COMPLETE LEFT-INVARIANT AFFINE STRUCTURES ON THE OSCILLATOR GROUP

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Abstract. *The goal of this paper is to provide a method, based on the theory of extensions of left-symmetric algebras, for classifying left-invariant affine structures on a given solvable Lie group of low dimension. To better illustrate our method, we shall apply it to classify all complete left-invariant affine structures on the oscillator group.*

1 Introduction

It is a well known result (see [1], [14]) that a simply connected Lie group G which admits a complete left-invariant affine structure, or equivalently G acts simply transitively by affine transformations on $\mathbb{R}^n$, must be solvable. It is also well known that not every solvable (even nilpotent) Lie group can admit an affine structure [2]. Now given a simply connected solvable Lie group G which can admit a complete left-invariant structure, it is important to classify all such possible structures on G.

Our goal in the present paper is to provide a method for classifying left-invariant affine structures on a given solvable Lie group of low dimension. Since the classification has been completely achieved up to dimension four in the nilpotent case (see [7], [10], [12]), we shall illustrate our method by applying it to classify all complete left-invariant affine structures on the remarkable solvable and non-nilpotent four-dimensional Lie group O_4, known as the *oscillator group*. Recall that O_4 can be viewed as a semidirect product of the real line with the Heisenberg group. Recall also that the Lie algebra $\mathcal{O}_4$ of O_4 (that we shall call *oscillator algebra*) is the Lie algebra with generators e_1, e_2, e_3, e_4, and with nonzero brackets

$$[e_1, e_2] = e_3, \ [e_4, e_1] = e_2, \ [e_4, e_2] = -e_1.$$

Since left-invariant affine structures on a Lie group G are in one-to-one correspondence with left-symmetric structures on its Lie algebra $\mathcal{G}$ [10], we shall carry out the classification of complete left-invariant affine structures on O_4 in terms of complete (in the sense of [17]) left-symmetric structures on $\mathcal{O}_4$.

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The paper is organized as follows. In Section 2, we recall the notion of extensions of Lie algebras and its relationship to the notion of $\mathcal{G}$-kernels. In Section 3, we give some necessary definitions and notations and basic results on left-symmetric algebras and their extensions. In Section 4, we consider the special case where the Lie algebra $\mathcal{G}$ is $\mathcal{O}_4$.We observe that any left-symmetric product on $\mathcal{O}_4$ can be obtained by extension of a left-symmetric product on the Heisenberg algebra $\mathcal{H}_3$ according to a short sequence of l.s. algebras of the form

$$0 \rightarrow A_3 \xrightarrow{i} A_4 \xrightarrow{\pi} \mathbb{R} \rightarrow 0,$$

where A_3 and A_4 are viewed as left-symmetric algebras whose associated Lie algebras are $\mathcal{H}_3$ and $\mathcal{O}_4$, respectively. In Section 5, we show that the Lie algebra associated to A_3 is isomorphic to the Lie algebra $\mathcal{E}(2)$ of the group of Euclidean motions of the plane. We show that, up to left-symmetric isomorphism, there is a unique complete left-symmetric structure on $\mathcal{E}(2)$, and we use this to carry out all complete left-symmetric structures on $\mathcal{O}_4$. We find that, up to left-symmetric isomorphism, there exist exactly two non-isomorphic complete left-symmetric structures on $\mathcal{O}_4$. By using the Lie group exponential maps, we deduce the classification of all complete left-invariant affine structures on the oscillator group O_4 in terms of simply transitive actions of subgroups of the affine group $Aff\left(\mathbb{R}^4\right) = GL\left(\mathbb{R}^4\right) \ltimes \mathbb{R}^4$ (see Theorem 18 below).

Throughout this paper, all considered vector spaces, Lie algebras, and left-symmetric algebras are supposed to be over the filed $\mathbb{R}$. We also suppose that all considered Lie groups are connected and simply connected.

2 Extensions of Lie algebras

Recall that a Lie algebra $\widetilde{\mathcal{G}}$ is an extension of the Lie algebra $\mathcal{G}$ by the Lie algebra $\mathcal{A}$ if there exists a short exact sequence of Lie algebras

$$0 \rightarrow \mathcal{A} \xrightarrow{i} \widetilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G} \rightarrow 0. \tag{1}$$

In other words, if we identify the elements of $\mathcal{A}$ with their images in $\widetilde{\mathcal{G}}$ via the injection i, then $\mathcal{A}$ is an ideal in $\widetilde{\mathcal{G}}$ such that $\widetilde{\mathcal{G}}/\mathcal{A} \cong \mathcal{G}$.

Two extensions $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$ are called equivalent if there exists an isomorphism of Lie algebras φ such that the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{A} & \xrightarrow{i_1} & \widetilde{\mathcal{G}}_1 & \xrightarrow{\pi_1} & \mathcal{G} & \longrightarrow 0 \\
& id_{\mathcal{A}} \downarrow & & \downarrow \varphi & & \downarrow id_{\mathcal{G}} & \\
0 \longrightarrow & \mathcal{A} & \xrightarrow{i_2} & \widetilde{\mathcal{G}}_2 & \xrightarrow{\pi_2} & \mathcal{G} & \longrightarrow 0
\end{array}$$

commutes.

The notion of extensions of a Lie algebra $\mathcal{G}$ by an abelian Lie algebra $\mathcal{A}$ is well known (see for instance, the books [5] and [9]). In light of [16], we shall describe here the notion of extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by a Lie algebra $\mathcal{A}$ which is not necessarily abelian.

Suppose that a vector space extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by another Lie algebra $\mathcal{A}$ is known, and we want to define a Lie structure on $\widetilde{\mathcal{G}}$ in terms of the Lie structures of $\mathcal{G}$ and $\mathcal{A}$. Let $\sigma : \mathcal{G} \to \widetilde{\mathcal{G}}$ be a section, that is, a linear map such that $\pi \circ \sigma = id$. Then the linear map $\Psi : (a, x) \mapsto i(a) + \sigma(x)$ from $\mathcal{A} \oplus \mathcal{G}$ onto $\widetilde{\mathcal{G}}$ is an isomorphism of vector spaces.

For (a, x) and (b, y) in $\mathcal{A} \oplus \mathcal{G}$, define a commutator on $\widetilde{\mathcal{G}}$ by

$$\begin{aligned} \left[i(a) + \sigma(x), i(b) + \sigma(y)\right] &= i([a,b]) + [\sigma(x), i(b)] \\ &\quad + [i(a), \sigma(y)] + [\sigma(x), \sigma(y)] \end{aligned} \tag{2}$$

Since $\sigma(x)$ lies in $i(\mathcal{A})$ for all $x \in \mathcal{G}$, we shall denote $\sigma(x) = i(\overline{x})$ with $\overline{x} \in \mathcal{A}$. Now we define a linear map $\phi : \mathcal{G} \to End(\mathcal{A})$ by

$$\phi(x)\, a = [\overline{x}, b],$$

that is, $\phi(x) = (ad_{\overline{x}})_{|\mathcal{A}}$. On the other hand, since

$$\pi([\sigma(x), \sigma(y)]) = \pi(\sigma([x,y])),$$

it follows that there exists an alternating bilinear map $\omega : \mathcal{G} \times \mathcal{G} \to \mathcal{A}$ such that

$$[\sigma(x), \sigma(y)] = \sigma[x,y] + \omega(x,y).$$

In summary, by means of the isomorphism above, $\widetilde{\mathcal{G}} \cong \mathcal{A} \oplus \mathcal{G}$ and its elements may be denoted by (a, x) with $a \in \mathcal{A}$ and x is simply characterized by its coordinates in $\mathcal{G}$. The commutator defined by (2) is now given by

$$[(a,x), (b,y)] = ([a,b] + \phi(x)\, b - \phi(y)\, a + \omega(x,y), [x,y]), \tag{3}$$

for all $(a, x) \in \widetilde{\mathcal{G}} \cong \mathcal{A} \oplus \mathcal{G}$.

Now, it is easy to see that this is actually a Lie bracket (i.e, it verifies the Jacobi identity) if and only if the following three conditions are satisfied

1. $\phi(x)[b,c] = [\phi(x)\, b, c] + [b, \phi(x)\, c]$,
2. $[\phi(x), \phi(y)] = \phi([x,y]) + ad_{\omega(x,y)}$,
3. $\omega([x,y], z) - \omega(x, [y,z]) + \omega(y, [x,z]) = \phi(x)\,\omega(y,z) + \phi(y)\,\omega(z,x) + \phi(z)\,\omega(x,y)$.

