On the intersection homology with twisted coefficients of toric varieties and the homology with twisted coefficients of the torus

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Abstract

We show that if $\underline{\mathbf{V}}$ is a local-system for intersection homology on a toric variety $\mathbf{X}$, and $\underline{\mathbf{K}}$ is its maximal constant sub system, then for any perversity $\bar{p}$, $IH_*^{\bar{p}}(\mathbf{X}, \underline{\mathbf{V}}) = \{0\} \Leftrightarrow H_*(\mathcal{T}^n; \underline{\mathbf{V}}) = \{0\} \Leftrightarrow \mathrm{rank}(\underline{\mathbf{K}}) = 0$.

We describe conditions which are necessary and sufficient for the inclusion $i : \underline{\mathbf{K}} \hookrightarrow \underline{\mathbf{V}}$ to induce an isomorphism on the homology of the torus $\mathcal{T}^n$ and, equivalently, on the intersection homology of $\mathbf{X}$. We show further that it is sufficient but not necessary that $\underline{\mathbf{V}}$ be defined by a unitary representation of the fundamental group of $\mathcal{T}^n$. We give the dimensional restrictions under which some isomorphism $H_*(\mathcal{T}^n; \underline{\mathbf{V}}) \cong H_*(\mathcal{T}^n; \underline{\mathbf{K}})$ exists for any $\underline{\mathbf{V}}$.

1 Introduction

Toric varieties were first defined in the early 70's ([Mu], [De]) as complex algebraic varieties. If they were seen at the time as anything more than a technical tool designed to help deal with a specific problem (the desingularization of symmetric varieties), than it was as a nice generalization of projective space, in which some varieties embed more naturally than in $\mathbf{P}^n$.

In the past decade toric varieties have shown up in a broad variety of disciplines such as Symplectic Geometry (in the study of the moment map of a torus action) and Representation Theory (the toric varieties associated to Weyl chamber decompositions of $\mathbf{R}^n$), and their intersection homology with twisted coefficients has been related by the work of Gel'fand Kapranov and

Zelevinsky [GKZ] (and the Riemman-Hilbert correspondence) to the theory of generalized hypergeometric functions. But nowhere, it seems, have they played so prominent a role as in Combinatorics, by means of fascinating links between the topology of toric varieties and the combinatorial theory of rational convex polytopes. We expand on this below.

In the early 80's R. MacPherson gave (but unfortunately did not publish) a topological description of the (compact) toric variety associated to a (complete) rational fan in $\mathbf{R}^{n\dagger}$ (see the definitions below). Far reaching results can be proved relying only on the topology and one quickly realizes that the loss of the information contained in the algebraic structure is, for many purposes, a small price to pay for the dramatic increase in accessibility for the non-specialist in Algebraic Geometry.

Some important milestones in the history of the toric variety – convex polytope connection: Any n-dimensional convex polytope $\Delta \subset \mathbf{R}^n$ with $0 \in \operatorname{int}\Delta$ and whose vertices have rational coordinates generates a complete rational fan (to each proper face $F \in \Delta$ corresponds the cone $\{tx \mid t \in [0,\infty), x \in F\}$) and hence gives rise to a (compact, projective) toric variety $\mathbf{X} = \mathbf{X}_\Delta$, which is a rational homology manifold *iff* Δ is simplicial. In his 1978 survey article [Da], Danilov calculates the cohomology ring (over $\mathbf{Q}$ or $\mathbf{C}$) of $\mathbf{X}_\Delta$ for simplicial Δ, and shows that the Betti numbers are precisely the components of the h-vector $h(\Delta)$, an important combinatorial invariant of Δ (the ring structure depends also on the specific embedding in $\mathbf{R}^n$). In this context, the well known Dehn-Somerville relations $h_i(\Delta) = h_{n-i}(\Delta)$ are none other than Poincaré duality on $\mathbf{X}_\Delta$. Using these facts, the hard Lefshetz theorem and his own newly introduced methods from Commutative Algebra, Stanley ([St1]) succeeded to prove the necessity of McMullen's conditions ([McM]) thus settling the almost century old problem of classifying face vectors, for the case of simple (dually simplicial) convex polytopes.

The combinatorial properties of a non simple/simplicial polytope are far more complex, as are the the topological properties – and in particular the nature of the singularities – of the associated toric variety. In 1988 McConnell showed ([McC]) that if Δ is not simplicial then the rational homology betti numbers of $\mathbf{X}_\Delta$ are *not* combinatorial invariants. However in

†More recently, M. Davis and T. Januskiewicz ([DJ]) have generalized MacPherson's definition by replacing the underlying fan by a "characteristic function". M. Audin ([Au]) gives a description of nonsingular toric varieties which naturally exhibits the symplectic structure when one exists.

the early 80's, shortly after the introduction of intersection homolgy theory ([GM]), the middle perversity groups $IH_*^{\bar{m}}(\mathbf{X}_\Delta, \mathbf{Q})$ for general Δ were calculated on at least two independant occasions, once by R. MacPherson and once by J. Bernstein and A. Khovanskii and indeed the ranks of these groups were found to be combinatorial invariants of Δ[‡] which (necessarily) coincide with Danilov's calculation when $\mathbf{X}$ is rationally nonsingular. These calculations used very heavy characteristic-p machinery. MacPherson's calculation was first published in 1987 by Stanley ([St2]) who used it to define generalized h-vectors and generalized Dehn-Somerville relations for general convex polytopes. As opposed to the simplicial case, these results do not suffice to classify face-vectors of general polytopes.

We take the next step in this sequence of generalisations by introducing an arbitrary local coefficient system $\underline{\mathbf{V}}$ for intersection homology on $\mathbf{X}$ (which can be thought of as a local system $\underline{\mathbf{V}}$ on the n-torus T^n). It is reasonable to expect that the resulting Betti numbers - at least for cleverly chosen local systems - will be new combinatorial invariants of convex polytopes, and since a version of Poincaré duality holds for intersection homology with twisted coefficients, this will yield new restrictions on face-vectors/flag-vectors.

