Helix Theory

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Abstract.

This is a detailed review of the helix theory, which describes exceptional sheaves and exceptional bases for the derived categories of coherent sheaves on Fano varieties. We explain systematically all basic ideas and constructions concerned with exceptional objects. Projective spaces and Del Pezzo surfaces are considered especially extensive. Some arithmetical interactions with the mirror symmetry phenomenon are discussed as well. This paper also may be considered as a necessary complement to the book [HuLe], which completely ignores rich structures staying behind the zero-dimensional moduli spaces.

Contents

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§1. Instead of introduction: non symmetric orthogonal linear algebra.

1.1. Unimodular Mukai lattices. An unimodular *Mukai lattice* is a free $\mathbb{Z}$-module $M \cong \mathbb{Z}^n$, of finite rank, equipped with a bilinear form $M \times M \longrightarrow \mathbb{Z} : v, w \longmapsto \langle v, w \rangle$, which may be neither symmetric nor skew symmetric but has to be unimodular, i. e. produce $\mathbb{Z}$-module isomorphism

$$M \xrightarrow{m \mapsto \langle m, * \rangle} Hom_{\mathbb{Z}}(M, \mathbb{Z}) .$$

An integer base $\{e_1, e_2, \ldots, e_n\} \subset M$ is called *exceptional* or *semiorthonormal*, if its Gram matrix $\chi_{ij} \stackrel{\text{def}}{=} \langle e_i, e_j \rangle$ is upper triangular with units on the main diagonal, i. e. $\langle e_i, e_j \rangle = 0$ for all $i > j$ and $\langle e_i, e_i \rangle = 1$ for all i. In general, an unimodular Mukai lattice does not have a semiorthonormal base at all: the unimodularity just means that $\det(\chi_{ij}) = \pm 1$ for any integer base of M (see n° 1.4.4 for general facts about non symmetric bilinear forms).

However, unimodular Mukai lattices equipped with semiorthonormal bases appear in the following two well known 'mirror symmetric' theories. The first is the Picard - Lefschetz theory, where M is a lattice of vanishing cycles in the middle cohomologies of a smooth level hypersurface for a holomorphic function near an isolated critical point. This lattice comes with Seifert's (non symmetric) form and its exceptional bases are usually called 'distinguished' (see [AVG]). The second theory deals with algebraic coherent sheaves on a Fano variety X. Here $M = K_0(X)$ is the Grothendieck ring equipped with the Euler form

$$\langle E, F \rangle = \sum (-1)^\nu \dim Ext^\nu(E, F)$$

and semiorthonormal bases are constructed from *exceptional objects* of the derived category of coherent sheaves $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$. In the present lectures we discuss this coherent theory (but at the last paragraph we make few remarks towards a bypass to Picard - Lefschetz).

1.2. What is the helix theory. The derived category $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$ may be considered as a kind of 'module' over a kind of 'ring' whose role is played by the triangulated category of graded vector spaces with direct sums and tensor products as the ring operations. Grothendieck's K_0 functor transforms this intuitive module structure to the ordinary one on the Mukai lattice. The helix theory lifts the basic concepts of orthogonal linear algebra (duality, orthogonal decompositions, e. t. c.) as well as some special effects (like the braid group action on the semiorthonormal bases) from the level of $K_0(X)$ onto the level of $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$. The main difficulty here is that we have to represent all linear combinations used to define/compute the sings in $K_0(X)$ by some actual canonical morphisms in $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$. Fortunately, it is possible almost everywhere as we will see in the next paragraph.

1.3. The paper is organized as follows. We finish this introduction by recalling some basic constructions from the (non symmetric) orthogonal linear algebra. In §2 we discuss the helices and orthogonal decomposition technique in general linear triangulated category setup. Then we will consider particular versions of the helix theory: on projective spaces in §3 and on Del Pezzo surfaces in §4. In this connection, in §4, §5, we discuss also the geometry of rigid sheaves and some concepts of 'stability'. §6 is devoted to transitivity of the braid group action on exceptional collections of sheaves over Del Pezzo surfaces. In the last §7 we make few remarks on interactions between the Helix theory and the Diophantine approximations on the one side and rigid local systems on $\mathbb{P}_1 \setminus \{$ some points $\}$ on the other side.

1.4. Linear algebra draught. Before we begin an abstract rubbish on orthogonal decompositions in $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$, let us outline some basic concepts of the integer non symmetric orthogonal linear algebra on the level of Mukai lattices.

1.4.1. Mutations and orthogonal projectors. Let M be a Mukai lattice and $e \in M$ have $\langle\, e\,,\, e\,\rangle = 1$. We write

$$^{\perp}\!\langle e\rangle \overset{\text{def}}{=} \{m \in M \mid \langle\, m\,,\, e\,\rangle = 0\} \quad \text{ and } \quad \langle e\rangle^{\perp} \overset{\text{def}}{=} \{m \in M \mid \langle\, e\,,\, m\,\rangle = 0\}$$

for its left and right orthogonals. The maps

$$\begin{aligned} M &\xrightarrow{L_e} \langle e\rangle^{\perp} \;:\; m \longmapsto \langle\, e\,,\, m\,\rangle \cdot e - m \\ M &\xrightarrow{R_e} {}^{\perp}\!\langle e\rangle \;:\; m \longmapsto \langle\, m\,,\, e\,\rangle \cdot e - m \end{aligned} \tag{1-1}$$

are called *left* and *right* mutations w. r. t. e. Up to the sign, they coincide with the left orthogonal projection onto $\langle e\rangle^{\perp}$ and the right orthogonal projection onto $^{\perp}\!\langle e\rangle$, that is, $\forall\, m\in M$ we have

$$\langle\, m\,,\, x\,\rangle = -\,\langle\, L_e m\,,\, x\,\rangle \quad \forall\, x\in\langle e\rangle^{\perp} \qquad \text{and} \qquad \langle\, x\,,\, m\,\rangle = -\,\langle\, x\,,\, R_e m\,\rangle \quad \forall\, x\in{}^{\perp}\!\langle e\rangle \;. \tag{1-2}$$

We call L_e and R_e *left* and *right mutations* with respect to e.

1.4.2. Duality and orthogonal decomposition. Let $\{e_0, e_1, \dots, e_n\} \subset M$ be an semiorthonormal base. Its *left* and *right dual* semiorthonormal bases $\{^{\vee}e_i\}$ and $\{e_i^{\vee}\}$ are defined by the orthogonality relations:

$$(-1)^i\,\langle\, {}^{\vee}e_i\,,\, e_{n-j}\,\rangle = (-1)^{n-j}\,\langle\, e_{n-j}\,,\, e_i^{\vee}\,\rangle = \delta_{ij} \;. \tag{1-3}$$

By (1-1) and (1-2), they can be constructed using consequent mutations as follows

$$\begin{aligned} e_i^{\vee} &= L_{e_0} L_{e_1} \dots L_{e_{n-i-1}}\, e_{n-i} \\ {}^{\vee}e_i &= R_{e_n} R_{e_{n-1}} \dots R_{e_{n-i+1}}\, e_{n-i} \end{aligned}$$

For any $m = \sum x_\nu e_\nu \in M$ the coordinates x_ν can be computed as

$$x_\nu = (-1)^{n-\nu}\,\langle\, {}^{\vee}e_{n-\nu}\,,\, m\,\rangle = (-1)^{\nu}\,\big\langle\, m\,,\, e_{n-\nu}^{\vee}\,\big\rangle \;.$$

In §2 this formula will turn to a representation of any object $m \in Ob\,\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$ as a canonical contraction of some functorial complex of exceptional objects; this gives natural spectral sequences for computing the values of cohomology functors at m.

If $E \subset M$ is a sublattice generated by some semiorthonormal collection $\{e_0, e_1, \dots, e_n\}$, then there are left and right orthogonal projections $M \longrightarrow E$ as well as the left orthogonal projection $M \xrightarrow{L_E} E^{\perp}$ and the right orthogonal projection $M \xrightarrow{R_E} {}^{\perp}E$. The last two generalize (1-1) and take

$$\begin{aligned} L_E \;:\; m &\longmapsto m - \sum_{\nu} (-1)^{n-\nu}\,\langle\, {}^{\vee}e_{n-\nu}\,,\, m\,\rangle \cdot e_\nu \\ R_E \;:\; m &\longmapsto m - \sum_{\nu} (-1)^{\nu}\,\big\langle\, m\,,\, e_{n-\nu}^{\vee}\,\big\rangle \cdot e_\nu \end{aligned} \tag{1-4}$$

where $\{{}^{\vee}e_\nu\}$ and $\{e_\nu^{\vee}\}$ are left and right dual to $\{e_\nu\}$ bases for E.

1.4.3. Canonical Serre operator. There exist a canonical linear automorphism $M \xrightarrow{\varkappa} M$ such that

$$\langle\, m_1\,,\, m_2\,\rangle = \langle\, m_2\,,\, \varkappa m_1\,\rangle \quad \forall\, m_1, m_2 \in M$$

Its matrix is expressed through the Gram matrix as $\varkappa = \chi^{-1}\chi^t$. If $\{e_0, e_1, \ldots, e_n\}$ and $\{{}^{\vee}e_0, {}^{\vee}e_1, \ldots, {}^{\vee}e_n\}$ be a pair of dual exceptional bases for M, then

$$\varkappa : m \longmapsto \sum_\nu (-1)^\nu \,\langle\, m\,,\, {}^{\vee}e_{n-\nu}\,\rangle \cdot e_\nu \tag{1-5}$$

Indeed, the orthogonality relations (1-3) show that if $\{\varepsilon_0^{\vee}, \varepsilon_1^{\vee}, \ldots, \varepsilon_n^{\vee}\}$ is the right dual base to $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n\}$, then $\{\varkappa\varepsilon_0, \varkappa\varepsilon_1, \ldots, \varkappa\varepsilon_n\}$ is the right dual base to $\{\varepsilon_0^{\vee}, \varepsilon_1^{\vee}, \ldots, \varepsilon_n^{\vee}\}$, that is, writing $\varkappa\varepsilon_\nu = e_\nu$ we get

$$m = \sum_\nu (-1)^\nu \left\langle\, m\,,\, \varepsilon_{n-\nu}^{\vee}\,\right\rangle \cdot \varepsilon_\nu \xmapsto{\;\varkappa\;} \sum_\nu (-1)^\nu \left\langle\, m\,,\, \varepsilon_{n-\nu}^{\vee}\,\right\rangle \cdot \varkappa\varepsilon_\nu =$$
$$= \sum_\nu (-1)^\nu \,\langle\, m\,,\, {}^{\vee}e_{n-\nu}\,\rangle \cdot e_\nu$$

In the categorical setup $\varkappa$ is realized up to a sign by the Serre - Verdier dualizing functor.

1.4.4. Digression: non degenerated bilinear forms over $\mathbb{C}$. From the standpoint of the invariant theory, the map

$$\chi \longmapsto \varkappa = \chi^{-1}\chi^t\,,$$

which sends bilinear forms to linear operators, is clearly $GL(M)$-invariant. If we extend M to a vector space $\overline{M} = \mathbb{C} \underset{\mathbb{Z}}{\otimes} M$ over $\mathbb{C}$, then one can prove that this map separates the orbits, i. e. two forms are equivalent iff the corresponding canonical operators are conjugate. Moreover, its image may be explicitly described, say, in terms of all Jordan normal forms that are possible for $\varkappa$ (see [Go5] for more details and further references). This leads to the following classification of vector spaces V equipped with non degenerated bilinear forms up to isometries[1]. We call V *decomposable*, if it splits into biorthogonal direct sum of proper subspaces:

$$V = V_1 \oplus V_2\,, \quad \text{where } \langle\, V_1\,,\, V_2\,\rangle = \langle\, V_2\,,\, V_1\,\rangle = 0\,,\; V_1 \neq 0\,,\; V_2 \neq 0\,.$$

Clearly, each V is a biorthogonal direct sum of indecomposable spaces. Indecomposable space is isometric either to the space $W_k(\lambda)$ of dimension $2k$ with the form

$$\begin{pmatrix} 0 & I \\ I_\lambda & 0 \end{pmatrix}\,, \quad \text{where } I = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}\,,\; I_\lambda = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \lambda \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & \lambda & & 0 \end{pmatrix}\,,\; \lambda \neq \pm 1$$

[1] i. e. form preserving linear isomorphisms

(non isometric to each other for different λ) or to the space U_n of dimension n with the form

$$\begin{pmatrix} & & & & 1 \\ & 0 & & -1 & 1 \\ & & 1 & -1 & \\ & \iddots & 1 & & \\ & & & 0 & \\ \iddots & \iddots & & & \end{pmatrix}.$$

1.5. Braid group action. Let the standard generators $g_1, g_2, \dots, g_{n-1}$ of B_n act on the semiorthonormal collections $\{e_1, e_2, \dots, e_n\} \subset M$ as

$$g_i : \begin{cases} e_\nu \longmapsto e_\nu \quad \text{for } \nu \neq i\,,\ (i+1) \\ e_i \longmapsto e_{i+1} - \langle\, e_i\,,\, e_{i+1}\,\rangle\ e_i \\ e_{i+1} \longmapsto e_i \end{cases}$$

In other words, we transpose the pair of consequent vectors (e_i, e_{i+1}) and then restore the broken semiorthonormality by the standard orthogonalization procedure, which can go in two directions:

$$\begin{tikzcd} & (e_i, e_{i+1}) \arrow[d] & \\ & (e_{i+1}, e_i) \arrow[dl, "g_i"'] \arrow[dr, "g_i^{-1}"] & \\ (\, e_{i+1} - \langle\, e_i\,,\, e_{i+1}\,\rangle\ e_i\,,\, e_i\,) & & (\, e_{i+1}\,,\, e_i - \langle\, e_i\,,\, e_{i+1}\,\rangle\ e_{i+1}\,) \end{tikzcd} \tag{1-6}$$

If we leave e_i and change e_{i+1} by $e_{i+1} - \langle\, e_i\,,\, e_{i+1}\,\rangle\ e_i$, then we get g_i written above. If leave e_{i+1} and replace e_i by $e_i - \langle\, e_i\,,\, e_{i+0}\,\rangle\ e_{i+1}$, then we get the inverse transformation g_i^{-1} actually. A direct computation shows that the braid group relations

$$g_i g_j = g_j g_i \text{ for } |i - j| > 1$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

hold for this action. So, in order to describe all the semiorthonormal bases of M we have to study the braid group orbits. An other natural group acting on M is the isometry group *Isom*(M), of all form preserving linear automorphisms.

1.5.1. CONJECTURE. *If $\overline{M} = \mathbb{C} \underset{\mathbb{Z}}{\otimes} M$ is isometric to U_n (see n° 1.4.4), then, up to the signs of basic vectors, a semidirect product Isom$(M) \leftthreetimes B_n$ acts transitively on the set of all semiorthonormal bases in M.*

This conjecture is not proved yet but it is verified for a number of algebraic geometric examples, where $M = K_0(X)$. It is closely connected with classical problems from the geometry of numbers. Postponing this subject to the last paragraph, we mention just two basic examples here.

1.6. Example: $K_0(\mathbb{P}_2)$ and the Markoff numbers. As we will see in n° 7.2, there exists a unique (up to isometric isomorphism over $\mathbb{Z}$) rank 3 lattice $M \cong \mathbb{Z}^3$ that is indecomposable over $\mathbb{C}$ and admits semiorthonormal base. This is $M = K_0(\mathbb{P}_2)$. Moreover, elements of any semiorthonormal Gram matrix

$$(\langle e_i\,,\,e_j\rangle) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

satisfy *the triple Markoff equation*

$$x^2 + y^2 + z^2 = xyz\,, \tag{1-7}$$

(see details in [Go5], [GoRu] and n° 7.2 below). So, there is a $1-1$ correspondence between the isometry classes of semiorthonormal bases and triples of the the triple Markoff numbers (i. e. the the solutions of (1-7)). Up to the signs, all the solutions of (1-7) are obtained from the 'initial' one $(3,3,3)$ by natural quadratic transformations

$$\begin{tikzcd} & (x, xz-y, z) & \\ & (x,y,z) \arrow[u] \arrow[dl] \arrow[dr] & \\ (yz-x,y,z) & & (x,y,xy-z) \end{tikzcd}$$

predicted by the Vieta theorem (see [Ca]). The initial triple $(3,3,3)$ corresponds to 'the exceptional base' $\{\mathcal{O}\,,\mathcal{T}(-1)\,,\mathcal{O}(1)\}$ for $K_0(\mathbb{P}_2)$ and the above transformations may be organized into B_3 - action on the triples of the solutions for (1-7) in such a way that the correspondence in question would B_3 - equivariant (see n° 7.2 below). So, up to signs of basic vectors, B_3 acts transitively on the isometry classes of exceptional bases of M.

Originally, the Markoff numbers appear more than 100 years ago in view of the following phenomenon. For any totally real[1] quadratic form $q(x,y)$ *a homogeneous minimum* is defined as

$$\mu(q) = \frac{1}{\sqrt{|\det(q)|}} \min_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |\,q(x,y)\,|\ .$$

Then, there exists a unique (up to $SL_2(\mathbb{Z})$-action) form q_1 whose $\mu(q_1)$ is maximal among all totally real q's. Outside the $SL_2(\mathbb{Z})$-orbit of q_1 there is again a unique up to $SL_2(\mathbb{Z})$-action form q_2 with maximal $\mu(q_2)$ and so on. The sequence $\mu(q_1)$, $\mu(q_2)$, $\dots$, of these isolated minima, is called *the Markoff spectrum* and the corresponding forms q_i are known as *the Markoff forms*. The Markoff forms are in $1-1$ correspondence with the solutions of the Markoff equation, which appears initially as a tool for building the Markoff forms (see [Ma]).

It turns out that under the above correspondence between semiorthonormal bases of $M = K_0(\mathbb{P}_2)$ and triple Markoff numbers the Markoff forms appear naturally as the restrictions of the quadratic form $q(v) = \langle\,v\,,\,v\,\rangle$ from M onto rank 2 sublattices $\langle e\rangle^{\perp}$ for all $e \in M$ that can be included in a semiorthonormal base of M.

[1]this means that q is an (irreducible over $\mathbb{Q}$) product of two linear forms with real coefficients

1.7. Example: semiorthonormal basises of rank two Mukai lattices. Let M have a semiorthonormal base $\{e_0, e_1\}$ with the Gram matrix

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} , \quad \text{where } h > 0 . \tag{1-8}$$

Consider two operations L, R taking a semiorthonormal pair $\{u, v\}$ to

$$L(\{u,v\}) = \{L_v(u), v\} \text{ , where } L_v(u) = \langle\, u \,,\, v \,\rangle\, v - u$$
$$R(\{u,v\}) = \{u, R_u(v)\} \text{ , where } R_u(v) = \langle\, u \,,\, v \,\rangle\, u - v$$

They are inverse to each other, preserve the Gram matrix, and up to a sign coincide with the left and right mutations (1-6), which define an action of the braid group $B_2 = \mathbb{Z}$ on semiorthonormal bases of M. Applying R and L to the initial base $\{e_0, e_1\}$, we get a series of unit vectors $\{e_i\}_{i\in\mathbb{Z}}$ such that $R(\{e_{i-1}, e_i\}) = \{e_i, e_{i+1}\} = L(\{e_{i+1}, e_{i+2}\})$, or equivalently, $e_{i-1} + e_{i+1} = he_i \;\; \forall\, i \in \mathbb{Z}$.

1.7.1. PROPOSITION. *All semiorthonormal bases of M, up to a sign, are exhausted by $\{\pm e_i, \pm e_{i+1}\}$. If $h \geq 2$, then $e_i \neq e_j$ for $i \neq j$ and each $e \in M$ such that $\langle\, e \,,\, e \,\rangle = 1$ coincides with some $\pm e_i$, $i \in \mathbb{Z}$; the isometry group[1] $Isom(M) \cong \mathbb{Z}_2 \times \mathbb{Z}$. If $h = 1$, then there are exactly 6 vectors $e \in M$ such that $\langle\, e \,,\, e \,\rangle = 1$: up to a sign, they are $e_0 = u$, $e_1 = v$ and $e_2 = v - u$ and the isometry group is $Isom(M) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, i. e. all semiorthonormal bases of M, up to a sign, are exhausted by*

$$\{e_0, e_1\} \underset{L}{\overset{R}{\rightleftarrows}} \{e_1, e_2\} \underset{L}{\overset{R}{\rightleftarrows}} \{e_2, e_0\} \underset{L}{\overset{R}{\rightleftarrows}} \{e_0, e_1\} \; ;$$

In the both cases, $Isom(M)$ is generated by $-Id$ and $\varrho : \{e_0, e_1\} \longmapsto \{e_1, e_2\}$.

PROOF. Identify M with the integer sublattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ spanned by the cartesian base $\{e_0, e_1\}$. Then $v = xe_0 + ye_1 \in M$ with $\langle\, v \,,\, v \,\rangle = 1$ give all integer solutions of the equation $x^2 + hxy + y^2 = 1$. If $h = 1$, then all $x, y \in \mathbb{Z}$ such that

$$x^2 - xy + y^2 = (x - y)^2 + xy = 1$$

are (up to a sign) $(1, 0)$, $(0, 1)$, and $(1, 1)$. This implies the second part of the proposition.

Now, let $h \geq 2$. The operator $\varrho : \{e_0, e_1\} \longmapsto \{e_1, e_2\}$ has the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix} \in SL_2(\mathbb{Z})$$

It preserves the bilinear form, commutes with L, R, and sends $\{e_i, e_{i+1}\} \longmapsto \{e_{i+1}, e_{i+2}\}$ for all $i \in \mathbb{Z}$. So, $e_i = \varrho e_{i-1} = \varrho^i e_0 = z_i e_0 - z_{i+1} e_1$, where the integer numbers z_i form a sequence

$$\ldots , (1 - h^2), \; -h \, - 1, \; 0, \; 1, \; h, \; (h^2 - 1), \; \ldots$$

such that $z_0 = 1$, $z_1 = 0$, $hz_i = z_{i-1} + z_{i+1}$. This sequence is skew symmetric w. r. t. $i = 1$, i. e. $z_i = -z_{2-i}$. Since

$$z_{i+1}^2 - z_i z_{i+2} = \det \begin{pmatrix} z_i & z_{i+1} \\ -z_{i+1} & -z_{i+2} \end{pmatrix} = \det \varrho^i = 1$$

[1] in fact, $Isom(M)$ is nothing but the group of integer units in the quadratic extension $\mathbb{Q}[\sqrt{h^2 - 4}]$

and
$$\frac{z_{i+1}}{z_{i+2}} - \frac{z_i}{z_{i+1}} = \frac{1}{z_{i+1}z_{i+2}} = \frac{1}{z_{i+1}^2 - 1}\ ,$$
z_i are monotone increasing with $\lim\limits_{i\to\pm\infty} z_i = \pm\infty$ and $\{e_i, e_{i+1}\}$ form a base for $M\ \ \forall\, i\in\mathbb{Z}$. At the same time, When i runs through $\mathbb{Z}$ from $-\infty$ up to $+\infty$, the point $e_i = z_i e_0 - z_{i+1}e_1$ runs consequently all integer points on the right upper branch of this hyperbola, because there are no integer points inside a parallelogram spanned by e_i and e_{i+1}. This gives all the statements of the proposition and also leads to the corollary from n° 1.7.2 below. $\square$

1.7.2. COROLLARY. *For any $v\in M$ with $\langle\, v\,,\ v\,\rangle > 0$ there exists a unique $i\in\mathbb{Z}$ such that $v = ae_i + be_{i+1}$ with integer a, b of the same sign* and $a \neq 0$. $\square$

§2. Semiorthogonal decompositions in linear triangulated categories.

2.1. Tensoring and convolutions. A triangulated category $\mathscr{D}$ is called *linear*, if functor
$$Hom^\bullet(X,Y) \overset{\text{def}}{=} \bigoplus_\nu Hom^\nu(X,Y)$$
takes values in the triangulated category of finite dimensional graded vector spaces. We will use *the tensor product* $V^\bullet\otimes W^\bullet$ and *the dualization* $V^\bullet \longmapsto V^{\times\bullet}$ defined for graded vector spaces by the usual rules
$$Gr^p(V^\bullet\otimes W^\bullet) \overset{\text{def}}{=} \bigoplus_{\alpha+\beta=p} Gr^\alpha(V^\bullet)\otimes Gr^\beta(W^\bullet) \quad ; \qquad Gr^p(V^{\times\bullet}) \overset{\text{def}}{=} Gr^{-p}(V^\bullet)$$

If $V^\bullet$ is a graded vector space and X is an object of a linear triangulated category $\mathscr{D}$, then the tensor product $V^\bullet\otimes X$ is defined by the universal property
$$Hom^\bullet(Y, V^\bullet\otimes X) = V^\bullet\otimes Hom^\bullet(Y,X) \quad \forall\, Y\in Ob\,\mathscr{D}$$
and can be constructed as $V^\bullet\otimes X = \bigoplus\limits_\alpha V^\alpha\otimes X[-\alpha]$, where $V^\alpha\otimes X[-\alpha]$ is $\dim V^\alpha$-times direct sum $X[-\alpha]$ with itself. A covariant exact (or cohomology) functor $\mathrm{Cov}^\bullet$ between linear categories is called *linear* if $\mathrm{Cov}^\bullet(V^\bullet\otimes X) = V^\bullet\otimes\mathrm{Cov}^\bullet(X)$. Similarly, a contravariant *linear* functor $\mathrm{Cnt}^\bullet$ must satisfy the condition $\mathrm{Cov}(V^\bullet\otimes X) = V^{\times\bullet}\otimes\mathrm{Cov}(X)$. For example, $Hom^\bullet(*,*)$ is bilinear, i. e.
$$Hom^\bullet(W^\bullet\otimes X, V^\bullet\otimes Y) = W^{\times\bullet}\otimes V^\bullet\otimes Hom^\bullet(X,Y)\ .$$
This implies, in particular, that the identity endomorphism from
$$End(Hom^\bullet(X,Y)) = Hom^\bullet(Hom^\bullet(X,Y)\otimes X, Y) = Hom^\bullet(X, Hom^{\times\bullet}(X,Y)\otimes Y)$$

induces two canonical convolutions

$$\begin{array}{rcl} j^*(X,Y) & : & Hom^\bullet(X,Y)\otimes X \longrightarrow Y \\ j_*(X,Y) & : & X \longrightarrow Hom^{\times\bullet}(X,Y)\otimes Y \ . \end{array} \tag{2-1}$$

2.2. Exceptional objects and mutations. An object $E \in Ob\,\mathscr{D}$ is called *exceptional*, if $Hom^\bullet(E,E)$ is 1-dimensional algebra generated by the identity map. For an exceptional E and any $X \in \mathscr{D}$ *a left mutation* L_EX and *a right mutation* R_EX of X w.r.t. E are defined by the distinguished triangles

$$L_EX \xrightarrow{k^*} Hom^\bullet(E,X)\otimes E \xrightarrow{j^*} X \xrightarrow{i^*} L_EX[1] \tag{2-2}$$

$$R_EX[-1] \xrightarrow{i_*} X \xrightarrow{j_*} Hom^{\times\bullet}(X,E)\otimes E \xrightarrow{k_*} R_EX \tag{2-3}$$

extending the canonical maps $j^* = j^*(E,X)$ and $j_* = j_*(X,E)$ from (2-1). Applying $Hom^\bullet(E,*)$ to the triangle (2-2), we get the orthogonality relation

$$Hom^\bullet(E, L_EX) = 0 \tag{2-4}$$

since $Hom^\bullet(E,*)$ takes $j^*(E,X)$ to the identity map. Similarly, $Hom^\bullet(*,E)$ applied the triangle (2-3) takes $j_*(E,X)$ to the identity map and gives the orthogonality relation

$$Hom^\bullet(R_EX, E) = 0 \tag{2-5}$$

Full triangulated subcategories

$$E^\perp \overset{\text{def}}{=} \{X \in Ob\,\mathscr{D} \mid Hom^\bullet(E,X) = 0\}$$

$${}^\perp E \overset{\text{def}}{=} \{X \in Ob\,\mathscr{D} \mid Hom^\bullet(X,E) = 0\}$$

are called *right* and *left orthogonals* to E in $\mathscr{D}$. Taking $E' \in Ob\,{}^\perp E$, $E'' \in Ob\,E^\perp$ and applying $Hom^\bullet(E',*)$ and $Hom^\bullet(*,E'')$ to the both mutation triangles (2-2) and (2-3), we get for any $X \in Ob\,\mathscr{D}$ the following bifunctor isomorphisms

$$Hom^\bullet(E',X) = Hom^\bullet(E',R_EX)[-1] = Hom^\bullet(E',L_EX)[1] = Hom^\bullet(L_EE',L_EX) \tag{2-6}$$

$$Hom^\bullet(X,E'') = Hom^\bullet(L_EX,E'')[-1] = Hom^\bullet(R_EX,E'')[1] = Hom^\bullet(R_EX,R_EE'') \tag{2-7}$$

(the right identities follow from the left ones, because of $L_EX \in Ob\,E^\perp$ and $R_EX\,Ob\,{}^\perp E$). So, L_E and R_E are shifted 'orthogonal projections' onto the right and left orthogonals of E. More precisely, we have

2.2.1. PROPOSITION. *$\mathscr{D} \xrightarrow{X\mapsto R_EX[-1]} {}^\perp E$ is the right adjoint functor*[1] *to the inclusion ${}^\perp E \hookrightarrow \mathscr{D}$ and $\mathscr{D} \xrightarrow{X\mapsto L_EX[1]} E^\perp$ is the left adjoint functor to the inclusion $E^\perp \hookrightarrow \mathscr{D}$. They satisfy the identities $L_E\circ R_E = L_E[1]$ and $R_E\circ L_E = R_E[-1]$ and their restrictions*

$$L_E|_{{}^\perp E} : {}^\perp E \longrightarrow E^\perp \quad \text{and} \quad R_E|_{E^\perp} : E^\perp \longrightarrow {}^\perp E$$

[1] Note that the third term of a distinguished triangle is not *canonicaly* defined, in general, by the other two terms. In our case L_EX and R_EX are *unique up to unique* isomorphism, because of the orthogonality conditions $Hom^\bullet(E,L_EX) = Hom^\bullet(R_EX,E) = 0$. Some generalizations can be founded in [BBD]

are inverse to each other isomorphisms between these subcategories.