Remark 1 *We see that condition (1) above is equivalent to say that $\phi(x)$ is a derivation of $\mathcal{A}$, and condition (3) is equivalent to the fact that ω is a 2-cocycle (i.e., $\delta_\phi \omega = 0$, where δ_ϕ refers to the coboundary operator corresponding to the action ϕ). Condition (2) indicates clearly that if $\mathcal{A}$ is supposed to be abelian, then $\mathcal{G}$ becomes an $\mathcal{A}$-module in a natural way, because in this case the linear map $\phi : \mathcal{G} \to End(\mathcal{A})$ given by $\phi(x)\, a = [\overline{x}, b]$ is well defined. Namely, ϕ does not depend on the choice of the section σ, given that $\mathcal{A}$ is abelian.*

As we mentioned in Remark 1, $\mathcal{G}$ is actually acting by derivations, that is, $\phi : \mathcal{G} \to Der(\mathcal{A})$. If now $\sigma' : \mathcal{G} \to \widetilde{\mathcal{G}}$ is another section, then $\sigma' - \sigma = \tau$ for some linear map $\tau : \mathcal{G} \to \mathcal{A}$, and it follows that the corresponding morphism and 2-cocycle are, respectively, $\phi' = \phi + ad \circ \tau$ and $\omega' = \omega + \delta_\phi \tau + \frac{1}{2}[\tau, \tau]$, where ad stands here and below, if there is no ambiguity, for the adjoint representation in $\mathcal{A}$ (Recall here that $\frac{1}{2}[\tau,\tau](x,y) = [\tau(x), \tau(y)]$). Therefore, $\omega' - \omega$ is a 2-coboundary if and only if $[\tau(x), \tau(y)] = 0$ for all $x, y \in \mathcal{G}$. Equivalently, $\omega' - \omega$ is a 2-coboundary if and only if τ has its range in the center $Z(\mathcal{A})$ of $\mathcal{A}$. In that case, we get $\omega' - \omega = \delta_\phi \tau \in B^2_\phi(\mathcal{G}, Z(\mathcal{A}))$, the group of 2-coboundaries for $\mathcal{G}$ with values in $Z(\mathcal{A})$.

To overcome all these difficulties, we proceed as follows. Let $C^2(\mathcal{G}, \mathcal{A})$ be the abelian group of all 2-cochains, i.e. alternating bilinear mappings $\mathcal{G} \times \mathcal{G} \to \mathcal{A}$. For a given $\phi : \mathcal{G} \to Der(\mathcal{A})$, let $T_\phi \in C^2(\mathcal{G}, \mathcal{A})$ be defined by

$$T_\phi(x,y) = [\phi(x), \phi(y)] - \phi([x,y]), \quad \text{for all } x, y \in \mathcal{G}.$$

If there exists some $\omega \in C^2(\mathcal{G}, \mathcal{A})$ such that $T_\phi = ad \circ \omega$ and $\delta_\phi \omega = 0$, then the pair (ϕ, ω) is called a *factor system for* $(\mathcal{G}, \mathcal{A})$. Let $Z^2(\mathcal{G}, \mathcal{A})$ be the set of all factor systems for $(\mathcal{G}, \mathcal{A})$. It is shown in ([16], Theorem II.7) that the equivalent classes of extensions of a Lie algebra $\mathcal{G}$ by a Lie algebra $\mathcal{A}$ are in one-to-one correspondence with the elements of the quotient space $Z^2(\mathcal{G}, \mathcal{A}) / C^1(\mathcal{G}, \mathcal{A})$, where $C^1(\mathcal{G}, \mathcal{A})$ is the space of linear maps from $\mathcal{G}$ into $\mathcal{A}$. Note that if we assume that $\mathcal{A}$ is abelian, then we meet the well known result (see for instance [4]) stating that for a given action $\phi : \mathcal{G} \to End(\mathcal{A})$, the equivalent classes of extensions of $\mathcal{G}$ by $\mathcal{A}$ are in one-to-one correspondence with the elements of the second cohomology group

$$H^2_\phi(\mathcal{G}, \mathcal{A}) = Z^2_\phi(\mathcal{G}, \mathcal{A}) / B^2_\phi(\mathcal{G}, \mathcal{A}).$$

In the present paper, we shall be concerned with the special case where $\mathcal{A}$ is non-abelian and $\mathcal{G}$ is the field $\mathbb{R}$, and hencefore the cocycle ω is identically zero.

Remark 2 *It is worth noticing that the construction above is closely related to the notion of $\mathcal{G}$-kernels (considered for Lie algebras firstly in [15]). On* $\left\{\phi : \mathcal{G} \to Der(\mathcal{A}) : T_\phi = ad \circ \omega, \text{ for some } \omega \in C^2(\mathcal{G}, \mathcal{A})\right\}$*, define an equivalence relation by $\phi \sim \phi'$ if and only if $\phi' = \phi + ad \circ \tau$, for some linear map $\tau : \mathcal{G} \to \mathcal{A}$. The equivalence class $[\phi]$ of ϕ is called a $\mathcal{G}$-*kernel*. It turns out that if$\mathcal{A}$ is abelian, then a $\mathcal{G}$-kernel is nothing but a $\mathcal{G}$-module. By considering the quotient morphism $\Pi : Der(\mathcal{A}) \to Out(\mathcal{A}) = Der(\mathcal{A}) / ad_{\mathcal{A}}$, and remarking that $\Pi \circ ad \circ \tau = 0$ for any linear map $\tau : \mathcal{G} \to \mathcal{A}$, we can naturally associate to each $\mathcal{G}$-kernel $[\phi]$ the morphim $\phi = \Pi \circ [\phi] : \mathcal{G} \to Out(\mathcal{A})$.*

3 Extensions of left-symmetric algebras

The notion of a *left-symmetric algebra* (or *l.s. algebra* in short) arises naturally in various areas of mathematics and physics. It originally appeared in the works of Vinberg [18] and Koszul [11] concerning convex homogeneous cones and bounded homogeneous domains, respectively. It also appears, for instance, in connection with Yang-Baxter equation and integrable hydrodynamical systems (cf. [3], [8], [13]).

A left-symmetric algebra $(A, .)$ is a finite-dimensional algebra A in which the products, for all $x, y, z \in A$, satisfy the identity

$$(xy)\, z - x\, (yz) = (yx)\, z - y\, (xz)\,, \tag{4}$$

where here and frequently during this paper we simply write xy instead of $x \cdot y$.

It is clear that an associative algebra is a l.s. algebra. Actually, for a l.s. algebra A, if $(x, y, z) = (xy)\, z - x\, (yz)$ is the associator of x, y, z, then we see that (4) is equivalent to $(x, y, z) = (y, x, y)$ This means that l.s. algebras are natural generalizations of associative algebras.

Now if A is a l.s. algebra, then the commutator

$$[x, y] = xy - yx \tag{5}$$

defines a structure of Lie algebra on A, called the *associated Lie algebra*. On the other hand, if $\mathcal{G}$ is a Lie algebra with a l.s. product $\cdot$ satisfying

$$[x, y] = x \cdot y - y \cdot x,$$

then we say that the l.s. structure is *compatible* with the Lie structure on $\mathcal{G}$.

Suppose now we are given a Lie group G with a left-invariant flat affine connection ∇, and define a product $\cdot$ on the Lie algebra $\mathcal{G}$ of G by

$$x \cdot y = \nabla_x y, \tag{6}$$

for all $x, y \in \mathcal{G}$. Then, the conditions on the connection ∇ for being flat and torsion-free are now equivalent to the conditions (4) and (5), respectively.

Conversely, suppose that G is a simply connected Lie group with Lie algebra $\mathcal{G}$, and suppose that $\mathcal{G}$ is endowed with a l.s. product $\cdot$ which is compatible with the Lie bracket of $\mathcal{G}$. We define an operator ∇ on $\mathcal{G}$ according to identity (6), and then we extend it by left-translations to the whole Lie group G. This clearly defines a left-invariant flat affine structure on G. In summary, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, the left-invariant flat affine structures on G are in one-to-one correspondence with the l.s. structures on $\mathcal{G}$ compatible with the Lie structure.

Let A be a l.s. algebra, and let the left and right multiplications L_x and R_x by the element x be defined by $L_x y = x \cdot y$ and $R_x y = y \cdot x$. We say that A is *complete* if R_x is a nilpotent operator, for all $x \in A$. It turns out that, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, the complete left-invariant flat affine structures on G are in one-to-one correspondence with the complete l.s. structures on $\mathcal{G}$ compatible with the Lie structure. It is also known that an n-dimensional simply connected Lie group admits a complete left-invariant flat affine structure if and only if it acts simply transitively on $\mathbb{R}^n$ by affine transformations (see [10]). A simply connected Lie group which is acting simply transitively on $\mathbb{R}^n$ by affine transformations must be solvable according to [1], but it is worth noticeable that there exist solvable (even nilpotent) Lie groups which do not admit affine structures (see [2]).

