New Results in the Theory of the Classical Riemann Zeta-Function

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Abstract. The results which are presented here can be divided into two groups. The results relating to the first group are associated with the construction of an operator acting in a Hilbert space such that the Riemann hypothesis is equivalent to the problem of the existence of an eigenvector with eigenvalue -1 for this operator and is connected with properties of the corresponding dynamical system . The results relating to the second group are associated with the behavior of the Riemann ξ-function and its derivatives at the point $s = 1/2$. It is proved that if at least one even derivative of the function $\xi(s)$ at the point $s = 1/2$ were not positive, the Riemann hypothesis on the zeros of the classical zeta-function $\zeta(s)$ would be false. However, it was also proved that all the even derivatives at the point $s = 1/2$ are strictly positive and the asymptotics for the values of the even derivatives at the same point as the order of the derivative tends to infinity was found. These results permit to show that the Riemann hypothesis does not hold for an arbitrary sharp approximation of $\zeta(s)$ satisfying the same functional equation as $\zeta(s)$.

1. Introduction

Recently we obtained some new results in the theory of the classical Riemann zeta-funtion $\zeta(s)$ associated with the Riemann hypothesis ([**1**]-[**6**]). The results can be divided into two groups. The results relating to the first group are associated with the construction of an operator acting in a Hilbert space such that the Riemann hypothesis is equivalent to the problem of the existence of an eigenvector with eigenvalue -1 for this operator and is connected with properties of the corresponding dynamical system ([**1**], [**2**]). The results relating to the second group ([**3**]-[**6**]) are associated with the behavior of the Riemann ξ-function and its derivatives at the point $s = 1/2$. It is proved that if at least one even derivative of the function $\xi(s)$ at the point $s = 1/2$ were not positive, the Riemann hypothesis on the zeros of the classical zeta-function $\zeta(s)$ would be false. However, it was also proved that all the even derivatives at the point $s = 1/2$ are strictly positive and the asymptotics for the values of the even derivatives at the same point as the order of the derivative tends to infinity was found. These results permit to show that the Riemann hypothesis

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does not hold for an arbitrary sharp approximation of $\zeta(s)$ satisfying the same functional equation as the function $\zeta(s)$ ([**6**]). Here we describe the results in more details in the following five sections. A part of results were generalized to the case of Dirichlet L-Functions ([**7**]–[**9**]).

2. The connection of the Riemann hypothesis on the zeros of the function $\zeta(s)$ with the spectrum of an operator acting in a Hilbert space

The idea that the Riemann hypothesis on the zeros of the Riemann zeta-funtion $\zeta(s)$ ist connected with the spectrum of a certain operator goes back to Hilbert. In this section we state that the Riemann hypothesis is equivalent to the problem of the existence of an eigenvector with eigenvalue $\lambda = -1$ for a certain operator acting in a Hilbert space and given by means of an infinite Jacobi matrix. This operator is represented as the sum of a self-adjoint operator and an operator defined with the help of a bidiagonal matrix with elements decreasing according to a certain law.

We consider the Hilbert space l whose elements are the one-sided sequence $x = (x_1, x_2, ...)$ of complex numbers satisfying the condition $\sum_{i=1}^{\infty} |x_i|^2 < \infty$, with the inner product $(x, y) = \sum_{i=1}^{\infty} x_i \bar{y}_i$, where $y = (y_1, y_2, ...)$.

We introduce an operator $A = A(s)$ that depends on a complex number $s \neq 1$, acts in l, and is given by means of the infinite matrix $A = (a_{kj})$ $(k, j = 1, 2, ...)$ with elements

$$a_{kj} = \begin{cases} 1 & \text{if} \quad k - j = -1, \\ p_k & \text{if} \quad 0 \le k - j \le 1, \\ 0 & \text{if} \quad |k - j| > 1, \end{cases}$$

where $p_k = h_k / h_{k-1}$,

$$h_k = h_k(s) = \begin{cases} \frac{1}{k^s} - \frac{k^{1-s} - (k-1)^{1-s}}{1-s} & \text{if} \quad k \ge 2, \\ (s-1)^{-1} & \text{if} \quad k = 1, \\ 1 & \text{if} \quad k = 0. \end{cases}$$

The operator A carries $x = (x_1, x_2, ...)$ into $x' = (x'_1, x'_2, ...)$, where $x'_k = \sum_{j=1}^{\infty} a_{kj} x_j$ $(k = 1, 2, \ldots)$.

REMARK 2.1. As $k \to \infty$

$$p_k = 1 - \frac{s+1}{k} - O(\frac{1}{k^2}), \; h_k = \frac{1}{(k+1)^s} - \frac{1}{k^s} + \frac{s}{2k^{1+s}} + O(\frac{1}{k^{2+s}}),$$

where $\left|O(\frac{1}{k^2})\right| < \frac{C}{k^2}, \left|O(\frac{1}{k^{2+s}})\right| < \frac{C}{k^{2+s}}$, and C does not depend on k. If $0 < \Re[s] < 1$, then $h_k(s) \neq 0$.