PROOF. The functoriality of the both projectors $L_E[1]$ and $R_E[-1]$ follows form the relations $Hom^\bullet(E',X) = Hom^\bullet(E', R_E X[-1])$ and $Hom^\bullet(X,E'') = Hom^\bullet(L_E X[1], E'')$, which are the parts of (2-6) and (2-7) and mean that the both functors are adjoint to the corresponding inclusions. Applying L_E and R_E to the mutation triangles (2-3) and (2-2), we get the identities $L_E \circ R_E = R_E[1]$ and $R_E \circ L_E = L_E[-1]$, because of $R_E E = L_E E = 0$. Further, (2-6) and (2-7) imply that both restrictions $L_E|_{{}^\perp E} : {}^\perp E \longrightarrow E^\perp$ and $R_E|_{E^\perp} : E^\perp \longrightarrow {}^\perp E$ preserve $Hom^\bullet(*,*)$'s. Finally, $R_E L_E E'' = R_E E''[-1] = E''$, if $E'' \in Ob\, E^\perp$, and $L_E R_E E' = L_E E'[1] = E'$, if $E' \in Ob\, {}^\perp E$. □

2.2.2. PROPOSITION. *$Hom^{\times\,\bullet}(L_E X, E) = Hom^\bullet(E, R_E X)[-1]$ functorially on X, E.*

PROOF. Applying $Hom^\bullet(*,E)$ to (2-2) and $Hom^\bullet(E,*)$ to (2-3), we get the following distinguished triangles of graded vector spaces

$$\begin{array}{c} Hom^\bullet(L_E X, E)[-1] \longrightarrow Hom^\bullet(X,E) \xrightarrow{\lambda} Hom^{\times\,\bullet}(E,X) \longrightarrow Hom^\bullet(L_E X, E) \\ Hom^\bullet(E, R_E X) \longleftarrow Hom^{\times\,\bullet}(X,E) \xleftarrow{\varrho} Hom^\bullet(E,X) \longleftarrow Hom^\bullet(E, R_E X)[-1] \end{array}$$

where the linear operators λ, ϱ satisfy for all $\varphi \in Hom^\bullet(X,E)$, $\psi \in Hom^\bullet(E,X)$ the identity

$$\varrho\psi(\varphi)\cdot \mathrm{Id}_E = \varphi\circ\psi = \lambda\varphi(\psi)\cdot \mathrm{Id}_E\,,$$

because of the definition of the canonical convolutions (2-1). So, these triangles are mapped to each other by the $\times$-functor from n° 2.1. □

2.3. Mutations of exceptional pairs. An ordered pair $\{E_1, E_2\}$ of exceptional objects is called *exceptional* or *semiorthogonal*, if $Hom^\bullet(E_2, E_1) = 0$. Applying (2-6) and (2-7) to the category $\mathscr{D}$ generated[1] by an exceptional pair $\{E_1, E_2\}$, we see that the pairs

$$L\{E_1,E_2\} \overset{\text{def}}{=} \{L_{E_1}E_2, E_1\} \qquad \text{and} \qquad R\{E_1,E_2\} \overset{\text{def}}{=} \{E_2, R_{E_2}E_1\}$$

are exceptional as well and generate the same category $\mathscr{D}$. They are called the *left mutation* and *right mutation* of the initial pair $\{E_1, E_2\}$. Since $R_{E_1}E_2[-1] = E_2$ and $L_{E_2}E_1[1] = E_1$, we have

$$\begin{array}{c} Hom^{\times\,\bullet}(L_{E_1}E_2, E_1) = Hom^\bullet(E_1, R_{E_1}E_2[-1]) = Hom^\bullet(E_1,E_2) \\ Hom^\bullet(E_2, R_{E_2}E_1) = Hom^{\times\,\bullet}(L_{E_2}E_1[1], E_1) = Hom^{\times\,\bullet}(E_1,E_2) \end{array}$$

by (2-6), (2-7). Hence, the triangles

$$\begin{array}{c} L_{E_1}E_2 \longrightarrow Hom^\bullet(E_1,E_2)\otimes E_1 \longrightarrow E_2 \longrightarrow L_{E_1}E_2[1] \\ L_{E_1}E_2 \longrightarrow Hom^{\times\,\bullet}(L_{E_1}E_2, E_1)\otimes E_1 \longrightarrow R_{E_1}L_{E_1}E_2 \longrightarrow L_{E_1}E_2[1] \end{array}$$

coincide canonically, that is $R\circ L\,\{E_1,E_2\} = \{E_1,E_2\}$. Similarly, $L\circ R\,\{E_1,E_2\} = \{E_1,E_2\}$. So, R and L act on the set of all exceptional pairs as inverse transformations.

[1] a category *generated* by a collection of objects $E_1, E_2, \dots, E_m$ always means the smallest full triangulated subcategory containing all 'linear combinations' $\bigoplus_\nu V_\nu^\bullet \otimes E_\nu$, where $V_\nu^\bullet$ are graded vector spaces

2.4. Exceptional collections and braid group action. An ordered collection of exceptional objects $\{E_1, E_2, \ldots, E_k\}$ is called *exceptional* or *semiorthogonal*, if

$$Hom^\bullet(E_j, E_i) = 0 \qquad \forall\, i < j\ .$$

2.4.1. Mutations of an exceptional collection $\{E_0, E_1, \ldots, E_k\}$ are defined as follows. For $1 \le i \le k$ a *left mutation* L_i replaces i-th pair of consequent elements $\{E_{i-1}, E_i\}$ by its left mutation $\{L_{E_{i-1}}E_i, E_{i-1}\}$. A *right mutation* R_i replaces the same pair by its right mutation $\{E_i, R_{E_i}E_{i-1}\}$.

2.4.2. PROPOSITION. *The mutations preserve the exceptionality and satisfy the relations*

$$L_iR_i = R_iL_i = Id \tag{2-8}$$

$$L_iL_j = L_jL_i \text{ for } |i-j| > 1 \tag{2-9}$$

$$L_{i+1}L_iL_{i+1} = L_iL_{i+1}L_i \text{ for } 1 < i < k \tag{2-10}$$

So, the braid group B_{k+1} acts by the mutations on exceptional collections of length $(k+1)$.

PROOF. Since the mutations L_i and R_i make changes only inside the category generated by the pair $\{E_{i-1}, E_i\}$, the exceptionality of the muted collection as well as the relation (2-8) follows from the discussion in the previous section. The relations (2-9) are evident. The

$$\begin{tikzcd}
 & \{A, B, C\} \arrow[dl, "L_1"'] \arrow[dr, "L_2"] & \\
\{L_AB,\ A,\ C\} \arrow[d, "L_2"'] & & \{A,\ L_BC,\ B\} \arrow[d, "L_1"] \\
\{L_AB,\ L_AC,\ A\} \arrow[dr, "L_1"'] & & \{L_AL_BC,\ A,\ B\} \arrow[dl, "L_2"] \\
 & \{L_{L_AB}L_AC,\ A,\ L_AB\} &
\end{tikzcd}$$

Fig. A. The triangle relation (2-10) for the mutations

triangle equation (2-10) means that the diagram in fig. A on the page 11 is commutative for any exceptional triple $\{A, B, C\}$. The key point is to check the coincidence

$$L_AL_BC = L_{L_AB}L_AC$$

prescribed by the lower right arrow L_2. To do this we apply the exact linear functor L_A to the canonical triangle

$$L_BC \longrightarrow Hom^\bullet(B, C) \otimes B \longrightarrow C$$

and remember that $Hom^\bullet(B, C) = Hom^\bullet(L_AB, L_AC)$ by (2-6). So, we have the exact triangle

$$L_AL_BC \longrightarrow Hom^\bullet(L_AB, L_AC) \otimes L_AB \longrightarrow L_BC$$

as required. $\square$

2.5. Subcategories $[E_i \,\dots\, E_j]$. We write $[E_1 \,\dots\, E_k]$ for the full triangulated subcategory generated by an exceptional collection $\{E_1, E_2, \dots, E_k\}$ in a linear triangulated category $\mathscr{D}$. For any $X \in Ob\,\mathscr{D}$ consider compositions of its consequent mutations w. r. t. elements of the exceptional collection $E_1, E_2, \dots, E_k$

$$\begin{aligned} L_{[E_1 \dots E_k]} &: X \longmapsto L_{E_1} L_{E_2} \dots L_{E_k} X \\ R_{[E_1 \dots E_k]} &: X \longmapsto R_{E_k} R_{E_{k-1}} \dots R_{E_1} X \end{aligned}$$

Applying inductively (2-6) and (2-7) to the consequent mutations, we get for all $X \in Ob\,\mathscr{D}$, $E' \in Ob\,{}^{\perp}[E_1 \,\dots\, E_k]$, $E'' \in Ob\,[E_1 \,\dots\, E_k]^{\perp}$ the functorial isomorphisms

$$\begin{aligned} Hom^{\bullet}(E', X) = Hom^{\bullet}(E', R_{[E_1 \dots E_k]}X)[-k] = Hom^{\bullet}(E', L_{[E_1 \dots E_k]}X)[k] = \\ = Hom^{\bullet}(L_{[E_1 \dots E_k]}E', L_{[E_1 \dots E_k]}X) \end{aligned} \tag{2-11}$$

$$\begin{aligned} Hom^{\bullet}(X, E'') = Hom^{\bullet}(L_{[E_1 \dots E_k]}X, E'')[-k] = Hom^{\bullet}(R_{[E_1 \dots E_k]}X, E'')[k] = \\ = Hom^{\bullet}(R_{[E_1 \dots E_k]}X, R_{[E_1 \dots E_k]}E'') \end{aligned} \tag{2-12}$$

The proposition from n° 2.2.1 is obviously extended as follows.

2.5.1. PROPOSITION. *The functor* $\mathscr{D} \xrightarrow{X \mapsto L_{[E_1 \dots E_k]}X[k]} [E_1 \,\dots\, E_k]^{\perp}$ *is left adjoint for the inclusion* $[E_1 \,\dots\, E_k]^{\perp} \hookrightarrow \mathscr{D}$ *and the functor* $\mathscr{D} \xrightarrow{X \mapsto R_{[E_1 \dots E_k]}X[-k]} {}^{\perp}[E_1 \,\dots\, E_k]$ *is right adjoint for the inclusion* ${}^{\perp}[E_1 \,\dots\, E_k] \hookrightarrow \mathscr{D}$. *They satisfy the relations*

$$R_{[E_1 \dots E_k]} L_{[E_1 \dots E_k]} = R_{[E_1 \dots E_k]}[-k] \quad \text{and} \quad L_{[E_1 \dots E_k]} R_{[E_1 \dots E_k]} X = L_{[E_1 \dots E_k]} X[k] \tag{2-13}$$

and their restrictions

$$\begin{aligned} L_{[E_1 \dots E_k]}|_{{}^{\perp}[E_1 \dots E_k]} &: \quad {}^{\perp}[E_1 \,\dots\, E_k] \longrightarrow [E_1 \,\dots\, E_k]^{\perp} \\ R_{[E_1 \dots E_k]}|_{[E_1 \dots E_k]^{\perp}} &: \quad [E_1 \,\dots\, E_k]^{\perp} \longrightarrow {}^{\perp}[E_1 \,\dots\, E_k] \end{aligned}$$

give inverse isomorphisms between the left and right orthogonals to $[E_1 \,\dots\, E_k]$.

PROOF. Indeed, (2-11) implies that

$$Hom^{\bullet}(E', X) = Hom^{\bullet}(E', L_{[E_1 \dots E_k]}X)[k] = Hom^{\bullet}(E', R_{[E_1 \dots E_k]} L_{[E_1 \dots E_k]}X)\,,$$

that is $R_{[E_1 \dots E_k]} L_{[E_1 \dots E_k]}$ is the right adjoint functor to the inclusion ${}^{\perp}[E_1 \,\dots\, E_k] \hookrightarrow \mathscr{D}$, i. e. it has to coincide with $R_{[E_1 \dots E_k]}[-k]$. The second relation in (2-13) is proved similarly. Other statements are proved using (2-11) (2-12) precisely like in n° 2.2.1. □

2.5.2. COROLLARY. *The functors* $L_{[E_1 \dots E_k]}$ *and* $R_{[E_1 \dots E_k]}$ *depend only on the subcategory* $[E_1 \,\dots\, E_k]$ *but not on the particular choice of its semiorthonormal base* $E_1, E_2, \dots, E_k$ *).* □

2.6. Dual exceptional bases. Let a linear triangulated category $\mathscr{D}$ be generated by an exceptional collection $\{E_0, E_1, \dots, E_n\}$. We will refer such a collection as an *exceptional base* of $\mathscr{D}$. Two collections of objects $\{{}^{\vee}E_0, {}^{\vee}E_1, \dots, {}^{\vee}E_n\}$ and $\{E_0^{\vee}, E_1^{\vee}, \dots, E_n^{\vee}\}$ are called respectively *a left dual* base and *a right dual* base if all $Hom^{\alpha}({}^{\vee}E_{\beta}, E_{\gamma})$ and $Hom^{\alpha}(E_{\beta}, E_{\gamma}^{\vee})$ vanish except only for

$$Hom^{k}\,(\,{}^{\vee}E_k, E_{n-k}\,) = Hom^{n-k}\,(\,E_{n-k}, E_k^{\vee}\,) = \mathbb{C}$$

Dual bases exist and can be constructed explicitly by consequent mutations as follows:

$$\begin{aligned} E_k^\vee &\stackrel{\text{def}}{=} L_{[E_0 \dots E_{n-k-1}]} E_{n-k} = L_{E_0} L_{E_1} \dots L_{E_{n-k-1}} E_{n-k} \\ {}^\vee E_k &\stackrel{\text{def}}{=} R_{[E_{n-k+1} \dots E_n]} E_{n-k} = R_{E_n} R_{E_{n-1}} \dots R_{E_{n-k+1}} E_{n-k} \end{aligned} \tag{2-14}$$

Indeed, $L_{[E_0 \dots E_{n-k-1}]} E_{n-k} \in [E_0 \, \dots \, E_{n-k-1}]^\perp$ and by (2-11) we also have

$$\begin{aligned} Hom^\bullet(E_\nu, L_{[E_0 \dots E_{n-k-1}]} E_{n-k}) &= Hom^\bullet(E_\nu, E_{n-k})[-(n-k)] = \\ &= \begin{cases} 0\,, \text{ if } \nu > (n-k) \\ \mathbb{C} \text{ in degree } (n-k) \text{ for } \nu = (n-k) \end{cases} \end{aligned}$$

So, $E_k^\vee = L_{[E_0 \dots E_{n-k-1}]} E_{n-k}$ satisfies the orthogonality conditions. The arguments for ${}^\vee E_k = R_{[E_{n-k+1} \dots E_n]} E_{n-k}$ are quite similar. In fact, the dual bases are uniquely up to unique isomorphism defined by the above orthogonality conditions. This follows from

2.6.1. PROPOSITION. *If $\{{}^\vee E_0, {}^\vee E_1, \dots, {}^\vee E_n\}$ and $\{E_0^\vee, E_1^\vee, \dots, E_n^\vee\}$ satisfy the dual base orthogonality conditions, then ${}^\vee E_k$ represents the covariant functor*

$$X \longmapsto Hom^\bullet(E_{n-k}, L_{[E_0 \dots E_{n-k}]} X)$$

and $E_k^\vee$ represents the contravariant functor

$$X \longmapsto Hom^{\times\,\bullet}(E_{n-k}, L_{[E_0 \dots E_{n-k}]} X)[-n]\,.$$

In particular, for any $X \in Ob\,\mathscr{D}$ there are functorial isomorphisms

$$Hom^\bullet({}^\vee E_k, X)[n] = Hom^{\times\,\bullet}(X, E_k^\vee) \tag{2-15}$$

PROOF. Since $L_{[E_0 \dots E_n]} X \in [E_0 \, \dots \, E_n]^\perp = 0$, applying $Hom^\bullet({}^\vee E_k, *)$ and $Hom^\bullet(*, E_k^\vee)$ to the left mutation triangles

$$Hom^\bullet(E_\nu, L_{[E_{\nu+1} \dots E_n]} X) \otimes E_\nu \longrightarrow L_{[E_{\nu+1} \dots E_n]} X \longrightarrow L_{[E_\nu \dots E_n]} X[1] \tag{2-16}$$

consequently for $\nu = 0,\, 1,\, \dots\, ,\, (n-k-1)$, we get

$$Hom^\bullet({}^\vee E_k, L_{[E_\nu \dots E_n]} X) = Hom^\bullet(L_{[E_\nu \dots E_n]} X, E_k^\vee) = 0 \quad \text{ for } 0 \le \nu \le (n-k)\,.$$

Hence, using (2-11) and applying $Hom^\bullet({}^\vee E_k, *)$ to (2-16) written for $\nu = n-k$, we get

$$\begin{aligned} Hom^\bullet({}^\vee E_k, X) &= Hom^\bullet({}^\vee E_k, L_{[E_{n-k+1} \dots E_n]} X)[k] = \\ &= Hom^\bullet(E_{n-k}, L_{[E_{n-k+1} \dots E_{n-k}]} X) \otimes Hom^\bullet({}^\vee E_k, E_{n-k}) = \\ &= Hom^\bullet(E_{n-k}, L_{[E_{n-k+1} \dots E_{n-k}]} X) \end{aligned}$$

because of $Hom^\bullet({}^\vee E_k, E_{n-k}) = \mathbb{C}[-k]$. Similarly, applying $Hom^\bullet(*, E_k^\vee)$ to the same triangle and using (2-12), we get

$$\begin{aligned} Hom^\bullet(X, E_k^\vee) &= Hom^\bullet(L_{[E_{n-k+1} \dots E_n]} X, E_k^\vee)[-k] = \\ &= Hom^{\times\,\bullet}(E_{n-k}, L_{[E_{n-k+1} \dots E_{n-k}]} X) \otimes Hom^\bullet(E_{n-k}, E_k^\vee)[-n] = \\ &= Hom^{\times\,\bullet}(E_{n-k}, L_{[E_{n-k+1} \dots E_{n-k}]} X)[-n] \end{aligned}$$

because of $Hom^\bullet(E_{n-k}, E_k^\vee) = \mathbb{C}[-(n-k)]$. □

2.6.2. COROLLARY. *If $\mathscr{D} = [E_0 \,\ldots\, E_n]$ is generated by an exceptional collection, then there are the following isomorphisms of functors:*

$$\begin{tikzcd} Hom^\bullet({}^\vee E_{n-k}, X)[n] \arrow[r, equal] \arrow[d, equal] & Hom^\bullet(E_k, L_{[E_{k+1}\,\ldots\,E_n]}X)[n] \arrow[d, equal] \\ Hom^{\times\,\bullet}(R_{[E_0\,\ldots\,E_{k-1}]}X, E_k) \arrow[r, equal] & Hom^{\times\,\bullet}(X, E^\vee_{n-k}) \end{tikzcd} \tag{2-17}$$

PROOF. The upper and right identities was proved just before. If we change those arguments by their 'right' version, that is, apply $Hom^\bullet({}^\vee E_k, *)$ and $Hom^\bullet(*, E^\vee_k)$ to the triangle

$$R_{[E_0\,\ldots\,E_{k-1}]}X \longrightarrow Hom^{\times\,\bullet}(R_{[E_0\,\ldots\,E_{k-1}]}X, E_k) \otimes E_k \longrightarrow R_{[E_0\,\ldots\,E_k]}X$$

and use (2-11), (2-12), and the identities

$$Hom^\bullet({}^\vee E_{n-k}, R_{[E_0\,\ldots\,E_k]}X) = Hom^\bullet(R_{[E_0\,\ldots\,E_k]}X, E^\vee_{n-k}) = 0$$

which come by induction on $[E_\nu \,\ldots\, E_n]$ starting from $\nu = 0$, for which

$$R_{[E_0\,\ldots\,E_n]}X \in {}^\perp[E_0 \,\ldots\, E_n] = 0\ ,$$

then we get

$$\begin{aligned} &Hom^{\times\,\bullet}(R_{[E_0\,\ldots\,E_{k-1}]}X, E_k) = \\ &\qquad = Hom^{\times\,\bullet}(R_{[E_0\,\ldots\,E_{k-1}]}X, E_k) \otimes Hom^\bullet({}^\vee E_{n-k}, E_k)[n-k] = \\ &\qquad\qquad = Hom^\bullet({}^\vee E_{n-k}, R_{[E_0\,\ldots\,E_{k-1}]}X)[n-k] = Hom^\bullet({}^\vee E_k, X)[n] \end{aligned}$$

$$\begin{aligned} &Hom^\bullet(R_{[E_0\,\ldots\,E_{k-1}]}X, E_k) = \\ &\qquad = Hom^\bullet(R_{[E_0\,\ldots\,E_{k-1}]}X, E_k) \otimes Hom^\bullet(E_k, E^\vee_{n-k})[k] = \\ &\qquad\qquad = Hom^\bullet(R_{[E_0\,\ldots\,E_{k-1}]}X, E^\vee_{n-k})[k] = Hom^\bullet(X, E^\vee_{n-k}) \end{aligned}$$

$\square$

2.7. Canonical Postnikov systems.. We continue to assume that $\mathscr{D}$ is generated by an exceptional collection $\{E_0, E_1, \ldots, E_n\}$ and write $V^\bullet_k$ for the graded vector spaces (2-17)

$$\begin{aligned} V^\bullet_k &\stackrel{\text{def}}{=} Hom^\bullet({}^\vee E_{n-k}, X)[n] = Hom^{\times\,\bullet}(X, E^\vee_{n-k}) = \\ &\qquad = Hom^\bullet(E_k, L_{[E_{k+1}\,\ldots\,E_n]}X)[n] = Hom^{\times\,\bullet}(R_{[E_0\,\ldots\,E_{k-1}]}X, E_k)\ , \end{aligned}$$

So, the triangles defining the consequent right mutations of X and the consequent left mutations of $X[n]$ through $[E_0 \,\ldots\, E_n]$ can be written as

$$\begin{aligned} &V_\nu \otimes E_\nu[-1] \xrightarrow{k_\nu} R_{[E_0\,\ldots\,E_\nu]}X[-1] \xrightarrow{i_\nu} R_{[E_0\,\ldots\,E_{\nu-1}]}X \xrightarrow{j_\nu} V_\nu \otimes E_\nu \\ &V_\nu \otimes E_\nu \xrightarrow{j^{\nu+1}} L_{[E_{\nu+1}\,\ldots\,E_n]}X[n] \xrightarrow{i^{\nu+1}} L_{[E_\nu\,\ldots\,E_n]}X[n+1] \xrightarrow{k^{\nu+1}} V_\nu \otimes E_\nu[1]\ , \end{aligned} \tag{2-18}$$

where $\nu = 0, 1, \ldots, n$. We arrange them into a big diagram in fig. B on the page 15. At this diagram all oriented triangles along the left and right vertical borders are distinguished, the morphisms $i_\bullet$ and $i^\bullet$ shown by broken arrows have degree 1, and all triangles and rhombuses in the central column are commutative. So, there is a complex (functorial in X)

$$0 \longrightarrow V_0^\bullet \otimes E_0 \longrightarrow V_1^\bullet \otimes E_1 \longrightarrow \cdots \longrightarrow V_{n-1}^\bullet \otimes E_{n-1} \longrightarrow V_n^\bullet \otimes E_n \longrightarrow 0 \tag{2-19}$$

$$\begin{tikzcd}
R_{[E_0\,\dots\,E_n]}X = 0 \arrow[dd, dashed, "i_n"'] & & X[n] \arrow[dd, dashed, "i^0"] \\
& V_n \otimes E_n \arrow[ul, "k_n"'] \arrow[ur, "j^0"] & \\
R_{[E_0\,\dots\,E_{n-1}]}X \arrow[ur, "j_n", "\sim"'] \arrow[dd, dashed, "i_{n-1}"'] & & L_{E_n}X[n] \arrow[ul, "k^0"'] \arrow[dd, dashed, "i^1"] \\
& V_{n-1} \otimes E_{n-1} \arrow[uu, "d_{n-1}"] \arrow[ul, "k_{n-1}"'] \arrow[ur, "j^1"] & \\
R_{[E_0\,\dots\,E_{n-2}]}X \arrow[ur, "j_{n-1}"] \arrow[dd, dashed, "i_{n-2}"'] & & L_{[E_{n-1}\,\dots\,E_n]}X[n] \arrow[ul, "k^1"'] \arrow[dd, dashed, "i^2"] \\
& V_{n-2} \otimes E_{n-2} \arrow[uu, "d_{n-2}"] \arrow[ul, "k_{n-2}"'] \arrow[ur, "j^2"] & \\
\vdots \arrow[dd, dashed, "i_2"'] & \vdots & \vdots \arrow[dd, dashed, "i^{n-2}"] \\
& V_2 \otimes E_2 & \\
R_{[E_0\,\dots\,E_1]}X \arrow[ur, "j_2"] \arrow[dd, dashed, "i_1"'] & & L_{[E_2\,\dots\,E_n]}X[n] \arrow[ul, "k^{n-2}"'] \arrow[dd, dashed, "i^{n-1}"] \\
& V_1 \otimes E_1 \arrow[uu, "d_1"] \arrow[ul, "k_1"'] \arrow[ur, "j^{n-1}"] & \\
R_{E_0}X \arrow[ur, "j_1"] \arrow[dd, dashed, "i_0"'] & & L_{[E_1\,\dots\,E_n]}X[n] \arrow[ul, "k^{n-1}"'] \arrow[dd, dashed, "i^n"] \\
& V_0 \otimes E_0 \arrow[uu, "d_0"] \arrow[ul, "k_0"'] \arrow[ur, "j^n"] & \\
X \arrow[ur, "j_0"] & & 0 = L_{[E_0\,\dots\,E_n]}X[n] \arrow[ul, "k^n"']
\end{tikzcd}$$

Fig. B. Canonical Postnikov systems.

whose differentials $V_p^\bullet \otimes E_p \xrightarrow{d^p} V_{p+1}^\bullet \otimes E_{p+1}$ equal $d^p = k^{-p-1} \circ j^{-p} = j_{n+p+2} \circ k_{n+p+1}$. The left mutations produce a canonical *left Postnikov system*[1], which naturally identifies X with a canonical (left) convolution of this complex. Similarly, the right mutations produce a canonical (right) convolution[2] of this complex and identify it with X as well. Let us outline again, that the both convolutions *naturally* depend on X.