Using purely topological definitions and techniques, we study the intersection homology $IH_*(\mathbf{X}; \underline{\mathbf{V}})$ and the ordinary homology $H_*(T^n; \underline{\mathbf{V}})$. We show that the two are related and express relations between them in terms of the maximal constant subsystem $\underline{\mathbf{K}} \subseteq \underline{\mathbf{V}}$, whose fiber is the fixed point set of the action of the fundamental group of T^n on the fiber of $\underline{\mathbf{V}}$. We first prove that if either of these two homologies vanishes in all degrees then the other does as well. This occurs in particular for all rank 1 local systems except the trivial one. We then list necessary and sufficient conditions for either (equivalently both) of these homologies to reduce naturally to homology with coefficients in $\underline{\mathbf{K}}$, and show that it is sufficient but not necessary that $\underline{\mathbf{V}}$ be defined by a unitary representation of $\pi_1 T^n$. We show further that each of these theorems reduces to simple Linear Algebra. All of our results concerning intersection homology hold for any perversity. The aforementioned characteristic-p methods do not apply to other than middle perversity.

The main theorems are stated in section 3 and proved in section 5 after we describe in section 4, the E^2 term of a collapsing spectral sequence for the product with a torus. In section 6 we take another look at the torus,

[‡]In this case the Betti numbers depend on more intricate combinatorial properties of Δ, not just on the number of faces in each dimension, and can be conveniently expressed in terms of the "flag vector" ([BK]).

and give precise dimensional restrictions under which $H_*(\mathcal{T}^n;\underline{\mathbf{V}})$ is always isomorphic to $H_*(\mathcal{T}^n;\underline{\mathbf{K}})$ although the isomorphism may not be induced by the inclusion $\underline{\mathbf{K}} \hookrightarrow \underline{\mathbf{V}}$. In the last section we discuss, with examples, the non-existence of a certain long exact sequence for intersection homology which arises in one of the main theorems.

2 Definitions and Notation

Let $\Sigma \subseteq \mathbf{R}^n$ be a complete rational fan, *i.e* a decomposition of $\mathbf{R}^n$ as a finite complex of closed, convex, polyhedral cones, each with apex 0, and each generated by lattice ponts $v_1, \ldots, v_k \in \mathbf{Z}^n$. Take as the dual complex $\mathcal{P}$, the polyhedral cell decomposition of the unit ball in $\mathbf{R}^n$ which has one unique n-cell, and which, when restricted to S^{n-1}, is dual to the cell decomposition obtained by intersecting S^{n-1} with the cones of Σ. We denote by $\hat{\sigma}$ the cell in $\mathcal{P}$ dual to the cone $\sigma \in \Sigma$. Each codimension k cone $\sigma \in \Sigma$ spans a codimension k subspace of $\mathbf{R}^n$ which (since it has a rational basis) maps under the projection $\mathbf{R}^n \to \mathbf{R}^n/\mathbf{Z}^n = \mathcal{T}^n$ to a codimension k subtorus $\mathcal{T}_\sigma \subseteq \mathcal{T}^n$.

Definition 2.1 The topological toric variety $\mathbf{X} = \mathbf{X}_\Sigma$ is obtained from $\mathcal{P} \times \mathcal{T}^n$ by modding out $\hat{\sigma}\backslash\partial\hat{\sigma} \times \mathcal{T}^n$ by the action of $\mathcal{T}_\sigma$ on $\mathcal{T}^n$, for each $\sigma \in \Sigma$. (For more details, see [FY]).

Denote by $\pi : \mathbf{X} \to \mathcal{P}$ the natural projection. $\mathbf{X}$ has a natural stratification

$$\mathbf{X} = \mathbf{X}_{2n} \supset \mathbf{X}_{2n-2} \supset \cdots \supset \mathbf{X}_2 \supset \mathbf{X}_0$$

where for each $1 \leq i \leq n$, $\mathbf{X}_{2i} = \pi^{-1}(\cup\{\hat{\sigma} \in \mathcal{P} | \dim \hat{\sigma} \leq i\})$. For any k-cell $\hat{\sigma} \in \mathcal{P}$, $\pi^{-1}(\hat{\sigma}\backslash\partial\hat{\sigma}) \cong (\mathbf{C}^*)^k$. We identify the "non-singular" open stratum $\mathbf{X}\backslash\mathbf{X}_{2n-2} = \pi^{-1}(\text{int}\mathcal{P})$ with $(\mathbf{C}^*)^n$ so that the maximal compact torus $\mathcal{T}^n \subset (\mathbf{C}^*)^n$ is equal to $\pi^{-1}(0)$.

It is a consequence of the allowability conditions ([GM],[Mac]) that a local system for intersection homology on an m-dimensional stratified pseudomanifold $\mathbf{Y}$ need only be defined on the non-singular open stratum $\mathbf{Y}\backslash\mathbf{Y}_{m-2}$. Denote by $\underline{\mathbf{V}}$ a local system of finite rank on $(\mathbf{C}^*)^n$. We may assume without loss of generality, that $\underline{\mathbf{V}}$ is the trivial extension to $(\mathbf{C}^*)^n$ of a local system on $\mathcal{T}^n$, which we also denote by $\underline{\mathbf{V}}$. In our setting, the fiber $\mathbf{V}_t$ over any point $t \in \mathcal{T}^n$ is a (finite dimensional) vector space over a field F and as usual, is endowed with the discrete topology.

Throughout the paper we will use the same notation for a local system on a space and for its restriction to a subspace. The meaning will be clear from the context.

Fix a base point $t_0 \in \mathcal{T}^n$ and write $\mathbf{V} = \mathbf{V}_{t_0}$. Corresponding to any basis of $\mathbf{Z}^n = \pi_1(\mathcal{T}^n, t_0)$ there are n commuting monodromies $T_1, \ldots, T_n \in GL(\mathbf{V})$. Let $\underline{\mathbf{K}} \subseteq \underline{\mathbf{V}}$ be the (maximal constant) subsystem whose fiber $\mathbf{K}_t$ is the fixed point set of the $\pi_1(\mathcal{T}^n, t)$-action on $\mathbf{V}_t$ and denote by $\mathbf{K}$ the fiber $\mathbf{K}_{t_0} = \cap_{i=1}^n \ker(T_i - I)$.