We close this section by fixing some notations which we will be using in what follows. For a l.s. algebra A, we can easily check that the subset

$$T(A) = \{x \in A : L_x = 0\}$$

is a two-sided ideal in A. Geometrically, if G is a Lie group which acts simply transitively on $\mathbb{R}^n$ by affine transformations then $T(\mathcal{G})$ corresponds to the set of central translational elements in G, where $\mathcal{G}$ is endowed with the complete l.s. product corresponding to the action of G on $\mathbb{R}^n$. It has been conjectured in [1] that every nilpotent Lie group G which acts simply transitively on $\mathbb{R}^n$ by affine transformations contains a central translation, but this turned out to be false (see [6]).

We discussed in the last section the problem of extension of a Lie algebra by another Lie algebra. Similarly, we shall briefly discuss in this section the problem of extension of a l.s. algebra $\tilde{A}$ by another l.s. algebra A. To our knowledge, the notion of extensions of l.s. algebras has been considered for the first time in [10], to which we refer for more details.

Suppose that a vector space extension $\tilde{A}$ of a l.s algebra A by another l.s. algebra E is given. We want to define a l.s. structure on $\tilde{A}$ in terms of the l.s structures given on A and E. In other words, we want to define a l.s. product on $\tilde{A}$ for which E becomes a two-sided ideal in $\tilde{A}$ such that $\tilde{A}/E \cong A$; or equivalently,

$$0 \to E \to \tilde{A} \to A \to 0$$

becomes a short exact sequence of l.s algebras.

Theorem 3 ([10]) *There exists a l.s. structure on $\tilde{A}$ extending a l.s. algebra A by a l.s. algebra E if and only if there exist two linear maps λ, $\rho : A \to End(E)$ and a bilinear map $g : A \times A \to E$ such that, for all $x, y, z \in A$ and $a, b \in E$, the following conditions are satisfied.*

(i) $\lambda_x(a \cdot b) = \lambda_x(a) \cdot b + a \cdot \lambda_x(b) - \rho_x(a) \cdot b,$

(ii) $\rho_x([a, b]) = a \cdot \rho_x(b) - b \cdot \rho_x(a),$

(iii) $[\lambda_x, \lambda_y] - \lambda_{[x,y]} = L_{g(x,y)-g(y,x)},$

(iv) $\left[\lambda_x, \rho_y\right] + \rho_y \circ \rho_x - \rho_{x \cdot y} = R_{g(x,y)},$

(v) $g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)) - g([x, y], z)$

$-\rho_z(g(x, y) + g(y, x)) = 0.$

If the conditions of the Theorem 3 are fulfilled, then the extended l.s. product on $\tilde{A} \cong A \times E$ is given by

$$(x,a) \cdot (y,b) = \big(x \cdot y, a \cdot b + \lambda_x (b) + \rho_y (a) + g(x,y)\big). \tag{7}$$

It is remarkable that if the l.s. product of E is trivial, then the conditions of the Theorem 3 simplify to the following two conditions:

(i) $[\lambda_x, \lambda_y] = \lambda_{[x,y]}$, i.e. λ is a representation of Lie algebras,

(ii) $\left[\lambda_x, \rho_y\right] + \rho_y \circ \rho_x - \rho_{x \cdot y} = 0.$

In this case, E becomes an A-bimodule and the extended product given in (7) simplifies too.

Let A be a l.s. algebra, and suppose that an A-bimodule V (which is not necessarily trivial) is known. We denote by $L^p(A,V)$ the space of all p-linear maps from A to V, and we define two coboundary operators $\delta_1 : L^1(A,V) \to L^2(A,V)$ and $\delta_2 : L^2(A,V) \to L^3(A,V)$ as follows : For a linear map $h \in L^1(A,V)$ we set

$$\delta_1 h(x,y) = \rho_y(h(x)) + \lambda_x(h(y)) - h(x \cdot y), \tag{8}$$

and for a bilinear map $g \in L^2(A,V)$ we set

$$\begin{aligned}\delta_2 g(x,y,z) \;\; = \;\; & g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) \\ & -g([x,y], z) - \rho_z(g(x,y) + g(y,x)).\end{aligned} \tag{9}$$

It may be verified that $\delta_2 \circ \delta_1 = 0$. Therefore, if we set $Z^2_{\lambda,\rho}(A,V) = \ker \delta_2$ and $B^2_{\lambda,\rho}(A,V) = \operatorname{Im} \delta_1$, we can define a notion of second cohomology for the actions λ and ρ by simply setting $H^2_{\lambda,\rho}(A,V) = Z^2_{\lambda,\rho}(A,V) / B^2_{\lambda,\rho}(A,V)$.

As in the case of extensions of Lie algebras, we can prove that for given linear maps λ, $\rho : A \to End(V)$, the equivalent classes of extensions of A by V are in one-to-one correspondence with the elements of the second cohomology group $H^2_{\lambda,\rho}(A,V)$.

3.1 Central extensions of l.s. algebras

The notion of central extensions known for Lie algebras may analogously be defined for l.s. algebras. Let $\tilde{A}$ be a l.s. extension of a l.s algebra A by another l.s. algebra E, and let $\widetilde{\mathcal{G}}$ be the Lie algebra associated to $\tilde{A}$. We say that the extension $0 \to E \xrightarrow{i} \tilde{A} \xrightarrow{\pi} A \to 0$ is *central* if $i(E) \subseteq Z\left(\widetilde{\mathcal{G}}\right)$, where $Z\left(\widetilde{\mathcal{G}}\right)$ is the center of the Lie algebra $\widetilde{\mathcal{G}}$.

In particular, when E is a trivial A-bimodule (i.e. $\lambda = \rho = 0$), we deduce that the extension $0 \to E \xrightarrow{i} \tilde{A} \xrightarrow{\pi} A \to 0$ is central if and only if $i(E) \subseteq C\left(\tilde{A}\right)$, where

$$C\left(\tilde{A}\right) = T\left(\tilde{A}\right) \cap Z\left(\widetilde{\mathcal{G}}\right) = \left\{x \in \tilde{A} : x \cdot y = y \cdot x = 0, \quad \text{for all } y \in \tilde{A}\right\},$$

where $T\left(\tilde{A}\right)$ is the two-sided ideal of $\tilde{A}$ defined above. In particular, if $i(E) = C\left(\tilde{A}\right)$ we say that the extension is *exact*. It is easy to verify (see [10]) that the extension is exact if and only if $I_{[g]} = 0$, where

$$I_{[g]} = \{x \in A : x \cdot y = y \cdot x = 0 \text{ and } g(x,y) = g(y,x) = 0, \text{ for all } y \in A\}.$$

We note here that $I_{[g]}$ is well defined because any other element in $[g]$ has the form $g + \delta_1 h$ with $\delta_1 h(x,y) = -h(x \cdot y)$ (since we have here $\lambda = \rho = 0$).

Remark 4 *We notice that, in [10], the center of $\tilde{A}$ is defined to be the ideal $C\left(\tilde{A}\right)$, and an extension is called central exactly whenever $i(E) \subseteq C\left(\tilde{A}\right)$. This implies that, for a central extension in the sense of [10], the A-bimodule E is always trivial.*

Given a l.s. algebra A and a trivial A-bimodule E, we denote a central extension $0 \to E \to \tilde{A} \to A \to 0$ corresponding to a cohomology class $[g] \in H^2(A,E)$ by $\left(\tilde{A},[g]\right)$. Let $\left(\tilde{A},[g]\right)$ and $\left(\tilde{A}',[g']\right)$ be two central extensions of A by E, and let $\alpha \in Aut(E) = GL(E)$ and $\eta \in Aut(A)$, where $Aut(E)$ and $Aut(A)$ are the groups of l.s. automorphisms of E and A, respectively. It is clear that, given $h \in L^1(A,E)$, then the linear mapping $\psi : \tilde{A} \to \tilde{A}'$ defined by

$$\psi(x,a) = (\eta(x), \alpha(a) + h(x))$$

is an isomorphism provided $g'(\eta(x), \eta(y)) = \alpha(g(x,y)) + \delta_1 h(x,y)$ for all $(x,y) \in A \times A$, i.e. $\eta^*[g'] = \alpha_*[g]$. This allows us to define an action of the group $G = Aut(E) \times Aut(A)$ on $H^2(A,E)$ by setting

$$(\alpha,\eta).[g] = \alpha_* \eta^* [g], \tag{10}$$

or equivalently, $(\alpha,\eta).g(x,y) = \alpha(g(\eta(x),\eta(y)))$ for all $x,y \in A$.