2.7.1. Orthogonal decomposition. It follows from the diagram on fig. B on the page 15 that the orthogonal projector $\mathscr{D} \xrightarrow{X \mapsto L_{[E_{k+1} \dots E_n]} X[n-k]} [E_{k+1} \dots E_n]^\perp$ from n° 2.5.1 takes its value inside the category $[E_0 \dots E_k]$, because the left mutation $L_{[E_{k+1} \dots E_n]}X$ is nothing but the canonical contraction of the complex (2-19) truncated on the right:

$$0 \longrightarrow V_0^\bullet \otimes E_0 \longrightarrow V_1^\bullet \otimes E_1 \longrightarrow \cdots \longrightarrow V_{k-1}^\bullet \otimes E_{k-1} \longrightarrow V_k^\bullet \otimes E_k \longrightarrow 0$$

Since $[E_0 \dots E_k] \subset [E_{k+1} \dots E_n]^\perp$, we have the coincidence

$$[E_{k+1} \dots E_n]^\perp = [E_0 \dots E_k] \tag{2-20}$$

Similarly, the right mutation $R_{[E_0 \dots E_k]}X$ equals the canonical contraction of the complex (2-19) truncated on the left:

$$0 \longrightarrow V_{k+1}^\bullet \otimes E_{k+1} \longrightarrow V_{k+2}^\bullet \otimes E_{k+2} \longrightarrow \cdots \longrightarrow V_{n-1}^\bullet \otimes E_{n-1} \longrightarrow V_n^\bullet \otimes E_n \longrightarrow 0$$

So, the projector $\mathscr{D} \xrightarrow{X \mapsto R_{[E_0 \dots E_k]} X[-k-1]} {}^\perp[E_0 \dots E_k]$ maps $\mathscr{D}$ into $[E_{k+1} \dots E_n]$, that is

$${}^\perp[E_0 \dots E_k] = [E_{k+1} \dots E_n] \tag{2-21}$$

Moreover, for each $k = 0, 1, \dots, n$ we have two natural transformations of functors:

$$R_{[E_0 \dots E_k]}X[-k-1] \xrightarrow{i_k \circ \dots \circ i_0} X \xrightarrow{i^{n-k-1} \circ \dots \circ i^0} L_{[E_{k+1} \dots E_n]}X[n-k] \tag{2-22}$$

2.7.2. COROLLARY. *The triangle* (2-22) *is distinguished and gives a functorial orthogonal decomposition of any object* $X \in \mathrm{Ob}\,\mathscr{D}$ *by the full triangulated subcategories* ${}^\perp[E_0 \dots E_k] = [E_{k+1} \dots E_n]$ *and* $[E_{k+1} \dots E_n]^\perp = [E_0 \dots E_k]$.

---

[1] the Postnikov system notion was introduced in this context in [Kp], see also [GeMa, p. 257], although our indexing style here is slightly different — we proceed from the following observation: if we consider the first two terms of a distinguished triangle

$$E^0 \xrightarrow{d_0} E^1 \longrightarrow \mathrm{Con}(d_0)$$

in a homotopy category of complexes as a bicomplex with $E^{0,0}$ situated at the origin, then the convolution of this complex coincides with the *twisted* cone $\mathrm{Con}(d_0)[-1]$; so, taking consequent cones from the left to the right in

$$\begin{tikzcd}
E^0 \arrow[rr] \arrow[dr, equal] & & E^1 \arrow[rr] \arrow[dr] & & E^2 \arrow[rr] & & \cdots \arrow[rr] & & E^n \arrow[dr] & \\
& T^0 \arrow[ur] & & T^1[1] \arrow[ll, dashed, "{[1]}"] \arrow[ur] & & \cdots \arrow[ll, dashed, "{[1]}"] & & T^{n-1}[n-1] \arrow[ll, dashed, "{[1]}"] \arrow[ur] & & T^n[n] \arrow[ll, dashed, "{[1]}"]
\end{tikzcd}$$

we get the total complex T^n twisted by n

[2] or a *right Postnikov system*, see loc.cit.

Proof. Consider any $Y \in Ob\,\mathscr{D}$ introduced by distinguished triangle

$$Y[-1] \longrightarrow R_{[E_0\,\dots\,E_k]}X[-k-1] \xrightarrow{i_k\circ\cdots\circ i_0} X \xrightarrow{\varphi} Y \tag{2-23}$$

For any $Z' \in Ob\,{}^{\perp}[E_0\,\dots\,E_k] = [E_{k+1}\,\dots\,E_n]$ we have the functorial isomorphism

$$Hom^\bullet(Z, R_{[E_0\,\dots\,E_k]}X[-k-1]) \xrightarrow[\quad]{\psi\mapsto i_k\circ\cdots\circ i_0\circ\psi} Hom^\bullet(Z', X)$$

given exactly by left multiplication by $i_k \circ \cdots \circ i_0$ (the functor $X \mapsto R_{[E_0\,\dots\,E_k]}X[-k]$ is right adjoint to the inclusion ${}^{\perp}[E_0\,\dots\,E_k] \hookrightarrow \mathscr{D}$, see n° 2.5.1). Hence, $Hom^\bullet(Z', Y) = 0$ for any such Z', i. e. $Y \in Ob\,[E_{k+1}\,\dots\,E_n]^{\perp} = [E_0\,\dots\,E_k]$. Now, taking $Z'' \in Ob\,[E_{k+1}\,\dots\,E_n]^{\perp} = [E_0\,\dots\,E_k]$ and applying $Hom^\bullet(*, Z'')$ to the triangle (2-23), we get an isomorphism

$$Hom^\bullet(Y, Z'') \xrightarrow[\quad]{\psi\mapsto\psi\circ\varphi} Hom^\bullet(X, Z'')$$

induced by right multiplication by φ. This means that the correspondence $X \mapsto Y$ is the left adjoint functor to the inclusion $[E_{k+1}\,\dots\,E_n]^{\perp} \hookrightarrow \mathscr{D}$, that is, the morphism $X \xrightarrow{\varphi} Y$ coincides with the morphism $X \xrightarrow{i^{n-k-1}\circ\cdots\circ i^0} L_{[E_{k+1}\,\dots\,E_n]}X[n-k]$ (see n° 2.5.1). □

2.7.3. Spectral sequences. If we apply to the Postnikov system on fig. B on the page 15 a covariant cohomology functor $\mathrm{Cov}^\bullet$ with values in some Abelian category, we get a bigraded coupled pair

$$\begin{tikzcd} D^{\bullet,\bullet} \arrow[rr, "\mathrm{Cov}^\bullet(i^\bullet)"] & & D^{\bullet,\bullet} \arrow[dl, "\mathrm{Cov}^\bullet(j^\bullet)"] \\ & E^{\bullet,\bullet} \arrow[ul, "\mathrm{Cov}^\bullet(j^\bullet)"] & \end{tikzcd}$$

with $E^{p,q} = \mathrm{Cov}^q(V_p^\bullet \otimes E_p) = \bigoplus_{\alpha+\beta=q} V^\alpha \otimes \mathrm{Cov}^\beta(E_p)$ and $D^{p,q} = \mathrm{Cov}^q(R_{[E_0\,\dots\,E_{p-1}]}X)$. The morphisms $\mathrm{Cov}^\bullet(i.)$, $\mathrm{Cov}^\bullet(j.)$, and $\mathrm{Cov}^\bullet(k.)$ have bidegrees $(-1,1)$, $(0,0)$, and $(1,0)$ respectively. The grading is bounded by $0 \le p, q \le n$. So, there exists a spectral sequence (see [We, sec. 5.9]), which has E_1-term

$$E_1^{p,q} = \bigoplus_{\alpha+\beta=q} V_p^\alpha \otimes \mathrm{Cov}^\beta(E_p) \tag{2-24}$$

where $V_\alpha = Hom^{n+\alpha}({}^\vee E_{n-p}, X) = Hom^{-\alpha}(X, E_{n-p}^\vee)^*$. Its limit terms $E_\infty^n = \bigoplus_{p+q=n} E_\infty^{p,q}$ are graded objects associated with some decreasing filtration

$$\mathrm{Cov}^n(X) = F^0\mathrm{Cov}^n(X) \supset F^1\mathrm{Cov}^n(X) \supset F^2\mathrm{Cov}^n(X) \supset \cdots \supset F^n\mathrm{Cov}^n(X) \supset 0\,.$$

on $\mathrm{Cov}^n(X)$.

Dually, to compute the values of a contravariant cohomology functor $\mathrm{Cnt}^{p+q}(X)$ from $\mathscr{D}$ to an Abelian category, there is a spectral sequence situated in the strip $-n \le p \le 0$, $0 \le q \le n$ and having

$$E_1^{p,q} = \bigoplus_{\alpha+\beta=q} V_{-p}^{\times\alpha} \otimes \mathrm{Cnt}^\beta(E_{-p})\,. \tag{2-25}$$

where $V_{-p}^{\times\alpha} = \mathit{Hom}^{\alpha}(X, E_{n+p}^{\vee}) = \mathit{Hom}^{-n-\alpha}({}^{\vee}E_{n+p}, X)^*$. It converges to $\mathrm{Cnt}^{p+q}(X)$ in the same sense as above.

2.8. Helices. If $\mathscr{D}$ is generated by an exceptional collection $\{E_0, E_1, \dots, E_n\}$, then an infinite collection $\{E_i\}_{i\in\mathbb{Z}}$ defined by prescription

$$E_{i+n+1} = R_{[E_{i+1}\dots E_{i+n}]}E_i \qquad \text{and} \qquad E_{i-n-1} = L_{[E_{i-n}\dots E_{i-1}]}E_i$$

is called *a helix* of *a period* $(n+1)$ with *a foundation* $\{E_0, E_1, \dots, E_n\}$. Each helix is uniquely recovered from any collection of $(n+1)$ its consequent elements $\{E_i, E_{i+1}, \dots, E_{i+n}\}$ and all such collections are called *the helix foundations* as well. The braid group B_{n+1} acts on the helices of period $(n+1)$ as follows. For $i = 1, \dots, n$ mutations L_i and R_i replace all the pairs

$$\{E_{i-1+k(n+1)}, E_{i+k(n+1)}\}\ , \quad \text{where} \quad k\in\mathbb{Z}$$

by its left and right mutations. The same arguments as in n° 2.4 show that the muted helix is again a helix.

2.8.1. Serre duality. By (2-17), for any X and any $i \in \mathbb{Z}$ we have the canonical *Serre duality* isomorphism

$$\mathit{Hom}^{\bullet}(E_i, X)[n] = \mathit{Hom}^{\times\bullet}(X, E_{i-n-1}) \qquad \forall\, i\in\mathbb{Z}\ . \tag{2-26}$$

Applying it twice, we also get the *periodicity* isomorphism

$$\mathit{Hom}^{\bullet}(E_i, E_j) = \mathit{Hom}^{\bullet}(E_{i-n-1}, E_{j-n-1}) \qquad \forall\ i, j \in \mathbb{Z}\,. \tag{2-27}$$

These isomorphisms mean that shifting an exceptional base by the helix period we get the Serre duality $\mathscr{D} \xrightarrow{\varkappa} \mathscr{D}$. More precisely, consider two consequent helix foundations, say

$$E_{-n-1},\ E_{-n},\ \dots,\ E_{-2},\ E_{-1} \quad \text{and} \quad E_0,\ E_1,\ \dots,\ E_{n-1},\ E_n$$

Then a collection $F_0, F_1, \dots, F_n$ with

$$\begin{aligned} F_0 &= E_{-1} = L_{[E_0\dots E_{n-1}]}E_n \\ F_n &= E_0 = R_{[E_{-n}\dots E_{-1}]}E_{-n-1} \\ F_k &= R_{[E_{-k}\dots E_{-1}]}E_{-k-1} = L_{[E_0\dots E_{n-k-1}]}E_{n-k} \end{aligned}$$

is simultaneously the left dual base for $E_{-n-1}, E_{-n}, \dots, E_{-2}, E_{-1}$ and the right dual base for $E_0, E_1, \dots, E_{n-1}, E_n$. Now let us present an object $X \in \mathit{Ob}\,\mathscr{D}$ as a canonical convolution of the complex (2-19)

$$0 \longrightarrow V_0^{\bullet}\otimes E_0 \longrightarrow V_1^{\bullet}\otimes E_1 \longrightarrow \cdots \longrightarrow V_{n-1}^{\bullet}\otimes E_{n-1} \longrightarrow V_n^{\bullet}\otimes E_n \longrightarrow 0$$

where $V_k^{\bullet} = \mathit{Hom}^{\times\bullet}(X, F_{n-k})$. This complex together with the convolution procedure is completely defined by some tensor from the space

$$\bigoplus_p \mathit{Hom}^0\left(\, V_p^{\bullet}\otimes E_p\,,\, V_{p+1}^{\bullet}\otimes E_{p+1}\,\right) = \bigoplus_{p,\alpha} \mathit{Hom}^{-\alpha}(V_p^{\bullet}, V_{p+1}^{\bullet})\otimes \mathit{Hom}^{\alpha}(E_p, E_{p+1})$$

Since $Hom^{\bullet}(E_p, E_{p+1}) = Hom^{\bullet}(E_{p-n-1}, E_{p-n})$, the same tensor produces a canonical convolution of the complex

$$0 \longrightarrow V_0^{\bullet} \otimes E_{-n-1} \longrightarrow V_1^{\bullet} \otimes E_{-n} \longrightarrow \cdots \longrightarrow V_{n-1}^{\bullet} \otimes E_{-2} \longrightarrow V_n^{\bullet} \otimes E_{-1} \longrightarrow 0$$

We denote this convolution by $\varkappa(X)$.

2.8.2. PROPOSITION. *$Hom^{\bullet}(Y, X) = Hom^{\times\,\bullet}(\varkappa(X), Y)[n]$ as bifunctors on X, Y.*

PROOF. $Hom^{\bullet}(Y, X)$ is computed by the functorial spectral sequence (2-24) with

$$E_1^{p,q} = \bigoplus_{\alpha} V_p^{\alpha} \otimes Hom^{q-\alpha}(Y, E_p)$$

By (2-26), it is canonically dual to the spectral sequence (2-25)

$$E_1^{p,q} = \bigoplus_{\alpha} V^{\times\alpha} \otimes Hom^{n-q+\alpha}(E_{-p-n-1}, Y)$$

which computes $Hom^{\bullet}(\varkappa(X), Y)[n]$. □

2.8.3. COROLLARY. *Each helix $\{E_i\}_{i\in\mathbb{Z}}$ of period $(n+1)$ is periodic in the sence that $E_{i-n-1} = \varkappa(E_n)$, where $\mathscr{D} \xrightarrow{\varkappa} \mathscr{D}$ is the Serre dualizing functor defined above.* □

§3. Helix theory on $\mathbb{P}_n$.

In this paragraph we always suppose $\mathscr{D} = \mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n)$ being a (bounded) derived category of coherent sheaves on the (complex) projective space. Historically, the helix theory was firstly developed in this category (see [Go1], [GoRu]).

3.1. Introductory example: invertible sheaves. The simplest helix on $\mathbb{P}_n$ consists of invertible sheaves $E_n = \mathscr{O}(n)$, $i \in \mathbb{Z}$; its period is $(n+1)$. To see this, consider the exceptional collection $\{\mathscr{O}, \mathscr{O}(1), \ldots, , \mathscr{O}(n)\}$. The consequent right mutations of $\mathscr{O}$ lay in the Abelian core $\mathscr{C}oh\,(\mathbb{P}_n) \subset \mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n)$, since the distinguished triangles (2-3) are reduced to the exact triples of sheaves. Indeed, the external powers of the Euler exact triple

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O} & \longrightarrow & V \otimes \mathscr{O}(1) & \longrightarrow & \mathscr{T} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathscr{T} & \longrightarrow & \Lambda^2 V \otimes \mathscr{O}(2) & \longrightarrow & \Lambda^2\mathscr{T} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \Lambda^{k-1}\mathscr{T} & \longrightarrow & \Lambda^k V \otimes \mathscr{O}(k) & \longrightarrow & \Lambda^k\mathscr{T} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \Lambda^{n-1}\mathscr{T} & \longrightarrow & \Lambda^n V \otimes \mathscr{O}(n) & \longrightarrow & \mathscr{O}(n+1) & \longrightarrow & 0
\end{array}$$

where $\mathbb{P}_n = \mathbb{P}(V)$, $V = H^0(\mathscr{O}(1))^*$, and $\mathscr{T} = \mathscr{T}_{\mathbb{P}_n}$ is the tangent bundle, have

$$\Lambda^k V = \mathit{Hom}(\mathscr{O}(k)\,,\,\Lambda^k\mathscr{T}) = \mathit{Hom}(\,\Lambda^{k-1}\mathscr{T},\,\mathscr{O}(k)\,)^*$$

and the maps coincide with the canonical convolutions (2-1). So,

$$R_{[\mathscr{O}(1)\dots\mathscr{O}(k)]}\mathscr{O} = \Lambda^k\mathscr{T} = L_{[\mathscr{O}(k+1)\dots\mathscr{O}(n)]}\mathscr{O}(n+1)\,.$$

and these relations are clearly 'equivariant' w. r. t. the twists. By the Serre theorem, $\mathscr{C}\mathit{oh}\,(\mathbb{P}_n)$ is spanned by invertible sheaves. Hence, $\{\mathscr{O},\,\mathscr{O}(1),\,\dots\,,\,,\mathscr{O}(n)\}$ is an exceptional base for $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}\mathit{oh}}(\mathbb{P}_n)$. In particular, the shift along the helix by the helix period should coincide with the dualizing functor (see n° 2.8.1), i. e. with the (anti)canonical twist. This agrees with the previous computation. Further, consider the following pair of dual bases for $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}\mathit{oh}}(\mathbb{P}_n)$

$$\begin{array}{lll} \{E_0, E_1, \dots, E_n\} = \{\,\Lambda^n\Omega(n)\,,\,\Lambda^{n-1}\Omega(n-1),\,\dots\,,\,\mathscr{O}\} & & (\text{with } E_p = \Lambda^{n-p}\Omega(n-p)) \\ \{{}^{\vee}E_0, {}^{\vee}E_1, \dots, {}^{\vee}E_n\} = \{\,\mathscr{O}\,,\,\mathscr{O}(1)\,,\,\dots\,,\,\mathscr{O}(n)\,\} & & (\text{with } {}^{\vee}E_p = \mathscr{O}(p)) \end{array}$$

and apply the functorial spectral sequence (2-24) to compute 'the sheaves of cohomology' functor

$$\mathscr{H}^{\bullet}\;:\;\mathscr{D}^{\mathrm{b}}_{\mathscr{C}\mathit{oh}}(\mathbb{P}_n) \longrightarrow \mathscr{C}\mathit{oh}\,(\mathbb{P}_n)$$

which takes a complex of coherent sheaves to its sheaves of cohomologies. For any 'pure' object $F \in \mathit{Ob}\,\mathscr{C}\mathit{oh}\,(\mathbb{P}_n)$ we have

$$\mathscr{H}^0(F) = F \quad \text{ and } \quad \mathscr{H}^{\nu}(F) = 0 \text{ for } \nu \neq 0\,,$$

and in (2-24) we obtain $V_i^{\alpha} = \mathit{Ext}^{n+\alpha}(\mathscr{O}(n-p), F) = H^{n+\alpha}(F(p-n))$. So, there is a spectral sequence with

$$E_1^{pq} = H^{n+q}(F(p-n)) \otimes \Lambda^{n-p}\Omega(n-p)$$

situated in the rectangle $0 \leq p \leq n$, $-n \leq q \leq 0$ and converging to zero outside the main diagonal $p + q = 0$. The limit graded sheaf $\bigoplus\limits_{p=0}^{n} E_{\infty}^{p,-p}$ at the main diagonal is associated with some decreasing filtration on F. This is nothing but the well known Beilinson spectral sequence (maybe, in slightly unusual grading).

3.2. Strictness and purity. All known helices on $\mathbb{P}_n$ have two extra features standing out the general setup of the previous paragraph. Namely, they are 'pure' and 'strict', that is, up to shifts through the derived category grading each helix consist of coherent (even locally free) sheaves satisfying $\mathit{Hom}^{\nu}(E_i, E_j) = 0$ for all $i < j$ and $\nu \neq 0$. Let us discuss these properties in more details.

3.2.1. Strict exceptional bases. An exceptional collection $\{E_0, E_1, \dots, E_n\} \subset \mathscr{D}$ is called *strict*, if $\mathit{Hom}^q(E_i, E_j) = 0$ for all $q \neq 0$ and $i < j$. A helix is called *strict*, if it has a strict foundation.

3.2.2. PROPOSITION. *If a helix is strict, then all its foundations are strict and the strictness is preserved by the helix mutations.*

PROOF. The first statement follows immediately from the periodicity (2-27) and Serre duality (2-26) isomorphisms:

$$\mathit{Hom}^{\bullet}(E_i, E_j) = \mathit{Hom}^{\bullet}(E_{i+n+1}, E_{j+n+1}) = \mathit{Hom}^{\times\bullet}(E_j, E_{i-n-1})[n]$$

To prove the second statement, we need some cohomology estimations. Let E be exceptional and X, Y — any objects of $\mathscr{D}$. The long exact sequences for $Hom^{\bullet}(Y,*)$ and $Hom^{\bullet}(*,Y)$ applied to the left and right mutation triangles:

$$X[-1] \longrightarrow L_E X \longrightarrow Hom^{\bullet}(E,X)\otimes E \longrightarrow X$$
$$X \longrightarrow Hom^{\times\bullet}(X,E)\otimes E \longrightarrow R_E X \longrightarrow X[1]$$

give the following implications:

$$Hom^{>k}(E,Y) = Hom^{>k}(X,Y) = 0 \quad\Longrightarrow\quad Hom^{>k}(L_E X,Y) = 0 \qquad (3\text{-}1)$$
$$Hom^{<k}(Y,E) = Hom^{<k}(Y,X) = 0 \quad\Longrightarrow\quad Hom^{<k}(Y,L_E X) = 0 \qquad (3\text{-}2)$$
$$Hom^{>k}(Y,E) = Hom^{>k}(Y,X) = 0 \quad\Longrightarrow\quad Hom^{>k}(Y,R_E X) = 0 \qquad (3\text{-}3)$$
$$Hom^{<k}(E,Y) = Hom^{<k}(X,Y) = 0 \quad\Longrightarrow\quad Hom^{<k}(R_E X,Y) = 0 \qquad (3\text{-}4)$$

So, the smallest index ν for which $Hom^{\nu}(Y,X)\neq 0$ does not decrease under the left mutations of X and the right mutations of Y as well as the greatest index ν for which $Hom^{\nu}(Y,X)\neq 0$ does not increase under the right mutations of X and the left mutations of Y.

Now consider, say, the right mutation $(E_0,E_1)\longmapsto(E_1,R_{E_1}E_0)$. It is enough to show that all the exceptional pairs $(E_\alpha,R_{E_1}E_0)$ with $-(n-2)\le\alpha\le -1$ and $\alpha=1$ as well as all the exceptional pairs $(R_{E_1}E_0,E_\beta,)$ with $2\le\beta\le n$ and $\beta=n+2$ are strict. The above estimations show that $Hom^{>0}(E_\alpha,R_{E_1}E_0)=Hom^{>0}(R_{E_1}E_0,E_\beta,)=0$. If we write

$$R_{E_1}E_0 = L_{[E_2\,\dots\,E_n]}E_{n+1} = L_{E_2}L_{E_3}\,\cdots\,L_{E_n}E_{n+1}$$

then by (3-1) and (3-2) we get $Hom^{<0}(E_\alpha,R_{E_1}E_0)=Hom^{<0}(R_{E_1}E_0,E_\beta)=0$ as well. □

3.2.3. COROLLARY. *$Hom^{>0}(E_i,E_j)=Hom^{<n}(E_j,E_i)=0$ for all $i<j$ in any strict helix $\{E_i\}_{i\in\mathbb{Z}}$.*

PROOF. By the Serre duality, it is enough to show that $Hom^{>0}(E_i,E_j)=0$ for $j\ge i$. This holds inside any foundation, i. e. for $i\le j\le i+n$; then we use an increasing induction on $j-i$: writing $E_j=R_{[E_{j-n}\,\dots\,E_{j-1}]}E_{j-n-1}$ and applying (3-3) to the consequent right mutations, we get inductively $Hom^{>0}(E_i,E_j)=0$ for all $j\ge(i+n+1)$. □

3.2.4. Pure exceptional objects. An object $X\in Ob\,\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n)$ is called *pure*, if it is isomorphic to $F[k]$ for some coherent sheaf F and $k\in\mathbb{Z}$. Since an exceptional coherent sheaf has trivial unobstructed deformation theory with the one point moduli space, it has to be homogeneous. In particular, exceptional sheaves in $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n)$ are locally free. There is also following very useful general criterium of purity established in [Po]:

3.2.5. PROPOSITION. *Let $E\in Ob\,\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$ be an object of the bounded derived category of coherent sheaves on a smooth algebraic variety X. If[1] $R\mathscr{H}om(E,E)\in\mathscr{D}^{\le 0}$, then E is pure and locally free.*

PROOF. It is the 'main lemma' of the paper [Po]. Its proof as well as some reformulations see there on the p.p. 5–6. □

[1]recall that $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)^{\mathrm{opp}}\times\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)\xrightarrow{R\mathscr{H}om(*,*)}\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$ is the derived functor for the *sheaves of local homomorphisms*: if $0\longrightarrow P_n\longrightarrow\ \cdots\ \longrightarrow P_1\longrightarrow P_0\longrightarrow F\longrightarrow 0$ is a locally free resolution for $F\in Ob\,\mathscr{C}oh(\mathbb{P}_n)$, then $R\mathscr{H}om(F,E)$ is represented by a complex of sheaves $\mathscr{H}om(P_{-p},F)$

3.2.6. COROLLARY. *Any strict helix in $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n)$ consist of locally free sheaves simultaneously subjected to the same grading translation. Moreover, non zero spaces $Hom^{\nu}(E_i,E_j)$ can appear only among $Hom^0(E_i,E_j)$ with $i>j$ and $Hom^n(E_i,E_j)$ with $j<i-n$.*

PROOF. We write $H^{\bullet}$ for the global cohomology spaces[1], $\mathscr{H}^{\bullet}\in Ob\,\mathscr{C}oh\,(\mathbb{P}_n)$ — for the sheaves of cohomologies, and put $\mathscr{E}_{ij}\overset{\text{def}}{=}R\mathscr{H}om(E_i,E_j)$. Let us compute $Hom^{\nu}(E_i,E_{j+k(n+1)})$ for $k\gg 0$. By the corollary from n° 3.2.3, these spaces vanish for all $\nu>0$. On the other hand, $Hom^{\nu}(E_i,E_j)=H^{\nu}(\mathscr{E}_{ij})$ can be computed using the spectral sequence with $E_1^{pq}=H^p\left(\mathscr{H}^q(\mathscr{E}_{ij})\right)$ for the composition of derived functors (see [GeMa, p. 207, 219]). Since $E_{j+k(n+1)}=E_j\otimes\omega^{*\otimes k}$ and ω^* is ample, this spectral sequence has $E_\infty=E_1$ for k large enough. In other words,

$$Hom^{\nu}(E_i,E_{j+k(n+1)})=H^0\left(\mathscr{H}^{\nu}(\mathscr{E}_{ij}\otimes\omega^{*\otimes k})\right)\,.$$

Moreover, this space is non zero, if $\mathscr{E}_{ij}\neq 0$. Hence, $\mathscr{E}_{ij}\in\mathscr{D}^{\leq 0}\ \forall\ i,j\in\mathbb{Z}$. Taking $i=j$ and applying the proposition from n° 3.2.5, we see that E_i are shifted locally free sheaves. Since we also have $\mathscr{E}_{ij}=R\mathscr{H}om(\mathscr{E}_{ji},\mathscr{O})\in\mathscr{D}^{\geq 0}$ as soon $\mathscr{E}_{ji}\in\mathscr{D}^{\leq 0}$, we actually have $\mathscr{E}_{ij}\in\mathscr{D}^{0}=Ob\,\mathscr{C}oh\,(\mathbb{P}_n)\ \forall\ i,j\in\mathbb{Z}$. Hence, $Hom^{<0}(E_i,E_j)=H^{<0}(\mathscr{E}_{i,j})=0$ and, by the Serre duality, $Hom^{>n}(E_i,E_j)=0$ as well. Together with n° 3.2.3 this completes the proof. □

3.2.7. Open problems. We have seen that strict helices consist of locally free sheaves, have very nice cohomology properties, and remains to be strict after mutations. There are no examples of non strict helices on $\mathbb{P}_n$ except for trivial ones obtained from the strict helices by shifting elements through the derived category grading. Also, there are no examples of exceptional objects different from (maybe shifted) locally free sheaves. These facts seems being of the same nature. The non-existence in the both cases could be deduced from the following (also open)

3.2.8. CONJECTURE. *Any exceptional object in $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n)$ can be included in a strict helix.*

3.3. Strict helices. If we restrict ourselves by the strict helix area, then all the theory can be pulled down from $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n)$ into $\mathscr{C}oh\,(\mathbb{P}_n)$. In what follows we always suppose that strict bases and/or helices are shifted to be consisting of pure sheaves situated in degree zero. Then for any $i<j$ canonical distinguished triangles

$$\begin{gathered}R_{E_j}E_i[-1]\xrightarrow{\ i_\bullet\ }E_i\xrightarrow{\ j_\bullet\ }Hom^{\times\,\bullet}(E_i,E_j)\otimes E_j\xrightarrow{\ k_\bullet\ }R_{E_j}E_i\\ L_{E_i}E_j\xrightarrow{\ k^\bullet\ }Hom^{\bullet}(E_i,E_j)\otimes E_i\xrightarrow{\ j^\bullet\ }E_j\xrightarrow{\ i^\bullet\ }L_{E_i}E_j[1]\end{gathered}$$

turn to usual exact triples of locally free sheaves

$$\begin{gathered}0\longrightarrow E_i\longrightarrow Hom^{*}(E_i,E_j)\otimes E_j\longrightarrow R_{E_j}E_i\longrightarrow 0\\ 0\longrightarrow L_{E_i}E_j\longrightarrow Hom(E_i,E_j)\otimes E_i\longrightarrow E_j\longrightarrow 0\end{gathered}\tag{3-5}$$

So, without loss of generality we could define the mutations inside a strict helix in terms of pure sheaves as it was done originally in [GoRu]. In particular, the left and right dual bases to any strict exceptional base consist of locally free sheaves as well.