We call $\underline{\mathbf{V}}$ *unitary* if each T_i is unitary.

We will supress the perversity $\bar{p}$ from any discussion of intersection homology which is independent of the perversity.

3 Statement of results

Theorem 3.1

$$IH_*(\mathbf{X}; \underline{\mathbf{V}}) = \{0\} \Leftrightarrow H_*(\mathcal{T}^n; \underline{\mathbf{V}}) = \{0\} \Leftrightarrow \mathbf{K} = \{0\}.$$

Note: There are natural isomorphisms $\mathbf{K} \cong H_n(\mathcal{T}^n; \underline{\mathbf{K}}) \cong H_n(\mathcal{T}^n; \underline{\mathbf{V}})$.

Theorem 3.2 Let $i : \underline{\mathbf{K}} \hookrightarrow \underline{\mathbf{V}}$ be the inclusion. The following are equivalent:
(i) i induces an isomorphism $i_* : IH_*(\mathbf{X}; \underline{\mathbf{K}}) \to IH_*(\mathbf{X}; \underline{\mathbf{V}})$.
(ii) i induces an isomorphism $i_* : H_*(\mathcal{T}^n; \underline{\mathbf{K}}) \to H_*(\mathcal{T}^n; \underline{\mathbf{V}})$.
(iii) i induces an isomorphism $i_* : IH_0(\mathbf{X}; \underline{\mathbf{K}}) \to IH_0(\mathbf{X}; \underline{\mathbf{V}})$.
(iv) i induces an isomorphism $i_* : H_0(\mathcal{T}^n; \underline{\mathbf{K}}) \to H_0(\mathcal{T}^n; \underline{\mathbf{V}})$.
(v) $\underline{\mathbf{V}}$ splits as a direct sum of sub local systems : $\underline{\mathbf{V}} = \underline{\mathbf{K}} \oplus \underline{\mathbf{M}}$.
(vi) The short exact sequence of local systems

$$0 \to \underline{\mathbf{K}} \xrightarrow{i} \underline{\mathbf{V}} \to \underline{\mathbf{V}/\mathbf{K}} \to 0 \tag{1}$$

induces a long exact sequence of intersection homology :

$$\cdots \to IH_k(\mathbf{X}; \underline{\mathbf{K}}) \xrightarrow{i_*} IH_k(\mathbf{X}; \underline{\mathbf{V}}) \to IH_k(\mathbf{X}; \underline{\mathbf{V}/\mathbf{K}}) \to IH_{k-1}(\mathbf{X}; \underline{\mathbf{K}}) \to \cdots \tag{2}$$

Theorem 3.3 If $\underline{\mathbf{V}}$ is a unitary local system, then it splits as a direct sum $\underline{\mathbf{V}} = \underline{\mathbf{K}} \oplus \underline{\mathbf{M}}$ and hence all of the equivalent conditions in theorem 3.2 hold.

We prove these results in section 5.

Remarks

(*i*) $\mathbf{K} = \cap_{i=1}^{n} \ker(T_i - I)$ and $H_0(\mathcal{T}^n; \underline{\mathbf{V}}) = \mathbf{V} / \sum_{i=1}^{n}(T_i - I)(\mathbf{V})$, whence both theorem 3.1 and theorem 3.2 (part (*iv*)) reduce to elementary Linear Algebra.

(*ii*) We show in section 6 that it is not necessary that $\underline{\mathbf{V}}$ be unitary in order for the conditions of theorem 3.2 to hold.

(*iii*) It follows from thereom 3.2 that if $\underline{\mathbf{V}}$ is a non-trivial irreducible local system then the maps i_* are not isomorphisms.

(*iv*) It follows from the note following theorem 3.1 that if the inclusion $i' : \underline{\mathbf{K}'} \hookrightarrow \underline{\mathbf{V}}$ induces an isomorphism $i'_* : H_*(\mathcal{T}^n; \underline{\mathbf{K}'}) \to H_*(\mathcal{T}^n; \underline{\mathbf{V}})$ then $\underline{\mathbf{K}} \subseteq \underline{\mathbf{K}'}$.

(*v*) The ranks of the intersection homology groups (with field coefficients) of a general toric variety $\mathbf{X}$ are known (see for example [St2]), and since $\underline{\mathbf{K}}$ is a direct sum of trivial, 1-dimensional local systems, $\operatorname{rank} IH_i(\mathbf{X}; \underline{\mathbf{K}}) = \operatorname{rank} IH_i(\mathbf{X}; \mathsf{F}) \operatorname{rank}(\underline{\mathbf{K}})\ \forall i$.

(*vi*) $\pi_1(\mathbf{X}, t_0) = 0$ ([Da]) whence there are no non-trivial local systems for ordinary homology on $\mathbf{X}$.

4 On the product with a torus

Let $\mathbf{Y}$ be a topological space and let $\underline{\mathbf{V}}$ be a local system on $\mathbf{Y} \times \mathcal{T}^n$. We assume given a decomposition of the torus as $\mathcal{T}^n = \mathbf{S}^1 \times \mathcal{T}^{n-1}$, and a common base point so that the inclusions $\mathbf{Y} \hookrightarrow \mathbf{Y} \times \mathbf{S}^1 \hookrightarrow \mathbf{Y} \times \mathcal{T}^n$ make sense.

For any q, there is an induced local system $\underline{\mathbf{H_q}(\mathbf{Y}; \underline{\mathbf{V}})}$ on $\mathcal{T}^n$, whose fiber over any point $t' \in \mathcal{T}^n$ is $H_q(\mathbf{Y} \times \{t'\}; \underline{\mathbf{V}}\,|_{\mathbf{Y} \times \{t'\}})$, and similarly defined local systems over other relevant subspaces of $\mathbf{Y} \times \mathcal{T}^n$.

Theorem 4.1 For any $0 \leq k \leq n$,

$$H_k(\mathbf{Y} \times \mathcal{T}^n; \underline{\mathbf{V}}) \cong \bigoplus_{p+q=k} H_p(\mathcal{T}^n; \underline{\mathbf{H_q}(\mathbf{Y}; \underline{\mathbf{V}})}).$$

Proof: : by induction on n.