Denoting the set of all exact central extensions of A by E by

$$H^2_{ex}(A,E) = \left\{[g] \in H^2(A,E) : I_{[g]} = 0\right\},$$

and the orbit of $[g]$ by $G_{[g]}$, it turns out that the following result is valid (see [10]).

Proposition 5 *Let $[g]$ and $[g']$ be two classes in $H^2_{ex}(A,E)$. Then, the central extensions $\left(\tilde{A},[g]\right)$ and $\left(\tilde{A}',[g']\right)$ are isomorphic if and only if $G_{[g]} = G_{[g']}$. In other words, the classification of the exact central extensions of A by E is, up to l.s. isomorphism, the orbit space of $H^2_{ex}(A,E)$ under the natural action of $G = Aut(E) \times Aut(A)$.*

4 Left-symmetric structures on the oscillator algebra

Recall that the Heisenberg group H_3 is the 3-dimensional Lie group diffeomorphic to $\mathbb{R} \times \mathbb{C}$ with the group law

$$(v_1, z_2) \cdot (v_2, z_2) = (v_1 + v_2 + \frac{1}{2}\operatorname{Im}(\overline{z_1} z_2), z_1 + z_2),$$

for all $v_1, v_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$.

Let $\lambda > 0$, and let $G = \mathbb{R} \ltimes H_3$ be equipped with the group law

$$(t_1, v_1, z_1) \cdot (t_2, v_2, z_2) = (t_1 + t_2, v_1 + v_2 + \frac{1}{2}\operatorname{Im}(\overline{z_1} z_2 e^{i\lambda t}), z_1 + z_2 e^{i\lambda t}),$$

for all $t_1, t_2 \in \mathbb{R}$ and $(v_1, z_1), (v_2, z_2) \in H_3$. This is a 4-dimensional Lie group with Lie algebra $\mathcal{G}$ having a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$[e_1, e_2] = e_3, \ [e_4, e_1] = \lambda e_2, \ [e_4, e_2] = -\lambda e_1,$$

and all the other brackets are zero.

It follows that the derived series is given by

$$\mathcal{D}^1\mathcal{G} = [\mathcal{G}, \mathcal{G}] = span\{e_1, e_2, e_3\}, \ \mathcal{D}^2\mathcal{G} = span\{e_3\}, \ \mathcal{D}^3\mathcal{G} = \{0\},$$

and therefore $\mathcal{G}$ is a (non-nilpotent) 3-step solvable Lie algebra.

When $\lambda = 1$, G is known as the *oscillator group.* We shall denote it by O_4, and we shall denote its Lie algebra by $\mathcal{O}_4$ and call it the *oscillator algebra*.

The following useful lemma is easy to prove.

Lemma 6 *The oscillator algebra $\mathcal{O}_4$ contains only two proper ideals which are $Z(\mathcal{O}_4) \cong \mathbb{R}$ and $[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$.*

We want to classify all the complete real l.s. structures on $\mathcal{O}_4$. In what follows, let A_4 be a l.s. algebra whose associated Lie algebra is $\mathcal{O}_4$ and consider the subspace

$$C(A_4) = T(A_4) \cap Z(\mathcal{O}_4) = \{a \in A_4 : ax = xa = 0, \quad \text{for all } x \in A_4\},$$

where $Z(\mathcal{O}_4)$ is the center of the Lie algebra $\mathcal{O}_4$ and $T(A_4)$ is the two-sided ideal of A_4 defined in Section 3.

Since $\dim Z(\mathcal{O}_4) = 1$, it follows that $\dim C(A_4) \leq 1$. Consequently, we should distinguish two cases according to whether $C(A_4)$ is trivial or not. However, as we will see below (see Lemmas 10 and 11), the classification of complete l.s. structures on $\mathcal{O}_4$ will be reduced to the case where $C(A_4)$ is nontrivial.

First, we begin by observing that any arbitrary algebra A contains $A^2 = A\cdot A$ as a two-sided ideal. This allows us to consider the two-sided ideal A_4^2 in A_4 which is also an ideal of the Lie algebra $\mathcal{O}_4$. Second, by Lemma 6, the only proper ideals of $\mathcal{O}_4$ are $Z(\mathcal{O}_4) \cong \mathbb{R}$ and $[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$. It follows that the associated Lie algebra of A_4^2 is $\mathcal{H}_3$, since $A_4^2 \supseteq [\mathcal{O}_4, \mathcal{O}_4]$. We thus get a short exact sequence of l.s. algebras

$$0 \to A_3 \xrightarrow{i} A_4 \xrightarrow{\pi} \mathbb{R} \to 0, \tag{11}$$

where $A_3 = A_4^2$ is viewed here as a three-dimensional l.s. algebra whose associated Lie algebra is $\mathcal{H}_3$.

Since the Lie algebra associated to A_4 is required to be $\mathcal{O}_4$, the short sequence (11) yields a short exact sequence of Lie algebras of the form

$$0 \to \mathcal{H}_3 \to \mathcal{O}_4 \to \mathbb{R} \to 0.$$

Let $\phi : \mathbb{R} \to Der(\mathcal{H}_3)$ be a derivation of $\mathcal{H}_3$, which is completely determined by its value at 1.

Lemma 7 *In a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ satisfying $[e_1, e_2] = e_3$, the derivation $D = \phi(1)$ takes the following simplified form*

$$D = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & -a_1 & 0 \\ a_3 & b_3 & 0 \end{pmatrix},$$

with $a_1^2 + a_2 b_1 \neq 0$.

Proof. Put $D = \phi(1)$ in the form

$$D = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

relative to a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ satisfying $[e_1, e_2] = e_3$. First of all, we deduce from the identity $De_3 = [De_1, e_2] + [e_1, De_2]$ that $c_1 = c_2 = 0$ and $c_3 = a_1 + b_2$. On the other hand, since a skew-symmetric bilinear form from $\mathbb{R}\times\mathbb{R}$ into $\mathcal{H}_3$ is obviously identically zero, we deduce that $Z_\phi^2(\mathbb{R}, \mathcal{H}_3) = 0$; that is, the extensions of $\mathbb{R}$ by $\mathcal{H}_3$ are precisely the semidirect products of $\mathbb{R}$ by $\mathcal{H}_3$ with respect to derivations of $\mathcal{H}_3$. In that case, the extended Lie bracket defined by (3) on $\mathbb{R}\times\mathcal{H}_3$ is given by

$$[(a,x), (b,y)] = ([a,b], [x,y] + \phi(a)\, y - \phi(b)\, x),$$

for all $x, y \in \mathcal{H}_3$ and $a, b \in \mathbb{R}$. By setting $\widetilde{e}_i = (0, e_i)$, $1 \le i \le 3$, and $\widetilde{e}_4 = (1, 0)$, we obtain

$$\begin{aligned} [\widetilde{e}_1, \widetilde{e}_2] &= \widetilde{e}_3, \\ [\widetilde{e}_4, \widetilde{e}_1] &= a_1\widetilde{e}_1 + a_2\widetilde{e}_2 + a_3\widetilde{e}_3, \\ [\widetilde{e}_4, \widetilde{e}_2] &= b_1\widetilde{e}_1 + b_2\widetilde{e}_2 + b_3\widetilde{e}_3, \\ [\widetilde{e}_4, \widetilde{e}_3] &= c_3\widetilde{e}_3. \end{aligned}$$

Since $\mathcal{O}_4$ is unimodular, we deduce that $a_1+b_2+c_3=0$; and taking into account that $c_3=a_1+b_2$ we deduce that $c_3=0$ and $b_2=-a_1$. It is now clear that, in the sub-basis $\{\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3\}$ which still satisfies $[\widetilde{e}_1,\widetilde{e}_2]=\widetilde{e}_3$, and hencefore in the initial basis $\{e_1,e_2,e_3\}$, the derivation D takes the desired simplified form. The condition $a_1^2+a_2b_1\neq 0$ follows now from the fact that $a_1\widetilde{e}_1+a_2\widetilde{e}_2$ and $b_1\widetilde{e}_1-a_1\widetilde{e}_2$ must be linearly independent, since $\dim[\mathcal{O}_4,\mathcal{O}_4]=3$. ∎

On the other hand, it is not difficult to prove the following proposition (compare [7], Theorem 3.5).