[1] i. e. the derived functors for the global sections

3.3.1. Beilinson type spectral sequence. Let us compute a covariant cohomology functor

$$\mathscr{H}^\bullet : \mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n) \longrightarrow \mathscr{C}oh\,(\mathbb{P}_n)$$

which takes a complex $X \in Ob\,\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_n)$ to its cohomology sheaves $\mathscr{H}(X)$ and acts identically on the pure shaves. The canonical spectral sequence (2-24) written for strictly exceptional base $\{E_0, E_1, \ldots, E_n\}$ and $X = F \in Ob\,\mathscr{C}oh\,(\mathbb{P}_n)$ may be arranged to have

$$E_1^{pq} = Ext^q({}^\vee E_{n+p}, F) \otimes E_{n+p} \tag{3-6}$$

situated in the square $-n \le p \le 0$, $0 \le q \le n$. It converges to a graded sheaf $\oplus E_\infty^{p,-p}$ situated on the main diagonal and associated with some decreasing filtration of F.

3.3.2. Eilenberg - Moore type spectral sequence. A contravariant functor $Hom^\bullet(*, Y)$ also can be computed via n° 2.7.3. Namely, let F_1, F_2 be coherent sheaves and $\{E_0, E_1, \ldots, E_n\}$ be a strictly exceptional base. Then there is a spectral sequence with

$$E_1^{pq} = \bigoplus_{\beta-\alpha=q} Ext^\alpha({}^\vee E_{n-p}, F_1)^* \otimes Ext^\beta(E_{n-p}, F_2) \tag{3-7}$$

situated in the strip $0 \le p \le n$, $-n \le q \le n$ and converging to $Ext^{p+q}(F_1, F_2)$.

3.3.3. Cohomology of a sheaf w. r. t. a helix. For a strict helix $\{E_\nu\}$ and any coherent sheaf F on $\mathbb{P}_n$ consider a table whose cells are indexed by the pairs of integers $-\infty < p < +\infty$, $0 \le q \le n$ and the (p,q)-cell contains a vector space $Ext^q(E_p, F)$. We call this table *a left cohomology table of* F w. r. t. $\{E_\nu\}$. It is very important for cohomology calculations: for example, the both spectral sequences above are referred to this table. Since $E_{i+k(n+1)} = E_i \otimes \omega^{*\otimes k}$ and ω^* is ample, we have

$$Ext^{>0}(E_i, F) = 0 \quad \text{for} \quad i \ll 0 \tag{3-8}$$
$$Ext^{<n}(E_i, F) = 0 \quad \text{for} \quad i \gg 0 \tag{3-9}$$

In other words, the cohomology table vanishes far on the right except for the upper row and vanishes far on the left except for the lower row. The placement of non-zero entries in the middle part of the table encodes an important information about F.

3.3.4. m-regularity w. r. t. a helix. A coherent sheaf F is called *m-regular* w. r. t. a helix $\{E_i\}$, if $Ext^\nu(E_{m+\nu}, F) = 0$ for $\nu = 1, 2, \ldots, n$. In other words, m-regularity means that the cohomology table contains an increasing sequence of consequent zeros

$$\begin{array}{cccccc}
 & & & & & 0 = Ext^n(E_{m+n}, F) \\
 & & & & 0 & \\
 & & & \cdot^{\cdot^{\cdot}} & & \\
 & & 0 & & & \\
 & 0 & & & & \\
Hom(E_m, F) = ? & & & & &
\end{array}$$

For the simplest helix $E_i = \mathcal{O}(i)$ our notion of m-regularity turns (up to the sign of m) to the classical Mumford - Castelnuvo m-regularity. Main properties of Mumford - Castelnuvo m-regular sheaves remains to be true in our more general setup:

3.3.5. PROPOSITION. *If a coherent sheaf F is m-regular w. r. t. a helix $\{E_i\}$, then it is k-regular for any $k < m$ as well and the canonical map $Hom(E_m, F) \otimes E_m \xrightarrow{\text{can}} F$ is surjective.*

PROOF. The long exact sequence of $Ext^\bullet(*, F)$ applied to a left mutation triangle

$$0 \longrightarrow L_E X \longrightarrow Hom(E, X) \otimes E \longrightarrow X \longrightarrow 0$$

gives the implication

$$Ext^{>(k+1)}(X, F) = Ext^{>k}(E, F) = 0 \quad \Longrightarrow \quad Ext^{>k}(L_E X, F) = 0 \tag{3-10}$$

If F is m-regular, then it follows from this implication that $Ext^{>0}(L_{[E_{m+1} \dots E_{m+k-1}]} E_{m+k}, F) = 0$ for $2 \leq k \leq (n+1)$. Hence, the Beilinson type spectral sequence (3-6)

$$E_1^{pq} = Ext^q({}^\vee E_{-p}, F) \otimes E_{m+p} ,$$

which is written for the base $\{E_{m-n}, \dots, E_m\}$ whose left dual base $\{{}^\vee E_0, {}^\vee E_1, \dots, {}^\vee E_n\}$ consists of

$${}^\vee E_i = R_{[E_{m-i+1} \dots E_m]} E_{m-i} = L_{[E_{m+1} \dots E_{m+n-i}]} E_{m+n-i+1} \quad \text{for} \quad 0 \leq i \leq n ,$$

turns to an exact resolvent

$$\begin{gathered} 0 \longrightarrow Hom(E_{m+1}, F) \otimes E_{m-n} \longrightarrow Hom({}^\vee E_{n-1}, F) \otimes E_{m-n+1} \longrightarrow \cdots \\ \cdots \longrightarrow Hom({}^\vee E_1, F) \otimes E_{m-1} \longrightarrow Hom(E_m, F) \otimes E_m \xrightarrow{\text{can}} F \longrightarrow 0 \end{gathered}$$

(we have substituted ${}^\vee E_n = E_{m+1}$ and ${}^\vee E_0 = E_m$). This gives the second assertion. Similarly, if we write $E_{m+\nu-k} = L_{[E_{m+\nu-k+1} \dots E_{m+\nu-k+n}]} E_{m+\nu-k+n+1}$ for $k \geq 1$ and step by step apply implications (3-10) to the consequent left mutations, we get $Ext^\nu(E_{m+\nu-k}, F) = 0$ for all $\nu > 0$ inductively on k. This gives the first assertion. $\square$

§4. Helix theory on Del Pezzo surfaces.

4.1. Del Pezzo surfaces. We write S for a smooth regular complex projective surface S whose anticanonical class $-K_S$ is ample. In other words, S is either the projective plane $\mathbb{P}_2$, or 2-dimensional quadric $\mathbb{P}_1 \times \mathbb{P}_1$, or a blow-up of $\mathbb{P}_2$ at appropriate $d \leq 8$ points. In the last case we will always write S_d instead of S. We fix a sequence of the blow-up's

$$S_d \xrightarrow{\sigma_d} S_{d-1} \xrightarrow{\sigma_{d-1}} \cdots \xrightarrow{\sigma_2} S_2 \xrightarrow{\sigma_1} \mathbb{P}^2$$

and write h and $e_1, e_2, \dots, e_d$ for the proper preimage of the line on $\mathbb{P}_2$ and the proper preimages of consequently blow-up'ed points (so, e_d is rational and irreducible). The grope $\operatorname{Pic} S_d$ is generated by h and $e_1, \dots, e_d$ with $h^2 = 1$, $e_i^2 = -1$, $he_i = e_i e_j = 0 \ \forall \ i \neq j$. The canonical class of S_d equals $K_{S_d} = -3h + e_1 + \dots + e_d$.

4.1.1. Riemann - Roch formula. A bilinear version of the Grothendieck - Hirtzebruch - Riemann - Roch formula on an algebraic surface X looks like

$$\chi(E,F)=\sum_{\nu}\dim Ext^{\nu}(E,F)=$$
$$=rk(E)\,rk(F)\,\left(\chi(\mathcal{O}_X)+\frac{1}{2}\left(\mu_{\mathrm{can}}(F)-\mu_{\mathrm{can}}(E)\right)+q(F)+q(E)-\frac{c_1(E)\cdot c_1(F)}{rk(E)rk(F)}\right), \quad (4\text{-}1)$$

where $\mu_{\mathrm{can}}(E)=\dfrac{-K_X\cdot c_1(E)}{rk(E)}$ is called an (anticanonical) slope and $q(E)=\dfrac{c_1^2(E)-2\,c_2(E)}{2\,rk(E)}$. For our surfaces S the arithmetic genus $\chi(\mathcal{O}_S)=1$. The constrains $\chi(E,E)=\chi(F,F)=1$ lead to

$$q(E)=\frac{1}{2}\Big(\frac{c_1^2(E)+1}{rk^2(E)}-1\Big)$$

and simplify (4-1) for exceptional sheaves E, F to

$$\chi(E,F)=$$
$$=\frac{rk(E)\,rk(F)}{2}\left(\mu_{\mathrm{can}}(F)-\mu_{\mathrm{can}}(E)+\frac{1}{rk^2(E)}+\frac{1}{rk^2(F)}+\left(\frac{c_1(F)}{rk(F)}-\frac{c_1(E)}{rk(E)}\right)^2\right). \quad (4\text{-}2)$$

The helix theory on Del Pezzo surfaces has more complicated combinatorics than on $\mathbb{P}_n$. However, it also can be formulated in terms of pure sheaves and, moreover, the set of exceptional (even rigid) sheaves is completely described as well as the set of all helices.

We have to start with

4.1.2. THEOREM. *The bounded derived category of coherent sheaves $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(S)$ has an exceptional base, say the following:*
$\{\mathcal{O}\,,\,\mathcal{O}(1)\,,\,\mathcal{O}(2)\}$ *on* $\mathbb{P}_2$,
$\{\mathcal{O}\,,\,\mathcal{O}(1,0)\,,\,\mathcal{O}(0,1)\,,\,\mathcal{O}(1,1)\}$ *on* $\mathbb{P}_1\times\mathbb{P}_1$, *and*
$\{\mathcal{O}_{e_1}(-1),\,\dots\,,\,,\mathcal{O}_{e_d}(-1),\,\mathcal{O}_S,\,\mathcal{O}_S(h),\,\mathcal{O}_S(2h)\}$ *on* S_d.

PROOF. Let X be a smooth projective variety such that $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(X)$ admits an exceptional base $\{E_1,E_2,\dots,E_k\}$. There are two general results of D. O. Orlov (see [Or1]) about the derived categories $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(\mathbb{P}(\mathcal{E}))$ and $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(B_Y(X))$, where $\mathbb{P}(\mathcal{E})\xrightarrow{p}X$ is a projective vector bundle over X and $B_Y(X)\xrightarrow{\sigma}X$ is the blow up of X along a smooth submanifold $Y\subset X$. Namely, if $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(X)$ admits an exceptional base $\{E_1,E_2,\dots,E_k\}$, then $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(\mathbb{P}(\mathcal{E}))$ is generated by exceptional collection

$$\{\,E_1,\,\dots\,,\,E_k\,,\;E_1(1),\,\dots\,,\,E_k(1)\,,\;\dots\;,\;\dots\;,\;\dots\;,\;E_1(r),\,\dots\,,\,E_k(r)\,\},$$

where $r=rk(\mathcal{E})-1$ and twisting is made by the tensor powers of relative $\mathcal{O}_p(1)$. In particular, if $Y\subset X$ also admits an exceptional base for $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(Y)$, then there exists an exceptional base in a derived category $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(I)$, of sheaves on the exceptional divisor $I=\mathbb{P}\mathcal{N}_{Y/X}$. The blow up diagram

$$\begin{tikzcd} I=\mathbb{P}\mathcal{N}_{Y/X} \arrow[r, hook, "i"] \arrow[d, "\pi"'] & B_Y(X) \arrow[d, "\sigma"'] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

decomposes $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(B_Y(X))$ by two semiorthogonal full subcategories: $L\sigma^*\left(\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)\right)$ and the right orthogonal to $\mathscr{O}$ inside $Ri_*\left(\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(I)\right)$. This allows to combine exceptional bases of $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(I)$ and $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(X)$ to get an exceptional base for $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(B_Y(X))$. We refer to [Or1] for all the details. □

4.2. Cohomological properties of sheaves on S. A coherent sheaf F on S is called *rigid*, if $Ext^1(F,F)=0$. A rigid sheaf F is called *extra rigid*, if $Ext^2(F,F)=0$ as well. A sheaf F is called *simple*, if $Hom(E,E)=\mathbb{C}$. So, 'exceptional' is 'simple' plus 'extra rigid'. We write $TF\subset F$ for the torsion subsheaf, write $T^0F\subset TF$ for the torsion concentrated at points, and put $T^1F\overset{\text{def}}{=}TF/T^0F$ (comp. with [HuLe, sec. 1.1]). Detailed proofs of all the results listed below may be found in [KuOr].

4.2.1. Mukai lemmas were established in [Mu]. They hold on any smooth projective surface and follow immediately from the spectral sequence[1] with

$$E_1^{p,q}=\bigoplus_{\beta-\alpha=p} Ext^{p+q}(Gr^\alpha E, Gr^\beta E) \tag{4-3}$$

which computes $Ext^{p+q}(E,E)$ using a decreasing filtration $E=F^0\supset F^1\cdots\supset F^{n+1}=0$ with quotients $Gr^i=F^i/F^{i+1}$.

4.2.2. LEMMA. *Let $0\longrightarrow F\longrightarrow E\longrightarrow G\longrightarrow 0$ be an exact triple of coherent sheaves on a smooth projective surface such that $Hom(F,G)=Ext^2(G,F)=0$. Then*

$$\dim Ext^1(E,E)\geq \dim Ext^1(G,G)+\dim Ext^1(F,F)\,.$$

PROOF. Write (4-3) for 2-term filtration $0\subset F\subset E$ with $Gr^0=E/F$ and $Gr^1=F$. □

4.2.3. LEMMA. *Let E be a torsion free coherent sheaf on a smooth projective surface. Then*

$$\dim Ext^1(E,E)\geq \dim Ext^1(E^{**},E^{**})+2\,\mathrm{length}\left(E^{**}/E\right)\,.$$

PROOF. Sinse E is torsion free and we deal with a surface, the reflexive span E^{**}, of E, is locally free and contains E as a subsheaf. The factor E^{**}/E haz zero dimensional support. It remains to write down (4-3) for 2-term filtration $0\subset E\subset E^{**}$ with $Gr^0=E^{**}/E$ and $Gr^1=E$. □

4.2.4. COROLLARY. *Rigid torsion free sheaf is locally free on any projective surface.*

4.2.5. Cohomology estimations. The next series of statements is specific for the Del Pezzo surfaces S listed in n° 4.1.

4.2.6. LEMMA. *Let G, F be coherent sheaves on S. Then*

$$\dim\, Hom(F,G)\geq \dim\, Ext^2(G,F)\,.$$

PROOF. Choose an anticanonical section $\varphi\in H^0(\mathscr{O}_S(-K_S))$ such that the corresponding anticanonical curve $D\overset{\text{def}}{=}(\varphi)_0\in|-K_S|$ satisfies the conditions

$$\dim(D\cap Supp\, T^1G)<1 \quad\text{and}\quad D\cap Supp\, T^0G=\varnothing \tag{4-4}$$

[1] See [We] for details about all the spectral sequences we will use.

(such a curve exists, because $|-K_S|$ has no base points and $\dim Supp\, T^1G = 1$). If we apply $\mathscr{H}om(*\,,\,G)$ to the exact sequence

$$0 \longrightarrow \mathscr{O}_S \xrightarrow{\varphi} \mathscr{O}_S(-K_S) \longrightarrow \mathscr{O}_S(-K_S)|_D \longrightarrow 0$$

then we get an exact quadruple

$$0 \longrightarrow \mathscr{H}om(\mathscr{O}_S(-K_S)|_D, G) \longrightarrow G(K_S) \longrightarrow G \longrightarrow \mathscr{E}xt^1(\mathscr{O}_S(-K_S)|_D, G) \longrightarrow 0$$

because of $\mathscr{H}om(\mathscr{O}_S(-K_S), G) \cong \mathscr{O}_S(-K_S)^* \otimes G = G(K_S)$. By the choice of D (see (4-4)), we have

$$\mathscr{H}om(\mathscr{O}_S(-K_S)|_D, T^1G) = \mathscr{H}om(\mathscr{O}_S(-K_S)|_D, T^0G) = 0\ .$$

This implies $\mathscr{H}om(\mathscr{O}_S(-K_S)|_D, G) = \mathscr{H}om(\mathscr{O}_S(-K_S)|_D, TG) = 0$, i. e. the above quadruple turns to the exact triple

$$0 \longrightarrow G(K_S) \longrightarrow G \longrightarrow \mathscr{E}xt^1(\mathscr{O}_S(-K_S)|_D, G) \longrightarrow 0\ ,$$

which gives an inclusion $Hom(F, G(K_S)) \subset Hom(F, G)$. Hence, by the Serre duality,

$$\dim Ext^2(G, F) = \dim Hom(F, G(K_S) \le \dim Hom(F, G)$$

as required. □

4.2.7. LEMMA. *Let F be a coherent sheaf on S such that $Supp\, F \cap C \neq \varnothing$ for some curve $C \subset |-K_S|$ (this holds, for example, if $rkF > 0$). Then $\dim Hom(F, F) > \dim Ext^2(F, F)$.*

PROOF. Since $\mathscr{E}xt^1(\mathscr{O}(-K_S)|_D, G) \cong G \otimes \mathscr{O}_D$, taking in the previous proof $G = F$, we get an exact triple

$$0 \longrightarrow F(K_S) \longrightarrow F \longrightarrow F \otimes \mathscr{O}_D \longrightarrow 0$$

If we choose $D \in |-K_S|$ such that $F \otimes \mathscr{O}_D \neq 0$, then we get the non zero map

$$Hom(F, F) \longrightarrow Hom(F, F \otimes \mathscr{O}_D)\ ,$$

that is, the strict inclusion $Hom(F, G(K_S)) \subsetneq Hom(F, G)$, which implies the strict inequality $\dim Hom(F, F) > \dim Ext^2(F, F)$. □

4.2.8. COROLLARY. *Let G be a rigid sheaf on S. Then $T^0G = 0$ and both TG, G/TG are rigid as well.*

PROOF. Take $F = TE$, $G = E/TE$ in n° 4.2.2 and note that by n° 4.2.6

$$\dim Ext^2(FE/TE, TE) \ge \dim Ext^0(TE, FE/TE) = 0$$

because there are no global Hom's from a torsion sheaf to a torsion free one. □

4.3. Exceptional sheaves on S and exceptional objects in $\mathscr{D}^b_{\mathscr{C}oh}(S)$. Combining cohomology estimations of the previous section with Riemann - Roch formula, we immediately obtain full geometric characterization of exceptional objects in $\mathscr{D}^b_{\mathscr{C}oh}(S)$. Let us start with the exceptional torsion sheaves.

4.3.1. LEMMA. *$c_1^2(E) = -1$ for any exceptional torsion sheaf E on S; moreover, $E = \mathcal{O}_C(d)$ for some irreducible rational curve $C \subset S$ with $C^2 = -1$.*

PROOF. The Riemann - Roch formula (4-2) powered by the condition $rkE = 0$ gives

$$1 = \chi(E, E) = rk^2(E) + (rk(E) - 1)\, c_1^2(E) - 2\, rk(E)\, c_2(E) = -c_1^2(E).$$

Let $C = Supp\, E$. If C is irreducible, then E is locally free sheaf on C, because its torsion $T^0 E = 0$ by n° 4.2.8. Write r for its rank. Then $c_1(E) = rC$ and $c_1(E)^2 = r^2 C^2 = -1$, i. e. $r = 1$. The genus formula $2g - 2 = C \cdot (C + K_S)$ shows that $2g = C \cdot K_S + 1 < 1$, since $-K_S$ is ample. So, C is rational curve with $C^2 = -1$ and $E = \mathcal{O}_C(d)$ is a line bundle over C.

If C is reducible, consider one of its irreducible components $C_0 \subset C$ and restrict E from C onto C_0. We get an exact triple

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_0 \longrightarrow 0,$$

where $Supp\, E_0 = C_0$, $Supp\, E_1 = C_1 = C \setminus C_0$ and both E_0, E_1 have no torsion concentrated at points. So, $Hom(E_1, E_0) = Ext^2(E_0, E_1) = 0$ and by the Mukai lemmas we can write

$$\begin{aligned} \dim Ext^0(E, E) &= \dim Ext^0(E_0, E_0) + \dim Ext^0(E_1, E_1) + \chi(E_0, E_1) \\ \dim Ext^2(E, E) &= \dim Ext^2(E_0, E_0) + \dim Ext^2(E_1, E_1) + \chi(E_1, E_0)\ . \end{aligned}$$

Since E is exceptional, we also have

$$\begin{aligned} 1 = \chi(E, E) = \dim Ext^0(E_0, E_0) - \dim Ext^2(E_0, E_0) + \chi(E_0, E_1) + \\ + \dim Ext^0(E_1, E_1) - \dim Ext^2(E_1, E_1) - \chi(E_1, E_0). \end{aligned}$$

But the skew part of $\chi(E_0, E_1)$ has for exceptional torsion sheaves E_0, E_1 an extremely simple Riemann - Roch expansion (see (4-2))

$$\chi(E_0, E_1) - \chi(E_1, E_0) = -K_S \cdot \Big(rk(E_0)\, c_1(E_1) - rk(E_1)\, c_1(E_0) \Big) = 0.$$

That is,

$$1 = \big(\dim Ext^0(E_0, E_0) - \dim Ext^2(E_0, E_0) \big) + \big(\dim Ext^0(E_1, E_1) - \dim Ext^2(E_1, E_1) \big)\ .$$

By n° 4.2.7, this is possible only if either C_0 or C_1 has the zero intersection with $-K_S$. But $-K_S$ is ample, i.e., $-K_S C_0 > 0$ and $-K_S C_1 > 0$. Thus, C is irreducible. □

4.3.2. COROLLARY. *An exceptional sheaf on S is either locally free or a torsion sheaf $\mathcal{O}_C(d)$, where $C \subset S$ is irreducible rational curve with $C^2 = -1$.* □

4.3.3. THEOREM. *All exceptional objects in $\mathcal{D}^b_{\mathcal{C}oh}(S)$ are pure, i. e. are exhausted by exceptional coherent sheaves shifted through the derived category grading.*

PROOF. Let $E \in Ob\, \mathcal{D}^b_{\mathcal{C}oh}(S)$ be exceptional with the cohomology sheaves $\mathcal{H}^\nu = \mathcal{H}^\nu(E)$. The Eilenberg - Moore spectral sequence has the form (see[1] [GeMa, p. 263–264])

$$E_1^{pq} = \bigoplus_{\alpha - \beta = p} \mathrm{Ext}^{2p+q} \left(\mathcal{H}^\alpha, \mathcal{H}^\beta \right)$$

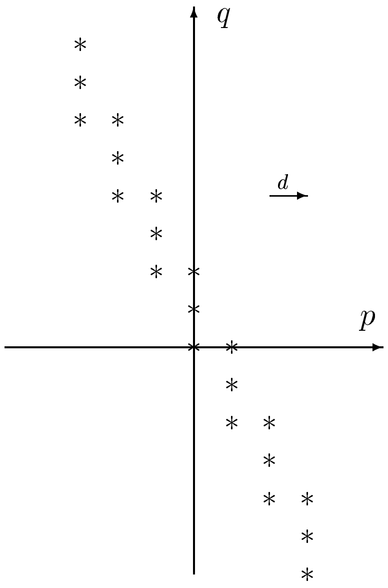

Fig. C. Eilenberg - Moore's $E_1^{pq} = \bigoplus_{\alpha-\beta=p} Ext^{2p+q}\left(\mathscr{H}^{\alpha}, \mathscr{H}^{\beta}\right)$

and converges to $Ext^{p+q}(E,E)$. All $Ext^1\left(\mathscr{H}^{\alpha}, \mathscr{H}^{\beta}\right)$-terms are situated on the line $2p+q=1$ and fail at once in E_∞ (see fig. C). In particular, $E_1^{01} = \oplus Ext^1\left(\mathscr{H}^{\nu}, \mathscr{H}^{\nu}\right) = 0$, i. e. all $\mathscr{H}^{\nu}$ are rigid sheaves. On the same time, the differential

$$d^{(-1,2)} : \oplus Hom\left(\mathscr{H}^{\nu}, \mathscr{H}^{\nu+1}\right) \longrightarrow \oplus\, Ext^2\left(\mathscr{H}^{\nu}, \mathscr{H}^{\nu}\right)$$

has to be an isomorphism and the differential

$$d^{(0,0)} : \oplus Hom\left(\mathscr{H}^{\nu}, \mathscr{H}^{\nu}\right) \longrightarrow \oplus\, Ext^2\left(\mathscr{H}^{\nu}, \mathscr{H}^{\nu-1}\right)$$

has to be an epimorphism with 1-dimensional kernel. But by n° 4.2.7 for each ν we have the inequality

$$\dim Ext^2\left(\mathscr{H}^{\nu}, \mathscr{H}^{\nu+1}\right) \leq \dim Ext^0\left(\mathscr{H}^{\nu+1}, \mathscr{H}^{\nu}\right)$$

and the strict inequality

$$\dim Ext^2\left(\mathscr{H}^{\nu}, \mathscr{H}^{\nu}\right) < \dim Ext^0\left(\mathscr{H}^{\nu}, \mathscr{H}^{\nu}\right).$$

So, there is at most one ν with non zero $\mathscr{H}^{\nu}$. □

[1] But be careful, because of indices misprints and two different indexing styles simultaneously used in this book during the spectral sequences discussion; we always follow the '*co*homology' indexation, where the differential in $E_1^{p,q}$ has bidegree $(1,0)$ and goes right down in the consequent terms E_2, E_3, …; it requires *decreasing* filtrations $\cdots \supset F^p \supset F^{p+1} \supset \cdots$, for example, the Eilenberg - Moore spectral sequence comes from the *canonical* filtration F^p_{can}, which has $Gr^p_{\text{can}} X = \mathscr{H}^{-p}(X)[p]$ and filters $Hom(X,Y)$ with

$$Gr^p Hom(X,Y) = \bigoplus_{\alpha-\beta=p} Hom(Gr^{\alpha}X, Gr^{\beta}Y)[p],$$

so, taking H^{p+q} we get Ext^{2p+q}; the good textbook on spectral sequences and coupled pairs machinery is [We]

4.4. Mutations of exceptional sheaves. Let (E_1, E_2) be an exceptional pair of sheaves on S. Since all exceptional objects in $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S)$ are pure, the mutations $L_{E_1}E_2$, $R_{E_2}E_1$ are pure sheaves and the triangles (2-2), (2-3) turns either into exact triples[1]

$$\begin{gathered} 0 \longrightarrow L_{E_1}E_2 \longrightarrow \mathit{Hom}(E_1,E_2)\otimes E_1 \longrightarrow E_2 \longrightarrow 0, \\ 0 \longrightarrow E_1 \longrightarrow \mathit{Hom}(E_1,E_2)^*\otimes E_2 \longrightarrow R_{E_2}E_1 \longrightarrow 0, \end{gathered} \tag{4-5}$$

or into more complicated sequences

$$\begin{gathered} 0 \longrightarrow \mathit{Hom}(E_1,E_2)\otimes E_1 \longrightarrow E_2 \longrightarrow L_{E_1}E_2[1] \longrightarrow \mathit{Ext}^1(E_1,E_2)\otimes E_1 \longrightarrow 0, \\ 0 \longrightarrow \mathit{Ext}^1(E_1,E_2)^*\otimes E_1 \longrightarrow R_{E_2}E_1[-1] \longrightarrow E_1 \longrightarrow \mathit{Hom}(E_1,E_2)^*\otimes E_2 \longrightarrow 0. \end{gathered} \tag{4-6}$$

In particular, we automatically have $\mathit{Ext}^2(E_1, E_2) = 0$. The mutations (4-5) are called *regular*. The mutations (4-6) are called *irregular*. We will prove later (see the proposition from n° 5.3.5) that at least one of the spaces $\mathit{Hom}(E_1, E_2)$, $\mathit{Ext}^1(E_1, E_2)$ should vanish for any exceptional pair on Del Pezzo surface, i. e. the quadruples (4-6) turn actually to exact triples

$$\begin{gathered} 0 \longrightarrow \mathit{Hom}(E_1,E_2)\otimes E_1 \longrightarrow E_2 \longrightarrow L_{E_1}E_2[1] \longrightarrow 0, \\ 0 \longrightarrow E_2 \longrightarrow L_{E_1}E_2[1] \longrightarrow \mathit{Ext}^1(E_1,E_2)\otimes E_1 \longrightarrow 0, \\ 0 \longrightarrow R_{E_2}E_1[-1] \longrightarrow E_1 \longrightarrow \mathit{Hom}(E_1,E_2)^*\otimes E_2 \longrightarrow 0, \\ 0 \longrightarrow \mathit{Ext}^1(E_1,E_2)^*\otimes E_1 \longrightarrow R_{E_2}E_1[-1] \longrightarrow E_1 \longrightarrow 0. \end{gathered} \tag{4-7}$$

4.4.1. COROLLARY. *If $\mathit{Hom}(E_1, E_2) \neq 0$, then each canonical contractions*

$$\mathit{Hom}(E_1,E_2)\otimes E_1 \longrightarrow E_2 \qquad E_1 \longrightarrow \mathit{Hom}(E_1,E_2)^*\otimes E_2$$

is either monomorphic or epimorphic.