For $n = 1$, the filtration $\mathbf{Y} \subset \mathbf{Y} \times \mathbf{S}^1$ gives rise to a spectral sequence in which $E^2_{p,q} = H_p(\mathbf{S}^1; \underline{\mathbf{H_q}(\mathbf{Y}; \underline{\mathbf{V}})})$. This spectral sequence obviously collapses at E^2.

Assume the theorem holds for all $m < n$.

$$\begin{aligned}
H_k(\mathbf{Y} \times T^n; \underline{\mathbf{V}}) &= H_k((\mathbf{Y} \times \mathbf{S}^1) \times T^{n-1}; \underline{\mathbf{V}}) \\
&\overset{ind.\ hyp.}{\cong} \bigoplus_{p+q=k} H_p(T^{n-1}; \underline{\mathbf{H_q}(\mathbf{Y} \times \mathbf{S}^1; \underline{\mathbf{V}})}) \\
&\overset{base\ case}{\cong} \bigoplus_{p+q=k} H_p(T^{n-1}; \underline{\bigoplus_{i+j=q} \mathbf{H_i}(\mathbf{S}^1; \underline{\mathbf{H_j}(\mathbf{Y}; \underline{\mathbf{V}})})}) \\
&\cong \bigoplus_{p+q=k} \bigoplus_{i+j=q} H_p(T^{n-1}; \underline{\mathbf{H_i}(\mathbf{S}^1; \underline{\mathbf{H_j}(\mathbf{Y}; \underline{\mathbf{V}})})}) \\
&= \bigoplus_{j=0}^{\dim \mathbf{Y}} \bigoplus_{p+i=k-j} H_p(T^{n-1}; \underline{\mathbf{H_i}(\mathbf{S}^1; \underline{\mathbf{H_j}(\mathbf{Y}; \underline{\mathbf{V}})})}) \\
&\overset{ind.\ hyp.}{\cong} \bigoplus_{j=0}^{\dim \mathbf{Y}} H_{k-j}(T^n; \underline{\mathbf{H_j}(\mathbf{Y}; \underline{\mathbf{V}})}) \\
&\cong \bigoplus_{p+q=k} H_p(T^n; \underline{\mathbf{H_q}(\mathbf{Y}; \underline{\mathbf{V}})}).
\end{aligned}$$

□

Remark: In [Bo] (proposition 2.1) it is shown that for any stratified pseudomanifold $\mathbf{Y}$, the "suspension" map $\xi \mapsto \xi \times \mathbf{R}$ induces an isomorphism $IH_*(\mathbf{Y}) \xrightarrow{\cong} IH^{BM}_{*+1}(\mathbf{Y} \times \mathbf{R})$, where the latter denotes intersection homology with closed (as opposed to compact) supports. There is a natural isomorphism $IH^{BM}_*(\mathbf{Y} \times \mathbf{R}) \cong IH_*(\mathbf{Y} \times S^1, \mathbf{Y} \times \{t\})$ (with $t \in S^1$), and one readily verifies (since $S^1 \backslash \{t\} \cong \mathbf{R}$) that the existence of these isomorphisms is not affected by the introduction of a local system. Thus, the proof of theorem 4.1 carries over for intersection homology, and in fact we can state the slightly more general relative case :

Theorem 4.2 Let $\mathbf{Y}$ be a stratified pseudomanifold, $\mathbf{Y}' \subset \mathbf{Y}$ a PL-subspace and $\underline{\mathbf{V}}$ a local system for intersection homology on $\mathbf{Y} \times T^n$. Then for any $0 \leq k \leq n$,

$$IH_k((\mathbf{Y}, \mathbf{Y}') \times T^n; \underline{\mathbf{V}}) \cong \bigoplus_{p+q=k} H_p(T^n; \underline{\mathbf{IH_q}(\mathbf{Y}, \mathbf{Y}'; \underline{\mathbf{V}})}).$$

□

5 Proofs of the main theorems

In the proofs we will make use of the following two lemmas.

Lemma 5.1 For any perversity $\bar{p}$

$$IH^{\bar{p}}_{2n}(\mathbf{X};\underline{\mathbf{V}}) \cong H_n(\mathcal{T}^n;\underline{\mathbf{V}}) \cong \mathbf{K}.$$

Proof: Let $\bar{q}$ be the perversity dual to $\bar{p}$ (i.e, for $2 \le k \le 2n$, $\bar{p}_k + \bar{q}_k = k-2$). Then

$$\begin{aligned} IH^{\bar{p}}_{2n}(\mathbf{X};\underline{\mathbf{V}}) &\cong (IH^{\bar{q}}_0(\mathbf{X};\underline{\mathbf{V}}^\star))^\star \\ &\cong (H_0((\mathbf{C}^*)^n;\underline{\mathbf{V}}^\star))^\star \\ &\cong (H_0(\mathcal{T}^n;\underline{\mathbf{V}}^\star))^\star \\ &\cong H_n(\mathcal{T}^n;\underline{\mathbf{V}}) \end{aligned}$$

(Here $\_^\star$ denotes the vector space dual). The first isomorphism follows from Poincaré duality for intersection homology. The second follows from the fact that 0 and 1 dimensional chains must be supported on the non-singular stratum. The last isomorphism follows from ordinary Poincaré duality. □

Lemma 5.2 If either $H_0(\mathcal{T}^n;\underline{\mathbf{V}}) = 0$ or $H_n(\mathcal{T}^n;\underline{\mathbf{V}}) = 0$, then $H_k(\mathcal{T}^n;\underline{\mathbf{V}}) = 0$ for all k.

Proof: The implication $H_0(\mathcal{T}^n;\underline{\mathbf{V}}) = 0 \Rightarrow H_*(\mathcal{T}^n;\underline{\mathbf{V}}) = \{0\}$ is a special case of a theorem of Dwyer ([Dw]). The same theorem, and two applications of Poincaré duality show that the vanishing of the homology in top degree implies the vanishing in all degrees. □

Proof of theorem 3.1: The equivalence on the right and the implication "$\Rightarrow$" on the left follow from lemmas 5.1 and 5.2.

