangular 2-face of P. Then there are exactly $n+1$ facets of P containing at least one of the points U, V and W. The whole triangle UVW is contained in exactly $n-2$ of these facets. Since P is simple, for each edge of UVW there exists a unique facet containing the edge and not containing UVW. Denote these facets by $\bar{u}, \bar{v}$ and $\bar{w}$ for the edges VW, UW and UV respectively. Denote by u, v and w the nodes of $\Sigma(P)$ corresponding to $\bar{u}, \bar{v}$ and $\bar{w}$ respectively. Then $\Sigma_U = <v, w, \Sigma_{UVW}>$, $\Sigma_V = <u, w, \Sigma_{UVW}>$ and $\Sigma_W = <u, v, \Sigma_{UVW}>$ (see Figure 4a). In particular, (III) implies that all these diagrams are parabolic.

Consider the edge of Σ_W joining u and v. By Lemma 1, either $[u, v] = 1$ or $[u, v] = 2$ or $[u, v] = \infty$.

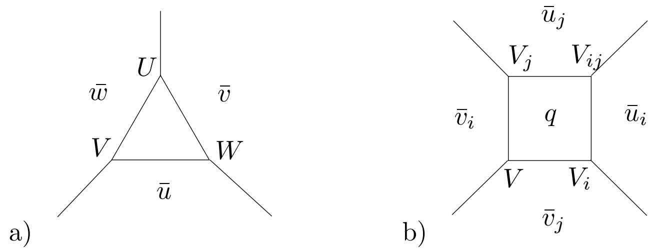


Figure 4: Notation for a triange (a) and for a quadrilateral (b).

Suppose that $[u, v] = \infty$. Then $\Sigma_W = <u, v, \Sigma_{UVW}>$ contains a dotted edge, in contradiction to the fact that Σ_W is parabolic. Thus, $[u, v] \neq \infty$ and, similarly, $[v, w] \neq \infty$ and $[u, w] \neq \infty$.

Suppose that u is a leaf of Σ_V and v is not a leaf of Σ_U. Then Lemma 2 shows that $<w, \Sigma_W> = <u, v, w, \Sigma_{UVW}>$ is one of the diagrams shown in Figure 3. No of these diagrams contains a node $w \neq u, v$, such that $<u, v, \Sigma_{UVW}>$ is parabolic. Thus, no of these diagrams corresponds to a triangle, and we may assume that either both u and v are the leaves of Σ_V and Σ_U respectively or none of them is.

Suppose that $[u, v] = 2$. It follows from Table 1 and the assumption $n > 5$ that either u or v is a leaf of Σ_W. Without loss of generality we can assume that u is a leaf. Then it is easy to see that we have one of the diagrams shown in Figure 5. Consider the case shown in Figure 5a. Since $[u, w] \neq \infty$, the diagram $<u, w, \Sigma_{UVW}> = \Sigma_V$ is elliptic, that is impossible by (ii). Consider the case shown in Figure 5b. If $[u, w] = 1$ then $<u, w, \Sigma_{UVW}> = \Sigma_V$ is elliptic, that is impossible. If $[u, w] = 2$ then $<u, v, w>$ is a parabolic diagram with only three nodes in contradiction to (iii).

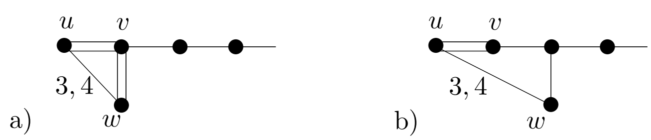


Figure 5: Possibilities for the case $[u, v] = 2$.

Suppose that $[u, v] = 1$. By Lemma 1, $\Sigma_W = <u, v, \Sigma_{UV}>$ coincides with one of the diagrams shown in Figure 2 (up to interchange of u and v). It is easy to see that Σ_{UV} contains no node $w \neq u, v$ such that $<u, v, \Sigma_{UV} \setminus w>$ is a parabolic diagram. Note that $\Sigma_{UV} = <w, \Sigma_{UVW}>$ and $<u, v, \Sigma_{UV} \setminus w> = <u, v, \Sigma_{UVW}> = \Sigma_W$. Thus, we have no parabolic diagram Σ_W, so $[u, v] \neq 1$.