4.5. Dual exceptional bases. Let an exceptional collection of sheaves $\{E_0, E_1, \dots, E_k\}$ be a base for $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S)$. Then its right dual base $\{E_0^\vee, E_1^\vee, \dots, E_k^\vee\}$ consists of shifted pure sheaves, i. e. we can write

$$E_i^\vee = L_{E_0}L_{E_1}\,\cdots\,L_{E_{k-i-1}}E_{k-i} = E_i^\times[\Delta_i^\times],$$

where $E_i^\times$ is an exceptional sheaf and $\Delta_i^\times$ equals a number of irregular left mutations (4-7) in this series. Symmetrically, we write

$${}^\vee E_i = R_{E_{k-i+1}}L_{E_{k-i+2}}\,\cdots\,L_{E_k}E_{k-i} = {}^\times E_i[-{}^\times\Delta_i]$$

where ${}^\times E_i$ is an exceptional sheaf and ${}^\times\Delta_i$ is number of irregular right mutations in this series (certainly, ${}^\times\Delta_i$ may differ from $\Delta_i^\times$). We have

$$\mathit{Ext}^{i-{}^\times\Delta_i}({}^\times E_i, E_{k-i}) = \mathit{Ext}^{k-i-{}^\times\Delta_i}(E_{k-i}, E_i^\times) = \mathbb{C}$$

and all other $\mathit{Ext}^\nu({}^\vee E_\alpha, E_\beta) = \mathit{Ext}^\nu(E_\alpha, E_\beta^\vee) = 0$.

[1] as it was on $\mathbb{P}_n$ considered in the previous paragraph

4.5.1. Beilinson-type spectral sequence written in terms of pure sheaves is situated in the strip $-k \le p \le 0$, $0 \le q \le k$ and has

$$E_1^{p,q} = Ext^{q-{}^{\times}\Delta_p}(E_{-p}, F) \otimes {}^{\times}E_{k+p}\,,$$

if the left dual base is used, or

$$E_1^{p,q} = Ext^{k-q-\Delta_p^{\times}}(F, E_{-p})^* \otimes E_{k+p}^{\times}$$

if you prefer the right one. The both depend functorially on a coherent sheaf F on S and converges to F in the usual sense: F admits a decreasing filtration with consequent quotients

$$E_{\infty}^{0,0}, \ E_{\infty}^{-1,1}, \ \dots\,, \ E_{\infty}^{-k,k}$$

and $E_{\infty}^{p,q} = 0$ for $p+q \ne 0$ at the same time.

4.5.2. Example. Consider the simplest helix spanned by the foundation

$$\{E_0, E_1, \dots, E_3\} = \{\mathscr{O}, \mathscr{O}(1,0), \mathscr{O}(0,1), \mathscr{O}(1,1)\}$$

on the quadric surface $S = \mathbb{P}_1 \times \mathbb{P}_1$ (we write $\mathscr{O}(a,b)$ for $p_1^*\mathscr{O}_{\mathbb{P}_1}(a) \underset{\mathscr{O}_S}{\otimes} p_2^*\mathscr{O}_{\mathbb{P}_1}(b)$, where

$$\mathbb{P}_1 \xleftarrow{\ p_1\ } \mathbb{P}_1 \times \mathbb{P}_1 \xrightarrow{\ p_2\ } \mathbb{P}_1$$

are the standard projections). Then the right dual base has, certainly, $E_0^{\times} = E_3 = \mathscr{O}(1,1)$ and $E_3^{\times} = E_3 \otimes \omega_S = \mathscr{O}(-1,-1)$. Further, $E_1^{\times} = \mathscr{O}(-1,0)$ because the corresponding mutation comes essentially from $\mathbb{P}_1 = \mathbb{P}(V)$:

$$0 \longrightarrow \mathscr{O}(-1,0) \longrightarrow V^* \otimes \mathscr{O} \longrightarrow \mathscr{O}(1,0) \longrightarrow 0$$

(in particular, it is regular, that is $\Delta_1^{\times} = 0$). Finally, $E_2^{\times} = \mathscr{O}(0,-1)$ with $\Delta_2^{\times} = 1$. Indeed, since the pair $\{E_1, E_2\} = \{\mathscr{O}(1,0), \mathscr{O}(0,1)\}$ is completely orthogonal, the first mutation $L_{E_1}E_2 = E_2$ is the trivial irregular one; the next mutation (through E_0) comes again from $\mathbb{P}_1$

$$0 \longrightarrow \mathscr{O}(0,-1) \longrightarrow V^* \otimes \mathscr{O} \longrightarrow \mathscr{O}(0,1) \longrightarrow 0$$

and is regular. Note that $\Delta_3^{\times} = 1$, because the straightforward computation of $E_3^{\times}$ requires the consequent mutations

$$\begin{gathered}
0 \longrightarrow \mathscr{O}(-1,1) \longrightarrow V^* \otimes \mathscr{O}(0,1) \longrightarrow \mathscr{O}(1,1) \longrightarrow 0 \\
0 \longrightarrow \mathscr{O}(-1,1) \longrightarrow \mathscr{T}_S(-1,-1) \longrightarrow \Lambda^2 V^* \otimes \mathscr{O}(1,0) \longrightarrow 0 \\
0 \longrightarrow \mathscr{O}(-1,-1) \longrightarrow S^2 V^* \otimes \mathscr{O} \longrightarrow \mathscr{T}_S(-1,-1) \longrightarrow 0
\end{gathered}$$

where the first and the third of are regular but the second is irregular and coincides with the canonical extension using

$$Ext^1(E_1, L_{E_2}E_3) = Ext^1(\mathscr{O}(1,0), O(-1,1)) = \Lambda^2 V^* = \mathbb{C}\,.$$

So, the Beilinson spectral sequence presents an arbitrary coherent sheaf F on $\mathbb{P}_1 \times \mathbb{P}_1$ by the functorial spectral sequence whose E_1 term is situated in the rectangle $-3 \le p \le 0$, $0 \le q \le 3$ and looks as

$H^2(F(1,1)) \otimes \mathscr{O}(-1,-1)$	$H^2(F(0,1)) \otimes \mathscr{O}(0,-1)$	0	0
$H^1(F(1,1)) \otimes \mathscr{O}(-1,-1)$	$H^1(F(0,1)) \otimes \mathscr{O}(0,-1)$	$H^2(F(1,0)) \otimes \mathscr{O}(-1,0)$	$H^2(F) \otimes \mathscr{O}$
$H^0(F(1,1)) \otimes \mathscr{O}(-1,-1)$	$H^0(F(0,1)) \otimes \mathscr{O}(0,-1)$	$H^1(F(1,0)) \otimes \mathscr{O}(-1,0)$	$H^1(F) \otimes \mathscr{O}$
0	0	$H^0(F(1,0)) \otimes \mathscr{O}(-1,0)$	$H^0(F) \otimes \mathscr{O}$

with zeros at all other cells.

4.6. Transitivity of the braid group action. The main advance of the Helix theory on Del Pezzo surfaces is given by the following theorem. Recall that an exceptional collection is called *full*, if it generates $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S)$ as a triangulated category.

4.6.1. THEOREM. *Any exceptional collection of sheaves on S may be extended to a full collection. The braid group acts transitively on the set of full exceptional collections of sheaves.*

In particular, this theorem implies that any exceptional sheaf on Del Pezzo surface may be constructed by the mutations starting from some standard initial collection, say one of listed in n° 4.1. The proof of this theorem was published in [Ku3] and [KuOr]. It is long enough: besides a number of pure combinatorial geometrical tricks, similar to the ones used in the integer linear programming, it uses essentially an interlaying between several *stability concepts*, which will be discussed briefly in the next paragraph. Then, in §6, we will discuss the proof of the theorem n° 4.6.1 itself.

§5. Del Pezzo surfaces: stability of rigid sheaves.

5.1. Slopes. Let $\mathscr{S}_{\mathrm{TF}}(X)$ be a set of all torsion free coherent sheaves on a smooth algebraic variety X. It is partially ordered by inclusions. A function $\tau : \mathscr{S}_{\mathrm{TF}}(X) \longrightarrow \Gamma$ with values in a partially ordered set Γ is called *a slope*, if it has the following properties:

swing quality: for any exact triple of torsion free sheaves

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

there is a triple of equivalences:

$$\begin{aligned} \tau(F) = \tau(E) &\iff \tau(E) = \tau(G) \\ \tau(F) \le \tau(E) &\iff \tau(E) \le \tau(G) \\ \tau(F) \ge \tau(E) &\iff \tau(E) \ge \tau(G) \end{aligned}$$

local finiteness: for any $r_0 \in \mathbb{N}$ and $\tau_0 \in \Gamma$ there are no infinite sequences of torsion free sheaves $G_1, G_2, G_3, \ldots$ with bounded ranks: $rkG_i \le r_0$ and monotone decreasing slopes $\tau(G_i) > \tau(G_{i+1})$ bounded below: $\tau(G^i) > \tau_0$;

week monotony: $\tau(F) \le \tau(E)$ for all $F \subset G$ of the same $rkF = rkG$.

5.1.1. The most common way for producing the slopes is to use an additive function of a coherent sheave divided by the rank. If Γ is an ordered Abelian group and $\alpha : K_0(X) \longrightarrow \Gamma$ is an additive function, then $\tau(E) \stackrel{\text{def}}{=} \alpha(E)/rk(E)$ is locally finite, because

$$|\tau(E) - \tau(F)| \ge \frac{1}{\max(rkE, rkF)^2},$$

and has the swing quality, since a sign of $\tau(E) - \tau(F)$ equals a sign of the determinant

$$\det\begin{pmatrix} \alpha(E) & \alpha(F) \\ rk(E) & rk(F) \end{pmatrix} = -\det\begin{pmatrix} (\alpha(E) - \alpha(F)) & \alpha(E) \\ (rk(E) - rk(F)) & rk(E) \end{pmatrix}.$$

The week monotony requires a more accurate choice of the additive function α.

5.1.2. Giesecker type slopes. For any ample divisor A and any coherent sheave E on X *a Hilbert polynomial* $h_{E,A} \in \mathbb{Q}[t]$ is defined by the property

$$h_{E,A}(n) \stackrel{\text{def}}{=} \chi\Big(E(nA)\Big) = \sum_{\nu=0}^{\dim X} \dim H^\nu\Big(E \otimes \mathcal{O}(nA)\Big) \qquad \forall\, n \in \mathbb{N}.$$

We order $\mathbb{Q}[t]$ by prescription $f \le g \iff f(n) \le g(n)\ \forall\, n \gg 0$ and define *a Giesecker slope* of a torsion free sheaf E as

$$\gamma_A(E) \stackrel{\text{def}}{=} \frac{h_{E,A}}{r(E)} \in \mathbb{Q}[t].$$

The Giesecker slope satisfies the *strict* version of the week monotony: if $F \subsetneq E$ are torsion free of the same rank, then $\gamma_A(F) < \gamma_A(E)$, since $\chi\big((E/F)(nA)\big) > 0$ for $E/F \ne 0$ by the Serre theorem. Any slope that satisfies this strict version of the week monotony is called *strict* (or *Giesecker type*) slope.

5.1.3. Mumford - Takemoto slope. For any 'NEF' divisor H on X define *a Mumford - Takemoto* slope of a torsion free sheaf E as

$$\mu_H(E) \stackrel{\text{def}}{=} \frac{c_1(E) \cdot H^{\dim X - 1}}{r(E)}.$$

This slope also satisfies little more precise property than the week monotony. Namely, if $F \subsetneq E$ are torsion free of the same rank, then $\mu_H(F) \le \mu_H(E)$ with the equality possible only if:

$$\operatorname{codim} \operatorname{supp}(E/F) \ge 1 \text{ (for 'NEF' } H), \tag{5-1}$$
$$\operatorname{codim} \operatorname{supp}(E/F) \ge 2 \text{ (for ample } H). \tag{5-2}$$

This follows from the 'NEF' property[1] and from the Nakai - Moishezon criterium, because of $c_1(E/F)$ is either zero or effective.

[1]By the definition, this means that $D \cdot H^{\dim X - 1} \ge 0$ for any effective divisor D.

5.2. Stability. Let τ be a slope. A torsion free sheaf E on X is called *τ-stable* (corr. *τ-semistable*), if for any subsheaf $F \subset E$ with $rkF < rkE$ the inequality

$$\tau(F) < \tau(E) \qquad (\text{corr. } \tau(F) \le \tau(E))$$

holds. By the swing quality, this inequalities are equivalent to the opposite inequalities

$$\tau(G) > \tau(E) \qquad (\text{corr. } \tau(G) \ge \tau(E))$$

satisfied for all torsion free quotients $G = E/F$. Indeed, the standard 'torsion removing' diagram

$$\begin{tikzcd} T & & \\ F' \arrow[u, two heads] \arrow[r, hook] & E \arrow[r, two heads] & G' \\ F \arrow[u, hook] \arrow[r, hook] & E \arrow[u, equal] \arrow[r, two heads] & G \arrow[u, two heads] \\ & & T \arrow[u, hook] \end{tikzcd},$$

where $T = \operatorname{Tors} G$, shows that it is enough to test (semi) stability only by subsheaves $F' \subset E$ with torsion free quotients $G' = E/F'$.

5.2.1. Properties of semistable sheaves. Let $F \xrightarrow{\varphi} E$ be a morphism between τ-semistable sheaves. The swing quality applied to the triple $\ker\varphi \hookrightarrow F \twoheadrightarrow \operatorname{im}\varphi$, taking into account the inclusion $\operatorname{im}\varphi \subset E$, leads to the following implications:

$$\begin{array}{rcl} \tau(F) > \tau(E) & \Longrightarrow & \varphi = 0; \\ \tau(F) = \tau(E) \text{ and } F \text{ is stable} & \Longrightarrow & \varphi \quad \text{is either zero or injective;} \\ \tau(F) = \tau(E) \text{ and } E \text{ is stable} & \Longrightarrow & \varphi \quad \text{is either zero} \\ & & \text{or surjective at generic point,} \end{array}$$

Moreover, if γ is a Giesecker type slope, then the third implication becomes stronger

$$\gamma(F) = \gamma(E) \text{ and } E \text{ is stable} \quad \Longrightarrow \varphi \text{ is either zero or an epimorphism.}$$

In particular, each stable sheaf E is *simple*, i. e. $Hom(E, E) = \mathbb{C}$.

Certainly, all invertible sheaves are stable just by the definition. Of course, semistablity is preserved under twists by invertible sheaves and tensoring by vector spaces. More generally, if an exact triple

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

consists of torsion free sheaves of the same slope, then E is semistable iff both F, G are.

A complete proof of these facts can be found in [Ku3].

Since any torsion free sheaf E on a projective variety is a quotient sheaf of a semistable sheaf $Hom(\mathscr{O}(nA), E) \otimes \mathscr{O}(nA)$, where A is an ample divisor and n is large enough, the slopes $\tau(G)$ of torsion free quotients $G = E/F$ are bounded below, say by $\tau(\mathscr{O}(nA))$.

5.2.2. Harder - Narasimhan filtration. Any torsion free sheaf E has a canonical[1] decreasing filtration
$$E = F^0 \supset F^1 \supset \cdots \supset F^n \supset F^{n+1} = 0$$
with semistable quotients $G^i = F^i/F^{i+1}$ of strictly increasing slopes $\tau(G^i) < \tau(G^{i+1})$. It is called the *Harder - Narasimhan filtration*.

It is constructed inductively as follows. Since the slopes of torsion free quotients $G = E/F$ are bounded below, the local finiteness property implies that there exists minimal such the slope τ_0. Let $G^1 = E/F^1$ be a torsion free sheaf of the maximal rank among all torsion free quotients E/F with $\tau(E/F) = \tau_0$. Then G^1 is semistable, has $\tau(G^1) > \tau(E) > \tau(F^1)$, and we can repeat the procedure with F^1 instead of E as soon F^1 is non semistable.

The uniqueness is established by a little more delicate induction going in the opposite direction. Namely, if $(G'_0, G'_1, G'_2, \dots, G'_m)$ and $(G''_0, G''_1, G''_2, \dots, G''_n)$ both satisfy the Harder - Narasimhan conditions, then $G'_m \subset E$ and $G''_n \subset E$ are the subsheaves of the same slope $\tau(G'_m) = \tau(G''_n)$. Indeed, an inequality, say $\tau(G'_m) > \tau(G''_n)$, implies that $Hom(G'_m, G''_i) = 0$ for all i by the semistability, i. e. $Hom(G'_m, E) = 0$. Further, an easy increasing induction on $i = 0, 1, \dots$ shows that $G'_m \subset F''_i$ for all $0 \le i \le n$, because $Hom(G'_m, G''_i) = 0$ implies that $G'_m \subset F''_i$ is (mod F''_{i+1})-congruent to zero. Similarly, $G''_n \subset F'_j$ for all $0 \le j \le m$. Taking $i = n$ and $j = m$, we get $G'_m = G''_n$ and can repeat the procedure for E/Q filtered with the quotients $(G'_0/Q, \dots, G'_{m-1}/Q)$ and $(G''_0/Q, \dots, G''_{n-1}/Q)$, where $Q = G'_m = G''_n$. We refer the readers to [HuLe] for details.

5.2.3. Jordan - Holder filtration. We need its rather more precise version than the usual one (comp. with [HuLe]). Let γ be a Giesecker type slope, i. e. satisfy the strict inequality $\gamma(F) < \gamma(E)$ for all torsion free $F \subsetneq E$ of the same rank. Then any γ-semistable torsion free sheaf E admits a decreasing filtration
$$E = F^0 \supset F^1 \supset F^2 \supset \cdots \supset F^n \supset F^{n+1} = 0 \tag{5-3}$$
whose quotients $G^i = F^i/F^{i+1}$ are semistable, have $\gamma(G^i) = \gamma(F^i) = \gamma(E)$, satisfy the orthogonality conditions:
$$Hom\left(F^{i+1}, G^i\right) = Hom\left(G^i, G^{i+1}\right) = Hom\left(G^{i+1}, G^i\right) = 0 \tag{5-4}$$
and each G^i for one's turn admits a filtration
$$G^i = F^{i,0} \supset F^{i,1} \supset F^{i,2} \supset \cdots \supset F^{i,m_i} \supset F^{i,m_i+1} = 0 \tag{5-5}$$
with γ-stable isotype quotients $Q^i = F^{i,\nu}/F^{i,\nu+1}$ (the same for all ν) with $\gamma(Q^i) = \gamma(E)$. We will call the filtration (5-3) *an isotype Jordan - Holder filtration*.

To construct it, consider a torsion free quotient Q^0, of E, with $\gamma(Q^0) = \gamma(E)$ and minimal possible rank. Obviously, it is stable. Write $\widetilde{F}^{1,0}$ for a kernel of the epimorphism $E \twoheadrightarrow Q^0$. It is semistable with $\gamma(\widetilde{F}^{1,0}) = \gamma(E)$. If $Hom(\widetilde{F}^{1,0}, Q^0) = 0$, we put $F^1 = \widetilde{F}^{1,0}$. Otherwise, there is an epimorphism $\widetilde{F}^{1,0} \twoheadrightarrow Q^0$. We write $\widetilde{F}^{1,1}$ for its kernel, which is semistable with $\gamma(\widetilde{F}^{1,1}) = \gamma(E)$. Repeating this procedure, we get a filtration
$$E = F^0 \supset \widetilde{F}^{1,0} \supset \widetilde{F}^{1,1} \supset \cdots \supset \widetilde{F}^{1,m_1}$$

[1] This means that the collection of factors $(G^0, G^1, \dots, G^n)$ is uniquely defined by E.

with stable isotype quotients $F^{1,\nu}/F^{1,\nu+1} = Q^0$ such that $Hom(\widetilde{F}^{1,m_1}, Q^0) = 0$. Then we put $F^1 = \widetilde{F}^{1,m_1}$ and apply whole of the previous procedure to F^1 instead of E. We get F^2 and so on. So, each quotient $G^i = F^i/F^{i+1}$ is filtered by $F^{i,\nu} = \widetilde{F}^{i,\nu}/F^{i+1}$ and this filtration has stable isotype quotients $Q^i = F^{i,\nu}/F^{i,\nu+1} = \widetilde{F}^{i,\nu}/\widetilde{F}^{i,\nu+1}$ such that $Hom(F^{i+1}, Q^i) = 0$. This implies the orthogonality conditions (5-4).

5.3. Stability of rigid sheaves on Del Pezzo surface. We write γ_{can} and μ_{can} for the Giesecker and Mumford - Takemoto slopes w. r. t. the anticanonical divisor $-K_S$ of a Del Pezzo surface S, that is

$$\gamma_{\text{can}}(E) = \frac{\chi(E \otimes \omega^{*\otimes n})}{rkE} \in \mathbb{Q}[n], \qquad \mu_{\text{can}}(E) = -\frac{c_1(E)\cdot K_S}{rkE} \in \mathbb{Q}.$$

The Riemann - Roch formula (4-1):

$$\gamma_{\text{can}}(E) = 1 + \frac{1}{2}\left(\mu_{\text{can}}(E) - \mu_{\text{can}}(\omega^{\otimes n})\right) + q(E) + q(\omega^{\otimes n}) - \frac{c_1(E)\cdot c_1(\omega^{\otimes n})}{rk(E)} =$$
$$= 1 + \frac{1}{2}nK_S^2 + \frac{1}{2}n^2K_S^2 + \frac{1}{2}\mu_{\text{can}}(E) + q(E) + \frac{c_1(E)\cdot K_S}{rkE}\,n$$

shows that $\mu_{\text{can}}(E)$ is the leading coefficient of the normalized Hilbert polynomial $\gamma_{\text{can}}(E)$ among the coefficients that do depend on E. So, μ_{can}-stability implies γ_{can}-stability and γ_{can}-semistability implies μ_{can}-semistability.

5.3.1. LEMMA. *Let E, F be μ_{can} - semistable sheaves on S. Then the inequality $\mu_{\text{can}}(E) > \mu_{\text{can}}(F) - K_S^2$ implies $Ext^2(F,E) = 0$.*

PROOF. By the Serre duality, $Ext^2(F,E) \cong Hom^*(E, F(K_s))$. On the other hand,

$$\mu_{\text{can}}(F(K_S)) = \frac{c_1(F(K_S))\cdot(-K_S)}{rk(F(K_S))} = \frac{-c_1(F)\cdot K_S - rk(F)K_S^2}{rk(F)} = \mu_{\text{can}}(F) - K_S^2 < \mu_{\text{can}}(E).$$

Using properties of semistable sheaves (n° 5.2.1), we obtain $Hom(E, F(K_S)) = 0$. □

5.3.2. COROLLARY. *Let E, F be γ_{can} - semistable sheaves on S. Then the inequality $\gamma_{\text{can}}(E) > \gamma_{\text{can}}(F(K_S))$ implies $Ext^2(F,E) = 0$.* □

5.3.3. PROPOSITION. *Locally free exceptional sheaves on S are μ_{can}-stable.*

PROOF. Let us use here a simple trick introduced by S. Zube in [Zu1]. Namely, restrict an exceptional vector bundle E from S onto a smooth elliptic curve $D \in |-K_S|$. Then applying $Hom(E, *)$ to the exact triple

$$0 \longrightarrow E(K_S) \longrightarrow E \longrightarrow E|_D \longrightarrow 0$$

we get $Hom_D(E|_D, E|_D) = Hom_S(E, E|_D) = \mathbb{C}$, because of

$$Ext^k(E, E(K_S))^* = Ext^{2-k}(E,E) = \begin{cases} \mathbb{C}, \text{ for } k = 2 \\ 0, \text{ for } k \neq 2 \end{cases}.$$

So, the restriction $E' = E|_D$ is a simple vector bundle over elliptic curve. By the Atiyah theorem (see [At] and [Ku1]), it is stable w. r. t. the slope $\mu(E) = \dfrac{\deg(E)}{rk(E)}$ on D. Since

$\mu(E') = \mu_{\mathrm{can}}(E)$, any destabilizing subsheaf for E on S would destabilize E' over D as well. Hence, E is μ_{can}-stable on S. □

5.3.4. COROLLARY. *Let E' be restriction of an exceptional bundle E from S onto a smooth elliptic curve $D \in |-K_S|$. Then $\mu(E') = \mu_{\mathrm{can}}(E)$ and E' is stable.* □

5.3.5. PROPOSITION. *Let $\{E_1, E_2\}$ be an exceptional pair of sheaves on S. Then at least one of the spaces $Hom(E_1, E_2)$, $Ext^1(E_1, E_2)$ is zero. Moreover, if all E_i are locally free, then*

$$\begin{aligned}
\mu_{\mathrm{can}}(E_1) = \mu_{\mathrm{can}}(E_2) &\iff Ext^1(E_1, E_2) = 0,\ Hom(E_1, E_2) = 0,\\
\mu_{\mathrm{can}}(E_1) < \mu_{\mathrm{can}}(E_2) &\iff Ext^1(E_1, E_2) = 0,\ Hom(E_1, E_2) \neq 0,\\
\mu_{\mathrm{can}}(E_1) > \mu_{\mathrm{can}}(E_2) &\iff Ext^1(E_1, E_2) \neq 0,\ Hom(E_1, E_2) = 0.
\end{aligned}$$

PROOF. By the corollary from n° 4.3.2 an exceptional sheaf on S is either locally free or a torsion sheaf $\mathcal{O}_e(d)$, where e is an exceptional curve.

Firstly, we suppose that both E_1 and E_2 are locally free sheaves. Consider the restriction sequence to a smooth elliptic curve $D \in |-K_S|$:

$$0 \longrightarrow E_1^* \otimes E_2(K_s) \longrightarrow E_1^* \otimes E_2 \longrightarrow (E_1^* \otimes E_2)|_D \longrightarrow 0\ .$$

Combining the Serre duality and the definition of exceptional pair, we get

$$Ext^q(E_1, E_2(K_S))^* \cong Ext^{2-q}(E_2, E_1) = 0\ \forall q\ .$$

Hence, the restriction sequence induces the isomorphism

$$Ext^q_S(E_1, E_2) \cong Ext^q_e(E_1|_D, E_2|_D)\ .$$

The previous corollary implies that $E_i|_D$ is a stable bundle on the elliptic curve D and $\mu(E_i|_D) = \mu_{\mathrm{can}}(E_i)$. Therefore, $Ext^1(E_1, E_2) = 0$ whenever $\mu_{\mathrm{can}}(E_1) < \mu(E_2)$. In the converse case, $Hom(E_1, E_2) = 0$.

In the case $\mu_{\mathrm{can}}(E_1) = \mu_{\mathrm{can}}(E_2)$ any nonzero map $E_1 \to E_2$ is isomorphism, whence $Hom(E_1, E_2) = 0$. In additional, Riemann-Roch formula (4-1) gives that $\chi(E_1, E_2) = 0$. This means that $Ext^1(E_1, E_2) = 0$ as well.

The Proposition holds if E_2 is a locally free sheaf and $E_1 = \mathcal{O}_e(d)$, since any map from a torsion sheaf to a locally free sheaf is zero.

Now, we assume that $E_1 = \mathcal{O}_{e_1}(d_1)$ and $E_2 = \mathcal{O}_{e_2}(d_2)$. If $e_1 \neq e_2$, then $Ext^q(E_1, E_2) = 0\ \forall q$. If $e_1 = e_2 = e$, then the Proposition follows from the equality:

$$Ext^q_S(E_1, E_2) = Ext^q_e(\mathcal{O}_e(d_1), \mathcal{O}_e(d_2)).$$

Finally, suppose that $E_1 = E$ is a locally free sheaf and $E_2 = \mathcal{O}_e(d)$. We have:

$$Ext^q_S(E_1, E_2) = Ext^q_e(E|_e, \mathcal{O}_e(d))\ .$$

Besides, by the Serre duality and by the definition of exceptional pair, for any q we obtain

$$0 = Ext^q_S(E_2, E_1)^* \cong Ext^{2-q}_S(E_1, E_2(K_S)) \cong Ext^{2-q}_e(E|_e, \mathcal{O}_e(d-1))$$

(in the last equality we use the relation: $eK_S = -1$). In particular, $Hom_e(E|_e, \mathscr{O}_e(d-1)) = Ext^1_e(E|_e, \mathscr{O}_e(d-1)) = 0$. Now we apply the functor $Ext_e(E|_e, \cdot)$ to the restriction sequence

$$0 \longrightarrow \mathscr{O}_e(d-1) \longrightarrow \mathscr{O}_e(d) \longrightarrow \mathscr{O}_x \longrightarrow 0$$

and get

$$Ext^1_S(E_1, E_2) \cong Ext^1_e(E|_e, \mathscr{O}_e(d)) \cong Ext^1(E|_e, \mathscr{O}_x) = 0.$$

□

5.4. Spectral sequences. More precise results than the above theorem may be obtained by a straightforward computations of $Ext^{p+q}(E,E)$ via spectral sequences associated with the Harder - Narasimhan and Jordan - Holder filtrations and using estimations from n° 4.2. Just to illustrate this approach, let us prove the μ_{can}-semistability of exceptional E without the above 'elliptic trick'.