Proposition 8 *Up to l.s. isomorphism, the complete l.s. structures on the Heisenberg algebra $\mathcal{H}_3$ are classified as follows: There is a basis $\{e_1,e_2,e_3\}$ of $\mathcal{H}_3$ relative to which the l.s. product is given by one of the following classes:*

(i) $e_1\cdot e_1=pe_3,\ e_2\cdot e_2=qe_3,\ e_1\cdot e_2=\frac{1}{2}e_3,\ e_2\cdot e_1=\frac{1}{2}e_3,$ *where* $p,q\in\mathbb{R}$.

(ii) $e_1\cdot e_2=me_3,\ e_2\cdot e_1=(m-1)\,e_3,\ e_2\cdot e_2=e_1,$ *where* $m\in\mathbb{R}$.

Remark 9 *It is noticeable that the l.s. products on $\mathcal{H}_3$ belonging to class (i) in Proposition 8 are obtained by central extensions (in the sense fixed in Section 3) of $\mathbb{R}^2$ endowed with some complete l.s. structure by $\mathbb{R}$ endowed with the trivial l.s. product. However, the l.s. products on A_3 belonging to class (ii) are obtained by central extensions of the nonabelian two-dimensional Lie algebra $\mathcal{G}_2$ endowed with its unique complete l.s. structure by $\mathbb{R}$ endowed with the trivial l.s. structure.*

Now we return to the short sequence (11). Let $\sigma:\mathbb{R}\to A_4$ be a section, and set $\sigma(1)=x_0\in A_4$. We define two linear maps $\lambda,\ \rho\in End\,(A_3)$ by putting $\lambda(y)=x_0\cdot y$ and $\rho(y)=y\cdot x_0$. Now if we put $\mathbf{e}=x_0\cdot x_0$, we get

$$\pi(\mathbf{e})=\pi(x_0\cdot x_0)=\pi(\sigma(1)\cdot\sigma(1))=\pi(\sigma(1))\cdot\pi(\sigma(1))=1\cdot 1=0.$$

This means that $\mathbf{e}=x_0\cdot x_0\in A_3$. Let $g:\mathbb{R}\times\mathbb{R}\to A_3$ be the bilinear map defined by $g(a,b)=\sigma(a)\cdot\sigma(a)-\sigma(a\cdot b)$. From the completeness of the l.s. structure on $\mathbb{R}$, we see that $g(a,b)=ab\mathbf{e}$, or equivalently $g(1,1)=\mathbf{e}$, and it is obvious too (using the notation of Section 3) to verify that $\delta_2 g=0$, i.e. $g\in Z^2_{\lambda,\rho}(\mathbb{R},A_3)$.

On the other hand, the extended l.s. product on $A_3\oplus\mathbb{R}$ given in (7) turns out to take here the simplified form

$$(x,a)\cdot(y,b)=(x\cdot y+a\lambda(y)+b\rho(x)+ab\mathbf{e},0),$$

for all $x,y\in A_3$ and $a,b\in\mathbb{R}$.

The conditions in Theorem 3 also simplify to the following conditions:

(a) $\lambda(x\cdot y)=\lambda(x)\cdot y+x\cdot\lambda(y)-\rho(x)\cdot y,$

(b) $\rho([x,y]) = x \cdot \rho(y) - y \cdot \rho(x)$,

(c) $[\lambda, \rho] + \rho^2 = R_{\mathbf{e}}$.

Now observe that $(x,a) \in T(A_4)$ if and only if $(x,a) \cdot (y,b) = (0,0)$ for all $(y,b) \in A_3 \oplus \mathbb{R}$, or equivalently, $x \cdot y + a\lambda(y) + b\rho(x) + ab\mathbf{e} = 0$ for all $(y,b) \in A_3 \oplus \mathbb{R}$. Since y and b are arbitrary, we conclude that this is also equivalent to say that $(L_x)_{|A_3} = a\lambda$ and $\rho(x) = a\mathbf{e}$. In particular, an element $x \in A_3$ belongs to $T(A_4)$ if and only if $(L_x)_{|A_3} = 0$ and $\rho(x) = 0$, or equivalently,

$$A_3 \cap T(A_4) = T(A_3) \cap \ker \rho \tag{12}$$

The following two lemmas will be crucial for the classification of l.s. structures on $\mathcal{O}_4$.

Lemma 10 *In the short sequence (11), if the l.s. algebra A_3 belongs to class (i) of Proposition 8, then the two-sided ideal $C(A_4)$ is nontrivial.*

Proof. Assume that A_3 belongs to class (i) of Proposition 8. Applying the formula in condition (b) above to e_3, we find that, in a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ satisfying $[e_1, e_2] = e_3$, the operator ρ has the form

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix},$$

with $\gamma_3 = p\beta_1 - q\alpha_2 + \frac{1}{2}(\alpha_1 + \beta_2)$. Since $D = \lambda - \rho$, we apply Lemma 7 to deduce that, relative to the same basis above, we have

$$\lambda = \begin{pmatrix} \alpha_1 + a_1 & \beta_1 + b_1 & 0 \\ \alpha_2 + a_2 & \beta_2 - a_1 & 0 \\ \alpha_3 + a_3 & \beta_3 + b_3 & \gamma_3 \end{pmatrix}.$$

Since $(L_{e_3})_{|A_3} = 0$ and $\mathbf{e} \in A_3$, then condition (c) above (when applied to e_3) gives

$$\gamma_3^2 e_3 = e_3 \cdot \mathbf{e} = 0,$$

which in turn obviously implies that $\gamma_3 = 0$, i.e. $\rho(e_3) = 0$. Hence, we get from (12) that $e_3 \in T(A_4)$. Since $Z(\mathcal{O}_4) = \mathbb{R}e_3$, we deduce that $C(A_4) = T(A_4) \cap Z(\mathcal{O}_4) \neq 0$, as required. ■

Lemma 11 *In the short sequence (11), the l.s. algebra A_3 could not belong to class (ii) of Proposition 8.*

Proof. Assume to the contrary that A_3 belongs to class (ii) of Proposition 8, i.e. there is a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ relative to which the l.s. product is given by : $e_1 \cdot e_2 = me_3$, $e_2 \cdot e_1 = (m-1)\, e_3$, $e_2 \cdot e_2 = e_1$, where m is a real number. Relative to that basis, put

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}.$$

Applying the formula in condition (b) above to e_3, we get

$$\gamma_1 = -\alpha_2,\ \gamma_2 = 0,\ \gamma_3 = m\beta_2 - (m-1)\,\alpha_1.$$

The same formula, when applied to e_1 or e_2, yields

$$(m-1)\,\gamma_1 = 0.$$

It follows that

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & -\alpha_2 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & m\beta_2 - (m-1)\,\alpha_1 \end{pmatrix},$$

with $(m-1)\,\alpha_2 = 0$. Since $\lambda = D + \rho$, Lemma 7 tells us that, in the same basis, we have

$$\lambda = \begin{pmatrix} \alpha_1 + a_1 & \beta_1 + b_1 & -\alpha_2 \\ \alpha_2 + a_2 & \beta_2 - a_1 & 0 \\ \alpha_3 + a_3 & \beta_3 + b_3 & m\beta_2 - (m-1)\,\alpha_1 \end{pmatrix},$$

with $(m-1)\,\alpha_2 = 0$ and $a_1^2 + a_2 b_1 \neq 0$.

Now, by applying the formula in condition (c) above to e_3, we have

$$\begin{aligned} \alpha_2\,(\alpha_2 + a_2) &= 0,\ \alpha_2\,(a_1 + m\beta_2 - (m-2)\,\alpha_1) = 0, \\ \alpha_2\,(\alpha_3 + a_3) &= (m\beta_2 - (m-1)\,\alpha_1)^2\,. \end{aligned} \tag{13}$$

Moreover, by applying the formula in condition (a) above to all products of the form $e_i \cdot e_j$, we get the following extra conditions

$$a_1 = 0,\ a_2 = -m\alpha_2,\ \beta_2 = \alpha_1,\ a_3 + \alpha_3 = mb_1 + (m-1)\,(b_1 + \beta_1)\,. \tag{14}$$

It is now easy to verify that (13) and (14) are not compatible (more precisely, we get a contradiction with the condition on D that $a_1^2 + a_2 b_1 \neq 0$). This terminates the proof of the lemma. ■

5 Classification

By Lemmas 10 and 11, any complete l.s. algebra A_4 whose associated Lie algebra is $\mathcal{O}_4$ contains a central translation, i.e. $C\,(A_4) \neq 0$. Since $\dim Z\,(\mathcal{O}_4) = 1$, we deduce that $\dim C\,(A_4) = 1$, i.e. $C\,(A_4)$ is isomorphic to the field $\mathbb{R}$ with the trivial l.s. product. It follows that A_4 may be obtained as an extension of a complete l.s. 3-dimensional algebra A_3 by the trivial l.s. algebra $\mathbb{R} \cong C(A_4)$. We therefore get a short exact sequence of l.s. algebras

$$0 \to \mathbb{R} \to A_4 \to A_3 \to 0, \tag{15}$$

which in turn yields a short exact sequence of Lie algebras of the form

$$0 \to \mathbb{R} \to \widetilde{\mathcal{G}} \to \mathcal{G}_3 \to 0, \tag{16}$$

where $\mathcal{G}_3$ denotes the associated Lie algebra to A_3 and $\widetilde{\mathcal{G}}$ is a (non specific) Lie algebra which extends $\mathcal{G}_3$ by $\mathbb{R}$.