For any $\sigma \in \Sigma$, set $\mathbf{X}_\sigma = \pi^{-1}(\sigma \cap \mathcal{P})$ and $\mathbf{X}_{\partial\sigma} = \partial\mathbf{X}_\sigma = \bigcup_{\tau \subset \partial\sigma} \mathbf{X}_\tau$.

If $\dim\sigma = n - m$ ($0 \le m \le n$) then there exists an m-torus $\mathcal{T}'_\sigma$, complementary to $\mathcal{T}_\sigma$ in $\mathcal{T}^n$ so that $\mathcal{T}^n = \mathcal{T}_\sigma \times \mathcal{T}'_\sigma$ and so that $(\mathbf{X}_\sigma, \mathbf{X}_{\partial\sigma}) = (\mathbf{X}'_\sigma, \mathbf{X}'_{\partial\sigma}) \times \mathcal{T}'_\sigma$, for some (any) $(n-m)$-dimensional toric variety $\mathbf{X}'$ associated to a fan $\Sigma' \subseteq \operatorname{span}\sigma \cong \mathbf{R}^{n-m}$ which contains σ. Furthermore, $\mathbf{X}'_\sigma$ is the topological cone $c\mathbf{X}'_{\partial\sigma}$, stratified by the cones on the strata of $\mathbf{X}'_{\partial\sigma}$ along with the apex of the cone as a (unique) 0-stratum.

Define a filtration of $\mathbf{X}$:

$$\mathcal{T}^n = \mathbf{X}^0 \subset \mathbf{X}^1 \subset \cdots \subset \mathbf{X}^n = \mathbf{X}, \text{ with}$$

$$\mathbf{X}^i = \bigcup_{dim\sigma=i} \mathbf{X}_\sigma, \ \forall i$$

and note that if $\sigma \neq \sigma'$ are m-cones in Σ, then $\mathbf{X}_\sigma \cap \mathbf{X}_{\sigma'} \subseteq \mathbf{X}^{m-1}$ and $\mathbf{X}_\sigma \cap \mathbf{X}^{m-1} = \mathbf{X}_{\partial\sigma}$. In the spectral sequence corresponding to this filtration we have

$$\begin{aligned} E^1_{p,q} &= IH_q(\mathbf{X}^p, \mathbf{X}^{p-1}; \underline{\mathbf{V}}) \\ &\cong \textstyle\bigoplus_{dim\sigma=p} IH_q(\mathbf{X}_\sigma, \mathbf{X}_\sigma \cap \mathbf{X}^{p-1}; \underline{\mathbf{V}}) \\ &= \textstyle\bigoplus_{dim\sigma=p} IH_q(\mathbf{X}_\sigma, \mathbf{X}_{\partial\sigma}; \underline{\mathbf{V}}) \\ &\cong \textstyle\bigoplus_{dim\sigma=p} IH_q((\mathbf{X}'_\sigma, \mathbf{X}'_{\partial\sigma}) \times \mathcal{T}'_\sigma; \underline{\mathbf{V}}). \end{aligned}$$

We show that $E^1_{p,*} = \{0\}$ by induction on p.
For $p = 0$, $IH_*(\mathbf{X}^0, \emptyset; \underline{\mathbf{V}}) = H_*(\mathcal{T}^n; \underline{\mathbf{V}}) = \{0\}$ is our hypothesis.
Assume the theorem holds for all $p < n - m$ and let $\sigma \in \Sigma$, $\dim \sigma = n - m$.
By the inductive hypothesis, for any $\tau \subset \sigma, \tau \neq \sigma$, $IH_*(\mathbf{X}_\tau, \mathbf{X}_{\partial\tau}; \underline{\mathbf{V}}) = \{0\}$.
Thus, when the filtration and the spectral sequence are restricted to $\mathbf{X}_{\partial\sigma}$, the E^1 term vanishes, and hence

$$IH_*(\mathbf{X}_{\partial\sigma}; \underline{\mathbf{V}}) = IH_*(\mathbf{X}'_{\partial\sigma} \times \mathcal{T}'_\sigma; \underline{\mathbf{V}}) = \{0\}. \tag{3}$$

We must show that $IH_*((c\mathbf{X}'_{\partial\sigma}, \mathbf{X}'_{\partial\sigma}) \times \mathcal{T}'_\sigma; \underline{\mathbf{V}}) = \{0\}$. By theorem 4.2

$$IH_k(\mathbf{X}'_{\partial\sigma} \times \mathcal{T}'_\sigma; \underline{\mathbf{V}}) \cong \bigoplus_{i+j=k} H_i(\mathcal{T}'_\sigma; \underline{\mathbf{IH_j}(\mathbf{X}'_{\partial\sigma}; \underline{\mathbf{V}})}), \text{ and} \tag{4}$$

$$IH_{k+1}((c\mathbf{X}'_{\partial\sigma}, \mathbf{X}'_{\partial\sigma}) \times \mathcal{T}'_\sigma; \underline{\mathbf{V}}) \cong \bigoplus_{i+j=k} H_i(\mathcal{T}'_\sigma; \underline{\mathbf{IH_{j+1}}(c\mathbf{X}'_{\partial\sigma}, \mathbf{X}'_{\partial\sigma}; \underline{\mathbf{V}})}). \tag{5}$$

By (3), all of the terms on the right hand side of (4) vanish.