We have to show that the Harder - Narasimhan filtration $E = F^0 \supset F^1 \cdots \supset F^{n+1} = 0$, for E, has just one term. Let $Gr^i = F^i/F^{i+1}$ be its quotients. Consider the spectral sequence with

$$E_1^{p,q} = \bigoplus_{\beta-\alpha=p} Ext^{p+q}(Gr^\alpha, Gr^\beta)$$

which converges to $Ext^{p+q}(E,E)$. It looks like in fig. D on page 39, because of

$$Ext^2(Gr^\beta, Gr^\alpha) = Hom(Gr^\alpha, Gr\beta) = 0 \text{ for } \alpha > \beta$$

by the semistability of Gr^ν and the lemma from n° 4.2.6. So, $E_2 = E_\infty$ and $E_1^{(0,1)} = E_\infty^{(0,1)} = 0$, that is, all Gr^ν are rigid, the differential

$$d^{(-1,2)} : \oplus Hom\left(Gr^\nu, Gr^{\nu-1}\right) \longrightarrow \oplus\, Ext^2\,(\,Gr^\nu, Gr^\nu\,)$$

has to be an isomorphism, and the differential

$$d^{(0,0)} : \oplus Hom\,(\,Gr^\nu, Gr^\nu\,) \longrightarrow \oplus\, Ext^2\left(Gr^\nu, Gr^{\nu+1}\right)$$

has to be an epimorphism with 1-dimensional kernel. This implies that there is at least one non zero Gr^ν exactly like in the proof of the theorem from n° 4.3.3.
Quite similar strategy allows to prove the following

5.4.1. LEMMA. *Suppose that a γ_{can}-semistable rigid sheaf F has a filtration*

$$F = F^1 \supset F^2 \supset \cdots \supset F^k \supset F^{k+1} = 0$$

with stable isotype quotients $E = F^i/F^{i+1}$ (the same for all i). Then E is exceptional and $F = k\,E$ in a sense that

$$F = \underbrace{E \oplus E \oplus \cdots \oplus E}_{k}\ .$$

PROOF. Since E is stable, $Hom(E,E) = \mathbb{C}$ and by the corollary from n° 5.3.2 we have $Ext^2(E,E) = 0$. So,

$$E_1^{p,q} = \bigoplus_{\beta-\alpha=p} Ext^{p+q}(Gr^\alpha, Gr^\beta)$$

which computes $Ext^{p+q}(E,E)$, looks like in fig. E on page 40.

The topmost left term $E_1^{-k,k+1} = Ext^1(E,E)$ fails into $E^1_\infty = Ext^1(F,F) = 0$. Hence E is rigid, that is, exceptional. □

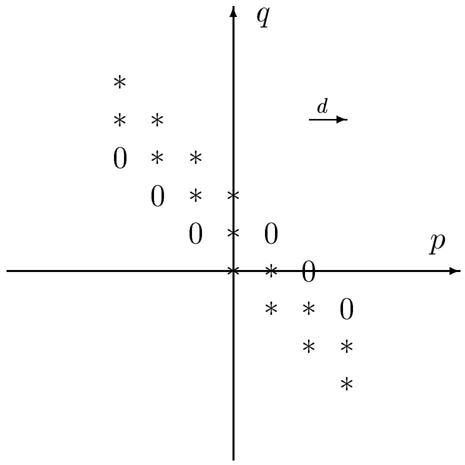

Fig. D. $E_1^{p,q} = \bigoplus\limits_{\beta-\alpha=p} Ext^{p+q}(Gr^{\alpha}, Gr^{\beta})$

5.5. Structure of rigid sheaves. In this section we apply the above technique to prove that any torsion free rigid sheaf E on a Del Pezzo surface S is a direct sum of exceptional locally free sheaves. The proof is nothing but a step by step analysis of the spectral sequences associated with Jordan - Holder and Harder - Narasimhan filtrations.

5.5.1. LEMMA. *A γ_{can}-semistable rigid sheaf E on a Del Pezzo surface is a direct sum of exceptional locally free sheaves.*

PROOF. Let $E = F^0 \supset F^1 \supset \cdots \supset F^n \supset F^{n+1} = 0$ be the isotype Jordan - Holder filtration constructed in n° 5.2.3. Each exact triple

$$0 \longrightarrow F^{i+1} \longrightarrow F^i \longrightarrow G^i \longrightarrow 0$$

satisfies the Mukai lemma n° 4.2.2, that is, $Hom(F^{i+1}, G^i) = Ext^2(G^i, F^{i+1}) = 0$, because of the filtration properties and the corollary from n° 5.3.2. Applying the Mukai lemma inductively, we get for all i

$$Ext^1(F^i, F^i) = Ext^1(G^i, G^i) = 0. \tag{5-6}$$

So, each isotype quotient G^i is rigid. The previous lemma (see n° 5.4.1) shows that G_i is a multiplicity of some exceptional vector bundle E_i:

$$G^i = k_i E_i = \underbrace{E_i \oplus E_i \oplus \cdots \oplus E_i}_{k_i} .$$

The corollary from n° 5.3.2 and properties of isotype Jordan - Holder filtration filtration, show that $Hom(G^{i+1}, G^i) = Hom(G^i, G^{i+1}) = 0$ and $Ext^2(G^i, G^j) = 0$ for all i, j. Therefore,

$$E_1^{p,q} = \bigoplus_{\beta-\alpha=p} Ext^{p+q}(Gr^{\alpha}, Gr^{\beta})$$

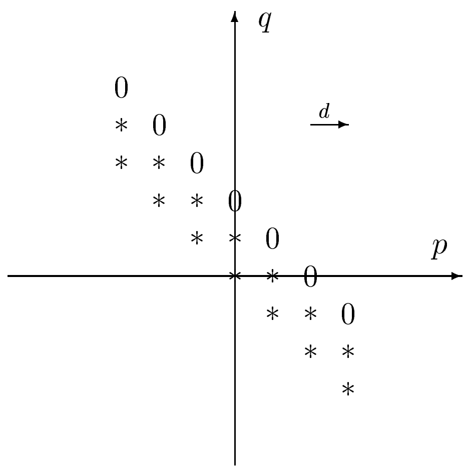

Fig. E. $E_1^{p,q} = \bigoplus_{\beta-\alpha=p} Ext^{p+q}(Gr^\alpha, Gr^\beta)$ for the lemma from n° 5.4.1

which computes $Ext^{p+q}(E,E)$, looks like in fig. F on page 41.

Again, the topmost left term $E_1^{-n,n+1} = Ext^1(G^n, G^0) \subset E_\infty^1 = Ext^1(E,E) = 0$. Hence, the exceptional locally free sheaves E_0 and E_n satisfy $Ext^1(E_n, E_0) = 0$. Since they are stable and have equal slopes, we also have either $E_0 = E_n$ or $Hom(E_0, E_n) = Hom(E_n, E_0) = 0$, and anyway, $Ext^2(E_0, E_n) = Ext^2(E_n, E_0) = 0$. The coincidence $\mu_{\text{can}}(E_0) = \mu_{\text{can}}(E_n)$ also means that 'the skew Euler characteristic' $\chi(E_n, E_0) - \chi(E_0, E_n)$ vanishes by the Riemann - Roch formula (n° 4.1.1). So, $Ext^1(E_0, E_n) = Ext^1(E_n, E_0) = 0$ and $Ext^1(G^0, G^n) = 0$ as well.

Now we can prove that E is a direct sum of its isotype quotients G^i's using the induction on the filtration length n. Case $n = 1$ was just considered above. Further, in the exact triple

$$0 \longrightarrow F^1 \longrightarrow E \longrightarrow G^0 \longrightarrow 0$$

F^1 is rigid, semistable, and decomposed as $F^1 = \bigoplus_{i=1}^{n} G^i$ by the inductive assertion. Since $Ext^1(G^0, G^n) = 0$, we have the splitting $E = E' \oplus G^n$ with γ_{can}-semistable rigid E' whose isotype Jordan - Holder filtration has smaller length, i. e. $E' = \bigoplus_{i=0}^{n-1} G^i$ by induction. □

5.5.2. THEOREM. *Any torsion free rigid sheaf E on a Del Pezzo surface S is a direct sum of exceptional locally free sheaves.*

PROOF. Consider a spectral sequence that computes $Ext^{p+q}(E,E)$ using the filtration of Harder - Narasimhan (see n° 5.2.2):

$$E = F^0 \supset F^1 \supset \cdots \supset F^n \supset F^{n+1} = 0$$

whose quotients $Gr^i = F^i/F^{i+1}$ are semistable and have strictly increasing slopes $\gamma_{\text{can}}(Gr^i) < \gamma_{\text{can}}(Gr^{i+1})$. The properties of semistable sheaves (see n° 5.2.1, n° 5.3.2) imply that

$$Hom(Gr^i, Gr^j) = 0 \text{ when } i > j \text{ and } \quad Ext^2(Gr^i, Gr^j) = 0 \text{ whenever } i \le j. \tag{5-7}$$

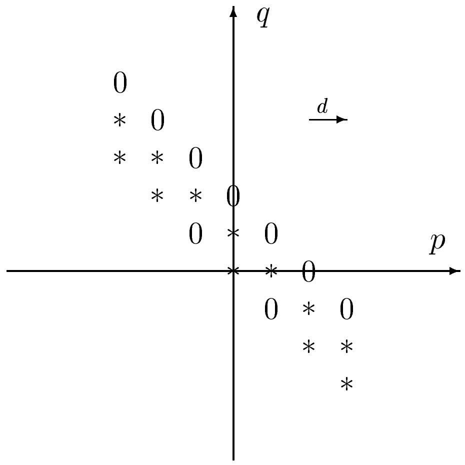

Fig. F. $E_1^{p,q} = \bigoplus_{\beta-\alpha=p} Ext^{p+q}(Gr^\alpha, Gr^\beta)$ for the lemma from n° 5.5.1

Therefore,

$$E_1^{p,q} = \bigoplus_{\beta-\alpha=p} Ext^{p+q}(Gr^\alpha, Gr^\beta)$$

looks like in fig. G on page 42.

First of all, we see that $\oplus Ext^1(Gr^i, Gr^i) = E_1^{0,1} = E_\infty^{0,1} \subset Ext^1(E,E) = 0$, that is, each quotient Gr^i is rigid and (by the lemma from n° 5.5.1) splits as $Gr^i = \bigoplus_\alpha E_i^\alpha$, where E_i^α are exceptional and have $\gamma_{\text{can}}(E_i^\alpha) = \gamma_{\text{can}}(Gr^i)$.

Secondary, we see that $\oplus Ext^1(Gr^i, Gr^{i-1}) = E_1^{-1,2} = E_\infty^{-1,2} \subset Ext^1(E,E) = 0$, that is,

$$Ext^1(Gr^i, Gr^{i-1}) = 0 \quad \forall\, i. \tag{5-8}$$

Let us show that $Ext^1(Gr^{i-1}, Gr^i) = 0 \;\forall\, i$ as well.

For this, it is enough to check that $Ext^1(E_{i-1}^\alpha, E_i^\beta) = 0 \;\forall\; i, \alpha, \beta$. Since $\gamma_{\text{can}}(E_{i-1}^\alpha) = \gamma_{\text{can}}(Gr^{i-1}) < \gamma_{\text{can}}(Gr^i) = \gamma_{\text{can}}(E_i^\beta)$, there are two possibilities: either $\mu_{\text{can}}(E_{i-1}^\alpha) = \mu_{\text{can}}(E_i^\beta)$ or $\mu_{\text{can}}(E_{i-1}^\alpha) < \mu_{\text{can}}(E_i^\beta)$. In the first case, the skewsymmetric part of the Euler characteristic $\chi(E_{i-1}^\alpha, E_i^\beta)$ vanishes and we use the same arguments as in the proof of the lemma from n° 5.5.1. In the second case, let us restrict the bundle $(E_i^\beta)^* \otimes E_{i-1}^\alpha$ to an elliptic curve $C \in |-K_S|$ and consider a long exact sequence of cohomologies

$$\cdots \longrightarrow Hom(E_i^\beta, E_{i-1}^\alpha) \longrightarrow Hom(E_i^\beta|_C, E_{i-1}^\alpha|_C) \longrightarrow$$
$$\longrightarrow Ext^1(E_i^\beta, E_{i-1}^\alpha(K_S)) \longrightarrow Ext^1(E_i^\beta, E_{i-1}^\alpha) \longrightarrow \cdots$$

which comes from the exact triple

$$0 \longrightarrow (E_i^\beta)^* \otimes E_{i-1}^\alpha \otimes K_S \longrightarrow (E_i^\beta)^* \otimes E_{i-1}^\alpha \longrightarrow (E_i^\beta)^* \otimes E_{i-1}^\alpha|_C \longrightarrow 0.$$

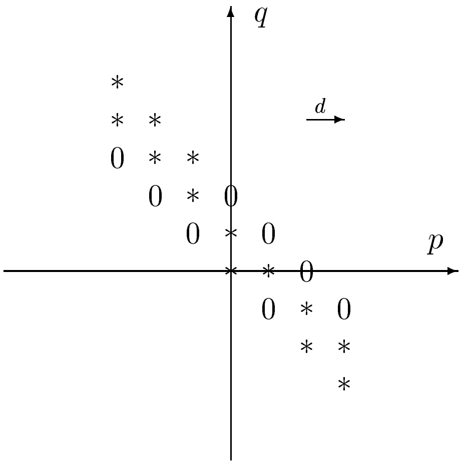

Fig. G. $E_1^{p,q} = \bigoplus_{\beta-\alpha=p} Ext^{p+q}(Gr^\alpha, Gr^\beta)$ for the theorem from n° 5.5.2

Since exceptional bundles are stabile, the corollary from n° 5.3.4 implies that

$$Hom(E_i^\beta, E_{i-1}^\alpha) = Hom(E_i^\beta|_C, E_{i-1}^\alpha|_C) = 0.$$

On the other side, (5-8) implies $Ext^1(E_i^\beta, E_{i-1}^\alpha) = 0$. So, we get $Ext^1\left(E_i^\beta, E_{i-1}^\alpha(K_S) \right) = 0$, or by the Serre duality, $Ext^1\left(E_{i-1}^\alpha, E_i^\beta \right) = 0$ as required.

Now we can prove that E is a direct sum of its Harder - Narasimhan quotients inductively on the length n of the Harder - Narasimhan filtration. The case $n = 1$ is already done. In general case, consider the first Harder - Narasimhan quotient:

$$0 \longrightarrow F^1 \longrightarrow E \longrightarrow Gr^1 \longrightarrow 0.$$

Semistablity of Gr^i's and the vanishing conditions (5-7) show that

$$Hom(F^1, Gr^1) = Ext^2(Gr^1, F^1) = 0.$$

Hence, F^1 is rigid by the Mukai lemma from n° 4.2.2 and we can suppose inductively that F^1 is a direct sum of its Harder - Narasimhan quotients: $F^1 = \bigoplus_{i=2}^{n} Gr^i$. Since $Ext^1(Gr^1, Gr^2) = 0$, the sheaf Gr^2 is a direct summand of E, that is, $E = E' \oplus Gr^2$ and we can apply the inductive assumption to E'. Now to finish the prove, it remains to apply the lemma from n° 5.5.1 to the Harder - Narasimhan quotients of E. $\square$

5.5.3. COROLLARY. *Let F be an extra rigid locally free sheaf on S. Then there exists an exceptional collection $(F_1, F_2, \dots, F_r)$ such that $F = \bigoplus_{i=1}^{r} y_i F_i$.*

PROOF. Since F is extra rigid, by the previous theorem, $F = \bigoplus_{i=1}^{r} y_i F_i$ and

$$Ext^{>0}(F_i, F_j) = 0 \quad \forall\, i, j\ .$$

So, if we renumber the bundles F_i in such a way that $\mu_{\text{can}}(F_i) \le \mu_{\text{can}}(F_{i+1})$, then they will form an exceptional collection, because $Hom(F_j, F_i) = 0$ will hold for $i < j$ by the proposition from n° 5.3.5. $\square$

5.5.4. Remark. If an extra rigid locally free sheaf F belongs to a subcategory $[E_i, \dots, E_j]$ generated by some subcollection of an exceptional base $\{E_1, E_2, \dots, E_m\}$ for $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S)$, then all exceptional direct summands F_ν in the decomposition $F = \bigoplus\limits_\nu y_\nu F_\nu$ have to belong to $[E_i, \dots, E_j]$ as well. Indeed, taking an other helix foundation generating $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S)$, we can suppose that $i = 1$ and $j = k$, say. That is, $F \in Ob\,[E_1 \dots E_k] = Ob\,[E_{k+1} \dots E_m]^\perp$. But

$$F \in Ob\,[E_{k+1} \dots E_m]^\perp \implies \text{ all } F_\nu \in Ob\,[E_{k+1} \dots E_m]^\perp$$

certainly.

§6. Del Pezzo surfaces: transitivity of mutations.

In this paragraph we prove the theorem from n° 4.6.1 for Del Pezzo surfaces S_d.

6.1. Terminology concerned with the braid group action. We call two exceptional collections *equivalent*, if they lay in the same braid group orbit. We say that an exceptional collection of sheaves $\{F_1, F_2, \dots, F_k\}$ *is obtained* from an exceptional collection $\{E_1, E_2, \dots, E_m\}$ by mutations, if $k \le m$ and the collection $\{F_1, F_2, \dots, F_k\}$ can be extended to some bigger exceptional collection $\{F_1, F_2, \dots, F_m\}$ equivalent to $\{E_1, E_2, \dots, E_m\}$. An exceptional collection is called *constructible*, if it is obtained by mutations from the initial collection given in the theorem n° 4.1.2:

$$\{\mathscr{O}_{e_1}(-1), \dots, , \mathscr{O}_{e_d}(-1), \mathscr{O}_{S_d}, \mathscr{O}_{S_d}(h), \mathscr{O}_{S_d}(2h)\}$$

where h and $e_1, e_2, \dots, e_d$ are, accordingly, the proper preimages of a line and consequently blow-up'ed points on $\mathbb{P}_2$.

It follows from n° 2.8.1 that a *full* exceptional collection $\{E_1, E_2, \dots, E_k\}$ is equivalent to $\{E_n(K_{S_d}), E_1, \dots, E_{n-1}\}$, because of $E_n(K_{S_d}) = L_{E_1} L_{E_2} \cdots L_{E_{n-1}} E_n$. In particular, shifting a helix foundation by the helix period, we get an equivalent collection, i. e. full exceptional collections

$$\{E_1, E_2, \dots, E_k\} \text{ and } \{E_1(xK_{S_d}), \dots, E_k(xK_{S_d})\} \tag{6-1}$$

are equivalent for all $x \in \mathbb{Z}$. The theorem from n° 4.6.1 can be reformulated now as

6.1.1. THEOREM. *Any exceptional collection on S_d is constructible.*

6.2. Proof of the theorem.. We will use an induction on d. For $d = 0$ we have $S_d = \mathbb{P}^2$. Historically, this case was essentially done by J.-M. Drézet and J. Le Potier in [DLP] two years before the helix theory birth. This was the first appearance of exceptional vector bundles (see also [Dr1]). The helix theory viewpoint on the description of exceptional bundles and helices on $\mathbb{P}_2$ is based on the arithmetical properties of Mukai lattice $K_0(\mathbb{P}_2)$ and will discuss it in details little bit later, in n° 7.2.

The rest of this paragraph will be devoted to the inductive step. Let us assume that $d > 0$ and the theorem holds for S_{d-1}. We will prove now that the theorem holds for S_d. We identify $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S_{d-1})$ with the left orthogonal ${}^{\perp}\mathscr{O}_{e_d}(-1) \subset \mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S_d)$ (see proof of the theorem from n° 4.1.2) via pull back under the blow up morphism $S_d \longrightarrow S_{d-1}$ with the exceptional divisor e_d. Then inductive hypothesis says that all exceptional collections laying in ${}^{\perp}\mathscr{O}_{e_d}(-1)$ can be obtained from the standard one. So, we have to mute an arbitrary exceptional collection to such one that contains $\mathscr{O}_{e_d}(-1)$ and has other elements from ${}^{\perp}\mathscr{O}_{e_d}(-1)$. This will be done in two steps.

6.2.1. The first step is a kind of 'integer simplex-method algorithm' modified to deal with objects of an Abelian category. We formulate it as

6.2.2. PROPOSITION. *Let $\{E_0, E_1, \dots, E_m\}$, $\{F_0, F_1, \dots, F_k\}$ be two exceptional collection on S_d such that all F_i and all E_j except for E_0 are locally free but E_0 is either locally free or equal $\mathscr{O}_{e_d}(-1)$. Suppose that $\{E_0, E_1, \dots, E_m\}$ can be extended to some exceptional base for $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S_d)$ and for some positive integer multiplicities α_ν, x_μ both the sums $F = \bigoplus_{i=0}^{k} \alpha_i F_i$, $E = \bigoplus_{i=1}^{m} x_i E_i$ are extra rigid and fit into an exact triple*

$$0 \longrightarrow \bigoplus_{i=1}^{m} x_i E_i \longrightarrow F \longrightarrow x_0 E_0 \longrightarrow 0 \,. \tag{6-2}$$

Then $\sum_{i=0}^{k} rk F_i \geq \sum_{i=0}^{m} rk E_i$ and $\{F_0, F_1, \dots, F_k\}$ is obtained from $\{E_0, E_1, \dots, E_m\}$ by mutations (in particular, $k \leq m$). Moreover, if $m = 1$ and E_0 is locally free, then the equality

$$\sum_{i=0}^{k} rk F_i = \sum_{i=0}^{1} rk E_i$$

implies that either $k = 1$ and $F_0 = E_0$, $F_1 = E_1$, or $k = 0$.

The proof consequently modifies 'the pivot' $\{E_0, E_1, \dots, E_m\}$ via simplex-method using arithmetics from n° 1.7 and induction on m. We give it in n° 6.3 below.

6.2.3. The second step reduces the general case of the theorem to the exceptional collections $\{F_0, F_1, \dots, F_k\}$ satisfying the previous proposition. Namely, in n° 6.4 we will prove

6.2.4. PROPOSITION. *Any exceptional collection on S_d can be obtained from an exceptional collection $\{F_0, F_1, \dots, F_k\}$ such that $F = \bigoplus_{i=0}^{k} F_i$ is an extra rigid locally free sheaf for which there is an exceptional collection $\{\mathscr{O}_{e_d}(-1), E_1, \dots, E_m\}$ such that the canonical morphism $F \xrightarrow{\text{can}} Hom^*(F, \mathscr{O}_{e_d}(-1)) \otimes \mathscr{O}_{e_d}(-1)$ is surjective and fits into exact triple:*

$$0 \longrightarrow \bigoplus_{i=1}^{m} x_i E_i \longrightarrow F \xrightarrow{\text{can}} x_0 \mathscr{O}_{e_d}(-1), \longrightarrow 0$$

where $\bigoplus_{i=1}^{m} x_i E_i$ is extra rigid and locally free as well.

6.2.5. Final part: use the inductive hypothezis. Two previous propositions show that any exceptional collection on S_d can be obtained from some exceptional collection of the form $\{\mathcal{O}_{e_d}(-1),\, E_1,\, \dots,\, E_m\}$. Since all the bundles E_i are orthogonal to $\mathcal{O}_{e_d}(-1)$, the exceptional collection $\{E_1, E_2, \dots, E_m\}$ is a pull-back of some exceptional collection on S_{d-1} under the blow up $S_d \xrightarrow{\sigma_d} S_{d-1}$ (comp. with the proof of the theorem from n° 4.1.2). So, by the inductive hypothesis, the collection $\{E_1, E_2, \dots, E_m\}$ is constructible inside the category ${}^{\perp}\mathcal{O}_{e_d}(-1)$, i. e. the whole 'pivot' $\{\mathcal{O}_{e_d}(-1),\, E_1,\, \dots,\, E_m\}$ is constructible on S_d. This finishes the proof of the main theorem n° 6.1.1.

So, it remains to prove the propositions from n° 6.2.2 and n° 6.2.4.

6.3. Proof of the proposition n° 6.2.2. Let us use an induction on m and consider firstly the base case $m = 1$.

6.3.1. Base of induction. For $m = 1$ the sequence (6-2) turns to exact triple

$$0 \longrightarrow x_1\, E_1 \longrightarrow F \longrightarrow x_0\, E_0 \longrightarrow 0\,. \tag{6-3}$$

Since the pair $\{E_0, E_1\}$ is assumed extendable to some exceptional base for $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(S_d)$, all exceptional summands F_i, of F, have to belong the subcategory $[E_0, E_1]$ (see n° 5.5.4).

If $Ext^1(E_0, E_1) = 0$, then $F = x_0\, E_0 \oplus x_1\, E_1$ and there is nothing to prove.

If $\dim Ext^1(E_0, E_1) = h > 0$, then the description of exceptional objects in the Mukai lattice $K_0\big([E_0, E_1]\big)$ (see n° 1.7) shows that F contains at most two direct summands F_0, F_1, which form an exceptional pair obtained from $\{E_0, E_1\}$ by mutations, i.e., $F = \alpha_0 F_0 \oplus \alpha_1 F_1$ with $\alpha_0 > 0,\ \alpha_1 \geq 0$. If we write the class of F_i in $K_0\big([E_0, E_1]\big)$ as $a_i e_0 + b_i e_1$, where e_i is the class of E_i, then

$$1 = a_i^2 + b_1^2 - h a_i b_i$$

implies that the integers a_i, b_i have the same sign. So, if the both $rkE_i > 0$, then (if $\alpha_1 > 0$) the equality $rkF_0 + rkF_1 = rkE_0 + rkE_1$ is possible only for $E_0 = F_0$ and $E_1 = F_1$. The case $m = 1$ is completely proved.

6.3.2. The inductive step consists of appropriate modification of the both collections E, F. This modification splits into a series of elementary transformations of two different kinds described in two lemmas below.

6.3.3. LEMMA. *Let F be an extra rigid sheaf and $\{E_0, E_1, \dots, E_m\}$ be an exceptional collection. Then an exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{m} x_i\, E_i \longrightarrow F \longrightarrow x_0\, E_0 \longrightarrow 0 \tag{6-4}$$

produces a commutative diagram

$$\begin{tikzcd}
x_1\, E_1 \arrow[r, hook] & E' \arrow[r, two heads] & x_0\, E_0 \\
\bigoplus_{i=1}^{m} x_i\, E_i \arrow[r, hook] \arrow[u, two heads] & F \arrow[r, two heads] \arrow[u, two heads] & x_0\, E_0, \arrow[u, equal] \\
\bigoplus_{i=2}^{m} x_i\, E_i \arrow[r, equal] \arrow[u, hook] & \bigoplus_{i=2}^{m} x_i\, E_i \arrow[u, hook] &
\end{tikzcd} \tag{6-5}$$

where E' is an extra rigid sheaf such that $Ext^q(\overset{m}{\underset{i=2}{\oplus}} x_i\, E_i, E') = 0\ \forall q$.

Proof. The diagram is induced by its left column and the middle row. Required orthogonality: $Ext^q(\overset{m}{\underset{i=2}{\oplus}} x_i\, E_i, E') = 0\ \forall q$ follows from the exceptionality of the collection $\{E_0, E_1, \dots, E_m\}$ after applying the functor $Ext(\overset{m}{\underset{i=2}{\oplus}} x_i E_i, *)$ to the first row of the diagramm.