Being solvable and unimodular, the three-dimensional Lie algebra $\mathcal{G}_3$ is necessarily isomorphic to one of the following Lie algebras

1. The abelian Lie algebra $\mathbb{R}^3$,
2. The Heisenberg algebra $\mathcal{H}_3$, i.e. the two-step nilpotent Lie algebra having a basis $\{e_1, e_2, e_3\}$ which satisfies $[e_1, e_2] = e_3$,
3. The Lie algebra $\mathcal{E}(2)$ of the group of Euclidean motions of the plane, i.e. the solvable Lie algebra having a basis $\{e_1, e_2, e_3\}$ which satisfies $[e_1, e_2] = e_3$ and $[e_1, e_3] = -e_2$,
4. The Lie algebra $\mathcal{E}(1,1)$ of the group of Lorentzian motions of the Minkowski plane, , i.e. the solvable Lie algebra having a basis $\{e_1, e_2, e_3\}$ which satisfies $[e_1, e_2] = e_3$ and $[e_1, e_3] = e_2$.

Proposition 12 *The Lie algebra $\mathcal{G}_3$ associated to A_3 is isomorphic to $\mathcal{E}(2)$.*

Proof. According to the discussion above, there are only four possibilities for A_3. As we will see below, a unimodular Lie algebra extension of a Lie algebra $\mathcal{G}$ by $\mathbb{R}$ is necessarily central. It follows that if, in sequence (16), we take $\widetilde{\mathcal{G}} = \mathcal{O}_4$ which is solvable but not nilpotent, then $\mathcal{G}_3$ could not be $\mathbb{R}^3$ or $\mathcal{H}_3$.

Now we wish to show that $\mathcal{G}_3$ could not be $\mathcal{E}(1,1)$ too. For, assume to the contrary that $\mathcal{G}_3$ is isomorphic to $\mathcal{E}(1,1)$. This means that $\mathcal{O}_4$ may be obtained as a central extension of $\mathcal{E}(1,1)$ by $\mathbb{R}$. As above, we consider a short exact sequence of the form

$$0 \to \mathbb{R} \to \widetilde{\mathcal{G}} \to \mathcal{E}(1,1) \to 0,$$

where $\widetilde{\mathcal{G}}$ is just a Lie algebra extension of $\mathcal{E}(1,1)$ by $\mathbb{R}$. In what follows, we shall make use of the notation of Section 2.

Let $\{e_1, e_2, e_3\}$ be a basis for $\mathcal{E}(1,1)$ satisfying $[e_1, e_2] = e_3$ and $[e_1, e_3] = e_2$. On $\mathbb{R}\times\mathcal{E}(1,1)$, the extended Lie bracket defined by (3) is now given by

$$[(a,x),(b,y)] = (\phi(x)\, b - \phi(y)\, a + \omega(x,y), [x,y]),$$

given that $\mathbb{R}$ is abelian, where $\phi : \mathcal{E}(1,1) \to End(\mathbb{R}) \cong \mathbb{R}$ and $\omega \in Z^2(\mathcal{E}(1,1), \mathbb{R})$. By setting $\widetilde{e}_i = (0, e_i)$, $1 \leq i \leq 3$, and $\widetilde{e}_4 = (1,0)$, we get

$$\begin{aligned}
[\widetilde{e}_1, \widetilde{e}_2] &= \widetilde{e}_3 + \omega(e_1, e_2)\, \widetilde{e}_4, \\
[\widetilde{e}_1, \widetilde{e}_3] &= \widetilde{e}_2 + \omega(e_1, e_3)\, \widetilde{e}_4, \\
[\widetilde{e}_2, \widetilde{e}_3] &= \omega(e_2, e_3)\, \widetilde{e}_4, \\
[\widetilde{e}_i, \widetilde{e}_4] &= a_i \widetilde{e}_4, \quad 1 \leq i \leq 3,
\end{aligned}$$

where $\phi(e_i) = a_i$, $1 \leq i \leq 3$.

Since $\widetilde{\mathcal{G}}$ is required to be unimodular (keep in mind that we are targeting $\mathcal{O}_4$), then $a_1 = a_2 = a_3 = 0$. This means that ϕ is identically zero, i.e. $\widetilde{\mathcal{G}}$ is a central extension of $\mathcal{E}(1,1)$ by $\mathbb{R}$. Actually, this is a general fact in the sense that any unimodular Lie algebra extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by $\mathbb{R}$ is necessarily central. Putting $\widetilde{e}_1' = \widetilde{e}_1$, $\widetilde{e}_2' = \widetilde{e}_2 + \omega(e_1, e_3)\widetilde{e}_4$, $\widetilde{e}_3' = \widetilde{e}_3 + \omega(e_1, e_2)\widetilde{e}_4$, and $\omega_{23} = \omega(e_2, e_3)$, we see that the new basis $\{\widetilde{e}_1', \widetilde{e}_2', \widetilde{e}_3', \widetilde{e}_4'\}$ satisfies

$$[\widetilde{e}_1', \widetilde{e}_2'] = \widetilde{e}_3', \;\; [\widetilde{e}_1', \widetilde{e}_3'] = \widetilde{e}_2', \;\; [\widetilde{e}_2', \widetilde{e}_3'] = \omega_{23}\widetilde{e}_4', \tag{17}$$

and all the other brackets are zero. It is now clear that, independently of the value of $\omega(e_2, e_3)$, these commutators do not yield $\mathcal{O}_4$. Hence, we have established that $\mathcal{G}_3$ is not isomorphic to $\mathcal{E}(1,1)$.

As we have seen up till now, the cases of $\mathbb{R}^3$, $\mathcal{H}_3$, and $\mathcal{E}(1,1)$ cannot occur. Next, we wish to show that $\mathcal{O}_4$ may be obtained as a central extension of $\mathcal{E}(2)$ by $\mathbb{R}$. For, we consider in a similar fashion as above a short exact sequence of the form

$$0 \to \mathbb{R} \to \widetilde{\mathcal{G}} \to \mathcal{E}(2) \to 0,$$

where $\widetilde{\mathcal{G}}$ is just a Lie algebra extension of $\mathcal{E}(2)$ by $\mathbb{R}$.

Let $\{e_1, e_2, e_3\}$ be a basis for $\mathcal{E}(2)$ satisfying $[e_1, e_2] = e_3$ and $[e_1, e_3] = -e_2$. Just as we could conclude above that there is a new basis$\{\widetilde{e}_1', \widetilde{e}_2', \widetilde{e}_3'\}$ for $\widetilde{\mathcal{G}}$ which satisfies (17), we obtain in the situation of $\mathcal{E}(2)$ a basis $\{\widetilde{e}_1', \widetilde{e}_2', \widetilde{e}_3', \widetilde{e}_4'\}$ satisfying

$$\left[\widetilde{e}', \widetilde{e}_2'\right] = \widetilde{e}_3', \;\; [\widetilde{e}_1', \widetilde{e}_3'] = -\widetilde{e}_2', \;\; [\widetilde{e}_2', \widetilde{e}_3'] = \omega_{23}\widetilde{e}_4',$$

where $\omega_{23} = \omega(e_2, e_3)$, and all the other brackets are zero.