By Lemma 1, the case $[u, v] = 0$ is also impossible. There are no more possibilities for $[u, v]$. Hence, no diagram Σ_{UVW} can be constructed, and P contains no triangular faces.

□

Note that an ideal Coxeter polytope in $\mathbb{H}^5$ may have a triangular 2-face. For example, the Coxeter diagram shown in Figure 6 determines a 5-dimensional ideal Coxeter simplex. All 2-faces of any simplex are triangles.

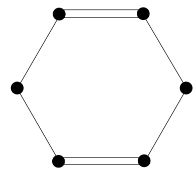


Figure 6: This diagram determines a 5-dimensional ideal Coxeter simplex.

Lemma 4. *Let V be a vertex of simple ideal Coxeter polytope P in $\mathbb{H}^n$, $n > 9$. Then V belongs to at most $n+3$ quadrilateral 2-faces.*

Proof. Let q be a quadrilateral 2-face with vertices V, V_i, V_j and V_{ij}. The 2-face q belongs to $n-2$ facets, each edge of q belongs to $n-1$ facets and each vertex belongs to n facets. Denote by $\bar{v}_i, \bar{u}_i, \bar{v}_j$ and $\bar{u}_j$ the facets not containing q and containing the edges VV_j, V_iV_{ij}, VV_i and V_jV_{ij} respectively (see Figure 4b). Denote by v_i, u_i, v_j and u_j the nodes of $\Sigma(P)$ corresponding to the facets $\bar{v}_i, \bar{u}_i, \bar{v}_j$ and $\bar{u}_j$ respectively.

Then $\Sigma_V = <v_i, v_j, \Sigma_q>$, $\Sigma_{V_i} = <v_j, u_i, \Sigma_q>$, $\Sigma_{V_j} = <v_i, u_j, \Sigma_q>$, and $\Sigma_{V_{ij}} = <u_i, u_j, \Sigma_q>$. See Figure 7 for an example of a quadrilateral.

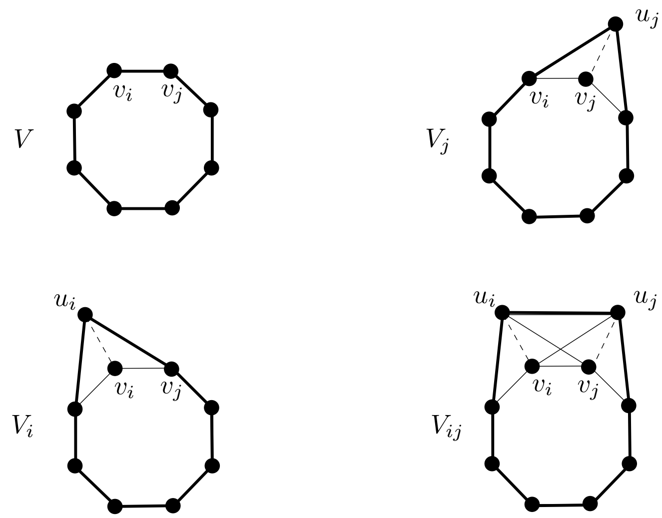


Figure 7: Example of a quadrilateral

Suppose that $\Sigma_V = \widetilde{A}_{n-1}$ and v_i and v_j are disjoint in Σ_V. Since $n > 8$, each of the vertices V_i, V_j, V_{ij} are of the type $\widetilde{A}_{n-1}$. Consider $<v_i, u_i, \Sigma_{VV_i}>$.