It follows from [Bo] (proposition 3.1) that for any stratified pseudomanifold $\mathbf{Y}$,

$$IH_{k+1}(c\mathbf{Y}, \mathbf{Y}) = \begin{cases} IH_k(\mathbf{Y}) & k \geq r \\ 0 & k < r \end{cases} \tag{6}$$

where the "cutoff point" r depends on the perversity and on the dimension of $\mathbf{Y}$, and the isomorphism for each $k \geq r$ is induced by the coning map $\xi \mapsto c\xi$. Since the non-singular open stratum of $\mathbf{Y}$ is a strong deformation retract of the non-singular open stratum of $c\mathbf{Y}$, any local system for intersection homology on $c\mathbf{Y}$ is equivalent to the trivial extension of one on $\mathbf{Y}$, and hence (6) continues to hold when a local system is introduced. Moreover, for the same reason, the isomorphism $IH_{j+1}(c\mathbf{X}'_{\partial\sigma}, \mathbf{X}'_{\partial\sigma}; \underline{\mathbf{V}}) \cong IH_j(\mathbf{X}'_{\partial\sigma}; \underline{\mathbf{V}})$ (for $j \geq r$) commutes with the action of the fundamental group of $\mathcal{T}'_\sigma$ whereby the local systems $\underline{\mathbf{IH_{j+1}}(\mathbf{cX}'_{\partial\sigma},\mathbf{X}'_{\partial\sigma};\underline{\mathbf{V}})}$ and $\underline{\mathbf{IH_j}(\mathbf{X}'_{\partial\sigma};\underline{\mathbf{V}})}$ are isomorphic for all $j \geq r$. It follows that $H_i(T'_\sigma;\underline{\mathbf{IH_{j+1}}(\mathbf{cX}'_{\partial\sigma},\mathbf{X}'_{\partial\sigma};\underline{\mathbf{V}})}) = \{0\}$ for all $j \geq r$. For $j < r$, the vanishing follows from the vanishing of $IH_{j+1}(c\mathbf{X}'_{\partial\sigma},\mathbf{X}'_{\partial\sigma};\underline{\mathbf{V}})$.

□

Proof of theorem 3.2: The following implications are immediate :
$(i) \Rightarrow (iii)$
$(ii) \Rightarrow (iv)$
$(v) \Rightarrow (vi)$.

"$(iv) \iff (iii)$": It follows from the allowability requirements in the definition of intersection homology, that 0 and 1 dimensional chains are not allowed to meet the positive-codimensional strata. Therefore $IH_0(\mathbf{X};\underline{\mathbf{V}}) \cong H_0((\mathbf{C}^*)^n;\underline{\mathbf{V}}) \cong H_0(T^n;\underline{\mathbf{V}})$. The naturailty of these isomorphisms implies the equivalence of (iv) and (iii).

"$(v) \Rightarrow (i),(ii)$": Denote by j the inclusion $\underline{\mathbf{M}} \hookrightarrow \underline{\mathbf{V}}$.

$$\underline{\mathbf{V}} = \underline{\mathbf{K}} \oplus \underline{\mathbf{M}} \Rightarrow \begin{cases} i_* \oplus j_* : H_*(T^n;\underline{\mathbf{K}}) \oplus H_*(T^n;\underline{\mathbf{M}}) \xrightarrow{\cong} H_*(T^n;\underline{\mathbf{V}}) \\ i_* \oplus j_* : IH_*(\mathbf{X};\underline{\mathbf{K}}) \oplus IH_*(\mathbf{X};\underline{\mathbf{M}}) \xrightarrow{\cong} IH_*(\mathbf{X};\underline{\mathbf{V}}). \end{cases}$$

$$\begin{aligned} \mathbf{K} \cong H_n(T^n;\underline{\mathbf{V}}) \cong H_n(T^n;\underline{\mathbf{K}}) \oplus H_n(T^n;\underline{\mathbf{M}}) \quad &\Rightarrow H_n(T^n;\underline{\mathbf{M}}) = 0 \\ &\Rightarrow H_*(T^n;\underline{\mathbf{M}}) = \{0\} \end{aligned}$$

and by theorem 3.1, $IH_*(\mathbf{X};\underline{\mathbf{M}}) = \{0\}$ as well.

"$(vi) \Rightarrow (i),(ii)$": Assume there is a long exact sequence as in (2), and consider its top end

$$0 \to IH_{2n}(\mathbf{X};\underline{\mathbf{K}}) \xrightarrow{i_*} IH_{2n}(\mathbf{X};\underline{\mathbf{V}}) \to IH_{2n}(\mathbf{X};\underline{\mathbf{V}/\mathbf{K}}) \to IH_{2n-1}(\mathbf{X};\underline{\mathbf{K}}) \to \cdots$$

Note that:
(a) i_* is an isomorphism since it is an injection on isomorphic vector spaces (lemma 5.1).
(b) The intersection homology with field coefficients of any toric variety vanishes in odd degrees. Thus by remark (v) at the end of section 3, $IH_{2n-1}(\mathbf{X};\underline{\mathbf{K}}) = 0$.
It follows that $IH_{2n}(\mathbf{X};\underline{\mathbf{V}/\mathbf{K}}) = 0$ and hence, by lemmas 5.1 and 5.2, $H_*(T^n;\underline{\mathbf{V}/\mathbf{K}}) = \{0\}$. Thus by theorem 3.1, every third term in the sequence is trivial, and (i) follows. (ii) follows as well since the short exact

sequence (1) always induces a long exact sequence

$$\cdots \to H_k(T^n;\underline{\mathbf{K}}) \xrightarrow{i_*} H_k(T^n;\underline{\mathbf{V}}) \to H_k(T^n;\underline{\mathbf{V}/\mathbf{K}}) \to H_{k-1}(T^n;\underline{\mathbf{K}}) \to \cdots.$$

"$(iv) \Rightarrow (v)$": Set $\mathbf{M} = Im(T_1 - I) + \cdots + Im(T_n - I) \subseteq \mathbf{V}$. $\mathbf{M}$ is invariant under each of the T_i's, whence there is a well defined sub local-system $\underline{\mathbf{M}} \subseteq \underline{\mathbf{V}}$. Now, $H_0(T^n;\underline{\mathbf{V}}) = \mathbf{V}/\mathbf{M}$. Thus if (iv) holds, then the inclusion $i : \mathbf{K} \hookrightarrow \mathbf{V}$ induces an isomorphism $\mathbf{K} \to \mathbf{V}/\mathbf{M}$, hence $\mathbf{V} = \mathbf{K} \oplus \mathbf{M}$ and (v) follows.

□

Proof of theorem 3.3: Call a subspace $\mathbf{U} \subseteq \mathbf{V}$ *invariant* if it is invariant under each of the T_i's, and if so denote $K_iU = \ker((T_i - I)|_{\mathbf{U}})$ and $M_iU = \mathrm{Im}((T_i - I)|_{\mathbf{U}})$. Assume $\underline{\mathbf{V}}$ is unitary, i.e. $\mathbf{V}$ is endowed with an inner product $<,>$ such that for each $1 \le i \le n$ and for all $v, w \in \mathbf{V}$, $< T_iv, T_iw > = < v, w >$. We need to show that the subspace $\mathbf{K} \subseteq \mathbf{V}$ has an invariant complement $\mathbf{M}$.