Further, applying $Ext(*, E_0)$ to the middle row, we get $Ext^{>0}(F, E_0) = 0$, and then, applying to the same row $Ext(F, *)$ and using the extra rigidness: $Ext^{>0}(F, F) = 0$, we get $Ext^2(F, \overset{m}{\underset{i=1}{\oplus}} x_i\, E_i) = 0$. Now, taking $Ext(F, *)$ in the middle column, we obtain the vanishing: $Ext^{>0}(F, E') = 0$. Finally, applying to the middle column $Ext(*, E')$, we get the required extra rigidness: $Ext^{>0}(E', E') = 0$. □

6.3.4. LEMMA. *Let F be an extra rigid sheaf and $\{E_0, E_1, \dots, E_m\}$ be an exceptional collection. Then an exact sequence*

$$0 \longrightarrow \overset{m}{\underset{i=2}{\oplus}} x_i\, E_i \longrightarrow F \longrightarrow x_0\, E_0 \oplus x_1\, E_1 \longrightarrow 0 \tag{6-6}$$

produces a commutative diagram

$$\begin{tikzcd}
 & x_0\, E_0 \arrow[r, equal] & x_0\, E_0 \\
\overset{m}{\underset{i=2}{\oplus}} x_i\, E_i \arrow[r, hook] \arrow[d, equal] & F \arrow[r, two heads] \arrow[u] & x_0\, E_0 \oplus x_1\, E_1 \arrow[u] \\
\overset{m}{\underset{i=2}{\oplus}} x_i\, E_i \arrow[r, hook] & E'' \arrow[r, two heads] \arrow[u, hook] & x_1\, E_1 \arrow[u, hook]
\end{tikzcd} \tag{6-7}$$

where E'' is an extra rigid sheaf such that $Ext^q(E'', x_0\, E_0) = 0\ \forall q$.

Proof. Completely similar to the previous one. □

6.3.5. Reduction to the case of locally free E_0. Let $E_0 = \mathcal{O}_{e_d}(-1)$. We are going to mute the collection $\{E_0, E_1, \dots, E_m\}$ into a new exceptional collection $\{E'_0, E'_1, \dots, E'_{m'}\}$, which also satisfies the proposition, i. e. represents F by the exact triple

$$0 \longrightarrow \overset{m'}{\underset{i=1}{\oplus}} x'_i\, E'_i \longrightarrow F \longrightarrow x'_0\, E'_0 \longrightarrow 0\,, \tag{6-8}$$

but has a locally free E'_0. To this aim, consider two exact sequences given by the lemma from n° 6.3.3:

$$\begin{gathered}
0 \longrightarrow \overset{m}{\underset{i=2}{\oplus}} x_i\, E_i \longrightarrow F \longrightarrow E' \longrightarrow 0, \\
0 \longrightarrow x_1\, E_1 \longrightarrow E' \longrightarrow x_0 \mathcal{O}_{e_d}(-1) \longrightarrow 0.
\end{gathered}$$

Let TE' be the torsion sheaf of E'. Since E_1 is locally free, $Hom(TE', x_1\, E_1) = 0$ and we have the following commutative diagram

$$\begin{tikzcd} x_1\,E_1 \arrow[r, hook] \arrow[d, equal] & E'/TE' \arrow[r, two heads] & G \\ x_1\,E_1 \arrow[r, hook] & E' \arrow[r, two heads] \arrow[u, two heads] & x_0\,\mathcal{O}_{e_d}(-1) \arrow[u, two heads] \\ & TE' \arrow[u, hook] \arrow[r, equal] & TE' \arrow[u, hook] \end{tikzcd} \tag{6-9}$$

where G has to be of the form $G = \underset{i}{\oplus}\,\beta_i\mathcal{O}_{e_d}(r_i) \oplus G'$ with $\dim Supp\, G' = 0$. Since E_1 is locally free, we have $Ext^1(G', x_1\,E_1) = 0$, i. e. G' is a direct summand of the torsion free sheaf E'/TE'. Hence, $G' = 0$ and $G = \underset{i}{\oplus}\,\beta_i\mathcal{O}_{e_d}(r_i)$, where all $r_i \geq -1$, certainly. Let us show that all $r_i \leq -1$ as well.

Indeed, since E'/TE' has no torsion, each $\mathcal{O}_{e_d}(r_i)$ is non trivially extended by E_1 in the first row of the diagram, that is, $Ext^1_{S_d}(\mathcal{O}_{e_d}(r_i), E_1) \neq 0\ \forall i$. On the other hand, the pair $\{\mathcal{O}_{e_d}(-1), E_1\}$ is exceptional, i. e. $\forall q$ we have

$$0 = Ext^q_{S_d}(E_1, \mathcal{O}_{e_d}(-1)) = Ext^q_{e_d}(E_1|_{e_d}, \mathcal{O}_{e_d}(-1))\,.$$

Therefore, $E_1|_{e_d} = \alpha\mathcal{O}_{e_d}$ and

$$\begin{aligned} Ext^1_{S_d}(\mathcal{O}_{e_d}(r_i), E_1)^* = Ext^1_{S_d}(E_1, \mathcal{O}_{e_d}(r_i)\otimes K_{S_d}) = Ext^1_{e_d}(E_1|_{e_d}, \mathcal{O}_{e_d}(r_i-1)) = \\ = Ext^1_{e_d}(\alpha\mathcal{O}_{e_d}, \mathcal{O}_{e_d}(r_i-1)) \neq 0 \end{aligned}$$

Thus, all $r_i \leq -1$ and $G = \beta\mathcal{O}_{e_d}(-1)$. So, the first column of the diagram (6-9) splits and we can write

$$E' = E'/TE' \oplus TE' = E'/TE' \oplus y_2\mathcal{O}_{e_d}(-1)\,.$$

Here E'/TE' is locally free and extra rigid (because E' is) and fits into exact sequence

$$0 \longrightarrow x_1\,E_1 \longrightarrow E'/TE' \longrightarrow (x_0 - y_2)\mathcal{O}_{e_d}(-1) \longrightarrow 0$$

Hence (see n° 5.5.4 and n° 1.7), $E'/TE' = y_0\,E_0' \oplus y_1\,E_1'$ for some exceptional pair of locally free sheaves $\{E_0', E_1'\}$ obtained from the pair $\{\mathcal{O}_{e_d}(-1), E_1\}$ by mutations. Since the sheaf

$$E' = y_0\,E_0' \oplus y_1\,E_1' \oplus y_2\mathcal{O}_{e_d}(-1) \in Ob\,[\mathcal{O}_{e_d}(-1), E_1]$$

is extra rigid, we have either $y_1 = 0$, or $y_2 = 0$, or $y_1 = y_2 = 0$. Anyway, $E' = y_0\,E_0' \oplus y\,\widetilde{E}$, where $\{E_0', \widetilde{E}\}$ is an exceptional pair obtained from the pair $\{\mathcal{O}_{e_d}(-1), E_1\}$ by mutations.

Now, let us note that we have the exact sequence

$$0 \longrightarrow \overset{m}{\underset{i=2}{\oplus}}\,x_i\,E_i \longrightarrow F \longrightarrow y_0\,E_0' \oplus y\,\widetilde{E} \longrightarrow 0 \tag{6-10}$$

for the exceptional collection $(E_0', \widetilde{E}, E_2, \dots E_m)$ obtained from $\{E_0, E_1, \dots, E_m\}$ by mutations. So, if $y = 0$, we have done. If $y > 0$, we apply to (6-10) the lemma from n° 6.3.4 and get exact sequences:

$$0 \longrightarrow E'' \longrightarrow F \longrightarrow y_0\,E_0' \longrightarrow 0,$$

$$0 \longrightarrow \overset{m}{\underset{i=2}{\oplus}}\,x_i\,E_i \longrightarrow E'' \longrightarrow y\widetilde{E} \longrightarrow 0$$

with extra rigid E'' such that $Ext^q_{S_d}(E'_0, E'') = 0\ \forall q$. Moreover, $E'' \subset F$ has no torsion, because F is locally free. By the corollary from n° 4.3.2, E'' is locally free, that is (see n° 5.5.4)

$$E'' = \overset{m'}{\underset{i=1}{\oplus}} x'_i\, E'_i$$

for some exceptional collection of locally free sheaves $\{E'_0, E'_1, \dots, E'_{m'}\}$. By the inductive hypothesis, $\{E'_1, E'_2, \dots, E'_{m'}\}$ is obtained from $\{\widetilde{E}, E_2, \dots, E_m\}$ by mutations. Hence, $\{E'_0, E'_1, \dots, E'_{m'}\}$ is obtained from $\{E_1, E_2, \dots, E_m\}$ as well and we have completely reduced the general case of the proposition n° 6.2.2 to the situation, where E_0 is locally free.

6.3.6. Proof of the proposition n° 6.2.2 for locally free E_0. Again, let us apply the lemma from n° 6.3.3 to the triple (6-8). We get two exact triples

$$0 \longrightarrow \overset{m}{\underset{i=2}{\oplus}} x'_i\, E'_i \longrightarrow F \longrightarrow E^{(1)} \longrightarrow 0, \tag{6-11}$$

$$0 \longrightarrow x'_1\, E'_1 \longrightarrow E^{(1)} \longrightarrow y_0\, E'_0 \longrightarrow 0. \tag{6-12}$$

Since E'_1 and E'_0 are locally free, the extra rigid sheaf $E^{(1)}$ is locally free as well. Remark n° 5.5.4 and the inductive hypothesis imply that $E^{(1)} = x^{(1)}_0\, E^{(1)}_0 \oplus x^{(1)}_1\, E^{(1)}_1$ for some exceptional pair $\{E^{(1)}_0, E^{(1)}_1\}$, which is obtained from $\{E'_0, E'_1\}$ by mutations and satisfies

$$rkE^{(1)}_0 + rkE^{(1)}_1 \geq rkE'_0 + rkE'_1. \tag{6-13}$$

So, $\{E^{(1)}_0, E^{(1)}_1, E'_2, \dots, E'_m\}$ is an exceptional collection obtained from $\{E'_0, E'_1, \dots, E'_m\}$ by mutations and satisfying

$$rkE^{(1)}_0 + rkE^{(1)}_1 + \sum_{i=2}^{m} rkE'_i \geq \sum_{i=0}^{m} rkE'_i.$$

By the inductive hypothesis, we can assume that $x^{(1)}_0 \neq 0$ and $x^{(1)}_1 \neq 0$. Applying the lemma from n° 6.3.4, we get two next exact sequences:

$$0 \longrightarrow E^{(2)} \longrightarrow F \longrightarrow x^{(1)}_0\, E^{(1)}_0 \longrightarrow 0$$

$$0 \longrightarrow \overset{m}{\underset{i=2}{\oplus}} x'_i\, E'_i \longrightarrow E^{(2)} \longrightarrow x^{(1)}_1\, E^{(1)}_1 \longrightarrow 0$$

with an extra rigid locally free $E^{(2)}$. Again, the inductive hypothesis and remark n° 5.5.4 show that $E^{(2)} = \overset{m_2}{\underset{i=1}{\oplus}} x^{(2)}_i\, E^{(2)}_i$, where $\{E^{(2)}_1, E^{(2)}_2, \dots, E^{(2)}_{m_2}\}$ is an exceptional collection of locally free sheaves obtained from $\{E^{(1)}_1, E'_2, \dots, E'_m\}$ and satisfying the inequality

$$\sum_{i=1}^{m_2} rkE^{(2)}_i \geq rkE^{(2)}_1 + \sum_{i=2}^{m} rkE'_i\,. \tag{6-14}$$

Thus, the extra rigid bundle F is included into

$$0 \longrightarrow \overset{m_2}{\underset{i=2}{\oplus}} x^{(2)}_i\, E^{(2)}_i \longrightarrow F \longrightarrow x^{(2)}_0\, E^{(2)}_0 \longrightarrow 0\,,$$

where $x_0^{(2)}\,E_0^{(2)} = x_0^{(1)}\,E_0^{(1)}$, the collection $\{E_0^{(2)}, E_1^{(2)}, \dots, E_{m_2}^{(2)}\}$ is obtained from the collection $\{E_0', E_1', \dots, E_m'\}$, and $\sum\limits_{i=0}^{m_2} rkE_i^{(2)} \geq \sum\limits_{i=0}^{m} rkE_i'$.

Repeating this two-step procedure, we construct for each $j = 1, 2, \dots$ an exceptional collection $(E_0^{(2j)}, \dots, E_{m_j}^{(2j)})$, which is obtained from $\{E_0', E_1', \dots, E_m'\}$ by mutations, satisfies inequality $\sum\limits_{i=1}^{m_j} rkE_i^{(2j)} \geq \sum\limits_{i=1}^{m_{j-1}} rkE_i^{(2j-2)}$, has $m_j \leq m_{j-1}$, and represents F by the exact triple

$$0 \longrightarrow \mathop{\oplus}_{i=1}^{m_j} x_i^{(2j)}\,E_i^{(2j)} \longrightarrow F \longrightarrow x_0^{(2j)}\,E_0^{(2j)} \longrightarrow 0\,. \tag{6-15}$$

As soon the number m_j becomes less than m, we can use the inductive hypothesis. So, we can assume that $m_j = m\ \forall j$. Since $\sum\limits_{i=0}^{m} rkE_i^{(2j)} \leq rkF$ by (6-15), we have to get a stabilization:

$$\sum_{i=0}^{m} rkE_i^{(2j)} = \sum_{i=0}^{m} rkE_i^{(2j-2)}$$

at some step j. Just to simplify the notations, assume that we got it already for $j = 1$. So, the inequality (6-13) turns to the equality

$$rkE_0^{(1)} + rkE_1^{(1)} = rkE_0' + rkE_1'.$$

As we have seen in n° 6.3.1, this means that $x_0^{(1)}\,E_0^{(1)} = y_0\,E_0'$, $x_1^{(1)}\,E_1^{(1)} = x_1'\,E_1'$, and (6-11) takes the form

$$0 \longrightarrow \mathop{\oplus}_{i=2}^{m} x_i'\,E_i' \longrightarrow F \longrightarrow y_0\,E_0' \oplus x_1'\,E_1' \longrightarrow 0\;.$$

Since $\mathop{\oplus}\limits_{i=1}^{m} x_i'\,E_i'$ is extra rigid, $Ext^1_{S_d}(x_1'\,E_1', \mathop{\oplus}\limits_{i=2}^{m} x_i'\,E_i') = 0$ and $x_1'\,E_1'$ is a direct summand of $F = \mathop{\oplus}\limits_{i=0}^{k} \alpha_i F_i$, that is, $x_1' = \alpha_\nu$ and $E_1' = F_\nu$ for some ν, say for $\nu = 0$. So, we obtain the exact triple

$$0 \longrightarrow \mathop{\oplus}_{i=2}^{m} x_i'\,E_i' \longrightarrow \mathop{\oplus}_{i=1}^{k} \alpha_i F_i \longrightarrow y_0\,E_0' \longrightarrow 0\;.$$

and can apply to it the inductive hypothesis, which says that the exceptional collection $\{F_1, F_2, \dots, F_k\}$ is obtained from $\{E_0', E_1', \dots, E_m'\}$ and has $\sum\limits_{i=1}^{k} rkF_i \leq \sum\limits_{i=2}^{m} rkE_i' + rkE_0'$. Hence, $\{F_0, F_1, \dots, F_k\}$ is obtained from $\{E_0', E_1', \dots, E_m'\}$ as well and satisfies inequality $\sum\limits_{i=0}^{k} rkF_i \leq \sum\limits_{i=0}^{m} rkE_i'$ as required. The proposition from n° 6.2.2 is completely proved.

6.4. Proof for the proposition from n° 6.2.4. First of all, we need some extra information about the restrictions of exceptional sheaves and exceptional pairs onto exceptional divisors. We collect it in two lemmas n° 6.4.1, n° 6.4.2 below. Then, in n° 6.4.3 we show that any exceptional collection can be obtained from an exceptional collection of locally free sheaves with extra rigid sum. After that we finish the proof in n° 6.4.6.

6.4.1. LEMMA. *Restricting a locally free exceptional sheaf to exceptional curve $e_d \subset S_d$, we get $E|_{e_d} = x_1 \mathscr{O}_{e_d}(s) \oplus y_1 \mathscr{O}_{e_d}(s+1)$, where $sx_1 + (s+1)y_1 = c_1(E) \cdot e_d$ and $x_1 + y_1 = rkE$.*

PROOF. Certainly, $E|_{e_d} = \sum x_i \mathscr{O}_e(i)$, because of $e_d \cong \mathbb{P}_1$. The exact triple

$$0 \longrightarrow E^* \otimes E \otimes \mathscr{O}(-e_d) \longrightarrow E^* \otimes E \longrightarrow (E^* \otimes E)|_{e_d} \longrightarrow 0$$

leads to exact sequence of cohomologies:

$$\cdots \longrightarrow Ext^1_{S_d}(E,E) \longrightarrow Ext^1_{S_d}(E|_{e_d}, E|_{e_d}) \longrightarrow Ext^2_{S_d}(E, E(-e_d)) \longrightarrow \cdots .$$

Here $Ext^1_{S_d}(E,E) = 0$, because E is exceptional. Since E is μ_{can}-stable and

$$\mu_{\mathrm{can}}(E(K_{S_d}+e_d)) = \mu_{\mathrm{can}}(E) + e_d \cdot K_{S_d} + K^2_{S_d} = \mu_{\mathrm{can}}(E) - 1 + K^2_{S_d},$$

we have $Ext^2(E, E(-e_d)) = Hom^*(E, E(K_{S_d}+e_d)) = 0$ as well[1]. So,

$$0 = Ext^1_{S_d}(E|_{e_d}, E|_{e_d}) = Ext^1_{e_d}(E|_{e_d}, E|_{e_d}) = \bigoplus_{i,j} Ext^1(\mathscr{O}_{e_d}(i), \mathscr{O}_{e_d}(j))$$

which implies $|i-j| \le 1 \ \forall i,j$. $\square$

6.4.2. LEMMA. *Let $\{E,F\}$ be an exceptional pair of locally free sheaves on S_d such that*

$$\textit{either} \qquad \mu_{\mathrm{can}}(F) - K^2_{S_d} < \mu_{\mathrm{can}}(E) < \mu_{\mathrm{can}}(F) \qquad \textit{or} \qquad \mu_{\mathrm{can}}(E) = \mu_{\mathrm{can}}(F)\ .$$

Then $E|_{e_d} = x_1\mathscr{O}_{e_d}(s) \oplus x_2\mathscr{O}_{e_d}(s+1)$ and $F|_{e_d}$ splits either as $y_1\mathscr{O}_{e_d}(s) \oplus y_2\mathscr{O}_{e_d}(s+1)$ or as $y_2\mathscr{O}_{e_d}(s+1) \oplus y_3\mathscr{O}_{e_d}(s+2)$; in the last case $F(K_{S_d})|_{e_d} = z_1\mathscr{O}_{e_d}(s) \oplus z_2\mathscr{O}_{e_d}(s+1)$.

PROOF. The long exact cohomology sequence for the triple

$$0 \longrightarrow E^* \otimes F(-e_d) \longrightarrow E^* \otimes F \longrightarrow (E^* \otimes F)|_{e_d} \longrightarrow 0$$

contains: $\cdots \longrightarrow Ext^1_{S_d}(E,F) \longrightarrow Ext^1_{e_d}(E_{e_d}, F_{e_d}) \longrightarrow Ext^2_{S_d}(E, F(-e_d)) \longrightarrow \cdots$. Here $Ext^1_{S_d}(E,F) = 0$ by n° 5.3.5. Since F and $E(K_{S_d}+e_d)$ are stable and $\mu_{\mathrm{can}}(E(K_{S_d}+e_d)) = \mu_{\mathrm{can}}(E) + 1 - K^2_S$, the space $Ext^2(E, F(-e_d)) = Hom^*(F, E(K_{S_d}+e_d))$ is non zero only if $K^2_S = 1$ and $F = E(K_{S_d}+e_d)$. In this case, the Riemann - Roch formula (see n° 4.1.1) applied to the equality $\chi(F,F) = \chi(F, E(K_{S_d}+e_d)) = 1$ gives $rkF = 1$, that is,

$$(E,F) = (\mathscr{O}_{S_d}(D), \mathscr{O}_{S_d}(D + K_{S_d} + e_d))$$

for some divisor D. Since in this case the pair (F,E) is exceptional as well, we have

$$(F \oplus E(K_{S_d}))|_{e_d} = \mathscr{O}_{e_d}(e_d \cdot D - 1) \oplus \mathscr{O}_{e_d}(e_d \cdot D - 2)$$

i. e. the lemma holds.

So, we can suppose that $Ext^1_{e_d}(E_{e_d}, F_{e_d}) = 0$. On the other side, the previous lemma predicts the splitting of the kind

$$E_{e_d} = x_1\mathscr{O}_{e_d}(s) \oplus x_2\mathscr{O}_{e_d}(s+1) \quad \text{and} \quad F_{e_d} = y_1\mathscr{O}_{e_d}(s') \oplus y_2\mathscr{O}_{e_d}(s'+1)$$

So, either $x_2 = 0$, $s' \ge s-1$ or $x_2 \ne 0$, $s' \ge s$. If we apply these estimations to the exceptional pair $\{F(K_{S_d}), E\}$, for which $F(K_{S_d})|_{e_d} = F_{e_d} \otimes \mathscr{O}_{e_d}(-1) = y_1\mathscr{O}_{e_d}(s'-1) \oplus y_2\mathscr{O}_{e_d}(s')$, then we get either $y_2 = 0$, $s \ge s'-2$ or $y_2 \ne 0$ and $s \ge s'-1$. $\square$

[1]By n° 5.2.1, any morphism from E to $E(K_{S_d}+e_d)$ is either zero or isomorphism (the last is possible only for $d = 8$, $K^2_{S_d} = 1$, but in this case it can be shown that $Ext^2(E, E(-e_d)) = 0$, comp. with the next proof).

6.4.3. Reduction to the extra rigid locally free collections. First of all, note that any exceptional collection $\{G_1, G_2, \dots, G_k\}$ can be obtained from an exceptional collection of locally free sheaves. Indeed, if all G_i are torsion sheaves, then $G_i = \mathcal{O}_{e_i}(d_i)$ by n° 4.3.2. But $\{\mathcal{O}_{e_1}(d_1), \dots, \mathcal{O}_{e_r}(d_r)\}$ is, clearly, obtained from the exceptional collection

$$\left\{ \mathcal{O}_{S_d}\left(-\sum_{i=1}^{r} d_i e_i \right), \mathcal{O}_{e_1}(d_1), \dots, \mathcal{O}_{e_r}(d_r) \right\},$$

where the first sheaf is locally free. As soon one of G_i's is locally free, we can assume[1] that this is G_1 but $G_2 = \mathcal{O}_{e_i}(d_i)$. The left mutation $\{L_{G_1}G_2, G_1, G_3, \dots, G_k\}$ has one more locally free $L_{G_1}G_2$ given by exact triple

$$0 \longrightarrow L_{G_1}G_2 \longrightarrow \mathit{Hom}(G_1, G_2) \otimes G_1 \longrightarrow G_2 \longrightarrow 0 .$$

Repeating this procedure, we remove all torsion sheaves.

Our next step is to make the collection extra rigid. To this aim, let us fix some extra terminology. We say that an exceptional pair $\{E, F\}$ is an *ext-pair*, if $\mathit{Ext}^1(E, F) \neq 0$. If $\mathit{Ext}^{\geq 0}_{S_d}(E, F) = 0$, we say that $\{E, F\}$ is a *hom-pair*. By n° 5.3.5, the first case is equivalent to inequolity $\mu_{\mathrm{can}}(E) > \mu_{\mathrm{can}}(F)$ and the last case — to $\mu_{\mathrm{can}}(E) \leq \mu_{\mathrm{can}}(F)$. An exceptional collection $\{G_1, G_2, \dots, G_k\}$ that consists of hom-pairs only is called a *hom-collection*. This happens if and only if the sheaf $\overset{k}{\underset{i=1}{\oplus}} G_i$ is extra rigid. For any exceptional collection of locally free sheaves $\sigma = \{G_1, G_2, \dots, G_k\}$ we put

$$\mu_0(\sigma) \overset{\mathrm{def}}{=} \min_i \{\mu_{\mathrm{can}}(G_i)\}, \qquad \mu_1(\sigma) \overset{\mathrm{def}}{=} \max_i \{\mu_{\mathrm{can}}(G_i)\} .$$

6.4.4. LEMMA. *Any exceptional collection on S can be obtained from an appropriate hom-collection ζ, of locally free sheaves, such that $\mu_1(\zeta) - \mu_0(\zeta) < K^2_{S_d}$.*

PROOF. As we have seen above, it is enough to obtain an arbitrary collection of locally free sheaves $\sigma = \{G_1, G_2, \dots, G_k\}$.

First of all, using induction on k, let us change σ by a hom-collection σ' that belongs to the same braid group orbit and has $\mu_0(\sigma) \leq \mu_0(\sigma')$, $\mu_1(\sigma') \leq \mu_0(\sigma)$. For $k = 2$ an ext-pair (G_1, G_2) can be replaced just by its left mutation $(L_{G_1}G_2, G_1)$, which is a hom-pair[2] and satisfies $\mu_{\mathrm{can}}(G_2) < \mu_{\mathrm{can}}(L_{G_1}G_2) < \mu_{\mathrm{can}}(G_1)$. In general case, we firstly replace first $k-1$ bundles in $\sigma = \{G_1, G_2, \dots, G_k\}$ by a hom-collection $\tau' = \{G'_1, G'_2, \dots, G'_{k-1}\}$ with

$$\mu_0(\sigma) \leq \mu_0(\tau) \leq \mu_0(\tau') \qquad \text{and} \qquad \mu_1(\tau') \leq \mu_1(\tau) \leq \mu_1(\sigma) .$$

If $\mu_{\mathrm{can}}(G'_{k-1}) \leq \mu_{\mathrm{can}}(G_k)$ we have done. If $\mu_{\mathrm{can}}(G'_{k-1}) > \mu_{\mathrm{can}}(G_k)$, then (G'_{k-1}, G_k) is an ext-pair and we pass to the left mutation $\sigma_1 = (G'_1, \dots, L_{G'_{k-1}}G_k, G'_{k-1})$, which has $\mu_{\mathrm{can}}(L_{G'_{k-1}}G_k) < \mu_{\mathrm{can}}(G'_{k-1})$, $\mu_0(\sigma) \leq \mu_0(\sigma_1)$, and $\mu_1(\sigma_1) \leq \mu_1(\sigma)$. If this collection is still not

[1] maybe, after a number of the replacements $\{G_1, G_2, \dots, G_k\}$ by $\{G_2, \dots, G_k, G_1(-K_{S_d})\}$

[2] Recall that $L_{G_1}G_2$ is defined as the canonical extension

$$0 \longrightarrow G_2 \longrightarrow L_{G_1}G_2 \longrightarrow \mathit{Ext}^1_{S_d}(G_1, G_2) \otimes G_1 \longrightarrow 0$$

.

a hom-collection, then we take the next left mutation $(G'_1, \dots, L_{G'_{k-2}} L_{G'_{k-1}} G_k, G'_{k-2}, G'_{k-1})$, and so on until we will get a required hom-collection $\sigma' = \{G'_1, G'_2, \dots, G'_k\}$, which has $\mu_0(\sigma') = \mu_{\mathrm{can}}(G'_1) \le \dots \le \mu_{\mathrm{can}}(G'_k) = \mu_1(\sigma')$.