COROLLARY 2.2. *The operator A can be represented as the sum of a self-adjoint operator having a Toeplitz matrix with non-zero elements equal to* 1 *and an operator with bidiagonal matrix $\tilde{A} = (\tilde{a}_{kj})$ such that $\tilde{a}_{kj} = -(s+1)/k + O(1/k^2)$ for $0 \le k - j \le 1$.*

THEOREM 2.3. *The function $\zeta(s)$ has a zero in the domain $0 < \Re[s] \neq 1/2$ if and only if the domain $0 < \Re[s] < 1$ contains an s such that the operator $A(s)$ actings in l has an eigenvector with eigenvalue $\lambda = -1$.*

For the proof see [**1**].

3. A representation of the Riemann zeta-function in the critical strip by means of an infinite product of matrices of order two, and a certain dynamical system

It is well known that the Riemann zeta-function $\zeta(s)$ can be represented as an Euler product only in the half-plane $\Re[s] > 1$. In this section we shall show that, in the critical strip $0 < \Re[s] < 1$, $\zeta(s)$ can be represented as an infinite product of concrete matrices of order two. This representation was first described in [**1**], although the existence part of the proof, which is stated in a lemma below, was published later in [**2**]. This lemma is not trivial: it fails, for example, when $s = 0$ for any positive integer $k \geq 2$, and the proof makes essential use of the condition $0 < \Re[s] < 1$. Theorem 3.3 and the main results of [**1**] enable us to construct a dynamical system which turns out to be related to the Riemann hypothesis in the following way: for each complex zero of $\zeta(s)$ not lying on the line $\Re[s] = 1/2$, there is a periodic trajectory of order two having a special form.

Lemma 3.1. *For an integer $k \geq 2$ consider the function*

$$h_k(s) = 1/k^s - (k^{1-s} - (k-1)^{1-s})/(1-s). \tag{1}$$

Then $h_k(s) \neq 0$ in the domain $0 < \Re[s] < 1;\ k = 2, 3, \ldots$.

For the proof see [**2**].

This lemma enables us to make the following definition.

Definition 3.2. For $k = 1, 2, \ldots$ and $0 < \Re[s] < 1$, consider the function $p_k(s) = h_k(s)/h_{k-1}(s)$, where $h_1(s) = (s-1)^{-1},\ h_0(s) = 1$, and for $k \geq 2$ the function $h_k(s)$ is given by (1), and define the matrix

$$Q_k = Q_k(s) = \begin{pmatrix} 0 & 1 \\ -p_k(s) & -1 - p_k(s) \end{pmatrix},$$

which depends on s.

Theorem 3.3. *The infinite products $Q'_\infty(s) = \lim_{k\to\infty} Q_{2k}(s)Q_{2k-1}(s)\ldots Q_1(s)$ and*

$Q''_\infty(s) = \lim_{k\to\infty} Q_{2k-1}(s)Q_{2k-2}(s)\ldots Q_1(s)$

are defined for $0 < \Re[s] < 1$ and have the form

$$Q'_\infty(s) = \begin{pmatrix} -\zeta(s)+1 & -\zeta(s) \\ \zeta(s)-1 & \zeta(s) \end{pmatrix}, Q''_\infty(s) = \begin{pmatrix} \zeta(s)-1 & \zeta(s) \\ -\zeta(s)+1 & -\zeta(s) \end{pmatrix},$$

The proof is contained in that of Theorem 2 in [**1**].

Let $\Pi = \{s : 0 < \Re[s] < 1\}$ denote the critical strip and l the Hilbert space of sequences $x = (x_1, x_2, \ldots)$ of complex numbers such that $||x|| \overset{def}{=} \sum_{\nu=1}^{\infty} |x_\nu|^2 < \infty$. Now we consider the operator $A(s)$ introduced in the previous section that depends on s and acts on l.

Definition 3.4. Consider the direct product $\Omega = \Pi \times l \times l$ and let T be the transformation of the space Ω defined as follows: if $(s, x, y) \in \Omega$ $(s \in \Pi, x \in l, y \in l)$, then $T(s, x, y) = (s', x', y')$, where $s' = 1 - s, x' = A(s')y, y' = A(s)x$.

Theorem 3.5. *$\zeta(s)$ has a zero $s_* \in \Pi$ with $\Re[s_*] \neq 1/2$ if and only if there is a point $(s_*, e, \delta) \in \Omega$ $(e \in l, \delta \in l)$ such that $||e|| + ||\delta|| \neq 0$ and $(s'_*, e', \delta') = T(s_*, e, \delta)$ implies that $e' = -\delta, \delta' = -e$.*

PROOF. By the definition of T, we can assume without loss of generality that $1/2 < \Re[s_*] < 1$. If $\zeta(s_*) = 0$, then we can take δ to be the all-zero sequence and e to be an eigenvector of the operator $A(s_*)$ corresponding to the eigenvalue $\lambda = -1$, which exists by Theorem 2.3. □

COROLLARY 3.6. *If* $\zeta(s_*) = 0,\ s_* \in \Pi,\ \Re[s_*] \neq 1/2$, *then the map* T *has a fixed point* $(s_*, e, \delta) \in \Omega$ *with* $||e|| + ||\delta|| \neq 0 : T^2(s_*, e, \delta) = (s_*, e, \delta)$.