Remark : One easily verifies that for every $1 \le i \le n$, $K_i\mathbf{V} = \ker(T_i - I)$ is orthogonal (with respect to $<,>$) to $M_iV = \mathrm{Im}(T_i - I)$, and that these are complementary dimensional, invariant subspaces. Thus $\mathbf{V}$ decomposes naturally as a direct sum $\mathbf{V} = K_i\mathbf{V} \oplus M_i\mathbf{V}$. Now write $\mathbf{V} = M_1\mathbf{V} \oplus K_1\mathbf{V}$, and repeatedly apply the same argument to the last direct summand each time, to obtain the following natural direct sum decomposition of $\mathbf{V}$:

$$\begin{array}{rcl} \mathbf{V} & = & M_1\mathbf{V} \\ & \oplus & M_2K_1\mathbf{V} \\ & \vdots & \\ & \oplus & M_nK_{n-1}\cdots K_1\mathbf{V} \\ & \oplus & K_nK_{n-1}\cdots K_1\mathbf{V} \end{array}$$

The last summand is equal to $\cap_{i=1}^n \ker(T_i - I) = \mathbf{K}$, and the desired invariant complement $\mathbf{M}$ is the direct sum of all the other summands. □

6 More on the homology of the torus; examples and counter examples

It is possible that $H_*(T^n;\underline{\mathbf{K}}) \cong H_*(T^n;\underline{\mathbf{V}})$, but that the inclusion $\underline{\mathbf{K}} \hookrightarrow \underline{\mathbf{V}}$ does not induce such an isomorphism, whereby conditions $(i) - (vi)$ of theorem 3.2 all fail. In this section we show that if $n = 1$ or $\mathrm{rank}(\underline{\mathbf{V}}) \le 2$, such an isomporphism always exists, whereas in all higher dimensions and ranks there exist counter examples. We then show that there exist non-unitary local systems satisfying the conditions of theorem 3.2.

Proposition 6.1 Let $\underline{\mathbf{V}}$ be a local system on $\mathcal{T}^n$. If $n = 1$ or $\mathrm{rank}(\underline{\mathbf{V}}) \leq 2$, then $H_*(\mathcal{T}^n; \underline{\mathbf{K}}) \cong H_*(\mathcal{T}^n; \underline{\mathbf{V}})$.

Proof:

Case 1: $\mathbf{n = 1}$. Let $T_1 \in GL(\mathbf{V})$ correspond to one of the two generators of $\pi_1(S^1, t)$. Then $H_1(S^1; \underline{\mathbf{V}}) = \ker(T_1 - I) = \mathbf{K} = H_1(S^1; \underline{\mathbf{K}})$. $H_0(S^1; \underline{\mathbf{V}}) = \mathrm{cok}(T_1 - I)$ which is isomorphic to $\ker(T_1 - I)$ and hence to $H_0(S^1; \underline{\mathbf{K}}) = \mathbf{K}$. But this isomorphism is not, in general, induced by the inclusion $i : \mathbf{K} \hookrightarrow \mathbf{V}$.

Example 6.2 Suppose that $\mathrm{rank}(\underline{\mathbf{V}}) = 2$, and that for a suitable basis of $\mathbf{V}$, $T_1(x, y) = (x + y, y)$. Then $\dim K = 1$ but the map $\mathbf{K} \rightarrow \mathrm{cok}(T_1 - I)$ induced by the inclusion $\mathbf{K} \hookrightarrow \mathbf{V}$ has rank 0.

Case 2: $\mathbf{rank}(\underline{\mathbf{V}})$= 1. Each of the monodromies $T_i \in GL(\mathbf{V})$, $1 \leq i \leq n$, is equal to multiplication by a constant $c_i \in \mathsf{F}$. There are only two possibilities: either $\forall i$, $c_i = 1$, in which case $\underline{\mathbf{K}} = \underline{\mathbf{V}}$, or for some i, $\ker(T_i - I) = \{0\} \Rightarrow \mathbf{K} = \{0\} \Rightarrow H_*(\mathcal{T}^n; \underline{\mathbf{V}}) = H_*(\mathcal{T}^n; \underline{\mathbf{K}}) = \{0\}$ (lemmas 5.1 and 5.2).

Case 3: $\mathbf{rank}(\underline{\mathbf{V}})$= 2. We prove this case by induction on n. The base case was taken care of in case 1. We will need the following

Lemma 6.3 $\dim H_n(\mathcal{T}^n; \underline{\mathbf{V}}) = \dim H_0(\mathcal{T}^n; \underline{\mathbf{V}})$.

The proof is an exercise in linear algebra. It suffices to show that if $A_1, \ldots, A_n$ are commuting 2×2 matrices then the $2 \times 2n$ matrix $(A_1\, A_2 \ldots A_n)$ and the the $2n \times 2$ matrix with the A_i's vertically alligned, have the same rank. □

Now denote $\beta_{i,n} = \dim_{\mathbf{Z}} H_i(\mathcal{T}^n; \mathbf{Z})$ and $k = \dim \mathbf{K}$, and assume that for all $m < n$ and for all $0 \leq i \leq m$, $\dim H_i(\mathcal{T}^m; \underline{\mathbf{V}}) = \dim H_i(\mathcal{T}^m; \underline{\mathbf{K}}) = \beta_{i,m} k$.