By (III), either $[v_i, u_i] = \infty$ or $[v_i, u_i] = 2$ (cf. Figure 1). The same statement holds for $[v_j, u_j]$. Since $\Sigma_{V_{ij}}$ is a parabolic diagram $\widetilde{A}_{n-1}$, we have $[u_i, u_j] = 0$ (see Figure 8). Then $\Sigma(P)$ contains two disjoint indefinite subdiagrams $v_iu_iw_i$ and $v_ju_jw_j$. Thus, any quadrilateral containing V corresponds to a pair of neighbouring nodes of Σ_V, and V belongs to at most n quadrilaterals.

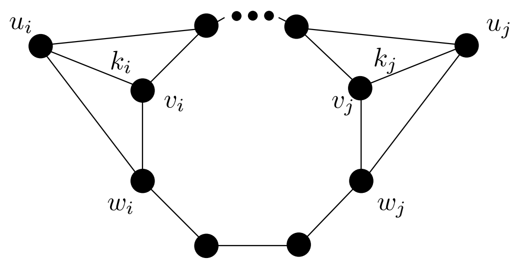


Figure 8: $v_iu_iw_i$ and $v_ju_jw_j$ are disjoint indefinite subdiagrams.
In this diagrams $k_i, k_j = 2$ or ∞.

From now on we assume that $\Sigma_V \neq \widetilde{A}_{n-1}$. Since $n > 9$, $\Sigma_V = \widetilde{B}_{n-1}, \widetilde{C}_{n-1}$ or $\widetilde{D}_{n-1}$. Define a *distance* $\rho(u, w)$ between two nodes u and w of connected graph as the number of edges in the shortest path connecting u and w.

Let x be a leaf of Σ_V. Denote by $\Sigma_V^{(5)}(x)$ a connected subdiagram of Σ_V spanned by five nodes closest to x in Σ_V (i.e., if $v_k \in \Sigma_V^{(5)}(x)$ and $v_l \notin \Sigma_V^{(5)}(x)$ then $\rho(x, v_k) \leq \rho(x, v_l)$. Note that for $\Sigma_V = \widetilde{B}_{n-1}, \widetilde{C}_{n-1}$ and $\widetilde{D}_{n-1}$ when $n \geq 9$ the diagram $\Sigma_V^{(5)}(x)$ is well-defined for any leaf x of Σ_V.

Denote by $L(\Sigma_V)$ the set of leaves of Σ_V. Define

$$\Sigma_V^{(5)} = \bigcup_{x_i \in L(\Sigma_V)} \Sigma_V^{(5)}(x_i)$$

(see Fig. 9). It is easy to see that if $n > 10$ then $\Sigma_V^{(5)}$ consists of two connected components.

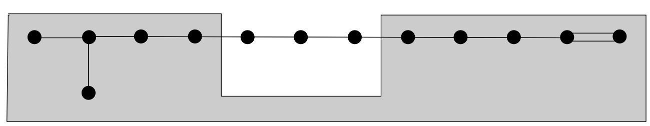
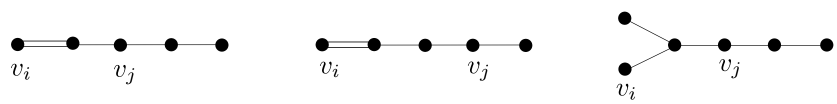


Figure 9: Subdiagram $\Sigma_V^{(5)}$ for $\Sigma_V = \widetilde{B}_{12}$.

Suppose that v_i and v_j do not belong to the same connected component of $\Sigma_V^{(5)}$ (v_i or v_j may lie in $\Sigma_V \backslash \Sigma_V^{(5)}$). By the same reason as in the case $\Sigma_V = \widetilde{A}_{n-1}$, nodes v_i and v_j are neighbours in Σ_V.