4. A new necessary condition for the validity of the Riemann hypothesis

Here we present a property of the Riemann function $\xi(s)$ which is obtained from the Riemann zeta-function $\zeta(s)$ via the equality $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, where $\Gamma(s)$ is Euler's gamma-function. Namely, the following theorem takes place.

THEOREM 4.1. *All the even derivatives of* $\xi(s)$ *at the point* $s = 1/2$ *are strictly positive.*

For the proof see ([**3**], Theorem 1).

REMARK 4.2. Because of the equality $\xi(s) = \xi(1-s)$ all the odd derivatives of $\xi(s)$ at the point $s = 1/2$ are equal to zero.

This theorem is very important in view of the next theorem which gives the necessary condition for the validity of the Riemann hypothesis.

THEOREM 4.3. *If at least one even derivative of the function* $\xi(s)$ *at the point* $s = 1/2$ *were not positive, then the Riemann hypothesis on the zeros of* $\zeta(s)$ *would be false: in this case there would exist a complex zero of* $\zeta(s)$ *that does not lie on the line* $\Re[s] = 1/2$.

For the proof see ([**3**], Theorem 2).

5. An asymptotic formula for the Taylor coefficients of the function $\xi(s)$

We represent here an asymptotic formula for the Taylor coefficients of the Riemann function $\xi(s)$ at the point $s = 1/2$ which is an entire function, and, by the well-known relation $\xi(s) = \xi(1-s)$, its Taylor series

$$\xi(s) = \sum_{r=0}^{\infty} \xi_r (s - \frac{1}{2})^{2r}$$

at $s = 1/2$ involves only even powers of $z = s - 1/2$. Finding an explicit asymptotic expression for the Taylor coefficients ξ_r of $\xi(s)$ as $r \to \infty$ is of interest both in itself and in relation to the Riemann hypothesis (see Theorem 4.3 above).

Theorem 5.1. *We have the following asymptotic expression as $r \to \infty$:*

$$\xi_r \sim \frac{2^{-(2r-2)}}{(2r)!} \left(\ln \frac{2r-2}{\pi} - \ln \ln \frac{2r-2}{\pi} + \beta \right)^{2r-2}$$
$$\times \exp \left(-(2r-2) \left(\ln \frac{2r-2}{\pi} \right)^{-1} e^{\beta} \right) (2r-2)^{1/4} 2r(2r-1)$$
$$\times \left(\ln \frac{2r-2}{\pi} \right)^{-1/4} \frac{\pi^{1/4}}{\sqrt{(r-1) \left(\frac{1}{(\ln \frac{2r-2}{\pi} - \ln \ln \frac{2r-2}{\pi})^2} + \frac{1}{\ln \frac{2r-2}{\pi}} \right)}},$$

where the function $\beta = \beta(r)$ satisfies the condition $\lim_{r\to\infty} \beta(r) = 0$, and e is the Euler's number.

For the proof see [**4**].

Corollary 5.2 (Theorem 3 in [**5**]). *The following equality holds*

$$\lim_{r\to\infty} \frac{(r!\xi_r)^{\frac{r+1}{r}}}{(r+1)!\xi_{r+1}} = e \ .$$

6. Refutation of the Riemann hypothesis about zeros for an arbitrary sharp approximation of zeta-function satisfying the same functional equation

The Riemann zeta-function $\zeta(s)$ is known to satisfy the following conditions:

(1) $\zeta(s)$ is an analytic function with a unique pole at $s = 1$, assuming real values for real values of s;
(2) all the real zeros of $\zeta(s)$ are attained at even negative values of s;
(3) $\zeta(s)$ satisfys the functional equation
$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),$$

where $\Gamma(s)$ is the gamma-function. We shall state a theorem implying that the Riemann hypothesis about zeros is violated for an arbitrarily sharp approximation $\zeta_\varepsilon(s)$ of $\zeta(s)$ satisfying conditions (1) – (3) (with $\zeta(s)$ replaced by $\zeta_\varepsilon(s)$). Namely, $\zeta_\varepsilon(s)$ has complex zeros not lying on the straight line $\Re[s] = 1/2$.

Theorem 6.1. *For any compact set K in the complex plane not containing the point $s = 1$ and for any $\varepsilon > 0$ there is a function $\zeta_\varepsilon(s)$ satisfying conditions above* (1) – (3) *such that* $\sup_{s\in K} |\zeta(s) - \zeta_\varepsilon(s)| \le \varepsilon$ *and $\zeta_\varepsilon(s)$ has four zeros $s_{1,2} = \sigma \pm it, s_{3,4} = 1 - \sigma \pm it$, where σ and t are real numbers such that $\sigma \ne 1/2$ and $t \ne 0$.*

The proof is contained in [**6**] and makes essential use of Theorem 5.1 and Corollary 5.2.

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