Now it is clear that if $\omega_{23} \neq 0$, then the Lie algebra spanned by $\widetilde{e}_1', \widetilde{e}_2', \widetilde{e}_3', \widetilde{e}_4'$ is isomorphic to $\mathcal{O}_4$. This means that $\mathcal{O}_4$ may be obtained as a central extension of $\mathcal{E}(2)$ by $\mathbb{R}$ corresponding to a cocycle $\omega \in Z^2(\mathcal{E}(2), \mathbb{R})$ such that $\omega(e_2, e_3) \neq 0$ with respect to a basis $\{e_1, e_2, e_3\}$ for $\mathcal{E}(2)$ satisfying $[e_1, e_2] = e_3$ and $[e_1, e_3] = -e_2$. ■

Remark 13 *It is remarkable that one can easily obtain $\mathcal{E}(1,1)$ by a central extension of the (unique) non-abelian two-dimensional Lie algebra $\mathcal{G}_2$ by $\mathbb{R}$. In contrast, $\mathcal{E}(2)$ cannot be obtained by central nor noncentral extensions of a two-dimensional Lie algebra by $\mathbb{R}$. It can however be obtained as an extension of $\mathbb{R}$ by $\mathbb{R}^2$. For, we consider a short exact sequence of the form*

$$0 \to \mathbb{R}^2 \to \mathcal{G} \to \mathbb{R} \to 0. \tag{18}$$

This is necessarily a semidirect extension, i.e. $\mathcal{G}$ is the semidirect product of $\mathbb{R}$ and $\mathbb{R}^2$ with respect to ϕ. To see this, recall from Section 2 that, since $\mathbb{R}^2$ is abelian then, for a given $\phi : \mathbb{R} \to End(\mathbb{R}^2)$, the equivalent classes of extensions of $\mathbb{R}$ by $\mathbb{R}^2$ are in one-to-one correspondence with the elements of the second cohomology group $H^2_\phi(\mathbb{R}, \mathbb{R}^2)$. Now, since a skew-symmetric bilinear form from

$\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}^2$ is obviously identically zero, we deduce that $Z^2_\phi\left(\mathbb{R},\mathbb{R}^2\right)=0$. In other words, the extensions of $\mathbb{R}$ by $\mathbb{R}^2$ are precisely the semidirect products of $\mathbb{R}$ by $\mathbb{R}^2$ with respect to representations of $\mathbb{R}$ in $\mathbb{R}^2$. Let $\phi:\mathbb{R}\to End\left(\mathbb{R}^2\right)$ be such a representation, and set

$$\phi(1)=\left(\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right).$$

On $\mathcal{G}=\mathbb{R}^2\rtimes_\phi\mathbb{R}$, the extended Lie bracket defined by (3) is now given by

$$[(x,a),(y,b)]=(\phi(a)\,y-\phi(b)\,x,0).$$

If e_1, e_2 is a basis for $\mathbb{R}^2$, then by putting $\widetilde{e}_i=(e_i,0)$, $1\leq i\leq 2$, and $\widetilde{e}_3=(0,1)$, we obtain

$$\begin{aligned}[\widetilde{e}_1,\widetilde{e}_2] &= 0,\\ [\widetilde{e}_3,\widetilde{e}_1] &= \alpha\widetilde{e}_1+\gamma\widetilde{e}_2,\\ [\widetilde{e}_3,\widetilde{e}_2] &= \beta\widetilde{e}_1+\delta\widetilde{e}_2,\end{aligned}$$

from which we deduce that $\mathcal{G}$ is unimodular if and only if $tr\left(ad_{\widetilde{e}_3}\right)=tr\left(\phi(1)\right)=\alpha+\delta=0$. If $\det(\phi(1))=\alpha\delta-\beta\gamma=0$, it is easy to see that $\mathcal{G}\cong\mathcal{H}_3$. If $\det(\phi(1))=\alpha\delta-\beta\gamma\neq0$, we can easily conclude that $\mathcal{G}\cong\mathcal{E}(2)$ when $\det(\phi(1))>0$, and $\mathcal{G}\cong\mathcal{E}(1,1)$ when $\det(\phi(1))<0$. We end this remark by noticing that if in the short sequence (18) we replace $\mathbb{R}^2$ with the non-abelian two-dimensional Lie algebra $\mathcal{G}_2$, then the resulting Lie algebra $\mathcal{G}$ is not unimodular.

We return to the short sequences (15) and (16). By Proposition 12, the Lie algebra associated to A_3 is isomorphic to $\mathcal{E}(2)$. We therefore conclude that all the complete l.s. structures on the oscillator algebra $\mathcal{O}_4$ may be obtained through extensions of the complete l.s. structures on $\mathcal{E}(2)$ by the trivial l.s. structure on the field $\mathbb{R}$. Hence, we must first determine all the complete l.s. structures on $\mathcal{E}(2)$. For, we will make use of the following lemma which we state without proof (compare [7], Theorem 4.1).

Lemma 14 *Up to l.s. isomorphism, there is a unique complete l.s. structure on $\mathcal{E}(2)$, which is given as follows: there is a basis $\{e_1,e_2,e_3\}$ of $\mathcal{E}(2)$ relative to which the nontrivial l.s. products are: $e_1\cdot e_2=e_3$, $e_1\cdot e_3=-e_2$.*

Before we continue, we fix the following notation. From now on, A_3 will denote the vector space $\mathcal{E}(2)$ endowed with the complete l.s. product given in Lemma 14. On the one hand, observe that we have the central extension of Lie algebras given by (16) for which the extended Lie bracket on $\mathcal{E}(2)\oplus\mathbb{R}$ is given by

$$[(x,a),(y,b)]=([x,y],\omega(x,y)), \tag{19}$$

with $\omega\in Z^2(\mathcal{E}(2),\mathbb{R})$. On the other hand, we have the extension of l.s. algebras given by (15) for which the extended l.s. product on $A_3\oplus\mathbb{R}$ is given by

$$(x,a)\cdot(y,b)=\left(x\cdot y,a\cdot b+b\lambda_x+a\rho_y+g(x,y)\right), \tag{20}$$

with λ, $\rho : A_3 \to End(\mathbb{R}) \cong \mathbb{R}$ and $g \in Z^2_{\lambda,\rho}(A_3, \mathbb{R})$. Note here that we have identified the value of λ (resp. ρ) at an element $x \in A_3$ with the corresponding real number λ_x (resp. ρ_x) via the isomorphism $End(\mathbb{R}) \cong \mathbb{R}$.

As we have noted in Section 3, $\mathbb{R} \cong C(A_4)$ is an A_3-bimodule, or equivalently, the conditions in Theorem 3 simplify to the following conditions:

(i) $[\lambda_x, \lambda_y] = \lambda_{[x,y]}$, that is, λ is a representation of Lie algebras,

(ii) $[\lambda_x, \rho_y] + \rho_y \circ \rho_x - \rho_{x \cdot y} = 0$.

By using (19) and (20), we deduce from

$$[(x, a), (y, b)] = (x, a) \cdot (y, b) - (y, b) \cdot (x, a),$$

that

$$\omega(x, y) = g(x, y) - g(y, x) \quad \text{and } \lambda = \rho. \tag{21}$$

Furthermore, the fact that (20) defines a l.s. product yields

$$\lambda_{x \cdot y} = \lambda_x \circ \lambda_y.$$

Applying the last identity to $e_i \cdot e_i$, $1 \leq i \leq 3$, we deduce that $\lambda = 0$. In other words, in the sense of Section 3, the extension A_4 is central.

By Proposition 5, the classification of the exact central extensions of A_3 by $\mathbb{R}$ is, up to l.s. isomorphism, the orbit space of $H^2_{ex}(A_3, \mathbb{R})$ under the natural action of $G = Aut(\mathbb{R}) \times Aut(A_3)$. Accordingly, we must compute $H^2_{ex}(A_3, \mathbb{R})$. Since $\mathbb{R}$ is a trivial A_3-bimodule, we see first from formulae (8) and (8) in Section 3 that the coboundary operator δ simplifies as follows:

$$\begin{aligned}
\delta_1 h(x, y) &= -h(x \cdot y), \\
\delta_2 g(x, y) &= g(x, y \cdot z) - g(y, x \cdot z) - g([x, y], z),
\end{aligned}$$

where $h \in L^1(A_3, \mathbb{R})$ and $g \in L^2(A_3, \mathbb{R})$. By Lemma 14, there is a basis e_1, e_2, e_3 of $\mathcal{E}(2)$ for which the only nonzero products in A_3 are $e_1 \cdot e_2 = e_3$ and $e_1 \cdot e_3 = -e_2$. Using the first formula above for δ_1, we get

$$\delta_1 h = \begin{pmatrix} 0 & h_{12} & h_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $h_{12} = -h(e_3)$ and $h_{13} = h(e_2)$. Similarly, using the second formula above for δ_2, we verify easily that if g is a cocycle (i.e. $\delta_2 g = 0$), then

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & 0 & g_{23} \\ 0 & -g_{23} & 0 \end{pmatrix},$$

where $g_{ij} = g(e_i, e_j)$. We deduce that, in the basis above, the class $[g] \in H^2(A_3, \mathbb{R})$ of a cocycle g may be represented by a matrix of the simplified form

$$g=\left(\begin{array}{ccc}\alpha & 0 & 0\\ 0 & 0 & \beta\\ 0 & -\beta & 0\end{array}\right).$$

In fact, we have $\beta\neq 0$. Indeed, if we return to the proof of Proposition 12, we see that, in the same basis above, the cocycle ω has the form

$$\omega=\left(\begin{array}{ccc}0 & \omega_{12} & \omega_{13}\\ -\omega_{12} & 0 & \omega_{23}\\ 0 & -\omega_{23} & 0\end{array}\right),\ \ \omega_{23}\neq 0,$$

and we therefore deduce, using the first identity in (21), that $2\beta=\omega_{23}\neq 0$.