Let $\mathcal{T}^n = S^1 \times \mathcal{T}^{n-1}$. By theorem 4.1

$$\dim H_i(\mathcal{T}^n; \underline{\mathbf{V}}) = \dim H_i(\mathcal{T}^{n-1}; \underline{\mathbf{H_0(S^1; \underline{V})}}) + \dim H_{i-1}(\mathcal{T}^{n-1}; \underline{\mathbf{H_1(S^1; \underline{V})}}).$$

Note that $\dim \mathbf{V} = 2$ implies that $\underline{\mathbf{H_0(S^1; \underline{V})}}$ and $\underline{\mathbf{H_1(S^1; \underline{V})}}$ also have rank ≤ 2. By the inductive hypothesis and lemma 6.3

$$\begin{aligned} \dim H_i(\mathcal{T}^{n-1}; \underline{\mathbf{H_0(S^1; \underline{V})}}) &= \beta_{i,n-1} \dim H_0(\mathcal{T}^{n-1}; \underline{\mathbf{H_0(S^1; \underline{V})}}) \\ &\overset{theorem\ 4.1}{=} \beta_{i,n-1} \dim H_0(\mathcal{T}^n; \underline{\mathbf{V}}) \\ &\overset{lemma\ 6.3}{=} \beta_{i,n-1} k. \end{aligned}$$

By a similar argument, $\dim H_{i-1}(\mathcal{T}^{n-1};\underline{\mathbf{H_1}(\mathbf{S}^1;\underline{\mathbf{V}})}) = \beta_{i-1,n-1}k$. Finally,

$$\beta_{i,n-1}+\beta_{i-1,n-1} = \binom{n-1}{i}+\binom{n-1}{i-1} = \binom{n}{i} = \beta_{i,n}. \qquad \square$$

The following example shows that without additional restrictions on $\underline{\mathbf{V}}$, the dimensional bounds given in proposition 6.1 are the best ones possible.

Example 6.4 Let $\underline{\mathbf{V}}$ be a rank 3 local system on $\mathcal{T}^2$ such that for suitably chosen bases of $\pi_1(\mathcal{T}^2,t_0)$ and of $\mathbf{V}$, the monodromies T_1 and T_2 are represented by the (commuting) matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The 3×6 matrix $(A_1 - I \;\; A_2 - I)$ has rank 1 whence $\dim H_0(\mathcal{T}^2;\underline{\mathbf{V}}) = 2$, whereas the 6×3 matrix obtained by vertically alligning A_1-I and A_2-I has rank 2, whence $\dim H_2(\mathcal{T}^2;\underline{\mathbf{V}}) = \dim \mathbf{K} = 1$. An analogous counter example can be constructed for any $n > 2$ by setting $T_3 = T_4 = \cdots = T_n = I$ and for any rank > 2 by direct summing with a constant coefficient system.

We conclude this section by observing that if $\underline{\mathbf{V}}$ satisfies the conditions of theorem 3.2, and $\underline{\mathbf{V}'}$ is any non-unitary local system for which $H_*(\mathcal{T}^n;\underline{\mathbf{V}'}) = \{0\}$ (eg when one of the momodromies acts as multiplication by a constant $c \neq 1$), then $\underline{\mathbf{V}} \oplus \underline{\mathbf{V}'}$ is non-unitary and satisfies the conditions of theorem 3.2.

7 On the non-exactness of the intersection homology sequence

In the non-existence of an exact sequence (2), intersection homology differs from ordinary homology. The intersection complex $IC_*(\_;\_)$ *is* a (covariant) functor of the second argument and hence the short exact sequence (1) does induce an (exact) sequence

$$0 \to IC_*(\_,\underline{\mathbf{K}}) \to IC_*(\_,\underline{\mathbf{V}}) \xrightarrow{\pi} IC_*(\_;\underline{\mathbf{V}/\mathbf{K}}).$$

However, because of the allowability requirements on the boundaries of chains, the last map in this sequence is not, in general, surjective. Given a chain $\xi \in IC_*(\mathbf{Y}; \underline{\mathbf{V}/\mathbf{K}})$ we may choose representatives in $\underline{\mathbf{V}}$ for each of the coefficients of ξ, thus obtaining an allowable "preimage" chain ξ' with coefficients in $\underline{\mathbf{V}}$. However, $\partial\xi$ might contain non-allowable summands with coefficients in $\underline{\mathbf{K}}$. Restated, this difference between ordinary homology and intersection homology lies in the fact that the natural map

$$IC_*(\mathbf{Y}; \underline{\mathbf{V}})/IC_*(\mathbf{Y}; \underline{\mathbf{K}}) \xrightarrow{\bar{\pi}} IC_*(\mathbf{Y}; \underline{\mathbf{V}/\mathbf{K}})$$

is not an isomorphism (it is injective but need not be surjective).

Example 7.1 Let $\mathbf{X} \cong S^2$ be the (unique) 2-dimensional toric variety, with two antipodal points p_1, p_2 as the 0-stratum, and let $\underline{\mathbf{V}}$ be the local system of example 6.2 (note that $\underline{\mathbf{V}/\mathbf{K}}$ is a rank 1 constant coefficient system). A sequence

$$0 \to IH_2(\mathbf{X}; \underline{\mathbf{K}}) \xrightarrow{i_*} IH_2(\mathbf{X}; \underline{\mathbf{V}}) \to IH_2(\mathbf{X}; \underline{\mathbf{V}/\mathbf{K}}) \to IH_1(\mathbf{X}; \underline{\mathbf{K}})$$

cannot be exact at $IH_2(\mathbf{X}; \underline{\mathbf{V}/\mathbf{K}})$, as it is easy to see that each of the first 3 terms is 1-dimensional whereas $IH_1(\mathbf{X}; \underline{\mathbf{K}}) = 0$. Indeed, let $\xi' \in IC_1(\mathbf{X}; \underline{\mathbf{V}/\mathbf{K}})$ be the chain with (constant) coefficient $(0,1) + \mathbf{K} \in \mathbf{V}/\mathbf{K}$ which is supported on a great circle in $\mathbf{X} \setminus \{p_1, p_2\}$, and let $\xi = c\xi' \in IC_2(\mathbf{X}; \underline{\mathbf{V}/\mathbf{K}})$ be the cone to p_1. Since $(0,1) \notin \mathbf{K} = \ker(T_1 - I)$, any preimage of ξ' would necessarily have a boundary point with non-trivial coefficient, and hence any proposed preimage of ξ will necessarily have a non-trivial boundary component meeting the 0-stratum, and this is not allowable since 1-chains may not meet the singular set. Thus neither π nor $\bar{\pi}$ are surjective.

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