Now, suppose that $\mu_1(\sigma') - \mu_0(\sigma') \ge K^2_{S_d}$ and take a number j such that $\mu_{\mathrm{can}}(G'_{j-1}) < \mu_{\mathrm{can}}(G'_j) = \mu_{\mathrm{can}}(G'_{j+1}) = \dots = \mu_{\mathrm{can}}(G'_k)$. The exceptional collection

$$\sigma'' = (G_j(K_{S_d}), \dots, G_k(K_{S_d}), G_1, \dots, G_{j-1})$$

lies in the same braid group orbit and has

$$\mu_{\mathrm{can}}(G'_j(K_{S_d})) = \dots = \mu_{\mathrm{can}}(G'_k(K_{S_d})) = \mu_{\mathrm{can}}(G'_k) - K^2_{S_d} ,$$

i. e. $\mu_1(\sigma'') - \mu_0(\sigma'') = \mu_1(\sigma') - \mu_0(\sigma') - K^2_{S_d}$. As above, we can replace σ'' by a new exceptional hom-collection σ''' that has $\mu_0(\sigma''') \le \mu_0(\sigma'')$, $\mu_1(\sigma''') \le \mu_1(\sigma'')$ and so on until we will get the required estimation for the difference $\mu_1 - \mu_0$. □

6.4.5. COROLLARY. *Any exceptional collection on S can be obtained from an exceptional hom-collection $\sigma = \{F_1, F_2, \dots, F_k\}$, of locally free sheaves, such that*

$$\bigoplus_{i=1}^{k} F_i \Big|_{e_d} = x_0 \mathscr{O}_{e_d}(s) \oplus x_1 \mathscr{O}_{e_d}(s+1) . \tag{6-16}$$

PROOF. Let $\zeta = \{G_1, G_2, \dots, G_k\}$ be an exceptional collection constructed in the previous proposition. Then each pair $\{G_i, G_j\} \subset \zeta$ satisfies the assumptions of the lemma from n° 6.4.2. Hence, for some $j \, : \, 1 \le j \le k$ and some $s \in \mathbb{Z}$, the sheaves G_ν are restricted on exceptional divisor e_d as

$$G_\nu|_{e_d} = \begin{cases} x_\nu \mathscr{O}_{e_d}(s) \oplus y_\nu \mathscr{O}_{e_d}(s+1), & \text{for } \nu < j \\ y_\nu \mathscr{O}_{e_d}(s+1) \oplus z_\nu \mathscr{O}_{e_d}(s+2), & \text{for } \nu \ge j \end{cases} .$$

Then the collection $\sigma = \{G_j(K_S), G_{j+1}(K_S), \dots, G_k(K_S), G_1, G_2, \dots, G_{j-1}\}$ is what we need. □

6.4.6. End of proof for the proposition from n° 6.2.4. To finish the proof, we have to find, for an exceptional collection $\sigma = \{F_1, F_2, \dots, F_k\}$ obtained in the previous corollary, another exceptional collection $\{\mathscr{O}_{e_d}(-1), E_1, \dots, E_m\}$ that satisfies the assumptions of the proposition from n° 6.2.2. To this aim, consider the canonical map

$$F \xrightarrow{can} Hom_{S_d}(F, \mathscr{O}_{e_d}(-1))^* \otimes \mathscr{O}_{e_d}(-1)$$

Since $F|_{e_d} = x_0 \mathscr{O}_{e_d}(-1) \oplus x_1 \mathscr{O}_{e_d}$ and $Hom_{S_d}(F, \mathscr{O}_{e_d}(-1))^* = Hom_{e_d}(F|_{e_d}, \mathscr{O}_{e_d}(-1))^*$, the restriction of *can* onto e_d is surjective. Hence, *can* is surjective on S_d as well and we have an exact triple

$$0 \longrightarrow E \longrightarrow F \xrightarrow{can} x_0 \mathscr{O}_{e_d}(-1) \longrightarrow 0 , \tag{6-17}$$

which gives the right mutation $E = R_{\mathscr{O}_{e_d}(-1)} F \in Ob\left({}^{\perp}\mathscr{O}_{e_d}(-1)\right)$. Applying $Ext(\,*\,, F)$ to (6-17), we get

$$\dots \longrightarrow Ext^q_{S_d}(F, F) \longrightarrow Ext^q_{S_d}(E, F) \longrightarrow Ext^{q+1}_{S_d}(x_0 \mathscr{O}_{e_d}(-1), F) \longrightarrow \dots ,$$

Since F is extra rigid and

$$
\begin{aligned}
Ext^2_{S_d}(\mathscr{O}_{e_d}(-1),F)^* = Ext^0_{S_d}(F,\mathscr{O}_{e_d}(-1)\otimes\mathscr{O}_{S_d}(K_{S_d})) = \\
= Ext^0_{S_d}(F,\mathscr{O}_{e_d}(-2)) = Ext^0_{e_d}(F|_{e_d},\mathscr{O}_{e_d}(-2)) = 0
\end{aligned}
$$

(the last equality follows from (6-16) with $s=-1$), we have $Ext^{\geq 0}_{S_d}(E,F)=0$, which implies that E is an extra rigid locally free sheaf (recall that $Hom^\bullet(E,E) = Hom^\bullet(E,F)$, because $L_{\mathscr{O}_{e_d}(-1)}$ is the right adjoint functor for the inclusion ${}^\perp\mathscr{O}_{e_d}(-1) \hookrightarrow \mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(S_d)$). Thus, $E=\bigoplus\limits_{i=1}^{m} x_i\,E_i$ for some exceptional collection $\{E_1,E_2,\dots,E_m\}$. This completes the proof. $\square$

§7. Helices and Riemann - Roch arithmetics.

7.1. Arithmetics of $K_0(\mathbb{P}_n)$. Let $M=K_0(\mathbb{P}_n)$ be the Mukai lattice of the projective space and $W=M\otimes\mathbb{Q}$ be its complexification. We will write

$$\langle\, E\,,\,F\,\rangle = \sum(-1)^\nu \dim Ext^\nu(E,F)$$

for the Euler scalar product and $\chi=(\,\chi_{ij}\,)=(\,\langle\,e_i\,,\,e_j\,\rangle\,)$ for its Gram matrix as soon a base $\{e_\nu\}$ is chosen.

7.1.1. Hilbert polynomials model. As a $\mathbb{Z}$-module, M can be identified with the module of *integer valued* polynomials $\{h\in\mathbb{Q}[t]\mid h(\mathbb{Z})\subset\mathbb{Z}\}$ by taking a class $[E]\in K_0(\mathbb{P}_n)$ to its Hilbert polynomial $h_E(k)=\langle\mathscr{O}(-k)\,,\,E\,\rangle$. Under this identification the structure sheaves concentrated on the linear subspaces

$$\{\mathscr{O}_{\mathbb{P}_n},\mathscr{O}_{\mathbb{P}_{n-1}},\dots,\mathscr{O}_{\mathbb{P}_1},\mathscr{O}_{\mathbb{P}_0}\}$$

go to the standard binomial $\mathbb{Z}$-basis

$$\{\gamma_n(t),\gamma_{n-1}(t),\dots,\gamma_0(t)\}$$

for integer valued polynomials, where we write

$$
\begin{aligned}
\gamma_n(t) &\stackrel{\text{def}}{=} h_{\mathscr{O}_{\mathbb{P}_n}}(t) = \binom{t+n}{n} = \frac{1}{n!}(t+1)(t+2)\cdots(t+n)\\
&\cdots\cdots\cdots\\
\gamma_k(t) &\stackrel{\text{def}}{=} h_{\mathscr{O}_{\mathbb{P}_k}}(t) = \binom{t+k}{k} = \frac{1}{k!}(t+1)(t+2)\cdots(t+k)\\
&\cdots\cdots\cdots\\
\gamma_1(t) &\stackrel{\text{def}}{=} h_{\mathscr{O}_{\mathbb{P}_1}}(t) = \binom{t+1}{1} = (t+1)\\
\gamma_0(t) &\stackrel{\text{def}}{=} h_{\mathscr{O}_{\mathbb{P}_0}}(t) \equiv 1
\end{aligned}
$$

Further, the twisting operator $T : [E] \longmapsto [E(1)]$ goes to the shift

$$T = e^{\frac{d}{dt}} : f(t) \longmapsto f(t+1)$$

and the restriction onto a hyperplane $1 - T^{-1} : [E] \mapsto [E|_{\mathbb{P}_{n-1}}]$ goes to the left difference operator

$$\nabla = 1 - e^{-\frac{d}{dt}} : f(t) \longmapsto f(t) - f(t-1)$$

In particular, the canonical Serre operator from n° 1.4.3 equals

$$\varkappa = (-1)^n T^{n+1} = (-1)^n e^{-(n+1)\frac{d}{dt}}$$

and is quasiunipotent. This implies (see [Go5]) that M is a lattice of U_n-type in the sense of n° 1.4.4.

7.1.2. Differential operators model. All $\mathbb{Z}$-linear maps $M \longrightarrow M$ that commute with the canonical operator form a subalgebra $\mathscr{A} \subset End_{\mathbb{Z}}(M)$ that goes to $\mathbb{Z}[\nabla]/\nabla^{n+1}$ under the identification from the previous section. Its rational version $\mathscr{A} \otimes \mathbb{Q} \subset End_{\mathbb{Q}}(W)$ is nothing but $\mathbb{Q}[D]/D^{n+1}$, where D acts on $W \subset \mathbb{Q}[t]$ as $\frac{d}{dt}$. This leads to an other useful model for M: sending a formal power series $A(D) \in \mathbb{Q}[[D]]$ to $A(D)\gamma_n$, we identify $\mathscr{A} = \mathbb{Z}[\nabla]/\nabla^{n+1}$ with $M \subset \mathbb{Q}[t]$ in such a way that the product of power series goes to the tensor product of sheaves and the involution $[E] \longmapsto [E^*]$ corresponds to the sign change $D \longmapsto -D$. In this terms

$$\langle\, A(D)\,,\, B(D)\,\rangle = A(-D)B(D)\gamma_n\,(0)$$

and $\langle\, A(-D)h_1\,,\, h_2\,\rangle = \langle\, h_1\,,\, A(D)h_2\,\rangle \;\forall\; h_1, h_2 \in M \subset \mathbb{Q}[t]$.

7.1.3. COROLLARY. *The rational isometry group of $K_0(\mathbb{P}_n) \otimes \mathbb{Q} \subset \mathbb{Q}[t]$ consists of differential operators $\pm e^{A(D)}$ $(\mathrm{mod}\ D^{n+1})$, where $A(D) \in \mathbb{Q}[[D]]$ is odd: $A(-D) = -A(D)$.* □

7.1.4. Remark. The precise description of the integer isometry group of $K_0(\mathbb{P}_n)$ for general n is still unknown. Some its invariants are described in [Go5]. The conjecture from n° 1.5.1 asserts that any semiorthonormal collection of $n+1$ Hilbert polynomials can be obtained from the simplest one $\{\gamma_n(t),\, \gamma_n(t+1),\, \dots\,,\, \gamma_n(t+n)\}$ by the isometries and mutations. It is proved only for $n = 2, 3$. The only known proof for $\mathbb{P}_3$ (see [No1]–[No3]) is essentially a hard handwork in the integer linear programming style with a lot of quadratic relations.

7.2. Towards Markoff. Now we are going to revisit the discussion started in n° 1.6 of the introduction. Let N be a free $\mathbb{Z}$-module of rank 3 equipped with an unimodular integer bilinear form. In accordance with general results in [Go5], the Jordan normal form of the corresponding canonical Serre operator $\varkappa$ on $V = N \otimes \mathbb{C}$ is either $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, which corresponds to U_3-type, or $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which corresponds to $U_2 \oplus U_1$ - type, or $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which corresponds to $U_1 \oplus W_2(\lambda)$ - type, where $\lambda \neq \pm 1$. These cases are separated by $\mathrm{tr}\,(\kappa)$, which equals 3, -1

and $1+\lambda+\lambda^{-1}$ respectively. If N admits a semiorthonormal base with the Gram matrix

$$\chi = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} . \tag{7-1}$$

then $\operatorname{tr}(\kappa) = \operatorname{tr}(\chi^{-1}\chi^t) = 3 - a^2 - b^2 - c^2 + abc$. So, $N \otimes \mathbb{C}$ is an irreducible space of type U_3 iff the numbers $\{a, b, c\}$ satisfy the triple Markoff equation:

$$a^2 + b^2 + c^2 = abc .$$

All its solutions (see [Ca]) are obtained from the initial solution $\{3, 3, 3\}$ by changing signs of any two numbers and changing one of three numbers via Vieta theorem: $a \mapsto bc - a$, or $b \mapsto ac - b$, or $c \mapsto ab - c$. To change signs of any two elements in the Gram matrix it is enough to change a sign of one of basic vectors. The mutations

$$\begin{aligned}
\{e_0, e_1, e_2\} &\longmapsto \{\langle e_0 , e_1 \rangle \cdot e_0 - e_1 , \ e_0 , \ e_2\} \\
\{e_0, e_1, e_2\} &\longmapsto \{e_0 , \ \langle e_1 , e_2 \rangle \cdot e_1 - e_2 , \ e_1\} \\
\{e_0, e_1, e_2\} &\longmapsto \{e_0 , \ e_2 , \ \langle e_1 , e_2 \rangle \cdot e_2 - e_1\} ,
\end{aligned}$$

of a semiorthonormal basis $\{e_0, e_1, e_2\}$, change the Gram matrix by the rules

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & a & ab-c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & ac-b & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & b & bc-a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

So, any semiorthonormal basis in a Mukai lattice of U_3-type can be transformed by the braid group action and changing signs of basic vectors to the semiorthonormal basis with Gram matrix

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

which appears in $K_0(\mathbb{P}_2)$ for the basis $\{\mathscr{O}, \mathscr{T}(-1), \mathscr{O}(1)\}$. We have established

7.2.1. PROPOSITION. *There exists a unique up to an integer isometry Mukai lattice of type U_3 that admits a semiorthonormal base. This lattice is $K_0(\mathbb{P}_2)$. Up to the signs of basic vectors and integer isometries, the braid group B_3 acts transitively on the semiorthonormal bases of $K_0(\mathbb{P}_2)$.* □

7.2.2. Markoff tree and helices on $\mathbb{P}_2$. The description of exceptional vector bundles on $\mathbb{P}_2$ first obtained in [DLP] can be presented as follows.

First of all, any exceptional vector bundle E is defined up to an isomorphism by its class $[E] \in M = K_0(\mathbb{P}_2)$. Indeed, if $[E'] = [E]$, then $Hom(E, E') = Ext^2(E, E') = 0$ by the stability but $\chi(E, E') = \chi(E, E) = 1 > 0$ by Riemann - Roch.

Further, 1-dimensional sublattice $\langle E \rangle \subset M$ spanned by an exceptional $[E]$ contains explicitly two vectors e with $\langle e, e \rangle = 1$, namely $[E]$ and $-[E] = [E[1]]$. Since any exceptional sheaf E is contained in some exceptional base $\{E, E_1, E_2\}$ of $\mathscr{D}^{\mathrm{b}}_{\mathscr{C}oh}(\mathbb{P}_2)$ (see [DLP], [GoRu]), the 2-dimensional sublattice ${}^{\perp}\langle E \rangle \subset M$ admits an exceptional base $\{e_i, e_2\}$ with $e_\nu = [E_\nu]$. We have seen in n° 1.7 that all exceptional bases of this lattice are exhausted up to a sign by the mutations[1] in the pair (E_1, E_2).

Reducing the Riemann - Roch expansion (4-2) for $\chi(E_1, E_2)$ modulo the orthogonality relations $\chi(E_1, E) = \chi(E_2, E) = 0$ we get immediately $\chi(E_1, E_2) = 3rk(E)$. In other words, the restriction of χ onto ${}^{\perp}\langle E \rangle$ depends only on rkE, and ranks r_0, r_1, r_2 in an exceptional triple $\{E_0, E_1, E_2\}$ satisfy the ordinary (non triple) Markoff equation

$$r_0^2 + r_1^2 + r_2^2 = 3r_0r_1r_2 \tag{7-2}$$

Moreover, it follows easily from n° 1.7.2 that if $rkE \geq 2$, then there exists a unique semiorthogonal base $\{E_1, E_2\}$ in ${}^{\perp}\langle E \rangle$ such that both $rkE_1, rkE_2 < rkE$. Hence, applying the mutations that strictly decrease the rank of a muted bundle, we transform in a unique way any exceptional triple to an exceptional triple of invertible sheafs, that is, to $\{\mathscr{O}(-1), \mathscr{O}, \mathscr{O}(1)\}$ modulo a twist.

In other words, starting with the initial exceptional triple $\{\mathscr{O}(-1), \mathscr{O}, \mathscr{O}(1)\}$ and applying only the mutations

$$\begin{tikzcd}
 & \{E_0, E_1, E_2\} \arrow[dl, "\mathscr{G}_1"'] \arrow[dr, "\mathscr{G}_2^{-1}"] & \\
\{E_1, R_{E_1}E_0, E_2\} & & \{E_0, L_{E_1}E_2, E_1\}
\end{tikzcd}$$

we get a binary tree whose vertices are labelled by exceptional triples $\{E_0, E_1, E_2\}$ in such a way that each twist class of exceptional triples is represented exactly by one vertex. This three has a symmetry w. r. t. the vertical axis. It takes $\{E_0, E_1, E_2\}$ to $\{E_2^*, E_1^*, E_0^*\}$. So, taking the right half of this three we get a three that classifies exceptional triples up the twists and the duality. It is called *the Markoff three*, because it classifies all positive solutions (r_0, r_1, r_2) of the Markoff equation (7-2) such that $r_1 \geq r_0, r_2$.

7.2.3. Markoff's problem and A. N. Tyurin's conjecture. A famous problem in the geometry of numbers is: are there two different solutions (r_0, r, r_2), (r_0', r, r_2') (with the same middle term r) of the Markoff equation (7-2) such that $r \geq r_0, r_2, r_0', r_2'$? This problem has more than 100 years history with a lot of incorrect proofs like for the great Fermat theorem. The positive answer (conjectured by Markoff) could give a far improvements in the Diophantine approximations. Our previous discussion shows that the Markoff problem is equivalent to the following question asked by A. N. Tyurin in 1980'th before the helix theory was developed: is an exceptional vector bundle on $\mathbb{P}_2$ uniquely up to twists and duality defined by its rank? Indeed, careful control over the slopes shows that any exceptional vector bundle E such that $\mu(E) = c_1(E)/rk(E)$ lies in $[0, 1/2]$ presented as a middle element of exceptional triple at precisely one vertex of the Markoff three. So, the Tyurin's question is: can the same

[1] in this case mutations and isometries essentially coincide

r appear as the middle term in two different Markoff triples (r_0, r, r_2) ordered in such a way that the middle term is maximal? The Markoff problem is still open ...

7.3. Helices and local systems. In this section we discuss an interaction between helices and rigid local systems over $\mathbb{P}_1 \setminus \{\text{some points}\}$ discovered and studied by V. Golyshev in a very recent paper [Gl]. We follow below his notations and terminology.

7.3.1. Rigid local systems. Let $U \overset{j}{\hookrightarrow} \mathbb{P}_1$ be an (analytic) open set with a finite complement $\mathbb{P}_1 \setminus U = \{x_0, x_1, \dots, x_n\}$. We fix $n+1$ basic loops $\{\gamma_0, \gamma_1, \dots, \gamma_n\} \subset \pi_1(U)$ coming from some base point in U and passing around the points x_ν. A rank r local system $\mathscr{L}$ over U is a locally trivial vector bundle of rank r over U equipped with a flat $GL_r(\mathbb{C})$-connection, or equivalently, an isomorphism class of representations $\pi_1(U) \xrightarrow{\varphi} GL_r(\mathbb{C})$ given by the monodromy of $\mathscr{L}$. In other words, a local system is defined by $n+1$ linear operators $\varphi(\gamma_\nu) \in GL_r(\mathbb{C})$ such that $\varphi(\gamma_0)\varphi(\gamma_1) \cdots \varphi(\gamma_n) = 1$. A local system given by a representation φ is called *rigid*, if any other representation φ' such that

$$\exists\, g_\nu \in GL_r(\mathbb{C}) \;:\; \varphi(\gamma_\nu) = g_\nu \varphi(\gamma_\nu) g_\nu^{-1} \;\forall\, \nu$$

is isomorphic to φ, i. e. $\exists\, g \in GL_r(\mathbb{C}) : \varphi(\gamma) = g\varphi(\gamma)g^{-1} \;\forall\, \gamma \in \pi_1(U))$.

An irreducible local system $\mathscr{L}$ of rank $r \geq 2$ is rigid iff $\chi\left(\mathbb{P}_1, j_*(\mathscr{L}^* \otimes \mathscr{L})\right) = 2$. Moreover, rigidness is a local property: it is sufficient to check it only for φ' closed enough to φ (see [Kt, sec. 1.1] for all the details).

A rigid irreducible local system φ such that all $\varphi(\gamma_\nu)$ are quasiuniptent[1] operators is called *a Katz local system*. A pull back $\mathscr{L}' = p^*\mathscr{L}$ of a Katz local system $\mathscr{L}$ along a covering $U' \xrightarrow{p} U$ is called *a quasi Katz local system*. Such the local systems was studied by N. Katz in [Kt], where among other things any such a system was realized geometrically in the fiber cohomologies of appropriate pencil of algebraic varieties.

7.3.2. LEMMA. *Any two complex coprime polynomials of the same degree with unit constant terms: $p(t) = 1 + p_1 t + \dots + p_{r+1} t^r$, $q(t) = 1 + q_1 t + \dots + q_{r+1} t^r$ define a unique irreducible rigid local system φ of rank $r = \deg p = \deg q$ on $\mathbb{P}_1\{\,0\,,\,1\,,\,\infty\,\}$ such that its local monodromies $T = \varphi(\gamma_0)$, $S = \varphi(\gamma_1)$ satisfy the conditions*

(a) *S is a quasireflection, i. e. $rk(1 - S) = 1$*
(b) *a tilt vector of S is cyclic for T,*
i. e. $im\,(1 - S)$, $T(im\,(1 - S))$, ... , $T^{r-1}(im\,(1 - S))$ span $\mathbb{C}^r$
(c) $\det(1 - t\,T) = q(t)$
(d) $\det(1 - t\,ST) = p(t)$

PROOF. Let us fix a base $\{v_1, v_2, \dots, v_r\}$ in $V = \mathbb{C}^r$ that, by the definition, is cyclic for T with the cyclic vector v_1, i. e. $v_\nu = T^{\nu-1} v_1$ for $1 \leq \nu \leq r$. Further, define $\xi \in V^*$ by the prescription $\xi(v_\nu) = h_\nu$, where h_ν are the starting coefficients of a formal power series

$$H(t) = 1 + \sum_{\nu \geq 1} h_\nu t^\nu \;\overset{\text{def}}{=}\; \frac{p(t)}{q(t)}$$

[1] that is, have a form $\zeta + \eta$, where η is nilpotent and ζ is diagonalizable and satisfies $\zeta^m = 1$ for some integer m

Then $S \stackrel{\text{def}}{=} 1 - \xi \otimes v_1$ and T satisfy the conditions (a)–(d), and rank 1 operator $1 - S = \xi \otimes v$ is uniquely recovered from p and q, because of obvious determinantal identity

$$1 + \sum_{\nu \geq 1} \xi(v) t^\nu = \frac{\det(1 - tTS)}{\det(1 - tT)}$$

This proves the existence of S, T, TS, and their uniqueness up to a simultaneous conjugation. If $W \subset V$ is invariant subspace, then $S|_W = \mathrm{Id}_W$, because of otherwise $(1 - S)W \neq 0$ and $W \supset \sum T^\nu (1 - S)W = V$. So, $T|_W = ST|_W$ have common eigenvalue and their characteristic polynomials can not be coprime. □

7.3.3. Mutations and transvections. Let M be a Mukai lattice, $\langle *, * \rangle$ be its unimodular bilinear form, and $\varepsilon = \pm 1$. We put

$$(v, w)_\varepsilon \stackrel{\text{def}}{=} \langle v, w \rangle + \varepsilon \langle w, v \rangle ,$$

which is symmetric for $\varepsilon = 1$ and skew symmetric for $\varepsilon = -1$. For any $e \in M$ such that $\langle e, e \rangle = 1$ a map

$$M \xrightarrow{\sigma_e} M \ : \ \sigma_e^\varepsilon(v) = v - (e, v)_\varepsilon e$$

is called *a transvection* w. r. t. e. This is, certainly, a quasireflection. We clearly have

$$\sigma_e^\varepsilon(v) = -L_e(v) - \varepsilon \langle v, e \rangle e \tag{7-3}$$

where $L_e(v) = \langle e, v \rangle e - v$ is the left mutation w. r. t. e (see n° 1.4.1). In particular, if $\{e_1, e_2\} \subset M$ is an exceptional pair, then $\sigma_{e_1}^\varepsilon(e_2) = -L_{e_1}(e_2)$.

7.3.4. LEMMA. *For any exceptional base* $\{e_0, e_1, \ldots, e_n\} \subset M$

$$\sigma_{e_0}^\varepsilon \sigma_{e_1}^\varepsilon \cdots \sigma_{e_n}^\varepsilon = (-1)^{n+1} \varepsilon \varkappa$$

where $M \xrightarrow{\varkappa} M$ *is the canonical Serre operator (see n° 1.4.3).*

PROOF. Consequently applying (7-3) and using (1-3) and (1-5), we get[1]:

$$\begin{aligned}
&\sigma_{e_0}^\varepsilon \sigma_{e_1}^\varepsilon \cdots \sigma_{e_n}^\varepsilon(v) = \\
&\quad = (-1)^{n+1} L_{e_0} L_{e_1} \cdots L_{e_n}(v) - \varepsilon \sum_\nu (-1)^\nu \langle v, e_{n-\nu} \rangle L_{e_0} L_{e_1} \cdots L_{e_{n-\nu-1}} e_{n-\nu} = \\
&\qquad\qquad = -\varepsilon \sum_\nu (-1)^\nu \langle v, e_{n-\nu} \rangle e_\nu^\vee = (-1)^{n+1} \varepsilon \varkappa(v) ,
\end{aligned}$$

because of $L_{e_0} L_{e_1} \cdots L_{e_n}(v) \in \langle e_0, e_1, \ldots, e_n \rangle^\perp = 0$. □

7.3.5. Quasi Katz local systems associated with helices. Let M be a Mukai lattice of type U_{n+1} (for example, $M = K_0(\mathbb{P}_n)$). Then (see [Go5]) the canonical operator on M has a form $\varkappa = (-1)^n \mathrm{Id} + \eta$, where and η is maximal nilpotent, that is, $\eta^{n+1} = 0$ but $\mathrm{im}\, \eta^n = w_0 \neq 0$. Let us put $\varepsilon = (-1)^{n+1}$. Then the (anti) symmetric bilinear form on $W = M \otimes \mathbb{C}$ considered in the previous section, i. e.

$$(v, w)_\varepsilon = \langle v, w \rangle + \varepsilon \langle w, v \rangle = (-1)^n \langle v, ((-1)^n \mathrm{Id} - \varkappa) w \rangle ,$$

[1] comp. with n° 1.4.3 and n° 2.8.1

has 1-dimensional kernel $\langle w_0\rangle$, spanned by w_0. We put $V \stackrel{\text{def}}{=} W/\langle w_0\rangle$, write $(*,*)$ for the non degenerate bilinear form induced on V by $(*,*)_\varepsilon$ and write $\sigma_e(v) = v - (e,v)\,e \pmod{w_0}$ for the corresponding transvections.

For any exceptional base $\overline{e} = \{e_0, e_1, \dots, e_n\} \subset M$ let $\mathscr{L}_{M,\overline{e}}$ be a local system on $U = \mathbb{P}_1 \setminus \{\,1\ \zeta,\ \dots\ ,\ \zeta^n,\ \infty\}$, where ζ is a complex primitive $(n+1)$-th root of unity, associated with a representation $\sigma : \pi_1(U) \longrightarrow GL(V)$ given by the prescription

$$\left.\begin{aligned} \sigma(\gamma_{\zeta^\nu}) &\stackrel{\text{def}}{=} \sigma_{e_\nu}\ :\ v \longmapsto v - (e_\nu, v)\cdot e_\nu \qquad \text{for } 0 \le \nu \le n \\ \sigma(\gamma_\infty) &\stackrel{\text{def}}{=} \Big(\sigma_{e_0}\sigma_{e_1}\ \cdots\ \sigma_{e_n}\Big)^{-1} \end{aligned}\right. \qquad \pmod{w_0}$$

7.3.6. PROPOSITION. *The local system $\mathscr{L}_{M,\overline{e}}$ constructed by the above rules for $M = K_0(\mathbb{P}_n)\otimes\mathbb{C}$ and $\overline{e} = \{\mathscr{O},\ \mathscr{O}(1),\ \dots\ ,\ \mathscr{O}(n)\}$ is quasi Katz.*

PROOF. Consider a local system $\mathscr{R}$ on $U' = \mathbb{P}_1 \setminus \{\,0\ ,\ 1\ ,\ \infty\,\}$ whose local monodromy around 0 is induced by twisting operator, which takes E to $E(1)$, and local monodromy around 1 is the transvection σ_{e_0} w. r. t. to the class $[\mathscr{O}]$. In other words, $\mathscr{R}$ is associated with the representation $\varrho : \pi_1(U') \longrightarrow GL(W)$ that sends

$$\left.\begin{aligned} \varrho(\gamma_0) &\stackrel{\text{def}}{=} T : [E] \longmapsto [E\otimes\mathscr{O}(1)] \\ \varrho(\gamma_1) &\stackrel{\text{def}}{=} \sigma_{e_0} : v \longmapsto v - (e_0, v)\cdot e_0 \end{aligned}\right. \qquad \pmod{w_0}$$

Note that T and $S = \sigma_{e_0}$ satisfy the conditions of lemma from n° 7.3.2, that is, $\mathscr{R}$ is rigid and irreducible. Now consider $(n+1)$-th degree cover

$$\mathbb{P}_1 \setminus \{0,\ 1\ \zeta,\ \dots\ ,\ \zeta^{n-1},\ \infty\} \xrightarrow{\ p\ } \mathbb{P}_1 \setminus \{\,0\ ,\ 1\ ,\ \infty\,\}$$

given by $z \longmapsto z^{n+1}$ in the standard affine chart near $\{0\} \in \mathbb{P}_1$. Then $p^*\mathscr{R}$ can be continued to the origin $\{0\} \subset \mathbb{P}_1$, because all the eigenvalues of $\varkappa = \varrho(\gamma_0)$ are some $(n+1)$-th roots of unity. Let us write $\mathscr{R}'$ for this continuation. Actually, $\mathscr{R}'$ coincides with $\mathscr{L}_{M,\overline{e}}$. Indeed, the monodromy around the infinity in the both cases is given by the canonical operator T^{n+1} and the monodromies around the roots of unity are S, TST^{-1}, T^2ST^{-2}, $\dots$, T^nST^{-n}, because of $\sigma_{e_\nu} = \sigma_{T^\nu e_0} = T^\nu\sigma_{e_0}T^{-\nu}$ $\square$

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