Suppose that v_i and v_j belong to the same connected component of $\Sigma_V^{(5)}$. Suppose that v_i and v_j are disjoint. A straightforward check of possibilites with use of properties (I)–(III) shows that if Σ_q is the diagram of a quadrilateral 2-face, then the connected component of $\Sigma_V^{(5)}$ is one of the following configurations (up to interchange of v_i and v_j):



Hence, the quadrilaterals containing V are encoded either by one of $n-1$ pair of neighbouring nodes of Σ_V or by one of two pairs of nodes for each of two connected components of $\Sigma_V^{(5)}$. Thus, the number of quadrilaterals containing A is less than or equal to $2+2+(n-1)=n+3$.

□

Lemma 5. *Let A be a vertex of a simple ideal Coxeter polytope P in $\mathbb{H}^9$. Then A belongs to at most 15 quadrilateral 2-faces.*

Proof. The existence of $\Sigma_V=\widetilde{E}_8$ course a lot of possibilities for the diagram $<v, v_i, \Sigma_{VV_i}>$. This leads to a large number of different diagrams $<v, v_i, \Sigma_q>$. To observe all these possibilities we use a case-by-case check organized as follows:

Step 1. We consider the cases $\Sigma_V=\widetilde{A}_8, \widetilde{B}_8, \widetilde{C}_8, \widetilde{D}_8$ and $\widetilde{E}_8$ separately.

Step 2. For each node v_i, $i=1,...,9$, of Σ_V we list all possible diagrams $<v_i, u_i, \Sigma_{VV_i}>$ such that $<u_i, \Sigma_{VV_i}>$ is parabolic and $<v_i, u_i, \Sigma_{VV_i}>$ satisfies properties (I)—(III). We call such a diagram $<v_i, u_i, \Sigma_{VV_i}>$ an *edge-pattern*. Clearly, any edge incident to V corresponds to some edge-pattern $<v_i, u_i, \Sigma_{VV_i}>$.

Some nodes v_i of Σ_V may admit several edge-patterns $<v_i, u_i, \Sigma_{VV_i}>$ (up to 8 edge-patterns for one of the nodes of $\widetilde{E}_8$). Denote the edge-patterns by $(v_i, u_i)_r$, $r=1,...,k_i$, where k_i is the number of patterns for the node v_i of Σ_V.

Step 3. For each edge-pattern $(v_i, u_i)_r$ we consider all edge-patterns $(v_j, u_j)_s$, $j \neq i$. We list all cases when v_i, u_i, v_j, u_j correspond to the facets of some quadrilateral 2-face q (where $\Sigma_q=\Sigma_V \setminus \{v_i, v_j\}$).

Step 4. For each node v_i, $1 \leq i \leq 9$, choose an edge-pattern $(v_i, u_i)_{r_i}$. Then compute the total number $Q(r_1, \dots, r_9)$ of quadrilaterals determined by $(v_i, u_i)_{r_i}$ and $(v_j, u_j)_{r_j}$ for $1 \leq i < j \leq 9$.

Step 5. Denote by $Q(\Sigma_V)$ the maximal value of $Q(r_1, \dots, r_9)$ for all $r_1, \dots, r_9$. It turns out that

$$Q(\widetilde{A}_8)=15,$$
$$Q(\widetilde{B}_8)=14,$$
$$Q(\widetilde{C}_8)=12,$$

$$Q(\widetilde{D}_8) = 15,$$
$$Q(\widetilde{E}_8) = 14.$$

Thus, for any type of Σ_V we obtain that V belongs to at most 15 quadrilateral 2-facets.

□

Remark. At step 4 of the algorithm above one should check a huge number of possibilities (more than 15000 cases for $\widetilde{E}_8$). This was done by computer.

3 Absence of simple ideal Coxeter polytopes in large dimensions.

Recall that α_i denotes the number of i-faces of a polytope P and $\alpha_k^{(i)}$ denotes the average number of i-faces of k-face of P.