We can now determine the extended l.s. structure on A_4. By setting $\widetilde{e}_i=(0,e_i)$, $1\leq i\leq 3$, and $\widetilde{e}_4=(1,0)$, and using formula (20), we find

$$\begin{array}{rcl}\widetilde{e}_1\cdot\widetilde{e}_1 & = & \alpha\widetilde{e}_4,\ \ \widetilde{e}_1\cdot\widetilde{e}_2=\widetilde{e}_3,\ \ \widetilde{e}_1\cdot\widetilde{e}_3=-\widetilde{e}_2,\\ \widetilde{e}_2\cdot\widetilde{e}_3 & = & \beta\widetilde{e}_4,\ \ \widetilde{e}_3\cdot\widetilde{e}_2=-\beta\widetilde{e}_4,\end{array}$$

with $\beta\neq 0$, and all the other products are zero. Now, it is clear that by setting $t=\frac{\alpha}{2\beta}$, $e_i=\widetilde{e}_i$ for $1\leq i\leq 3$, and $e_4=2\beta\widetilde{e}_4$, we obtain a one-parameter family (with parameter t) of l.s. structures on $\mathcal{O}(4)$. We have thus established the following result.

Theorem 15 *The complete l.s. structures on $\mathcal{O}(4)$ are classified as follows: There is a basis $\{e_1,e_2,e_3,e_4\}$ of $\mathcal{O}(4)$ relative to which, the nontrivial l.s. products are:*

$$\begin{array}{rcl}e_1\cdot e_1 & = & te_4,\ \ e_1\cdot e_2=e_3,\ \ e_1\cdot e_3=-e_2,\\ e_2\cdot e_3 & = & e_4,\ \ e_3\cdot e_2=e_4,\end{array}$$

where $t\in\mathbb{R}$. We denote $\mathcal{O}(4)$ endowed with one of these structures by $A_{4,t}$.

Our goal is classify the complete l.s. structures on $\mathcal{O}(4)$, up to l.s. isomorphisms. For this purpose, we recall first from Subsection 3.1 that the extension given by the short sequence (15) is exact, i.e. $i(\mathbb{R})=C(A_4)$, if and only if $I_{[g]}=0$, where

$$I_{[g]}=\{x\in A_3 : x\cdot y=y\cdot x=0 \text{ and } g(x,y)=g(y,x)=0, \text{ for all } y\in A_3\}.$$

Claim 16 *The extension $0\to\mathbb{R}\to A_4\to A_3\to 0$ is exact.*

Proof. To show that $I_{[g]}=0$, we take an arbitrary $x\in I_{[g]}$. By putting $x=ae_1+be_2+ce_3\in I_{[g]}$ and computing all the products $x\cdot e_i=e_i\cdot x=0$, we deduce that $x=0$. ■

Let now $A_{4,t}$ and $A_{4,t'}$ be two l.s. algebras as in Theorem 15. We know from Subsection 3.1 that $A_{4,t}$ is isomorphic to $A_{4,t'}$ if and only if the exists $(\alpha,\eta)\in Aut(\mathbb{R})\times Aut(A_3)$ such that for all $x,y\in A_3$, we have

$$g'(x,y)=\alpha(g(\eta(x),\eta(y))). \tag{22}$$

We have $Aut(\mathbb{R}) = \mathbb{R}^*$, and it is easy too to determine $Aut(A_3)$. Indeed, recall that the structure of A_3 is given by $e_1 \cdot e_2 = e_3$, $\quad e_1 \cdot e_3 = -e_2$, and let $\eta \in Aut(A_3)$ be given, in the basis $\{e_1, e_2, e_3\}$, by

$$\eta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

From the identity $\eta(e_3) = \eta(e_1 \cdot e_2) = \eta(e_1) \cdot \eta(e_2)$, we deduce that $c_1 = 0$, $c_2 = -a_1 b_3$, and $c_3 = a_1 b_2$. Similarly, we deduce from the identity $-\eta(e_2) = \eta(e_1 \cdot e_3) = \eta(e_1) \cdot \eta(e_3)$ that $b_1 = 0$, $b_2 = a_1 c_3$, and $c_3 = -a_1 c_2$. Since $\det \eta \neq 0$, we get $a_1 = \pm 1$, which in turn implies that $b_2 = \pm c_3$ and $b_3 = \pm c_2$. This means that we get finally

$$\eta = \begin{pmatrix} \varepsilon & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & -\varepsilon c_2 & \varepsilon b_2 \end{pmatrix},$$

with $\varepsilon = \pm 1$ and $b_2^2 + c_2^2 \neq 0$.

We shall now apply formula (22). For this we recall that, in the basis above, the class g corresponding to $A_{4,t}$ has the form

$$g = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

From $g'(e_1, e_1) = \alpha g(\eta(e_1), \eta(e_1))$, we get

$$t' = \alpha t.$$

From $g'(e_1, e_2) = \alpha g(\eta(e_1), \eta(e_2))$ we get

$$\varepsilon a_2 c_2 + a_3 b_2 = 0.$$

Similarly, from $g'(e_1, e_3) = \alpha g(\eta(e_1), \eta(e_3))$ we get

$$\varepsilon a_2 b_2 - a_3 c_2 = 0.$$

The two last identities yield $a_2 = a_3 = 0$, and from $g'(e_2, e_3) = \alpha g(\eta(e_2), \eta(e_3))$ we get

$$\alpha = \frac{\varepsilon}{b_2^2 + c_2^2}.$$

Hence, $A_{4,t}$ and $A_{4,t'}$ are isomorphic if and only if $t' = \alpha t$ for some real number $\alpha \neq 0$. Summarizing, we have established the following theorem which is our main result.

Theorem 17 *There exist, up to l.s. isomorphism, exactly two complete l.s. structures on $\mathcal{O}(4)$. These are described below (by their nontrivial products) in a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathcal{O}(4)$:*

(i) $A_{4,0}: e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_3 = -e_2,\, e_2 \cdot e_3 = e_4, \quad e_3 \cdot e_2 = e_4,$

(ii) $A_{4,1}: e_1 \cdot e_1 = e_4, \quad e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_3 = -e_2,\, e_2 \cdot e_3 = e_4, \quad e_3 \cdot e_2 = e_4.$

We note that the mapping $X \mapsto (L_X, X)$ is a Lie algebra representation of $\mathcal{O}_4$ in $\mathfrak{aff}\left(\mathbb{R}^4\right) = End\left(\mathbb{R}^4\right) \oplus \mathbb{R}^4$. By using the (Lie group) exponential maps, Theorem 17 can now be stated, in terms of simply transitive actions of subgroups of the affine group $Aff\left(\mathbb{R}^4\right) = GL\left(\mathbb{R}^4\right) \ltimes \mathbb{R}^4$, as follows.

Theorem 18 *Suppose that the oscillator group O_4 acts simply transitively by affine transformations on $\mathbb{R}^4$. Then, as a subgroup of $Aff\left(\mathbb{R}^4\right)$, O_4 is conjugate to the following subgroup*

$$G_{4,\varepsilon} = \left\{ \begin{array}{c} \begin{pmatrix} 1+\varepsilon\left(e^x-1\right) & 0 & 0 & 0 \\ 0 & \cos x & -\sin x & 0 \\ 0 & \sin x & \cos x & 0 \\ 0 & y\sin x + z\cos x & y\cos x - z\sin x & 0 \end{pmatrix} \\ \times \begin{bmatrix} x+\varepsilon\left(e^x - x - 1\right) \\ y\cos x - z\sin x \\ y\sin x + z\cos x \\ w + 2yz\cos x + \left(y^2 - z^2\right)\sin x \end{bmatrix}, \ x,y,z,w \in \mathbb{R} \end{array} \right\},$$

with $\varepsilon = 0$ or 1 (i.e. there are only two distinct conjugacy classes according to whether $\varepsilon = 0$ or 1).

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