We will need the following lemma:

Lemma 6. *Let P be an n-dimensional simple polytope and let l be the number of vertices of P. Then*

$$\frac{l}{\alpha_2} = \frac{2}{n(n-1)}\alpha_2^{(1)}. \tag{1}$$

Proof. Denote by m_i the number of i-angular 2-faces of P. Let us compute the total number N of vertices of 2-faces. Clearly, $N = \sum\limits_{i\geq 3} i \cdot m_i$. From the other hand, each pair of edges incident to one vertex of simple polytope determines a 2-face of the polytope. Thus, $N = l\frac{n(n-1)}{2}$, and we obtain the following equality

$$l\frac{n(n-1)}{2} = \sum_{i\geq 3} i \cdot m_i. \tag{2}$$

By definition,

$$\alpha_2^{(1)} = \frac{\sum\limits_{i\geq 3} i \cdot m_i}{\alpha_2}. \tag{3}$$

Combining (2) and (3), we obtain

$$\frac{l}{\alpha_2} = \frac{2}{n(n-1)}\frac{\sum\limits_{i\geq 3} i \cdot m_i}{\alpha_2} = \frac{2}{n(n-1)}\alpha_2^{(1)}.$$

□

Theorem 1. *There is no simple ideal Coxeter polytope in $\mathbb{H}^n$ for $n \geq 9$.*

Proof. We use the notation from Lemma 6. Recall, that $\alpha_2 = \sum\limits_{i\geq 3} m_i$. By Lemma 3, $m_3 = 0$. Using (3), we have

$$\alpha_2^{(1)} \geq \frac{1}{\alpha_2}(4m_4 + 5\sum_{i\geq 5} m_i) = \frac{1}{\alpha_2}(5\sum_{i\geq 4} m_i - m_4) = 5 - \frac{m_4}{\alpha_2}. \tag{4}$$

Consider Nikulin's estimate for $\alpha_2^{(1)}$:

$$\alpha_2^{(1)} < \binom{n-1}{n-2}\frac{\binom{[n/2]}{1} + \binom{[(n+1)/2]}{1}}{\binom{[n/2]}{2} + \binom{[(n+1)/2]}{2}} = 4\frac{n-1+\varepsilon}{n-2+\varepsilon}, \tag{5}$$

where $\varepsilon = 0$ if n is even and $\varepsilon = 1$ if n is odd.
Combining (4) with (5), we obtain

$$5 - \frac{m_4}{\alpha_2} \leq \alpha_2^{(1)} < 4\frac{n-1+\varepsilon}{n-2+\varepsilon}. \tag{6}$$

Denote by l the number of vertices of P. Denote by N_4 the total number of vertices of quadrilateral 2-faces. Clearly, $N_4 = 4m_4$. By Lemmas 4 and 5 each of l vertices is incident to at most $n+6$ quadrilaterals. Thus, $N_4 \leq l(n+6)$ and we have $4m_4 \leq l(n+6)$. In view of (1) and (5), we have

$$\frac{m_4}{\alpha_2} \leq \frac{1}{4}\frac{l(n+6)}{\alpha_2} = \frac{n+6}{4}\frac{2}{n(n-1)}\alpha_2^{(1)} <$$

$$< \frac{n+6}{2n(n-1)}\frac{4(n-1+\varepsilon)}{(n-2+\varepsilon)} = 2\frac{n+6}{n(n-1)}\frac{(n-1+\varepsilon)}{(n-2+\varepsilon)}. \tag{7}$$

Combining (6) and (7), we obtain

$$5 - \frac{4(n-1+\varepsilon)}{(n-2+\varepsilon)} < \frac{m_4}{\alpha_2} < 2\frac{n+6}{n(n-1)}\frac{(n-1+\varepsilon)}{(n-2+\varepsilon)}.$$

This implies

$$(n-6+\varepsilon)n(n-1) < 2(n+6)(n-1+\varepsilon).$$

This is equivalent to $n^2 - 8n - 12 < 0$ if n is even and to $n^2 - 8n - 7 < 0$ if n is odd. The first inequality has no solutions for $n \geq 10$, and the second one has no solutions for $n \geq 9$. So, the theorem is proved.

□

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